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A note on the uniqueness of weak solutions for the Navier-Stokes equations

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ABSTRACT. Consider the Navier-Stokes equations with the initial data $a \in L^2_\sigma(\mathbb{R}^d)$. Let u and v be two weak solutions with the same initial value a. If $\nabla u \in L^{\frac{2}{2-r}}\left((0,T)\,;\dot{X}_r(\mathbb{R}^d)^d\right)$ where $\dot{X}_r(\mathbb{R}^d)$ is the multiplier space (see the definition in the text), then we have $u = v$.

CONTENTS

1. Introduction

Consider the Navier-Stokes equations in $(0,T)\times\mathbb{R}^d$ with $0 < T < \infty$ and $d \geq 3$

$$\begin{aligned} \partial_t u + (u.\nabla)\,u - \Delta u + \nabla p &= 0, \qquad (x,t) \in \mathbb{R}^d \times (0,\infty), \\ \nabla.u &= 0, \qquad (x,t) \in \mathbb{R}^d \times (0,\infty), \\ u(x,0) &= a(x), \qquad x \in \mathbb{R}^d, \end{aligned} \tag{1.1}$$

where $u = u(x,t)$ is the velocity field, $p = p(x,t)$ is the scalar pressure and $a(x)$ with div $a = 0$ in the sense of distribution is the initial velocity field. For simplicity, we assume that the external force has a scalar potential and is included into the pressure gradient.

In their famous paper, Leray [**8**] and Hopf [**3**] constructed a weak solution u of (1.1) for arbitrary $a \in L^2_\sigma$. The solution is called the Leray-Hopf weak solution. In the general case the problem on uniqueness of Leray-Hopf' s weak solutions is still open question. Masuda [**9**] extended Serrin' s class for uniqueness of weak solutions

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and made it clear that the class $L^{\infty}\left((0,T);L^{d}\left(\mathbb{R}^{d}\right)\right)$ plays an important role for uniqueness of weak solutions. Kozono-Sohr [**5**] showed that the uniqueness holds in $L^{\infty}\left((0,T);L^{d}\right)$.

Foias [**1**] and Serrin [**10**] introduced the class $L^{\alpha}\left((0,\infty);L^{q}\right)$ and showed that under the additional assumption

$$u\in L^{\alpha}\left((0,\infty);L^{q}\right)\quad\text{for}\quad\frac{2}{\alpha}+\frac{d}{q}=1\quad\text{with}\quad q>d,$$

u is the only weak solution.

The purpose of this note is to improve the criterion on uniqueness of weak solutions to in the class $L^{\frac{2}{2-r}}\left((0,T)\,;\dot{X}_{r}(\mathbb{R}^{d})^{d}\right)$. We know that for every $a\in L^{2}_{\sigma}\left(\mathbb{R}^{d}\right)$, there is at least one weak solution u of (1.1) satisfying the energy inequality. Here we mean by the weak solution a function u in $u\in L^{\infty}\left((0,T)\,;L^{2}_{\sigma}\right)\cap L^{2}\left((0,T)\,;\dot{H}^{1}_{\sigma}\right)$ satisfying (1.1) in the sense of distributions (Definition 2). For more facts concerning uniqueness of weak solutions, we refer to a celebrated paper of Kozono and Sohr [**5**] (see also [**2**]).

Now, we give a description of the multiplier space $\dot{X}_{r}$ introduced recently by P.G. Lemarié-Rieusset in his work [**6**] (see also [**7**]). The space $\dot{X}_{r}$ of pointwise multipliers which map L^{2} into $\dot{H}^{-r}$ is defined in the following way

Definition 1. *For $0\leq r<\frac{d}{2}$, we define the homogeneous space $\dot{X}_{r}$ by*

$$\dot{X}_{r}=\left\{f\in L^{2}_{loc}:\ \forall g\in\dot{H}^{r}\quad fg\in L^{2}\right\}$$

where we denote by $\dot{H}^{r}\left(\mathbb{R}^{d}\right)$ the completion of the space $\mathcal{D}\left(\mathbb{R}^{d}\right)$ with respect to the norm $\|u\|_{\dot{H}^{r}}=\left\|(-\Delta)^{\frac{r}{2}}u\right\|_{L^{2}}$.

The norm of $\dot{X}_{r}$ is given by the operator norm of pointwise multiplication

$$\|f\|_{\dot{X}_{r}}=\sup_{\|g\|_{\dot{H}^{r}}\leq1}\|fg\|_{L^{2}}$$

We have the homogeneity properties : $\forall x_{0}\in\mathbb{R}^{d}$

$$\|f(.+x_{0})\|_{\dot{X}_{r}}=\|f\|_{\dot{X}_{r}}$$

$$\|f(\lambda .)\|_{\dot{X}_{r}}=\frac{1}{\lambda^{r}}\|f\|_{\dot{X}_{r}},\quad\lambda>0.$$

Additionally, for $0\leq r<\frac{d}{2}$, we have the following inclusion relations :

$$L^{\frac{d}{r}}\left(\mathbb{R}^{d}\right)\subset L^{\frac{d}{r},\infty}\left(\mathbb{R}^{d}\right)\subset\dot{X}_{r}\left(\mathbb{R}^{d}\right).$$

where $L^{p,\infty}$ denotes the usual Lorentz (weak L^{p}) space. For the definition and basic properties of Lorentz spaces $L^{p,q}$ we refer to [**11**].

2. Uniqueness theorem

Before turning our attention to uniqueness issues, we start with some prerequisites for our main result. Let

$$C^{\infty}_{0,\sigma}\left(\mathbb{R}^{d}\right)=\left\{\varphi\in\left(C^{\infty}_{0}\left(\mathbb{R}^{d}\right)\right)^{d}:\operatorname{div}\varphi=0\right\}\subseteq\left(C^{\infty}_{0}\left(\mathbb{R}^{d}\right)\right)^{d}.$$

The subspace

$$L^2_\sigma\left(\mathbb{R}^d\right)=\overline{C^\infty_{0,\sigma}\left(\mathbb{R}^d\right)}^{\|.\|_{L^2}}=\left\{u\in L^2\left(\mathbb{R}^d\right)^d:\text{div }u=0\right\}$$

obtained as the closure of $C^\infty_{0,\sigma}$ with respect to L^2-norm $\|.\|_{L^2}$. H^r_σ denotes the closure of $C^\infty_{0,\sigma}$ with respect to the norm

$$\|u\|_{H^r}=\|u\|_{L^2}+\left\|(1-\Delta)^{\frac{r}{2}}u\right\|_{L^2},\ \text{ for }\ r\geq 0.$$

Our definition of Leray-Hopf weak solutions (see e.g. [**5**]) now reads :

Definition 2 (weak solutions). *Let $a\in L^2_\sigma$ and $T>0$. A measurable function u is called a weak solution of (1.1) on $(0,T)$ if u satisfies the following properties*

(1): $u\in L^\infty\left((0,T);L^2_\sigma\right)\cap L^2\left((0,T);\dot{H}^1_\sigma\right)$ *for all $T>0$;*

(2): *$u(t)$ is continuous in time in the weak topology of L^2_σ with*

$$\langle u(t),\phi\rangle\to\langle a,\phi\rangle\quad as\quad t\to 0^+$$

for all $\phi\in L^2_\sigma$;

(3): *for any $0\leq s\leq t\leq T$, u satisfies the identity*

$$\int_s^t\left\{-\langle u,\partial_\tau\phi\rangle+\langle u.\nabla u,\phi\rangle+\langle\nabla u,\nabla\phi\rangle\right\}d\tau=-\langle u(t),\phi(t)\rangle+\langle u(s),\phi(s)\rangle\,, \tag{2.1}$$

for all $\phi\in H^1\left((s,t);H^1_\sigma\right)$. Here $\langle .,.\rangle$ denotes the scalar product and $\|.\|_{L^2}$ denotes the norm in $L^2\left(\mathbb{R}^d\right)^d$.

Remark 1. *For u and ϕ as above, the integral*

$$\int_0^T\langle u.\nabla u,\phi\rangle\,d\tau$$

is well defined since we have by the Sobolev inequality

$$\|u\|_{L^{\frac{2d}{d-2}}}\leq C\,\|\nabla u\|_{L^2}$$

that

$$\begin{aligned}\left|\int_0^T\langle u.\nabla u,\phi\rangle\,d\tau\right| &\leq \int_0^T\|u\|_{L^{\frac{2d}{d-2}}}\|\nabla u\|_{L^2}\|\phi\|_{L^d}\,d\tau\\ &\leq C\sup_{0<t<T}\|\phi\|_{L^d}\int_0^T\|\nabla u\|^2_{L^2}\,d\tau\end{aligned}$$

Existence of weak solutions has been established by Leray in [**8**] for initial velocity in $L^2_\sigma\left(\mathbb{R}^d\right)$. The result is the following

Theorem 1 (Leray - Hopf). *Let $T>0$. Then, for any given $a\in L^2_\sigma\left(\mathbb{R}^d\right)$, there exits at least one weak solution u to (1.1) on $(0,T)$ such that*

$$\|u(t)\|^2_{L^2}+2\int_0^t\|\nabla u(s)\|^2_{L^2}\,ds\leq\|a\|^2_{L^2}\,,\quad 0\leq t<T. \tag{2.2}$$

and

$$\|u(t)-a\|_{L^2}\to 0\quad as\quad t\to+0.$$

Let us introduced the class $L^s\left((0,T);L^\gamma\right)$ with the norm $\|.\|_{L^s((0,T);L^\gamma)}$

$$\|u\|_{L^s((0,T);L^\gamma)} = \left(\int_0^T \|u(t)\|_{L^\gamma}^s \, dt\right)^{\frac{1}{s}}.$$

The classical result on uniqueness of weak solutions in the class $L^s\left((0,T);L^\gamma\right)$ was given by Foias, Serrin and Masuda [**1**], [**10**], [**9**].

Theorem 2 (Foias-Serrin-Masuda). *Let $a \in L^2_\sigma\left(\mathbb{R}^d\right)$. Let u and v are two weak solutions of (1.1) on $(0,T)$. Suppose that u satisfies*

$$u \in L^s\left((0,T);L^\gamma\right) \quad \textit{for} \quad \frac{2}{s}+\frac{d}{\gamma}=1 \quad \textit{with} \ \ d<\gamma<\infty. \tag{2.3}$$

Assume that v fulfills the energy inequality (2.2) for $0 \le t < T$. Then we have $u = v$ on $[0,T)$.

Remark 2. *In Theorem 2, v not need belong to the class (2.3). On the other hand, every weak solution u with (2.3) fulfills the energy identity*

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds = \|a\|_{L^2}^2, \quad 0 \le t \le T. \tag{2.4}$$

It seems to be an interesting question whether every weak solution satisfies the energy inequality (2.2).

Remark 3. *The class (2.3) is important from the view point of scaling invariance for the Navier-Stokes equations. It can be easily seen that if is a pair of the solution to (1.1) on $\mathbb{R}^d \times (0,T)$, then so is the family $\{u_\lambda, p_\lambda\}_{\lambda>0}$ where*

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x,t) = \lambda^2 u(\lambda x, \lambda^2 t).$$

Scaling invariance means that there holds

$$\|u_\lambda\|_{L^s((0,\infty);L^\gamma)} = \left(\lambda^{1-\left(\frac{2}{s}+\frac{d}{\gamma}\right)} \|u\|_{L^s((0,\infty);L^\gamma)}\right) = \|u\|_{L^s((0,\infty);L^\gamma)} \quad \textit{for all} \ \ \lambda > 0$$

if and only if

$$\frac{2}{s}+\frac{d}{\gamma}=1.$$

We shall next deal with the critical case with $s = \infty$ and $\gamma = d$ in (2.3).

Theorem 3 (Masuda [**9**], Kozono-Sohr [**5**]). *Let $a \in L^2_\sigma\left(\mathbb{R}^d\right)$. Let u and v be two weak solutions of (1.1) on $(0,T)$. Suppose that*

$$u \in L^\infty\left((0,T);L^d\right) \tag{2.5}$$

and that v fulfills the energy inequality (2.2) for all $0 \le t < T$. Then we have $u = v$ on $[0,T)$.

Remark 4. *Masuda* [**9**] *proved that if $u \in L^\infty\left((0,T);L^d\right)$ is continuous from the right on $[0,T)$ in the norm of L^d, then there holds $u = v$ on $[0,T)$. Later on, Kozono-Sohr* [**5**] *showed that every weak solution in $L^\infty\left((0,T);L^d\right)$ of (1.1) on $(0,T)$ becomes necessarily continuous from the right in the norm of L^d.*

The same result holds when, for $\gamma = +\infty$, we replace the assumption

$$u \in L^2\left((0,T);L^\infty\right)$$

by the weaker assumption

$$\nabla u \in L^2\left((0,T);\dot{X}_1\left(\mathbb{R}^d\right)^d\right).$$

The replacement of hypothesis $u \in L^2\left((0,T);L^\infty\right)$ by $\nabla u \in L^2\left((0,T);\dot{X}_1\left(\mathbb{R}^d\right)^d\right)$ was recently discussed in a similar context by Gala [**2**]. Moreover, we have

Theorem 4 (Gala). *Let $a \in L^2_\sigma\left(\mathbb{R}^d\right)$ and let u, v be two weak solutions of (1.1) on $(0,T)$. Suppose that*

$$\nabla u \in L^2\left((0,T);\dot{X}_1\left(\mathbb{R}^d\right)^d\right) \tag{2.6}$$

and that v fulfills the energy inequality (2.2) for $0 \le t < T$. Then we have $u = v$ on $[0,T]$.

Remark 5. *By Theorem 2, every weak solution in $L^2\left((0,T);L^\infty\right)$ is unique.*

Our result on uniqueness of the weak solution now reads :

Theorem 5. *Let $a \in L^2\left(\mathbb{R}^d\right)^d$ with $\nabla.a = 0$ Assume that there exists a solution u for the Navier-Stokes equations on $(0,T)\times\mathbb{R}^d$ (for some $T \in (0,+\infty]$ with some initial data a so that*

$$u \in L^\infty\left((0,T);L^2_\sigma\left(\mathbb{R}^d\right)^d\right) \cap L^2\left((0,T);\dot{H}^1_\sigma\left(\mathbb{R}^d\right)^d\right),$$

and

$$\nabla u \in L^{\frac{2}{2-r}}\left((0,T);\dot{X}_r(\mathbb{R}^d)^d\right) \quad \text{for all } 0 \le r \le 1.$$

Then, u is the unique Leray-Hopf solution associated with a on $[0,T)$.

The following corollary, which is an immediate consequence of Theorem 5 gives a simpler sufficient condition in term of Lorentz spaces.

Corollary 1. *Let $a \in L^2\left(\mathbb{R}^d\right)^d$ with $\nabla.a = 0$ Assume that there exists a solution u for the Navier-Stokes equations on $(0,T)\times\mathbb{R}^d$ (for some $T \in (0,+\infty]$ with some initial data a so that*

$$u \in L^\infty\left((0,T);L^2_\sigma\left(\mathbb{R}^d\right)^d\right) \cap L^2\left((0,T);\dot{H}^1_\sigma\left(\mathbb{R}^d\right)^d\right),$$

and

$$\nabla u \in L^{\frac{2}{2-r}}\left((0,T);L^{\frac{d}{r},\infty}\left(\mathbb{R}^d\right)^d\right),$$

where $L^{p,\infty}$ denotes the usual Lorentz (weak L^p) space. Then, u is the unique Leray-Hopf solution associated with a on $[0,T)$.

The same result again holds when the assumption

$$\nabla u \in L^{\frac{2}{2-r}}\left((0,T);L^{\frac{d}{r},\infty}(\mathbb{R}^d)^d\right)$$

is replaced by

$$u \in L^{\frac{2}{2-r}}\left((0,T);L^{\frac{d}{r}}\left(\mathbb{R}^d\right)^d\right).$$

We are now in a position to proof the main result.

PROOF. Let v be another weak solution of (1.1) associated to a on $(0,T)$ (with associated pressure p) such that

$$v \in L^{\infty}\left((0,T);L^2_{\sigma}\left(\mathbb{R}^d\right)^d\right) \cap L^2\left((0,T);\dot{H}^1_{\sigma}\left(\mathbb{R}^d\right)^d\right)$$

and

$$\nabla u \in L^{\frac{2}{2-r}}\left((0,T);\dot{X}_r(\mathbb{R}^d)^d\right).$$

We consider the difference $w = u - v$ and we obtain

$$\begin{aligned} \|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2\, ds &\leq \|a\|_{L^2}^2\,, \\ \|v(t)\|_{L^2}^2 + 2\int_0^t \|\nabla v(s)\|_{L^2}^2\, ds &\leq \|a\|_{L^2}^2\,. \end{aligned}$$

On the other hand, we have

$$\langle u(t), v(t)\rangle + 2\int_0^t \langle \nabla u(s), \nabla v(s)\rangle\, ds = \|a\|_{L^2}^2 + \int_0^t \langle w.\nabla u, w\rangle (s) ds$$

for all $0 \leq t < T$. Combining the above inequalities, we obtain

$$\begin{aligned} & \|w(t)\|_{L^2}^2 + 2\int_0^t \|\nabla w(s)\|_{L^2}^2\, ds \\ = {} & \|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2\, ds + \|v(t)\|_{L^2}^2 \\ & +2\int_0^t \|\nabla v(s)\|_{L^2}^2\, ds - 2\langle u(t), v(t)\rangle - 4\int_0^t \langle \nabla u(s), \nabla v(s)\rangle\, ds \\ \leq {} & -2\int_0^t \langle w.\nabla u, w\rangle\, ds. \end{aligned} \tag{2.7}$$

We thus observe that by Young inequality

$$\left(a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b \leq a + b \text{ with } a, b \geq 0 \text{ and } 0 \leq \alpha \leq 1\right),$$

it follows that

$$\begin{aligned} \left|\int_0^t \langle w.\nabla u, w\rangle\right| ds &\leq \int_0^t \|w.\nabla u(s)\|_{L^2} \|w(s)\|_{L^2}\, ds \\ &\leq \int_0^t \|w(s)\|_{\dot{H}^r} \|\nabla u(s)\|_{\dot{X}_r} \|w(s)\|_{L^2}\, ds \\ &\leq \int_0^t \|w(s)\|_{L^2}^{1-r} \|\nabla w(s)\|_{L^2}^r \|\nabla u\|_{\dot{X}_r} \|w(s)\|_{L^2}\, ds \\ &\leq \int_0^t \left(\|w(s)\|_{L^2}^2 \|\nabla u\|_{\dot{X}_r}^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}} \left(\|\nabla w(s)\|_{L^2}^2\right)^{\frac{r}{2}} ds \\ &\leq \frac{1}{2}\int_0^t \|\nabla w\|_{L^2(\mathbb{R}^d)}^2\, ds + \frac{C}{2}\int_0^t \|w\|_{L^2(\mathbb{R}^d)^d}^2 \|\nabla u\|_{\dot{X}_r(\mathbb{R}^d)}^{\frac{2}{2-r}}\, ds. \end{aligned}$$

where we used the following ones ($0 \leq r \leq 1$)

$$\|\omega\|_{\dot{H}^r} = \frac{1}{(2\pi)^{\frac{d}{2}}} \left\||\xi|^r \widehat{\omega}\right\|_{L^2} \leq \|\omega\|_{L^2}^{1-r} \|\nabla\omega\|_{L^2}^r\,.$$

Hence by (2.7) there holds

$$\|w(t)\|_{L^2}^2 + \int_0^t \|\nabla w\|_{L^2}^2 \, d\tau \leq C \int_0^t \|w\|_{L^2(\mathbb{R}^d)^d}^2 \|\nabla v\|_{\dot{X}_r(\mathbb{R}^d)}^{\frac{2}{2-r}} \, d\tau$$

for all $t > 0$. Since $\nabla u \in L^{\frac{2}{2-r}}\left((0,T); \dot{X}_r(\mathbb{R}^d)^d\right)$ and since $w(0) = 0$, it follows from the Gronwall inequality that

$$\|w(t)\|_{L^2}^2 \leq \|w(0)\|_{L^2}^2 \exp\left(C \int_0^t \|\nabla u\|_{\dot{X}_r(\mathbb{R}^d)}^{\frac{2}{2-r}} \, ds\right),$$

and thus

$$\|w(t)\|_{L^2}^2 = 0, \quad 0 \leq t < T$$

and implies uniqueness of weak solutions. □

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