The Navier Stokes Equations with data in bmo^{-1}

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Abstract

The aim of this paper is to give a link between Leray mollified solutions of the 3D Navier-Stokes equations and Kato mild solutions for initial data in the adherence of test functions for the norm of bmo^{-1}.

Key words: Navier-Stokes equations; bmo^{-1} spaces; Lorentz spaces; Besov spaces

1 Introduction

The motion of a viscous incompressible fluid is modelled by the well-known Navier-Stokes equations (NS) which, in non-dimensional form, can be written as

$$u_t + (u \cdot \nabla)\, u + \nabla p = \Delta u$$
$$\nabla \cdot u = 0$$
$$u(0, \cdot) = u_0.$$

Here u and p are non-dimensional quantities corresponding to the velocity of the fluid and its pressure, u_0 is the initial data and for the sake of simplicity, the fluid is supposed to fill the whole space $\mathbb{R}^3$.

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There is a huge literature on this subject. Many theories were developed to tackle this problem and many results are already known. To summarize we can say that two specific theories on the existence of solutions to the 3 dimensional Navier-Stokes equations are known. The first one is due to Leray [1] and is based on energy estimates in the functional space $L^\infty\left(]0,T[,\ L^2(\mathbb{R}^3)\right) \cap L^2\left(]0,T[,\ \dot{H}^1(\mathbb{R}^3)\right)$. The second one was developed by Kato [2] who obtained *mild* solutions of (NS) in the space $\mathcal{C}\left(]0,T[,\ L^3(\mathbb{R}^3)\right)$. For a review of these results the reader is referred to [3] and [4].

One of the aims of this paper is to see whether it is possible to establish a link between these two different approaches. To do so, we will work with initial data in the space bmo^{-1}. This space was introduced the first time by Koch and Tataru in [5]. The norm of this space is defined by a kind of local L^2 estimates (see (5)), in the mind of weak solutions, but is also well suited for the study of fixed point mild solutions as remarked by Koch and Tataru [5].

We will first prove the following property of Leray mollified solutions (see Definition 2.2) and Kato mild ones (see Definition 2.1)

Theorem 1.1 *Let the initial data* $u_0 \in \overline{\mathcal{D}\left(\mathbb{R}^3\right)}^{bmo^{-1}}$. *There exists* $T > 0$ *such that the sequence* $\left(u_\epsilon\right)_{\epsilon>0}$ *of solution to the mollified Navier-Stokes equations constructed via the theory of Leray converges when* ϵ *tends to* 0 *to the mild solution given by Kato, for* $t \in]0,T[$.

We are also interested in the results of Cannone and Planchon [6] on existence result "à la Kato" for an initial data belonging to a Besov space. Our result can be stated as follows

Theorem 1.2 *Let* $T > 0$, $2 \le p < \infty$ *and* $3 < q \le \infty$ *such that*

$$\frac{2}{p} + \frac{3}{q} = 1,$$

we define the following spaces

$$u \in X_T(p,q) \Leftrightarrow \sup_{t\in]0,T[} t^{1/p}\left\|u(t)\right\|_{q,\infty} < \infty \quad \text{and} \quad t^{1/p}\left\|u(t)\right\|_{q,\infty} \xrightarrow[t\to 0]{} 0,$$

and $u \in Y_T(p,q) \Leftrightarrow$

$$\forall\ \lambda > 0,\ \left|\left\{t \in]0,T[,\ \left\|u(t)\right\|_{q,\infty} > \lambda\right\}\right| \le C(\lambda)\lambda^{-p} \le C_0\lambda^{-p}, \quad C(\lambda) \xrightarrow[\lambda\to 0]{} 0.$$

Then, if u *is a solution of the Navier-Stokes equations for* $(t,x) \in]0,T[\times\mathbb{R}^3$, *we have*

$$u \in X_T(p,q) \Leftrightarrow u \in Y_T(p,q).$$

In particular this theorem claims that every locally in time solutions to the Navier-Stokes equations which has properties similar to these of Cannone and Planchon belongs to this kind of Lorentz space (that we will call weak Lorentz space in the following, see section 2.3). We will also see that a solution u of (NS) which belongs to $Y_T(p,q)$ is originated by an initial data in bmo^{-1} for $p>2$ and that a stronger condition will be made on u to get the result for $p=2$.

The plan of the article is the following. In the first section, we will recall some preliminaries notions. Then, we will deal with the link between mild solutions and mollified ones for initial data in the adherence of test functions in bmo^{-1}. Finally, we will generalize the results of Cannone and Planchon.

2 Preliminaries

2.1 The mild and mollified Navier-Stokes equations

Using the Duhamel principle and the properties of the Leray projector $\mathbb{P}$ onto the divergence free vector field, the system (NS) is equivalent, in our framework, to the following fixed point problem

$$u = e^{t\Delta}u_0 - B(u,u), \tag{1}$$

where the bilinear operator B is defined by

$$B(u,v) = \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(u\otimes v)(\tau)\ d\tau. \tag{2}$$

and $e^{t\Delta}$ denotes as usual the heat kernel.

Definition 2.1 *A mild solution to the* (NS) *equations is a solution to* (1) *obtained via a fixed point procedure.*

The mollified solutions are constructed in a same way that mild solutions, but with a slightly different model. Indeed, instead of the term $u\otimes u$ involved into the (NS) equations, we will look to something smoother. Let $w\in\mathcal{D}(\mathbb{R}^3)$ with $w\geq 0$ and $\int_{\mathbb{R}^3} w(x)\ dx = 1$. Then, for $\epsilon>0$, the mollified Navier-Stokes equations, introduced by Leray, are given by the system (NS_ϵ)

$$u_t + ((u*w_\epsilon)\cdot\nabla)\,u + \nabla p = \Delta u$$

$$\nabla\cdot u = 0$$

$$u(0,\cdot) = u_0,$$

where $w_\epsilon = \dfrac{1}{\epsilon^3} w\left(\dfrac{x}{\epsilon}\right)$.

As for the (NS) system, (NS_ϵ) can be rewritten into the following fixed point problem

$$u_\epsilon = e^{t\Delta}u_0 - B_\epsilon(u_\epsilon, u_\epsilon), \tag{3}$$

where the bilinear operator B_ϵ is defined by

$$B_\epsilon(u,v) = \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla \cdot \left((u * w_\epsilon)\otimes u\right)(\tau)\, d\tau. \tag{4}$$

Definition 2.2 *The mollified solution to the (NS) equations is the sequence $(u_\epsilon)_{\epsilon>0}$ of solutions to the system (3) for every $\epsilon > 0$.*

2.2 The bmo^{-1} space

The space bmo^{-1} is defined as follows

Definition 2.3 *$bmo^{-1}(\mathbb{R}^n)$ is the space of tempered distributions f on $\mathbb{R}^n$ such that*

$$\sup_{0<t<T}\sup_{x_0\in\mathbb{R}^n}\frac{1}{t^{n/2}}\int_0^t\int_{|x-x_0|\le\sqrt{t}}\left|e^{s\Delta}f(x)\right|^2\, dsdx < \infty.$$

The norm on bmo^{-1} is defined by

$$\|f\|_{bmo^{-1}} = \left(\sup_{0<t<1}\sup_{x_0\in\mathbb{R}^n}\frac{1}{t^{n/2}}\int_0^t\int_{|x-x_0|\le\sqrt{t}}\left|e^{s\Delta}f(x)\right|^2\, dsdx\right)^{1/2}. \tag{5}$$

It was proved in [5] that the space bmo^{-1} consists the derivatives of functions in BMO, which is composed of locally integrable functions f such that

$$\sup_{B\in\mathcal{B}}\frac{1}{|B|}\int_B |f(x) - m_B f|\, dx < \infty,$$

with $\mathcal{B}$ being the collection of all balls $B(x_0,r),\ x_0\in\mathbb{R}^3,\ r>0$ and $m_B f = |B|^{-1}\int_B |f(x)|\, dx$.

For more details on the space bmo^{-1}, the reader is referred to [3, chap. 16] and [5]. We also need to introduce the space $\mathcal{E}_T$

Definition 2.4 *Let $T \in]0, +\infty]$. We will say that f is in the space $\mathcal{E}_T$ if*

$$\begin{cases} \sup\limits_{0<t<T} \sqrt{t}\|f(\cdot,t)\|_{L^\infty} < \infty, \\ \\ \sup\limits_{x_0\in\mathbb{R}^n} \sup\limits_{0<t<T} \left(t^{-n/2} \int_0^t \int_{|x-x_0|\leq\sqrt{t}} |f(x)|^2 \ dsdx \right)^{1/2}. \end{cases}$$

The norm on $\mathcal{E}_T$ is given by

$$\|f\|_{\mathcal{E}_T} = \sup_{0<t<T} \sqrt{t}\|f\|_{L^\infty} + \sup_{x_0\in\mathbb{R}^n} \sup_{0<t<T} \left(t^{-n/2} \int_0^t \int_{|x-x_0|\leq\sqrt{t}} |f(x)|^2 \ dsdx \right)^{1/2}.$$

2.3 The weak Lorentz spaces

We will first recall, the definition of the Lorentz space $L^{p,\infty}(\mathbb{R}^n)$, for $1 < p < \infty$.

Definition 2.5 *Let $1 < p < \infty$. A function $f \in L^1_{loc}(\mathbb{R}^n)$ is in the Lorentz space $L^{p,\infty}(\mathbb{R}^n)$ if and only if*

$$\forall \ \lambda > 0, \quad |\{x \in \mathbb{R}^n, \ |f(x)| > \lambda\}| \leq \frac{C}{\lambda^p}.$$

For $p = \infty$, we have $L^{\infty,\infty}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

Remark that we are not using the usual definition of the Lorentz spaces as in [7], but the definition of the Marcinkiewicz spaces $L^{p,*}$ which are equivalent to the Lorentz spaces $L^{p,q}$ with $q = \infty$ and more adapted to solve our problem.

We will now define a new class of Lorentz spaces

Definition 2.6 *Let $1 < p < \infty$, the weak Lorentz space $\widetilde{L}^{p,\infty}(]0,T[)$ is the adherence of functions in $L^\infty(]0,T[)$ for the norm in the Lorentz space $L^{p,\infty}(]0,T[)$, with the usual notations, we have*

$$\widetilde{L}^{p,\infty}(]0,T[) = \overline{L^\infty(]0,T[)}^{L^{p,\infty}}.$$

We will give some equivalent definitions of these spaces in the particular case we are looking. Indeed, in the following pages we will look, for a $T > 0$ and particular p and q, to solutions to (NS) in the space $\widetilde{L}^{p,\infty}\left(]0,T[, \ L^{q,\infty}(\mathbb{R}^3)\right)$ of functions f measurable on $]0,T[\times\mathbb{R}^3$ such that $\|f(\cdot)\|_{q,\infty} \in \widetilde{L}^{p,\infty}$.

Proposition 2.7 *Let $T > 0$, $1 < p < \infty$ and $1 < q \leq \infty$. The following properties are equivalent*

(1) $f \in \widetilde{L}^{p,\infty}\Big(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\Big)$.

(2) for all $\lambda > 0$, there exists a constant $C(\lambda)$, depending on λ, such that

$$C(\lambda) \xrightarrow[\lambda\to\infty]{} 0$$

and

$$|\{\|f(t)\|_{q,\infty} > \lambda\}| < \frac{C(\lambda)}{\lambda^p}$$

(3) for all $\epsilon > 0$, there exists $f_1 \in L^\infty(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))$ and $f_2 \in L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))$ such that

$$\|f_2\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \leq \epsilon \quad \textit{and} \quad f = f_1 + f_2.$$

Proof :

- 1) $\Rightarrow$ 2) : There exists a sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $L^\infty(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))$ such that

$$\|f_n - f\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \xrightarrow[n\to\infty]{} 0.$$

Let $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\|f_n - f\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \leq \left(\frac{\epsilon}{2^{p+1}}\right)^{1/p}.$$

So that, for every $\lambda > 0$, we have

$$\lambda^p\,|\{\|(f_{n_0} - f)(t)\|_{q,\infty} > \lambda\}| \leq \frac{\epsilon}{2^{p+1}}.$$

Then, we can write

$$|\{\|f(t)\|_{q,\infty} > \lambda\}| \leq \left|\left\{\|(f_{n_0} - f)(t)\|_{q,\infty} > \frac{\lambda}{2}\right\}\right| + \left|\left\{\|f_{n_0}(t)\|_{q,\infty} > \frac{\lambda}{2}\right\}\right|$$

$$\leq 2^p\,\frac{(\epsilon/2^{p+1} + C_1(\lambda))}{\lambda^p}.$$

For λ' great enough, we have $C_1(\lambda) = 0$ for all $\lambda \geq \lambda'$. So, we obtained

$$\forall\ \lambda \geq \lambda',\ \ \lambda^p\,|\{\|f(t)\|_{q,\infty} > \lambda\}| \leq \epsilon.$$

And so, we have

$$|\{\|f(t)\|_{q,\infty} > \lambda\}| < \frac{C(\lambda)}{\lambda^p},$$

with $C(\lambda)$ tends to zero when λ goes to inifinity.

- 2) $\Rightarrow$ 3) : Let $\epsilon > 0$ and $\lambda > 0$ sufficiently large to ensure $C(\lambda^{'}) \leq \epsilon$, for $\lambda^{'} \geq \lambda$. We denote $f_1 = f\mathbb{1}_{\{\|f\|_{q,\infty} \leq \lambda\}}$ and $f_2 = f\mathbb{1}_{\{\|f\|_{q,\infty} > \lambda\}}$. So that,

$$f = f_1 + f_2,\ f_1 \in L^{\infty}\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right) \text{ and } \|f_2\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \leq \epsilon.$$

- 3) $\Rightarrow$ 1) : By 3), we have for each $n \in \mathbb{N} \setminus \{0\}$ that there exists $f_1^n \in L^{\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))$ and $f_2^n \in L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))$ such that

$$f = f_1^n + f_2^n \text{ and } \|f_2^n\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \leq \frac{1}{n}.$$

  We denote by $(f^n)_{n \in \mathbb{N}\setminus\{0\}}$ the sequence of functions in $L^{\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))$ defined by

$$f^n = f_2^n,\ \forall\, n \in \mathbb{N} \setminus \{0\}.$$

  So, we get

$$\|f - f^n\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} = \|f_2^n\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \leq \frac{1}{n}.$$

  So that,

$$\|f - f^n\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \xrightarrow[n \to \infty]{} 0.$$

□

In the following results, we will stress the importance of this space. The first property of these spaces is that if we have two solutions of the Navier-Stokes equations belonging to the space $\widetilde{L}^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))$ (with a suitable scaling on p and q), which are equal at a certain time $\theta > 0$, then the definition of these spaces allow us to claim that the two solutions are equal on $]\theta, T[$. This result is in fact not known for arbitrary functions in $L^p(]0,T[,\ L^q(\mathbb{R}^3))$ or $L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))$. The result states as follows

Proposition 2.8 *Let $T > 0$ and u and v be two solutions to the (NS) equations belonging to the space $\widetilde{L}^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))$ with $2 < p < \infty$ and $3 < q < \infty$ such that*

$$\frac{2}{p} + \frac{3}{q} = 1.$$

Assume that there exists $\theta \in]0,T[$ such that $u(\theta) = v\theta$. Then, u and v are equal for $t \in]\theta, T]$.

Proof : Let $t_{0>0}$ and $\lambda > 0$. We can decompose u and v as follows

$$u = u_\lambda + u_\lambda^{'} \text{ and } v = v_\lambda + v_\lambda^{'},$$

where $u_\lambda = u\mathbb{1}_{\{t/\|u(t)\|_{q,\infty} > \lambda\}}$ and $v_\lambda = v\mathbb{1}_{\{t/\|v(t)\|_{q,\infty} > \lambda\}}$. By construction and the definition of the weak Lorentz spaces we have

$$\|u_\lambda^{'}\|_{L^{\infty}(]\theta,\theta+t_0[,\ L^{q,\infty})} \leq \lambda,\ \|u_\lambda\|_{L^{p,\infty}(]\theta,\theta+t_0[,\ L^{q,\infty})} \leq C(\lambda), \qquad (6)$$

and the same estimates hold true for v'_λ and v_λ. Then, we compute

$$\|u-v\|_{L^{p,\infty}(]\theta,\theta+t_0[,\ L^{q,\infty})} \leq \|B(u-v,u)\|_{L^{p,\infty}(]\theta,\theta+t_0[,\ L^{q,\infty})} + \|B(v,u-v)\|_{L^{p,\infty}(]\theta,\theta+t_0[,\ L^{q,\infty})}$$

$$\leq C_0\|u-v\|_{L^{p,\infty}(]\theta,\theta+t_0[,\ L^{q,\infty})} \times \Bigg[\|u_\lambda\|_{L^{p,\infty}(]\theta,\theta+t_0[,\ L^{q,\infty})} + \|v_\lambda\|_{L^{p,\infty}(]\theta,\theta+t_0[,\ L^{q,\infty})}$$

$$+t_0^{1/p}\|u'_\lambda\|_{L^{\infty}(]\theta,\theta+t_0[,\ L^{q,\infty})} + t_0^{1/p}\|v'_\lambda\|_{L^{\infty}(]\theta,\theta+t_0[,\ L^{q,\infty})}\Bigg].$$

Here we used Lemma 6.1 (we can note that the condition $p > 2$ is imposed here). Then, using (6) we get

$$\|u-v\|_{L^{p,\infty}(]\theta,\theta+t_0[,\ L^{q,\infty})} \leq C_0\left[2C(\lambda) + 2t_0^{1/p}\lambda\right].$$

We choose $\lambda > 0$ great enough to guarantee that $2C_0C(\lambda) < 1/4$ and $t_0 > 0$ small enough to that ensure $C_0t_0^{1/p} < 1/4$. Thus we get that there exists $\delta < 1$ such that

$$\|u-v\|_{L^{p,\infty}(]\theta,\theta+t_0[,\ L^{q,\infty})} \leq \delta\|u-v\|_{L^{p,\infty}(]\theta,\theta+t_0[,\ L^{q,\infty})}.$$

So, u and v are equal for $t \in]\theta, \theta+t_0[$, with t_0 independent of θ. Thus, by repeating the argument a finite number of time we conclude the proof (because there exists $n \in \mathbb{N}$ such that $T \leq \theta + nt_0$).

□

Then, in the following lemma we will give a sufficient condition for a function to belong to the space $\widetilde{L}^{p,\infty}\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right)$.

Lemma 2.9 *Let $T > 0$, $1 \leq p, q \leq \infty$. If u satisfies*

$$\begin{cases} \sup_{t\in]0,T[} t^{1/p}\|u(t)\|_{q,\infty} < \infty \\ \\ t^{1/p}\|u(t)\|_{q,\infty} \xrightarrow[t\to 0]{} 0 \end{cases},$$

then the function u belongs to the space $\widetilde{L}^{p,\infty}\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right)$.

Proof : For every $t \in]0,T[$,

$$\|u(t)\|_{q,\infty} \leq \frac{C}{t^{1/p}},$$

so that $u \in L^{p,\infty}\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right)$. Let $\epsilon > 0$. The convergence hypothesis implies that there exists $t_0 > 0$ such that, for all $t < t_0$

$$\|u(t)\|_{q,\infty} \leq \frac{\epsilon}{t^{1/p}}.$$

We define

$$u_1 = u\mathbb{1}_{\{t \geq t_0\}} \quad \text{and} \quad u_2 = u\mathbb{1}_{\{t<t_0\}}.$$

The function u_1 satisfies

$$\|u_1(t)\|_{q,\infty} \leq \frac{C}{t_0^{1/p}}.$$

Then,

$$u_1 \in L^{\infty}\left(]0,T[,\ L^{q,\infty}\right) \quad \text{and} \quad \|u_2\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty})},$$

and Proposition 2.7 implies $u \in \widetilde{L}^{p,\infty}\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right)$.

□

Remark 2.10 *In fact, as we will see in section 4, such a condition was already introduced by Cannone and Planchon in [4,6]. Moreover, the reverse result of Lemma 2.9 will be proved in section 4 for the case $p > 2$ and in section 5 for the limit case.*

2.4 The weak Besov spaces

We introduce two new classes of spaces

Definition 2.11 *Let $\alpha > 0$, $1 < q < \infty$. We denote by $\widetilde{B}_q^{-\alpha,\infty}$ the adherence of functions in L^q for the norm of $B_q^{-\alpha,\infty}$ and by $\widetilde{B}_{q,\infty}^{-\alpha,\infty}$ the adherence of functions in $L^{q,\infty}$ for the norm of $B_{q,\infty}^{-\alpha,\infty} = B_{L^{q,\infty}}^{-\alpha,\infty}$, that is to say*

$$\widetilde{B}_q^{-\alpha,\infty} = \overline{L^q}^{B_q^{-\alpha,\infty}} \quad \text{and} \quad \widetilde{B}_{q,\infty}^{-\alpha,\infty} = \overline{L^{q,\infty}}^{B_{q,\infty}^{-\alpha,\infty}}.$$

Functions in such spaces have the following properties

Lemma 2.12 *Let $\alpha > 0$ and $1 < q < \infty$. If $u \in \widetilde{B}_{q,\infty}^{-\alpha,\infty}$, then*

$$\begin{cases} \sup\limits_{0<t<1} t^{\alpha/2}\left\|e^{t\Delta}u\right\|_{q,\infty} < \infty \\ \\ t^{\alpha/2}\left\|e^{t\Delta}u\right\|_{q,\infty} \xrightarrow[t\to 0]{} 0. \end{cases}$$

Remark 2.13 *The reverse result is also true, but we don't need this fact. Thus, we will skip the proof.*

Proof of Lemma 2.12 We obviously have $u \in B^{-\alpha,\infty}_{q,\infty}$, so that

$$\sup_{t>0} t^{\alpha/2} \left\| e^{t\Delta} u \right\|_{q,\infty} < \infty.$$

As $u \in \widetilde{B}^{-\alpha,\infty}_{q,\infty}$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of functions in $L^{q,\infty}$ such that

$$\|u_n - u\|_{B^{-\alpha,\infty}_{q,\infty}} \xrightarrow[t\to 0]{} 0.$$

So, there exists $N > 0$ such that for every $n > N$,

$$\sup_{t>0} t^{\alpha/2} \left\| e^{t\Delta}(u_n - u) \right\|_{q,\infty} < \frac{\epsilon}{2}.$$

Then, for all $t > 0$ we have

$$t^{\alpha/2} \left(\|e^{t\Delta} u\|_{q,\infty} - \|e^{t\Delta} u_{N+1}\|_{q,\infty} \right) < \frac{\epsilon}{2}.$$

So,

$$t^{\alpha/2} \|e^{t\Delta} u\|_{q,\infty} < \frac{\epsilon}{2} + t^{\alpha/2} \|e^{t\Delta} u_{N+1}\|_{q,\infty} \leq \frac{\epsilon}{2} + C t^{\alpha/2} \|u\|_{q,\infty}.$$

Let $t_0 = \epsilon^{2/\alpha} \left(2C \|u\|_{q,\infty} \right)^{-2/\alpha}$, for every $t < t_0$ we have

$$t^{\alpha/2} \|e^{t\Delta} u\|_{q,\infty} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Which concludes the proof.

□

Remark 2.14 *Lemma* 2.12 *can also be proved in the case of* $u \in \widetilde{B}^{-\alpha,\infty}_{q}$, *and in that case we get*

$$\begin{cases} \sup_{t>0} t^{\alpha/2} \left\| e^{t\Delta} u \right\|_{q} < \infty \\ \\ t^{\alpha/2} \left\| e^{t\Delta} u \right\|_{q} \xrightarrow[t\to 0]{} 0. \end{cases}$$

3 Equivalence between mollified and mild solutions

In this section, we will prove that the solutions to the mollified Navier-Stokes equations and the mild solution are related each others in a certain sense. In the following, the constant $C > 0$ will refer to the constant of continuity of the bilinear term

$$B(u, v) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes v)(s)\, ds.$$

The result of Koch and Tataru (see [3, Chap. 16] and [5]) gives that there exists $C > 0$ such that

$$\forall\, u, v \in \mathcal{E}_T, \quad \|B(u,v)\|_{\mathcal{E}_T} \le C\, \|u\|_{\mathcal{E}_T}\, \|v\|_{\mathcal{E}_T}\,.$$

We will first prove an existence theorem for mild solutions to the (NS) equations initiated with an initial data in the closure of test functions for the norm of bmo^{-1}.

Theorem 3.1 *Let $u_0 \in \overline{\mathcal{D}(\mathbb{R}^3)}^{bmo^{-1}}$ such that $\nabla \cdot u_0 = 0$ and $T > 0$ small enough to ensure*

$$\left\| e^{t\Delta} u_0 \right\|_{\mathcal{E}_T} < \frac{1}{4C}.$$

Then, there exists a solution $u \in \overline{\mathcal{D}(]0,T] \times \mathbb{R}^3)}^{\mathcal{E}_T}$ of the Navier-Stokes equations.

Before giving the proof of the previous theorem, we first need to introduce this technical lemma.

Lemma 3.2 *Let $v \in \mathcal{D}\left(]0,T] \times \mathbb{R}^3\right)$ and $R > 0$ such that*

$$\text{supp } v \subset]0,T] \times \overline{B}(0,R),$$

where supp *denotes the supports of the function v and $\overline{B}(0,R)$ the closed ball of radius R centered at 0. Then, for $t \in]0,T]$ and $y \in \mathbb{R}^3$ such that $|y| \ge \lambda R$ for $\lambda > 1$, we have for some constant $C > 0$,*

$$|B(v,v)(t,y)| \le \frac{C}{(\lambda R)^4} \|v\|^2_{L^2([0,T]\times\mathbb{R}^3)}\,.$$

In particular, for $\varphi \in \mathcal{D}(\mathbb{R}^3)$ with support in $\overline{B}(0,R+1)$ such that $\varphi(x) = 1$ for $x \in B(0,R)$ and $\|\varphi\|_\infty = 1$, we have

$$\|(1-\varphi)B(v,v)\|_{\mathcal{E}_T} \le C\sqrt{T}\frac{1}{R^4}\|v\|^2_{L^2([0,T]\times\mathbb{R}^3)}.$$

Proof : Using the properties of the Oseen kernel (see [3, ch. 11]) we have

$$|B(v,v)(t,y)| \le C \int_0^t \int_{\mathbb{R}^3} \frac{1}{(t-s)^2 + (y-z)^4} |v(s,z)|^2 \; dz ds$$

$$\le \frac{C}{|y|^4}\|v\|^2_{L^2([0,T]\times\mathbb{R}^3)},$$

here we used that $|z| \leq R$ and $|y| \geq \lambda R$. The second part of the lemma is a direct consequence of the previous inequality and the definition of the norm in $\mathcal{E}_T$. Thus, the proof is completed.

□

Proof of Theorem 3.1 : We define the sequence of functions $(v_n)_{n\in\mathbb{N}}$ by

$$\begin{cases} v_n = v_0 - B\left(v_{n-1}, v_{n-1}\right), \quad \text{for } \; n \geq 1 \\ \\ v_0 = e^{t\Delta} u_0 \end{cases}$$

First, we want to prove that for all $n \in \mathbb{N}$, the functions v_n belong to the space $\overline{\mathcal{D}\left(]0,T] \times \mathbb{R}^3\right)}^{\mathcal{E}_T}$. To do that, we will use an induction reasoning on n.

- Let us start to prove the result for $n = 0$. By assumption, $u_0 \in \overline{\mathcal{D}(\mathbb{R}^3)}^{bmo^{-1}}$, so that there exists a sequence $u_0^m \in \mathcal{D}\left(\mathbb{R}^3\right)$ suth that

  $$\|u_0 - u_0^m\|_{bmo^{-1}} \xrightarrow[m\to\infty]{} 0.$$

  Recalling that
  $$\left\|e^{t\Delta} f\right\|_{\mathcal{E}_T} \leq C\|f\|_{bmo^{-1}},$$
  this result comes from the following estimate

  $$\|e^{t\Delta} f\|_{L^\infty} \leq C\frac{1}{\sqrt{t}} \sup_{x_0\in\mathbb{R}^n} \left(t^{-n/2} \int_0^{t/2} \int_{|x-x_0|\leq\sqrt{t/2}} |e^{s\Delta} f(x)|^2 \; dsdx\right)^{1/2}.$$

  The previous estimate is based on the following equality

  $$e^{t\Delta} f = \frac{2}{t} \int_0^{t/2} e^{(t-s)\Delta} e^{s\Delta} f \; ds.$$

  For the details of the proof, the reader is referred to [3, Chap. 16]. So, we obtain
  $$\left\|e^{t\Delta} u_0 - e^{t\Delta} u_0^m\right\|_{\mathcal{E}_T} \xrightarrow[m\to\infty]{} 0.$$
  So that, $v_0 \in \overline{\mathcal{D}\left(]0,T] \times \mathbb{R}^3\right)}^{\mathcal{E}_T}$.
- Let us assume that $v_{n-1} \in \overline{\mathcal{D}\left(]0,T] \times \mathbb{R}^3\right)}^{\mathcal{E}_T}$. To conclude the induction, it remains to prove that $v_n \in \overline{\mathcal{D}\left(]0,T] \times \mathbb{R}^3\right)}^{\mathcal{E}_T}$. First, recall that v_n is defined by
  $$v_n = e^{t\Delta} u_0 - B\left(v_{n-1}, v_{n-1}\right).$$
  This is sufficient to prove that $B\left(v_{n-1}, v_{n-1}\right) \in \overline{\mathcal{D}\left(]0,T] \times \mathbb{R}^3\right)}^{\mathcal{E}_T}$.

By the induction hypothesis, there exists a sequence $v_{n-1}^m \in \mathcal{D}\left(]0,T] \times \mathbb{R}^3\right)$ such that

$$\left\|v_{n-1} - v_{n-1}^m\right\|_{\mathcal{E}_T} \xrightarrow[m\to\infty]{} 0.$$

As v_{n-1}^m is compactly supported in time and space, $B(v_{n-1}^m, v_{n-1}^m)$ belongs to $\mathcal{C}^\infty\left(]0,T] \times \mathbb{R}^3\right)$ and is with compact support in time. Let $(\varphi_m)_{m\in\mathbb{N}}$ a sequence of functions in $\mathcal{D}\left(\mathbb{R}^3\right)$ such that for each $m \in \mathbb{N}$ $\|\varphi_m\|_\infty = 1$. Assume also that φ_m is with support in $\overline{B}(0, \lambda_m R_m + 1)$ and that $\varphi_m(x) = 1$ for $x \in B(0, \lambda_m R_m)$, where $R_m > 0$ is such that supp $v_{n-1}^m \subset]0,T] \times B(0, R_m)$ and $\lambda_m > m \left\|v_{n-1}^m\right\|_{L^2([0,T]\times\mathbb{R}^3)}^{1/2}$. For every $m \in \mathbb{N}$, we define

$$B^m\left(v_{n-1}, v_{n-1}\right) = \varphi_m \times B\left(v_{n-1}^m, v_{n-1}^m\right).$$

By construction, the functions $B^m\left(v_{n-1}, v_{n-1}\right)$ are in the space $\mathcal{D}\left(]0,T] \times \mathbb{R}^3\right)$. Then, we compute

$$\left\|B\left(v_{n-1}, v_{n-1}\right) - B^m\left(v_{n-1}, v_{n-1}\right)\right\|_{\mathcal{E}_T} \leq \left\|B\left(v_{n-1}, v_{n-1}\right) - B\left(v_{n-1}^m, v_{n-1}^m\right)\right\|_{\mathcal{E}_T}$$

$$+ \left\|\left(1 - \varphi_m\right) B\left(v_{n-1}^m, v_{n-1}^m\right)\right\|_{\mathcal{E}_T}.$$

Using the continuity in the space $\mathcal{E}_T$ of the bilinear form B (see [3, ch. 16] for the proof) and Lemma 3.2 we get

$$\left\|B\left(v_{n-1}, v_{n-1}\right) - B^m\left(v_{n-1}, v_{n-1}\right)\right\|_{\mathcal{E}_T}$$

$$\leq C\left\|v_{n-1} - v_{n-1}^m\right\|_{\mathcal{E}_T}\left[\left\|v_{n-1}\right\|_{\mathcal{E}_T} + \left\|v_{n-1}^m\right\|_{\mathcal{E}_T}\right] + C\sqrt{T}\frac{1}{\lambda_m R_m^4}\left\|v_{n-1}^m\right\|_{L^2([0,T]\times\mathbb{R}^3)}^2$$

$$\leq C\left\|v_{n-1} - v_{n-1}^m\right\|_{\mathcal{E}_T}\left[\left\|v_{n-1}\right\|_{\mathcal{E}_T} + \left\|v_{n-1}^m\right\|_{\mathcal{E}_T}\right] + C\sqrt{T}\frac{1}{m R_m^4}.$$

Thus, passing to the limit when m tends to $+\infty$, we get

$$\left\|B\left(v_{n-1}, v_{n-1}\right) - B^m\left(v_{n-1}, v_{n-1}\right)\right\|_{\mathcal{E}_T} \xrightarrow[m\to\infty]{} 0,$$

which concludes the induction.

We have just proven that

$$\forall\, n \in \mathbb{N}, \quad v_n \in \overline{\mathcal{D}\left(]0,T] \times \mathbb{R}^3\right)}^{\mathcal{E}_T}.$$

Then, we will prove by induction that for every $n \in \mathbb{N}$,

$$\left\|v_n\right\|_{\mathcal{E}_T} \leq 2\left\|e^{t\Delta}u_0\right\|_{\mathcal{E}_T}.$$

This is obviously clear for v_0. Let assume that this is true for a $n \in \mathbb{N}$. We compute

$$\|v_{n+1}\|_{\mathcal{E}_T} \leq \|v_0\|_{\mathcal{E}_T} + \|B(v_n, v_n)\|_{\mathcal{E}_T}$$

$$\leq \left\|e^{t\Delta}u_0\right\|_{\mathcal{E}_T} + 4C\left\|e^{t\Delta}u_0\right\|_{\mathcal{E}_T}^2 < 2\left\|e^{t\Delta}u_0\right\|_{\mathcal{E}_T}.$$

So, for every $n \in \mathbb{N}$, $\|v_n\|_{\mathcal{E}_T}$ is in the ball centered at 0, of radius $2\left\|e^{t\Delta}u_0\right\|_{\mathcal{E}_T}$. We have the following estimate

$$\|v_n - v_{n-1}\|_{\mathcal{E}_T} \leq \|B(v_{n-1}, v_{n-1}) - B(v_{n-2}, v_{n-2})\|_{\mathcal{E}_T}$$

$$\leq C\|v_{n-1} - v_{n-2}\|_{\mathcal{E}_T}\left[\|v_{n-1}\|_{\mathcal{E}_T} + \|v_{n-2}\|_{\mathcal{E}_T}\right]$$

$$\leq 4C\left\|e^{t\Delta}u_0\right\|_{\mathcal{E}_T}\|v_{n-1} - v_{n-2}\|_{\mathcal{E}_T}$$

Now, as $4C\left\|e^{t\Delta}u_0\right\|_{\mathcal{E}_T} < 1$, the Picard contraction principle concludes the proof.

□

We are now regarding the existence of mollified solutions to (NS) for an initial data in the same space that for the mild solution constructed in Theorem 3.1.

Theorem 3.3 *Let $u_0 \in \overline{\mathcal{D}(\mathbb{R}^3)}^{bmo^{-1}}$ such that $\nabla \cdot u_0 = 0$ and $T > 0$ small enough to ensure*

$$\left\|e^{t\Delta}u_0\right\|_{\mathcal{E}_T} < \frac{1}{4C}.$$

Then, for $\epsilon > 0$,there exists a solution $u_\epsilon \in \overline{\mathcal{D}(]0,T] \times \mathbb{R}^3)}^{\mathcal{E}_T}$ of the mollified Navier-Stokes equations.

Proof : The proof is similar to the previous one. We just need to estimate the $\mathcal{E}_T$ norm of the term $f * w_\epsilon$. We have

$$\|f * w_\epsilon\|_{L^\infty} \leq \|w_\epsilon\|_{L^1}\|f\|_{L^\infty},$$

and

$$t^{-3/2}\int_0^t \int_{|x-x_0|<\sqrt{t}} |f * w_\epsilon|^2 \, dsdx \leq Ct^{-3/2} \sup_{x_0 \in \mathbb{R}^3} \int_0^t \int_{|x-x_0|<\sqrt{t}} \|w_\epsilon\|_{L^1}^2 |f|^2 \, dsdx.$$

So that

$$\|f * w_\epsilon\|_{\mathcal{E}_T} \leq C\|f\|_{\mathcal{E}_T}.$$

□

Remark 3.4 *By classical arguments, we can prove that the mollified solutions can be extended globally in time.*

Now that local existence results are stated and proved for mild and mollified solutions to (NS) for initial data in the closure of test functions in the norm of the space bmo^{-1}, we are able to prove the following property relating these solutions each others.

Theorem 3.5 *Let* $u_0 \in \overline{\mathcal{D}(\mathbb{R}^3)}^{bmo^{-1}}$ *and* $T > 0$ *given by Theorem* 3.1*. Then the solutions* u_ϵ *to the mollified Navier-Stokes equations, obtained by Theorem* 3.3*, converge strongly for* $(t, x) \in]0, T[\times\mathbb{R}^3$ *to the mild solution* u *obtained by the Picard contraction principle, of Theorem* 3.1 *when* ϵ *tends to* 0*.*

Proof : We compute

$$u - u_\epsilon = B(u, u - u_\epsilon) + B\left((u - u_\epsilon) * w_\epsilon, u_\epsilon\right) + B\left(u - (u * w_\epsilon), u_\epsilon\right),$$

which gives

$$\|u - u_\epsilon\|_{\mathcal{E}_T} \leq \frac{2C\left\|e^{t\Delta}u_0\right\|_{\mathcal{E}_T}}{1 - 4C\left\|e^{t\Delta}u_0\right\|_{\mathcal{E}_T}}\|u - u * w_\epsilon\|_{\mathcal{E}_T}.$$

To conclude the proof, it remains to prove that

$$\|u - u * w_\epsilon\|_{\mathcal{E}_T} \xrightarrow[\epsilon\to 0]{} 0,$$

which is a consequence of the fact that $u \in \overline{\mathcal{D}([0, T] \times \mathbb{R}^3)}^{\mathcal{E}_T}$.

□

4 Cannone and Planchon solutions

In this section, we want to give some properties of solutions of (NS) equations belonging to the weak Lorentz space in time. We will see that we have a generalization of the results of Cannone and Planchon [4,6] and also that for our class of solutions the initial data is embedded in bmo^{-1}.

We first recall the result of Cannone and Planchon for initial data in the Besov spaces.

Theorem 4.1 *Let* $3 < q < \infty$, $\alpha = 1-(3/q)$, $u_0 \in \widetilde{B}_q^{-\alpha,\infty}$, *such that* $\nabla \cdot u_0 = 0$ *and*

$$\sup_{0<t<T} t^{\alpha/2} \|e^{t\Delta} u_0\|_q < \delta_q,$$

then, there exists $T > 0$ *and a solution* u *of* (NS) *such that*

$$\begin{cases} \sup_{t\in]0,T[} t^{\alpha/2} \|u(t)\|_q < \infty, \\ \\ t^{\alpha/2} \|u(t)\|_q \xrightarrow[t\to 0]{} 0. \end{cases}$$

For the proof of the previous theorem, the reader is refered to [4,6]. Then, we can easily prove that these solutions are embedded in our class of solution.

Theorem 4.2 *Let* $3 < q < \infty$ *and* u *the solution given by Theorem* 4.1, *then* $u \in \widetilde{L}^{p,\infty}\left(]0, T[,\ L^{q,\infty}(\mathbb{R}^3)\right)$, *with*

$$\frac{2}{p} + \frac{3}{q} = 1.$$

Proof : We have

$$\sup_{t\in]0,T[} t^{1/p} \left\| u(t,x) \right\|_q < \infty, \quad \lim_{t\to 0} t^{1/p} \|u(t)\|_q = 0.$$

The proof comes from Lemma 2.9, using that $L^q(\mathbb{R}^3) \subset L^{q,\infty}(\mathbb{R}^3)$.

□

We will now see under which condition on the initial data, we can obtain solutions to the (NS) equations in our class.

Theorem 4.3 *Let* $3 < q \leq \infty$, $2 \leq p < \infty$, *such that*

$$\frac{2}{p} + \frac{3}{q} = 1,$$

and $u_0 \in \widetilde{B}_{L^{q,\infty}}^{-\alpha,\infty}$ *with* $\alpha = 1 - (3/q)$, *such that* $\nabla \cdot u_0 = 0$. *Then, there exists* $T > 0$ *and a solution* u *of the Navier-Stokes equations in the space* $\widetilde{L}^{p,\infty}\left(]0, T[,\ L^{q,\infty}(\mathbb{R}^3)\right)$.

Proof :

We define the sequence of functions $(v_n)_{n\in\mathbb{N}}$ by

$$\begin{cases} v_n = e^{t\Delta}v_0 - B\left(v_{n-1}, v_{n-1}\right), \quad \text{for} \quad n \geq 1 \\ \\ v_0 = e^{t\Delta}u_0 \end{cases}$$

First, we want to prove that for all $n \in \mathbb{N}$, the function v_n belongs to the space $\widetilde{L}^{p,\infty}\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right)$. Then, we will use an induction argument on n.

- Let us start to prove the result for $n = 0$. By assumption, $u_0 \in \widetilde{B}^{-\alpha,\infty}_{L^{q,\infty}}$. By Lemma 2.12 we have

$$\begin{cases} \sup\limits_{t\in]0,T[} t^{1/p}\left\|v_0(t)\right\|_{q,\infty} < \sup\limits_{t>0} t^{1/p}\left\|e^{t\Delta}u_0\right\|_{q,\infty} < \infty \\ \\ t^{1/p}\left\|v_0(t)\right\|_{q,\infty} = t^{1/p}\left\|e^{t\Delta}u_0\right\|_{q,\infty} \xrightarrow[t\to 0]{} 0. \end{cases}$$

  So, Lemma 2.9 implies $v_0 \in \widetilde{L}^{p,\infty}\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right)$.
- Let us assume that $v_{n-1} \in \widetilde{L}^{p,\infty}\Big(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\Big)$.

  To conclude the induction, it remains to prove that $v_n \in \widetilde{L}^{p,\infty}\Big(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\Big)$. First, recall that v_n is defined by

$$v_n = e^{t\Delta}u_0 - B\left(v_{n-1}, v_{n-1}\right).$$

  This is sufficient to prove that $B\left(v_{n-1}, v_{n-1}\right) \in \widetilde{L}^{p,\infty}\Big(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\Big)$.

  Let $\epsilon > 0$. As $v_{n-1} \in \widetilde{L}^{p,\infty}\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right)$, there exist two functions $v^1_{n-1} \in L^{\infty}\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right)$ and $v^2_{n-1} \in L^{p,\infty}\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right)$ such that

$$\|v^2_{n-1}\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \leq \epsilon \quad \text{and} \quad v_{n-1} = v^1_{n-1} + v^2_{n-1}.$$

  We have

$$B(v_{n-1}, v_{n-1}) = \mathcal{U} + \mathcal{V},$$

  with

$$\mathcal{U} = B(v^1_{n-1}, v^1_{n-1}) + B(v^1_{n-1}, v^2_{n-1}) + B(v^2_{n-1}, v^1_{n-1}) \quad \text{and} \quad \mathcal{V} = B(v^2_{n-1}, v^2_{n-1}).$$

  Then, by Lemma 6.1, we get

$$\|\mathcal{U}\|_{L^{\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} < \infty$$

$$\|\mathcal{V}\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} < \epsilon^2.$$

  So, $B(v_{n-1}, v_{n-1}) \in \widetilde{L}^{p,\infty}\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right)$, which concludes the induction.

We have just proven by induction that

$$\forall\, n \in \mathbb{N}, \quad v_n \in \widetilde{L}^{p,\infty}\left(]0, T[,\ L^{q,\infty}(\mathbb{R}^3)\right).$$

Then, we will prove by induction that for every $n \in \mathbb{N}$,

$$\|v_n\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \leq 2\left\|e^{t\Delta}u_0\right\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}.$$

This is obviously clear for v_0. Let assume that this is true for a $n \in \mathbb{N}$. We compute

$$\begin{aligned}
\|v_{n+1}\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} &\leq \|v_0\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} + \|B(v_n, v_n)\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}\\
&\leq \left\|e^{t\Delta}u_0\right\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} + 4C\left\|e^{t\Delta}u_0\right\|^2_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}\\
&< 2\left\|e^{t\Delta}u_0\right\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}.
\end{aligned}$$

So, for every $n \in \mathbb{N}$, $\|v_n\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}$ is in the ball centered at 0, of radius $2\left\|e^{t\Delta}u_0\right\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}$. We have the following estimate

$$\begin{aligned}
\|v_n - v_{n-1}\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} &\leq \|B(v_{n-1}, v_{n-1}) - B(v_{n-2}, v_{n-2})\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}\\
&\leq C\,\|v_{n-1} - v_{n-2}\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}\Big[\|v_{n-1}\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}\\
&\qquad + \|v_{n-2}\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}\Big]\\
&\leq 4C\left\|e^{t\Delta}u_0\right\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}\|v_{n-1} - v_{n-2}\|_{\mathcal{E}_T}
\end{aligned}$$

Now, as $4C\left\|e^{t\Delta}u_0\right\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} < 1$, the Picard contraction principle concludes the proof.

□

We next want to show the reverse result. That is to say that a solution to the (NS) equations which belongs to our class of solution satisfies the same properties of the Cannone and Planchon solutions in the norm $L^{q,\infty}$ instead of the L^q one.

Theorem 4.4 *Let $3 < q < \infty$, $2 < p < \infty$, such that*

$$\frac{2}{p} + \frac{3}{q} = 1.$$

and $u \in \widetilde{L}^{p,\infty}\,(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))$, a solution to (NS), then

$$\begin{cases} \sup_{t\in]0,T[} t^{1/p}\|u(t)\|_{q,\infty} < \infty, \\ \\ t^{1/p}\|u(t)\|_{q,\infty} \xrightarrow[t\to 0]{} 0. \end{cases}$$

Before giving the proof, we need this preliminary lemma :

Lemma 4.5 *Let $3 < q < \infty$, $2 < p < \infty$, such that*

$$\frac{2}{p} + \frac{3}{q} = 1.$$

and $u \in \widetilde{L}^{p,\infty}\,(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))$ a solution to (NS). Then, for every $0 < \epsilon < 1$, there exists $0 < t_0 < T$ such that

$$\forall\ t \in]0,t_0],\ \|u(t)\|_{q,\infty} \leq \frac{\epsilon}{2C_0 t^{1/p}},$$

where C_0 denotes the constant of continuity of the bilinear term B for functions in the space $L^\infty\,(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))$ (see Lemma 6.1).

Proof :

Let $0 < \epsilon < 1$ and $0 < t_0 < 1$. Denote

$$\lambda_{t_0} = \frac{\epsilon}{4C_0 t_0^{1/p}}.$$

We choose t_0 small enough to ensure

$$C(\lambda_{t_0}) < \frac{\epsilon^p}{2\times 4^{2p} C_0^p}.$$

Let $t \leq t_0$, we have $\lambda_t \geq \lambda_{t_0}$, so that $C(\lambda_t) \leq C(\lambda_{t_0})$. Then, by definition of the weak Lorentz space, we have

$$\left|\left\{t \in]0,T[,\ \|u(t)\|_{q,\infty} > \lambda_t\right\}\right| < \frac{C(\lambda_t)}{\lambda_t^p} < \frac{t}{2\times 4^p}.$$

Thus, there exists θ such that

$$t - \frac{t}{4^p} \leq \theta \leq t \quad \text{and} \quad \|u(\theta)\|_{q,\infty} \leq \frac{\epsilon}{4C_0 t^{1/p}} \tag{7}$$

Let $T^* = \left(4C_0 \|u(\theta)\|_{q,\infty}\right)^{-p}$. Theorem 6.2 implies that there exists a solution $\tilde{u} \in L^\infty\left(]\theta, \theta + T^*[,\ L^{q,\infty}(\mathbb{R}^3)\right)$ to the (NS) equations. The condition (7) on $\|u(\theta)\|_{q,\infty}$ and the assumption $\epsilon < 1$ imply that $T^* > t$. So that $]\theta, \theta + T^*[\supset]\theta, \theta + t[\supset]\theta, t]$.
Then Proposition 2.8 implies that $u = \tilde{u}$ on $]\theta, t]$. So, for every $t \leq t_0$, there exists $0 < \theta < t$ such that $u \in L^\infty\left(]\theta, t],\ L^{q,\infty}(\mathbb{R}^3)\right)$ and

$$\forall\ s \in]\theta, t], \quad \|u(s)\|_{q,\infty} \leq 2\,\|u(\theta)\|_{q,\infty} \leq \frac{\epsilon}{2C_0 t^{1/p}}.$$

□

Now we are ready to prove our main result.

Proof of Theorem 4.4 :

The convergence

$$t^{1/p}\,\|u(t)\|_{q,\infty} \xrightarrow[t \to 0]{} 0,$$

is a direct consequence of Lemma 4.5.
Once again, Lemma 4.5 implies that there exists t_0, independent of u, such that

$$\forall\ t \in]0, t_0], \quad \|u(t)\|_{q,\infty} \leq \frac{1}{4C_0 t_0^{1/p}}.$$

If we denote $\tilde{u}(s) = u(t_0 + s)$ for every $s \in]0, T - t_0]$, we have that $\tilde{u}$ is a solution to (NS) such that $\tilde{u} \in \widetilde{L}^{p,\infty}\left(]0, T - t_0[,\ L^{q,\infty}(\mathbb{R}^3)\right)$. Then applying Lemma 4.5 to $\tilde{u}$ we obtain that

$$\forall\ s \in]0, t_0], \quad \|\tilde{u}(s)\|_{q,\infty} \leq \frac{1}{4C_0 t_0^{1/p}}.$$

So that for every $t \in]0, 2t_0]$, we have

$$\|u(t)\|_{q,\infty} \leq \frac{1}{4C_0 t_0^{1/p}}.$$

Hence, repeating this argument a finite number of time we get

$$\forall\ t \in]0, T], \quad \|u(t)\|_{q,\infty} \leq \frac{1}{4C_0 t_0^{1/p}} \leq \frac{T^{1/p}}{4C_0 t_0^{1/p}} \times \frac{1}{t^{1/p}}.$$

Which concludes the proof of Theorem 4.4.

□

Remark 4.6 *Here we recall the following embedding*

$$\forall\ 3 \le q < \infty,\ B_q^{-1+\frac{3}{q},\infty} \hookrightarrow bmo^{-1}.$$

For the proof, the reader is referred to [3, Proposition 20.3], where it was proved that $B_{M^{1,q}}^{-1+3/q,\infty} \subset bmo^{-1}$*. Thus, the obvious embedding* $L^q \subset M^{1,q}$ *gives the desired result, where* $M^{1,q}$ *denotes the Morrey-Campanato space. The same embedding holds true for the space* $B_{q,\infty}^{-1+\frac{3}{q},\infty}$*.*

The previous remark implies that for $p > 2$, the initial data in Theorem 4.3 belongs to the space bmo^{-1}.

5 The critical case for Cannone and Planchon solutions

To obtain similar results in the limit case $p = 2$ and $q = \infty$, we need to impose more regularity on the solutions of (NS). Indeed, to obtain Lemma 4.5 we need to obtain an uniqueness result. For that, we will make use of the theory of Leray solutions (see [3, Chap. 13 & 14]).

Lemma 5.1 *Let* u *a solution to the* (NS) *equations such that* u *is in the space* $L^\infty(]0,T[,\ L^2(\mathbb{R}^3)) \cap L^2(]0,T[,\ L^\infty(\mathbb{R}^3)) \cap L^2(]0,T[,\ H^1(\mathbb{R}^3))$*. Then, for every* $0 < \epsilon < 1$*, there exists* $0 < t_0 < T$ *such that*

$$\forall\ t \in]0, t_0],\ \|u(t)\|_\infty \le \frac{\epsilon}{2C_0\sqrt{t}},$$

where C_0 *denotes the constant of continuity of the bilinear term* B *for functions in the space* $L^\infty(]0,T[,\ L^\infty(\mathbb{R}^3))$ *(see Lemma* 6.1*).*

Proof : The beginning of the proof is similar to the one of Lemma 4.5. To be able to conclude the proof, we need to prove the equality of u and $\widetilde{u}$. To do so, we will use the theory of the Leray solutions (see [3, Chap. 14]). As $u \in L^2(]0,T[,\ L^\infty(\mathbb{R}^3)) \cap L^2(]0,T[,\ H^1(\mathbb{R}^3))$ according to chapter 14 of [3], we can say that u is a Leray solution and satisfies the energy equality. Moreover, as $u(\theta) \in L^\infty(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ and $\widetilde{u}$ is the mild solution associated to $u(\theta)$, we can prove that $\widetilde{u}$ is also a Leray solution. Thus, using the Serrin uniqueness theorem (see [3, Chap. 14]), we have $u = \widetilde{u}$ on $]\theta, t_0]$, which concludes the proof.

□

Theorem 5.2
Let $u \in L^\infty(]0,T[,\ L^2(\mathbb{R}^3)) \cap L^2(]0,T[,\ L^\infty(\mathbb{R}^3)) \cap L^2(]0,T[,\ H^1(\mathbb{R}^3))$*, a solution to* (NS)*, then*

$$\begin{cases} \sup_{t\in]0,T[} \sqrt{t}\|u(t)\|_\infty < \infty, \\ \\ \sqrt{t}\|u(t)\|_\infty \xrightarrow[t\to 0]{} 0. \end{cases}$$

Moreover, the initial data u_0 *is in the space* $bmo^{-1}(\mathbb{R}^3)$

Proof : The proof of the first part of the theorem is similar to the one of Theorem 4.4 and is a direct consequence of Lemma 5.1.
As $u \in L^2(]0,T[,\ \ L^\infty(\mathbb{R}^3))$ we can prove that $u \in \mathcal{E}_T$. So, we also have that $B(u,u) \in \mathcal{E}_T$. Hence, as $e^{t\Delta}u_0 = u + B(u,u)$, we get

$$e^{t\Delta}u_0 \in \mathcal{E}_T,$$

which concludes the proof.

□

6 Tools

6.1 Continuity of the bilinear term

In this section, we will give many results concerning the continuity of the bilinear term B involved into the Navier-Stokes equations.

Lemma 6.1 *Let* $T > 0$ *and*

$$B(u,v)(t,x) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes v)(s,x)\ ds.$$

Let $2 \le p < \infty$ *and* $3 < q \le \infty$ *such that*

$$\frac{2}{p} + \frac{3}{q} = 1.$$

Then, we have

(1) for $p \neq 2$ *(and so* $q \neq \infty$*), we have* $B : L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)) \times$

$L^{p,\infty}\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right) \to L^{p,\infty}\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right)$ *with*

$$\|B(u,v)\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \leq C\|u\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}\|v\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}.$$

(2) $B : L^{\infty}\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right)\times L^{p,\infty}\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right) \to L^{\infty}\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right)$ *with*

$$\|B(u,v)\|_{L^{\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \leq C\|u\|_{L^{p,\infty}(]0,T[,\ L^{\infty}(\mathbb{R}^3))}\|v\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}.$$

(3) $B : L^{\infty}\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right)\times L^{\infty}\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right) \to L^{\infty}\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right)$ *with*

$$\|B(u,v)\|_{L^{\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \leq CT^{\frac{1}{p}}\|u\|_{L^{\infty}(]0,T[,\ L^{\infty}(\mathbb{R}^3))}\|v\|_{L^{\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}.$$

(4) $B : L^{\infty}\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right)\times L^{p,\infty}\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right) \to L^{p,\infty}\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right)$ *with*

$$\|B(u,v)\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \leq CT^{\frac{1}{p}}\|u\|_{L^{p,\infty}(]0,T[,\ L^{\infty}(\mathbb{R}^3))}\|v\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}.$$

Proof :

(1) By the Hölder inequality and the properties of the heat semigroup, we have

$$\|B(u,v)(t)\|_{q,\infty} \leq \int_0^t (t-s)^{-\frac{1}{2}-\frac{3}{2q}}\, \|u(s)\|_{q,\infty}\, \|v(s)\|_{q,\infty}\ ds. \tag{8}$$

Then, by the Young inequality with

$$1+\frac{1}{p}=\frac{1}{p}+\frac{1}{p}+\frac{1}{r},$$

and using that

$$\frac{1}{r}=1-\frac{1}{p}=\frac{1}{2}+\frac{3}{2q},$$

we obtain

$$\|B(u,v)\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \leq C\|u\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}\|v\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}.$$

(2) Starting from (8) we get

$$\|B(u,v)\|_{L^{\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \leq \|u\|_{L^{\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \times \sup_{t\in]0,T[} \int_0^t (t-s)^{-\frac{1}{2}-\frac{3}{2q}}\, \|v(s)\|_{q,\infty}\ ds.$$

Then, by the Young inequality we obtain

$$\|B(u,v)\|_{L^{\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \leq C\|u\|_{L^{p,\infty}(]0,T[,\ L^{\infty}(\mathbb{R}^3))}\|v\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}.$$

(3) Using (8) and taking the supremum on t, we have

$$\|B(u,v)\|_{L^\infty(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \leq C\|u\|_{L^\infty(]0,T[,\ L^\infty(\mathbb{R}^3))}\|v\|_{L^\infty(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \times \int_0^t (t-s)^{-\frac{1}{2}-\frac{3}{2q}}\,ds.$$

As

$$-\frac{1}{2}-\frac{3}{2q} = -1+\frac{1}{p},$$

the following inequality holds true

$$\|B(u,v)\|_{L^\infty(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \leq CT^{\frac{1}{p}}\|u\|_{L^\infty(]0,T[,\ L^\infty(\mathbb{R}^3))}\|v\|_{L^\infty(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}$$

(4) Estimate (8) and the Young inequality implies

$$\|B(u,v)\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \leq C\|u\|_{L^{p,\infty}(]0,T[,\ L^\infty(\mathbb{R}^3))}\|v\|_{L^{p,\infty}(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \times \int_0^t (t-s)^{-\frac{1}{2}-\frac{3}{2q}}\,ds.$$

As

$$\int_0^t (t-s)^{-\frac{1}{2}-\frac{3}{2q}}\,ds = \int_0^t (t-s)^{-1+\frac{1}{p}}\,ds = t^{\frac{1}{p}} \leq T^{\frac{1}{p}}.$$

□

Theorem 6.2 *Let $3 < q \leq \infty$ and $u_0 \in L^{q,\infty}(\mathbb{R}^3)$. For $T > 0$, such that*

$$4T^{\frac{1}{p}}\|u_0\|_{q,\infty} < 1,$$

there exists a solution $u \in L^\infty\left(]0,T[,\ L^{q,\infty}(\mathbb{R}^3)\right)$ of the (NS) equations, which is unique in the ball centered at zero of radius $2\|u_0\|_{q,\infty}$.

Proof : We construct (e_n) by iteration

$$\begin{cases} e_{n+1} = e_0 - B(e_n,e_n) \\ e_0 = e^{t\Delta}u_0. \end{cases}$$

Let us prove by induction that for all $n \in \mathbb{N}$

$$\|e_n\|_{L^\infty(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \leq 2\|u_0\|_{q,\infty}.$$

- For $n = 0$, by asumption we have $\left\|e^{t\Delta}u_0\right\|_{L^\infty(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \leq \|u_0\|_{q,\infty} \leq 2\|u_0\|_{q,\infty}$.
- Assume that the estimate holds true for a certain $n \in \mathbb{N}$.

- Let us prove the estimate for $n+1$.

$$\|e_{n+1}\|_{L^\infty(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \leq \|e_0\|_{L^\infty(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} + T^{\frac{1}{p}}\|e_n\|^2_{L^\infty(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}$$

$$\leq \|u_0\|_{q,\infty} + 4T^{\frac{1}{p}}\|u_0\|^2_{q,\infty} \leq 2\|u_0\|_{q,\infty}.$$

Moreover,

$$\|e_{n+1}-e_n\|_{L^\infty(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \leq (4T^{\frac{1}{p}}\|u_0\|_q)\|e_n - e_{n-1}\|_{L^\infty(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}.$$

Hence,

$$\|e_{n+1}-e_n\|_{L^\infty(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))} \leq (4T^{\frac{1}{p}}\|u_0\|_{q,\infty})^n\|e_1 - e_0\|_{L^\infty(]0,T[,\ L^{q,\infty}(\mathbb{R}^3))}.$$

Since $4T^{\frac{1}{p}}\|u_0\|_{q,\infty} < 1$, we have that the sequence (e_n) tends to a limit e.

□

6.2 Young and the weak Lorentz spaces

We will prove a kind of Young inequality for the weak Lorentz spaces $\widetilde{L}^{p,\infty}$. First we start with the following characterization

Lemma 6.3 *Let $p_1 < p < p_2$ and $0 < \gamma < 1$ such that*

$$\frac{1}{p_2} = \frac{1}{p} - \gamma,$$

for $0 < \gamma < 1$. Then,

$$\overline{L^{p_1,\infty} \cap L^{p_2,\infty}} = \widetilde{L}^{p,\infty}.$$

Proof :

- Let $f \in \overline{L^{p_1,\infty} \cap L^{p_2,\infty}}$, there exists a sequence (f_n) in $\mathcal{D}(]0,T[)$, such that :

$$\|f - f_n\|_{p_1,\infty} \xrightarrow[n\to\infty]{} 0 \quad \text{et} \quad \|f - f_n\|_{p_2,\infty} \xrightarrow[n\to\infty]{} 0.$$

  We denote

$$a = \frac{p_2 - p}{p_2 - p_1} \quad \text{et} \quad b = \frac{p - p_1}{p_2 - p_1}.$$

Then a and b are such that

$$\begin{cases} a+b=1 \\ ap_1+bp_2=p \\ 0<a,b<1 \end{cases}$$

Let $\epsilon>0$, there exists $N_1>0$ and $N_2>0$,

$$\forall\ n>N_1,\quad a\lambda^{p_1}\left|\{|f-f_n|>\lambda\}\right\| \leq \frac{\epsilon}{2},$$

and

$$\forall\ n>N_1,\quad b\lambda^{p_2}\left|\{|f-f_n|>\lambda\}\right\| \leq \frac{\epsilon}{2}.$$

Moreover, the arithemtic and geometric comparison, (with $a_1=\lambda^{p_1},\ a_2=\lambda^{p_2},\ \alpha_1=a,\ \alpha_2=b$) gives :

$$\lambda^p \leq a\lambda^{p_1}+b\lambda^{p_2}.$$

Thus, for every $n>\max(N_1,N_2)$:

$$\sup_{\lambda>0}\lambda^p\mu\left(|f_n-f|>\lambda\right) \leq a\sup_{\lambda>0}\lambda^{p_1}\mu\left(|f_n-f|>\lambda\right)+b\sup_{\lambda>0}\lambda^{p_2}\mu\left(|f_n-f|>\lambda\right)$$

$$\leq \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

- Let $f\in\widetilde{L}^{p,\infty}(]0,T[)$, there exists $(f_n)_{n\in\mathbb{N}}$ in $L^\infty(]0,T[)$ such that :

$$\|f-f_n\|_{p,\infty}\xrightarrow[n\to\infty]{}0.$$

We denote $g_n=f_n*\rho_n$, où $\rho_n\in\mathcal{D}(]0,T[)$, $\|\rho_n\|_\infty=1$, supp $\rho_n\subset]\epsilon_n,T-\epsilon_n[$ with $\epsilon_n\xrightarrow[n\to\infty]{}0$. We get

$$\|f-g_n\|_{p_1,\infty}\leq\|f-f_n\|_{p_1,\infty}+\|f_n-f_n*\rho_n\|_{p_1,\infty}$$

$$\leq T^{\frac{1}{p_1}-\frac{1}{p}}\|f-f_n\|_{p,\infty}+T^{\frac{1}{p_1}}\|f_n-f_n*\rho_n\|_\infty$$
$$\xrightarrow[n\to\infty]{}0.$$

We are now looking for the $L^{p_2,\infty}$-norm. We have

$$\|f-g_n\|_{p_2,\infty}\leq\|f-f*\rho_n\|_{p_2,\infty}+\|(f-f_n)*\rho_n\|_{p_2,\infty}$$

$$\leq\|f-f*\rho_n\|_{p_2,\infty}+\|\rho_n\|_r\|f-f_n\|_{p,\infty},$$

where $1/r=1/p_2-1/p+1=1-\gamma$.
Thus, $f\in\overline{L^{p_1,\infty}\cap L^{p_2,\infty}}$.

□

Now, we can generalize the Young inequality

Theorem 6.4 *For* $1 < p, r, s < \infty$ *such that*

$$1 + \frac{1}{s} = \frac{1}{p} + \frac{1}{r},$$

we have

$$L^{r,\infty} * \widetilde{L}^{p,\infty} \subset \widetilde{L}^{s,\infty}.$$

Proof : Let $p_1 < p < p_2$. We have

$$L^{r,\infty} * L^{p_1,\infty} \subset L^{s_1,\infty}, \quad L^{r,\infty} * L^{p_2,\infty} \subset L^{s_2,\infty},$$

with

$$\frac{1}{s_1} + 1 = \frac{1}{r} + \frac{1}{p_1}, \quad \frac{1}{s_2} + 1 = \frac{1}{r} + \frac{1}{p_2}.$$

So,

$$s_1 < s < s_2.$$

Then,

$$L^{r,\infty} * \left(L^{p_1,\infty} \cap L^{p_2,\infty}\right) \subset \left(L^{s_1,\infty} \cap L^{s_2,\infty}\right).$$

The previous Lemma concludes the proof.

□

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