Lectures on Lie groups and geometry

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Abstract

These are the notes of the course given in Autumn 2007 and Spring 2011.

Two good books (among many):
Adams: Lectures on Lie groups (U. Chicago Press)
Fulton and Harris: Representation Theory (Springer)
Also various writings of Atiyah, Segal , Bott, Guillemin and Sternberg ....

Contents

1 Review of basic material

1.1 Lie Groups and Lie algebras

1.1.1 Examples

Definition

A Lie group is a group with G which is a differentiable manifold and such that multiplication and inversion are smooth maps. The subject is one which is to a large extent "known", from the theoretical point of view and one in which the study of Examples is very important.

Examples

- $\mathbf{R}$ under addition.
- $S^1 \subset \mathbf{C}$ under multiplication. This is isomorphic to $\mathbf{R}/\mathbf{Z}$.
- $GL(n, K)$ where $K = \mathbf{R}, \mathbf{C}, \mathbf{H}$.

  (Recall that the quaternions $\mathbf{H}$ form a 4-dimensional real vector space with basis $1, i, j, k$ and mupltiplication defined by $i^2 = j^2 = k^2 = -1, ij = -ji = k$. They form a non-commutative field.) We will use the notation $GL(V)$ where V is an n-dimensional K-vector space interchangeably.

  [A useful point of view is to take the complex case as the primary one. Consider an n-dimensional complex vector space V and an antilinear map $J : V \to V$ with $J^2 = \pm 1$. The case $J^2 = 1$ gives V a *real structure* so it is written as $V = U \otimes_{\mathbf{R}} \mathbf{C}$ for a real vector space U and the complex linear maps which commute with J give $GL(U) = GL(n, \mathbf{R}) \subset GL(V) = GL(n, \mathbf{C})$. The case $J^2 = -1$ only happens when V is even dimensional and gives V the structure of a quaternionic vector space. The complex linear maps which commute with J give $GL(n/2, \mathbf{H}) \subset GL(n, \mathbf{C})$.]

- $SL(n, \mathbf{R}), SL(n, \mathbf{C})$, the kernels of the determinant homomorphisms to $K^* = K \setminus \{0\}$.
- $O(n) \subset GL(n, \mathbf{R}), U(n) \subset GL(n, \mathbf{C}), Sp(n) \subset GL(n, \mathbf{H})$, the subgroups of matrices A such that $AA^* = 1$, where A^* is conjugate transpose. More invariantly, these are the linear maps which preserve appropriate positive forms. We also have $O(p, q), U(p, q), Sp(p, q)$ which preserve indefinite forms of signature (p, q).
- $SO(n) \subset O(n), SU(n) \subset U(n)$; fixing determinant 1.
- $O(n, \mathbf{C}), SO(n, \mathbf{C})$ defined in the obvious ways.

- $Sp(n, \mathbf{R}), Sp(n, \mathbf{C})$: maps which preserve non-degenerate *skew symmetric* forms on $\mathbf{R}^{2n}, \mathbf{C}^{2n}$ respectively. [Notation regarding $n, 2n$ differs in the literature.]

- The groups of upper triangular matrices, or upper triangular matrices with 1's on the diagonal.

If we ignored all abstraction and just said that we are interested in studying familiar examples like these then we would retain most of the interesting ideas in the subject.

Definition

A *right action* of a Lie group on a manifold M is a smooth map $M \times G \to M$ written $(m, g) \to mg$ such that $mgh = m(gh)$. Similarly for a left action.

Particularly important are linear actions on vector spaces, that is to say *representations* of G or homomorphisms $G \to GL(V)$.

1.1.2 The Lie algebra of a Lie group

Let G be a Lie group and set $\mathfrak{g} = TG_1$ the tangent space at the identity. Thus an element of $\mathfrak{g}$ is an equivalence class of paths g_t through the identity.

Example: if $G = GL(V)$ then $\mathfrak{g} = \mathrm{End}(V)$.

Now G acts on itself on the left by conjugation

$$Ad_g h = ghg^{-1}.$$

Then Ad_g maps 1 to 1 so acts on the tangent space giving the adjoint action

$$ad_g \in GL(\mathfrak{g}).$$

Thus we get a homomorphism $ad : G \to GL(\mathfrak{g})$ which has a derivative at the identity. This is a map, denoted by the same symbol

$$ad : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}).$$

There is a unique bilinear map $[\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

$$ad(\xi)(\eta) = [\xi, \eta].$$

Example If $G = GL(V)$ so $\mathfrak{g} = \mathrm{End}(V)$ then working from the definition we find that

$$[A, B] = AB - BA.$$

Definition A Lie algebra (over a commutative field k) is a k-vector space V and a bilinear map

$$[\ ,\ ] : V \times V \to V,$$

such that

$$[u, v] = -[v, u],$$

$$[[u,v],w]+[[v,w],u]+[[w,u],v]=0$$

for all u,v,w. This latter is called the *Jacobi identity.*

Proposition The bracket we have defined above makes $\mathfrak{g}$ into a Lie algebra.

We have to verify skew-symmetry and the Jacobi identity. For the first consider tangent vectors $\xi,\eta\in\mathfrak{g}$ represented by paths g_t,h_s. Then, from the definition, for fixed t

$$\frac{\partial}{\partial s}g_th_sg_t^{-1}=ad_{g_t}\eta.$$

(The derivative being evaluated at $s=0$.) Since the derivative of the inverse map $g\mapsto g^{-1}$ at the identity is -1 (Exercise!) we have

$$\frac{\partial}{\partial s}g_th_sg_t^{-1}h_s^{-1}=ad_{g_t}\eta-\eta.$$

Now differentiate with respect to t. From the definition we get

$$\frac{\partial^2}{\partial t\partial s}\left(g_th_sg^{-1}h_s^{-1}\right)=[\xi,\eta].$$

(Derivatives evaluated at $s=t=0$.) From the fact that the derivative of inversion is -1 we see that

$$\frac{\partial^2}{\partial t\partial s}\left(g_th_sg_t^{-1}h_s^{-1}\right)^{-1}=-[\xi,\eta],$$

then interchanging the roles of ξ,η and using the symmetry of partial derivatives we obtain $[\eta,\xi]=-[\xi,\eta]$ as required.

Another way of expressing the above runs as follows. In general if $F:M\to N$ is a smooth map between manifolds then for each $m\in M$ there ie an intrinsic first derivative $dF:TM_m\to TN_{F(m)}$ but not, in a straightforward sense, a second derivative. However if dF vanishes at m there is an intrinsic second derivative which is a linear map

$$d^2F:s^2TM_m\to TN_{F(m)},$$

(where s^2 denotes the second symmetric power). Define $K:G\times G\to G$ to be the commutator $K(g,h)=ghg^{-1}h^{-1}$. The first derivative of K at $(1,1)\in G\times G$vanishes since $K(g,1)=K(1,h)=1$. The second derivative is a linear map $s^2(\mathfrak{g}\oplus\mathfrak{g})=s^2(\mathfrak{g})\oplus\mathfrak{g}\otimes\mathfrak{g}\oplus s^2(\mathfrak{g})\to\mathfrak{g}$ and the same identity implies that it vanishes on the first and third summands so it can be viewed as a linear map $\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$. From the definition this map is the bracket, and the identity $K(g,h)^{-1}=K(h,g)$ gives the skew-symmetry.

For the Jacobi identity we invoke the naturality of the definition. Let $\alpha:G\to H$ be a Lie group homomorphism. This has a derivative at the identity

$$d\alpha:\mathfrak{g}\to\mathfrak{h},$$

and it follows from the definitions that

$$[d\alpha(\xi), d\alpha(\eta)]_{\mathfrak{h}} = d\alpha\left([\xi, eta]_{\mathfrak{g}}\right). \qquad (*)$$

Now apply this to the adjoint representation, viewed as a homomorphism $\alpha : G \to GL(\mathfrak{g})$. We know the bracket in the Lie algebra of $GL(\mathfrak{g})$ and the identity becomes

$$[\xi, [\eta, \theta]] - [\eta, [\xi, \theta]] = [[\xi, \eta], \theta],$$

for all $\xi, \eta, \theta \in \mathfrak{g}$. Re-arranging, using the skew-symmetry, this is the Jacobi identity.

There are other approaches to the definition of the the bracket on $\mathfrak{g}$. If M is any manifold we write $\mathrm{Vect}(M)$ for the set of vector fields on M. Then there is a *Lie bracket* making $\mathrm{Vect}(M)$ an infinite-dimensional Lie algebra. Suppose that a Lie group G acts on M on the right. The derivative of the action $M \times G \to M$ at a point $(m, 1)$ yields a map from $\mathfrak{g}$ to TM_m. Fix $\xi \in \mathfrak{g}$ and let m vary: this gives a vector field on M so we have a linear map

$$\rho : \mathfrak{g} \to \mathrm{Vect}(M)$$

the "infinitesimal action".

Proposition This is a Lie algebra homomorphism.

Now G acts on itself by right-multiplication and the image of this map is the set of left-invariant vector fields. So we can identify $\mathfrak{g}$ with the set of left invariant vector fields. It is clear that the Lie bracket of left invariant vector fields is left invariant so we can use this as an alternative definition of the bracket on $\mathfrak{g}$, that is we make the Proposition above a definition in the case of this action.

If one takes this route one needs to know the definition of the Lie bracket on vector fields. Again there are different approaches. One is in terms of the *Lie derivative.* For any space of tensor fields on which the diffeomorphism group $\mathrm{Diff}(M)$ acts we define the Lie derivative $\mathcal{L}_v(\tau) = \frac{d}{dt} f_t(\tau)$ where f_t is a 1-parameter family of diffeomorphisms with derivative v (all derivatives evaluated at $t = 0$). Then on vector fields

$$\mathcal{L}_v(w) = [v, w]. \qquad (**)$$

For the other approach one thinks of a vector field v as defining a differential operator $\nabla_v : C^\infty(M) \to C^\infty(M)$. Then the bracket can be defined by

$$\nabla_{[v,w]} = \nabla_v \nabla_w - \nabla_w \nabla_v. \qquad (***)$$

To understand the relation between all these different points of view it is often useful to think of the diffeomorphism group $\mathrm{Diff}(M)$ as an infinite dimensional Lie group. There are rigorous theories of such things but we only want to use the idea in an informal way. Then $\mathrm{Vect}(M)$ is interpreted as the tangent space at the identity of $\mathrm{Diff}(M)$ and the action of $\mathrm{Diff}(M)$ on vector fields is just the adjoint action. Then the definition (**) is the exact analogue of our definition of the bracket on $\mathfrak{g}$. The second definition (***) amounts to saying that $\mathrm{Diff}(M)$ has a representation on the vector space $C^\infty(M)$ and then using the commutator formula for the bracket on the endomorphisms of a vector space. From this viewpoint the identities relating the various constructions all amount to instances of (*).

Definition

A *one-parameter subgroup* in a Lie group G is a smooth homomorphism $\lambda : \mathbf{R} \to G$.

A one-parameter subgroup λ has a derivative $\lambda'(0) \in \mathfrak{g}$.

Proposition For each $\xi \in \mathfrak{g}$ there is a unique one-parameter subgroup λ_ξ with derivative ξ.

Given $\xi \in \mathfrak{g}$ let v_ξ be the corresponding left invriant vector field on G. The definitions imply that a 1 PS with derivative ξ is the same as an integral curve of this vector field which passes through the identity. By the existence theorem for ODE's there is an integral curve for a short time interval $\lambda : (-\epsilon, \epsilon) \to G$ and the multiplication law can be used to extend this to $\mathbf{R}$. Similarly for uniqueness. Define the exponential map

$$\exp : \mathfrak{g} \to G,$$

by $\exp(\xi) = \lambda_\xi(1)$. The definitions imply that the derivative at 0 is the identity from $\mathfrak{g}$ to $\mathfrak{g}$ and the inverse function theorem shows that exp gives a diffeommorphism from a neighbourhood of 0 in $\mathfrak{g}$ to a neighbourhood of 1 in G. We also have

$$\lambda_\xi(t) = \exp(t\xi).$$

When $G = GL(n, \mathbf{R})$ one finds

$$\exp(A) = 1 + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots.$$

The discussion amounts to the same thing as the solution of a linear system of ODE's with constant co-efficients

$$\frac{dG}{dt} = AG,$$

for a $n \times n$ matrix $G(t)$. The columns of $G(t)$ give n linearly independent solutions of the vector equation

$$\frac{dx}{dt} = Ax.$$

The exponential map can be used to prove a somewhat harder theorem than any we have mentioned so far.

Proposition 1 *Any closed subgroup of a Lie group is a Lie subgroup (i.e. a submanifold).*

We refer to textbooks for the proof. In particular we immediately see that the well-known matrix groups metioned above such as $O(n), U(n)$ are indeed Lie groups. Of course this is not hard to see without invoking the general theorem. It is also easy to identify the Lie algebras. For example the condition that $1+A$ is in $O(n)$ is $(1+A)(1+A^T) = 1$ which is

$$(A + A^T) + AA^T = 0.$$

When A is small the leading term is $A+A^T = 0$ and this is the equation defining the Lie algebra. So the Lie algebra of $O(n)$ is the space of skew-symmetric matrices. Similarly for he other examples.

When $n = 3$ we can write a skew-symmetric matrix as

$$\begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}$$

and the bracket $AB - BA$ becomes the cross-product on $\mathbf{R}^3$.

There is a fundamental relation between Lie groups and Lie algebras.

Theorem 1 *Given a finite dimensional real Lie algebra $\mathfrak{g}$ there is a Lie group $G = G_{\mathfrak{g}}$ with Lie algebra $\mathfrak{g}$ and the universal property that for any Lie group H with Lie algebra $\mathfrak{h}$ and Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{h}$ there is a unique group homomorphism $G \to H$ with derivative ρ.*

We will discuss the proof of this a bit later

In fact $G_{\mathfrak{g}}$ is the unique (up to isomorphism) connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Any other connected group with Lie algebra $\mathfrak{g}$ is a quotient of $G_{\mathfrak{g}}$ by a discrete normal subgroup.

Two Lie groups with isomorphic Lie algebras are called locally isomorphic..

Two other things we want to mention here.

Invariant quadratic forms

Given a representation $\rho : G \to GL(V)$ inducing $\rho : \mathfrak{g} \to \mathrm{End}(V)$ the map $\xi \mapsto -\mathrm{Tr}(\rho(\xi)^2)$ is a quadratic form on $\mathfrak{g}$, invariant under the adjoint action of G. The *Killing form* is the quadratic form defined in this way by the adjoint representation. In general these forms could be indefinite or even identically zero. If ρ is an orthogonal representation (preserving a Euclidean structure on V) then the quadratic form is ≥ 0 and if ρ is also faithful it is positive definite.

Complex Lie groups

We can define a *complex Lie group* in two ways which are easily shown to be equivalent

- A complex manifold with a group structure defined by holomorphic maps.

- A Lie group whose Lie bracket is complex bilinear with respect to a complex structure on the Lie algebra.

Any real Lie algebra $\mathfrak{g}$ has a complexification $\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$. It follows from the theorem above that, up to some complications with coverings, Lie groups can be complexified.

$$G \cong_{local} G' \subset G^{\mathbf{C}}.$$

If K is a real Lie subgroup of a complex Lie group G such that the Lie algebra of G is the complexification of the Lie algebra of K then K is called a *real form* of G.

Examples

- $SO(n)$ and $SO(p, n-p)$ are real forms of $SO(n, \mathbf{C})$.
- $Sp(n, \mathbf{R})$ and $Sp(n)$ are real forms of $Sp(n, \mathbf{C})$.
- $SU(n)$ and $SL(n, \mathbf{R})$ are real forms of $SL(n, \mathbf{C})$.
- $GL(n, \mathbf{H})$ is a real form of $GL(2n, \mathbf{C})$.

Study of the basic example, $SU(2)$

It is important to be familiar with the following facts. First examining the definitions one sees that $SU(2) \cong Sp(1)$. The Lie algebra of $SU(2)$ is three dimensional and its Killing form is positive definite so the adjoint action gives a homomorphism $SU(2) \to SO(3)$. it is easy to check that this is a 2:1 covering map with kernel $\{\pm 1\}$. So $SU(2), SO(3)$ are locally isomophic. We have seen that $SU(2)$ is the 3-sphere so it is simply connected and $\pi_1(SO(3)) = \mathbf{Z}/2$. This is demonstrated by the soup plate trick.

$SU(2)$ acts on $\mathbf{C}^2$ by construction and so on the projective space $\mathbf{CP}^1 = \mathbf{C} \cup \{\infty\} = S^2$. $SO(3)$ acts on $\mathbf{R}^3$ by construction and obviously acts on the 2-sphere seen as the unit sphere in 3-space. These actions are compatible with the covering map we have defined above. The action of $SU(2)$ is one way to define the *Hopf map* $h : S^3 \to S^2$. The 1-parameter subgroups in $SU(2)$ are all conjugate to

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

Thinking of $SU(2)$ as the 3-sphere these are the "great circles" through 1 and its antipodal point -1.

Some motivation for later theory

$Sp(2)$ acts on $\mathbf{H}^2$ and hence on the quaternionic projective line $\mathbf{HP}^1$. This is topologically the 4-sphere $S^4 = \mathbf{H} \cup \{\infty\}$. $SO(5)$ acts on $S^4 \subset \mathbf{R}^5$ and one can check that there is a 2-1 homomorphism $Sp(2) \to SO(5)$ under which these actions are compatible. In particular $Sp(2)$ and $SO(5)$ are locally isomorphic.

In general $\dim SO(2n+1) = \frac{1}{2}(2n+1)(2n) = n(2n+1)$ and $\dim Sp(n) = 3n + \frac{1}{2}4n(n-1) = 2n^2 + n$. So $SO(2n+1)$ and $Sp(n)$ have the same dimension and when $n = 1, 2$ they are locally isomorphic. Surprisingly, perhaps, they are not locally isomorphic when $n \geq 3$. Later in the course we will develop tools for proving and understanding this kind of thing.

1.2 Frobenius, connections and curvature

We have discussed vector fields on a manifold M, sections of the tangent bundle, and the Lie bracket. There is a dual approach using section of the cotangent bundle or more generally p forms $\Omega^p(M)$ and the exterior derivative $d : \Omega^p \to \Omega^{p+1}$. When $p = 1$ we have

$$d\theta(X, Y) = \nabla_X(\theta(Y)) - \nabla_Y(\theta(X)) - \theta([X, Y]),$$

so knowing d on 1-forms is the same as knowing the Lie bracket on vector fields. There is a simple formula for the Lie derivative on forms

$$\mathcal{L}_v \phi = (di_v + i_v d)\phi,$$

where $i_v : \Omega^p \to \Omega^{p-1}$ is the algebraic contraction operator.

If U is a fixed vector space we can consider forms with values in U: written $\Omega^p \otimes U$, the sections of $\Lambda^p T^*M \otimes U$. In particular we can do this when U is a Lie algebra $\mathfrak{g}$. The tensor product of the bracket $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ and the wedge product $\Lambda^1 \otimes \Lambda^1 \to \Lambda^2$ gives a quadratic map from $\mathfrak{g}$-valued 1-forms to $\mathfrak{g}$-valued 2-forms, written $[\alpha, \alpha]$.

Now let G be the Lie algebra of a Lie group G. There is a canonical $\mathfrak{g}$-valued left-invariant 1-form θ on G which satisfies the Maurer-Cartan equation

$$d\theta + \frac{1}{2}[\theta, \theta] = 0.$$

More explicitly,let e_i be a basis for $\mathfrak{g}$ and $[e_i, e_j] = \sum_k c_{ijk} e_k$. Let θ_i be the left-invariant 1-forms on G, equal to the dual basis of $\mathfrak{g}^*$ at the identity. Then one finds from (*) that

$$d\theta_k = -\frac{1}{2} \sum_{ij} c_{ijk} \theta_i \wedge \theta_j.$$

and the right hand side is $-\frac{1}{2}[\theta, \theta]$.

For a matrix group we can write $\theta = g^{-1} dg$.

1.2.1 Frobenius

Suppose we have a manifold N and a sub-bundle $H \subset TN$, a field of subspaces in the tangent spaces. Write π for the projection $TN \to TN/H$. Let X_1, X_2 be two sections of H the basic fact is that $\pi([X_1, X_2])$ at a point p depends only on the values of X_1, X_2 at p. Thus we get a tensor

$$\tau \in \Lambda^2 H^* \otimes TN/H.$$

Example Let $N \subset \mathbf{C}^n$ be a real hypersurface $(n > 2)$. Then $H = TN \cap (ITN)$ is a subbundle of TN. In this context the tensor above is called the *Levi form* and is important in several complex variables. For example suppose N' is another such submanifold and we have points $p \in N$ and $p' \in N'$. If the Levi form of N at p vanishes and that of N' at p' does not then there can be no holomorphic diffeomorphism from a neighbourhood of p to a neighbourhood of p' mapping N to N'.

The Frobenius theorem states that τ vanishes throughout N if and only if the field H is integrable. That is, through each point $p \in N$ there is a submanifold Q such that the tangent space at each point $q \in Q$ is the corresponding $H_q \subset TN_q$. The condition that $\tau = 0$ is the same as saying that the sections $\Gamma(H)$ are closed under Lie bracket. There is a dual formulation in terms of differential forms: if ψ is a form on N which vanishes when restricted to H then so does $d\psi$.

A basic example of the Frobenius theorem is given by considering a system of equations on $\mathbf{R}^n$

$$\frac{\partial f}{\partial x_i} = A_i(x, f),$$

where A_i are given functions. The equations can be regarded as defining a field H in $\mathbf{R}^n \times \mathbf{R}$ such that an integral submanifold is precisely the graph of a solution (at least locally). Suppose for simplicity that the A_i are just functions of x. Then the integrability condition is just the obvious one

$$\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} = 0.$$

Now consider a matrix version of this. So $A_i(x)$ are given $k \times k$ matrix-valued functions and we seek a solution of

$$\frac{\partial G}{\partial x_i} = GA_i.$$

For 1×1 matrices we can reduce to the previous case, at least locally, by taking logarithms. In general the integrability condition can be seem by computing

$$\frac{\partial^2 G}{\partial x_j \partial x_i}$$

imposing symmetry in i, j. Using

$$\frac{\partial G^{-1}}{\partial x_j} = -G^{-1} \frac{\partial G}{\partial x_j} G^{-1},$$

one finds the condition is

$$\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} + [A_i, A_j] = 0.$$

This is just the equation $dA + \frac{1}{2}[A, A] = 0$ if we define the 1-form $A = \sum A_i dx_i$ with values in the Lie algebra of $GL(k, \mathbf{R})$.

1.2.2 Bundles

Now consider a differentiable fibre bundle $p : \mathcal{X} \to M$ with fibre X. An *Ehresmann connection* is a field of subspaces as above which is complementary to the fibres. It can be thought of as an "infinitesimal trivialisation" of the bundle at each point of M. Given a path γ in M and a point $y \in p^{-1}(\gamma(0))$ we get a horizontal lift to a path $\tilde{\gamma}$ in $\mathcal{X}$ with $\tilde{\gamma}(0) = y$. (At least, this will be defined for a short time. If X is compact, say, it will be defined for all time.) In particular we get the notion of holonomy or parallel transport around loops in M.

In such a situation the quotient space $T\mathcal{X}/H$ at a point $y \in T\mathcal{X}$ can be identified with the tangent space V_y to the fibre (the vertical space). The horizontal space at y is identified with $TM_{p(y)}$. So we have $\tau(y) \in V_y \otimes \Lambda^2 T^* M_{p(y)}$.

We will be interested in *principal bundles*.

Definition A principal bundle over M with structure group G consists of a space P with a free right G action, an identification of the orbit space P/G with M such that $p : P \to M$ is a locally trivial fibre bundle, in a way compatible with the action.

This means that each point of M is contained in a neighbourhood U such that there is a diffeomorphism from $p^{-1}(U)$ to $U \times G$ taking the G action on $p^{-1}(U)$ to the obvious action on $U \times G$.

Example $S^1 \subset SU(2)$ acts by right multiplication on $SU(2)$ and the quotient space is S^2. This gives the Hopf map $S^3 \to S^2$ as a principal S^1 bundle.

Fundamental construction

Suppose that the Lie group G acts on the left on some manifold X. Then we can form

$$\mathcal{X} = P \times_G X$$

which is the quotient of $P \times X$ by identifying (pg, x) with (p, gx). Then $\mathcal{X} \to M$ is a fibre bundle with fibre X.

[**Remark** We can think of any fibre bundle as arising in this way if we are willing to take G to be the diffeomorphism group of the fibre.]

In particular we can apply this construction when we have a representation of G on a vector space and then we get a *vector bundle* over M.

Example

Let P be the frame bundle of M, so a point of P is a choice of a point of M and a basis for TM at this point. This is a principal $GL(n, \mathbf{R})$ bundle. Now $GL(n, \mathbf{R})$ has a representation on $\Lambda^p(\mathbf{R}^n)^*$ and the associated vector bundle is the bundle of p-forms.

Definition

A *connection* on a principal bundle P consists of a field of subspaces $H \subset TP$ (as before) invariant under the action of G.

In this context, the tensor τ is called the *curvature* of the connection.

Example

We can use this notion to analyse the well-known problem of the falling cat: a cat dropped upside down is able to turn itself over to land on its legs. For this we consider an abstract model K of the cat, so a position of the cat in space is a map $f : K \to \mathbf{R}^3$. Take the quotient of this space of maps by the translations. Then we get a space P on which the rotation group $SO(3)$ acts. The relevant maps do not have image in a line so the action is free and we get a principal $SO(3)$ bundle $P \to M = P/SO(3)$. (We could model K by a finite set, in which case P is an open subset of a product of a finite number of copies of $\mathbf{R}^3$ and so is a bona fide manifold.) The law of conservation of angular momentum defines a connection on P. By altering its geometry, the cat is able to impose a certain motion $\gamma(t)$ in M and the physical motion is the horizontal lift $\tilde{\gamma}(t)$ in P. By exploiting fact that the curvature of this connection does not vanish, the cat is able to find a path whose holonomy gives a rotation turning it the right way up.

If we have a connection on the principal bundle $P \to M$ we get an Ehresmann connection on any associated bundle $\mathcal{X}$. To see this consider the quotient map $TX \oplus TP \to T\mathcal{X}$ and take the image of $H \subset TP$.

Formalism

Write H_p as the kernel of the projection $\tilde{A}_p : TP_p \to V_p$ where V_p denotes the tangent space to the fibre. Now use the derivative of the action to identify V_p with $\mathfrak{g}$. Then $\tilde{A}$ is a $\mathfrak{g}$-valued 1-form on P. We have

1. $\tilde{A}$ is preserved by G, acting on P and by the adjoint action on $\mathfrak{g}$;

2. on each fibre, after any identification with G, $\tilde{A}$ is the Maurer-Cartan form θ

Set

$$F = dA + \frac{1}{2}[A, A].$$

This is a $\mathfrak{g}$-valued 2-form on P and is just the tensor τ in this context. It vanishes if and only if the field of subspaces H is integrable.

The direct sum decomposition

$$TP_p = H_p \oplus V_p$$

gives a decomposition

$$\Lambda^2 TP^* = \Lambda^2 TV_p^* \oplus \left(TH_p^* \otimes TV_p^*\right) \oplus \Lambda^2 TH_p^*.$$

By the second item above and the Maurer-Cartan equation the first component of F vanishes. One can also show that then invriance implies that the second component vanishes. This means that F can be viewed as a section of the bundle $\pi^*\Lambda^2 TM^* \otimes \mathfrak{g}$ over P. The transformation property (1) above means that F can also be viewed as a section of the bundle $\Lambda^2 TM^* \otimes adP$ over M where adP is the vector bundle over M with fibre $\mathfrak{g}$ associated to the adjoint action of G on $\mathfrak{g}$.

In particular, if $G = S^1$ and we fix an identification Lie$(S^1) = \mathbf{R}$ then the curvature is a 2-form on M. In fact it is a *closed* 2-form.

Choose a local trivialisation over $U \subset M$ by a section s of P and let $A = s^*(\tilde{A})$. This is a $\mathfrak{g}$-valued 1-form on U. Using this trivialisation to identify $P|_U$ with $U \times G$ we have

$$\tilde{A} = \theta + ad_{g^{-1}}(A). \qquad (****)$$

For a matrix group we can write this as

$$\tilde{A} = g^{-1}dg + g^{-1}Ag.$$

The same formula (****) can be read as saying that if we change the trivialisation by a map $g : U \to G$ then the connection 1-form, in the new trivialisation, is

$$g^*(\theta) + g^{-1}Ag.$$

In particular the statement that we can trivialise the connection locally if and only if the curvature vanishes is the same as:

Proposition

Let A be a $\mathfrak{g}$-valued 1-form on a ball U. We can write $A = g^*(\theta)$ for a map $g : U \to G$ if and only if $dA + \frac{1}{2}[A, A] = 0$.

Relation to the notion of a connection as a covariant derivative.

Let $\rho : G \to GL(n, \mathbf{R})$ be a representation and E the vector bundle associated to P and ρ. A section of E is the same as an equivariant map from P to $\mathbf{R}^n$. The covariant derivative of the section is defined by differentiating this map along the horizontal lifts of tangent vectors. In terms of a local trivialisation and local co-ordinates on M this boils down to defining

$$\nabla_i s = \frac{\partial s}{\partial x_i} + \rho(A_i)s,$$

where $A = \sum A_i dx_i$. From this point of view the curvature appears as the commutator

$$[\nabla_i, \nabla_j]s = \rho(F)s.$$

The construction of a Lie group from a Lie algebra

There is an attractive approach as follows. Suppose that G is a 1-connected Lie group with Lie algebra $\mathfrak{g}$. Then any group element can be joined to the identity by a path and any two such paths are homotopic. On the other a path $g : I \to G$ can be recovered from its derivative $g^*(\theta)$ which is a map $I \to \mathfrak{g}$. (This is the usual technique in mechanics.)

So, not having G *a priori* we consider maps $\gamma : [0,1] \to \mathfrak{g}$ and the relation $\gamma_0 \sim \gamma_1$ if there are $\Gamma_1, \Gamma_2 : I \times I \to \mathfrak{g}$ with

$$\Gamma_1(t,0) = \gamma_0(t)$$

$$\Gamma_1(t,1) = \gamma_1(t)$$

$$\Gamma_2(0,s) = \Gamma_2(1,s) = 0$$

$$\frac{\partial \Gamma_1}{\partial s} - \frac{\partial \Gamma_2}{\partial t} = [\Gamma_1, \Gamma_2].$$

The last is just the statement that $d\Gamma + \frac{1}{2}[\Gamma, \Gamma] = 0$ where $\Gamma = \Gamma_1 dt + \Gamma_2 ds$.

It is an interesting exercise to show that this is an equivalence relation, that there is a natural group structure on the space of equivalence classes and that the universal property holds. However it seems not so easy to show that what we have is a Lie group.

For another approach we work locally. There is a useful concept of a "Lie group germ", but having mentioned this we will just talk about groups with the understanding that we are ignoring global questions. Once we have constructed the Lie group germ we can use the approach above, or other global arguments, to obtain the group $G_{\mathfrak{g}}$.

If c_{ijk} are the structure constants of the Lie algebra then to construct the corresponding Lie group germ it is enough to construct a local frame of vector fields X_i on a neighbourhood of 0 in $\mathfrak{g}$ with $[X_i, X_j] = c_{ijk} X_k$. But this is also not so easy to do directly.

We can break the problem up by using the adjoint action. In general if H is a Lie group and $\mathfrak{k} \subset \mathfrak{h}$ a Lie subalgebra then a simple application of Frobenius constructs a Lie subgroup $K \subset H$. Now the adjoint action defines a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathrm{End}(\mathfrak{g})$ with image is a Lie subalgebra $\mathfrak{g}_0$ and we know that this corresponds to a Lie group G_0 by the above. The kernel of $\mathfrak{g} \to \mathfrak{g}_0$ is the centre $\mathfrak{z}$ of $\mathfrak{g}$. What we have to discuss is the constructions of central extensions of Lie groups.

Note: the treatment in lectures went a little wrong here so what follows is a corrected version

Choose a direct sum decomposition *as vector spaces* $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{z}$. There is a component of the bracket in $\mathfrak{g}$ which is a skew-symmetric bilinear map $\phi : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathfrak{z}$. The Jacobi identity implies that

$$\phi(a, [b,c]) + \phi(b, [c,a]) + \phi(c, [a,b]) = 0.$$

For simplicity, suppose that $\mathfrak{z} = \mathbf{R}$ so $\phi \in \Lambda^2 \mathfrak{g}_0^*$. Left translation gives a left-invriant 2-form $\tilde{\phi}$ on G_0 and the identity above is equivalent to saying that $d\tilde{\phi} = 0$.

Now suppose we have any manifold M with an action of a Lie group H. Suppose that F is a closed 2-form on M preserved by the action. By the Poincaré lemma we can write $F = dA$ (we emphasise again that this whole discussion is supposed to be local, in terms of germs etc.). We can regard A as a connection on the trivial $\mathbf{R}$-bundle P over M. (It is just as good to work with an S^1-bundle.) Now define a group $\tilde{H}$ of pairs $(h, \tilde{h})$ where h is in H and $\tilde{H} : p \to P$ is a lift of the action of h which preserves the connection. It is straightforward to show that we have an exact sequence

$$1 \to \mathbf{R} \to \tilde{H} \to H \to 1.$$

Putting these ideas together we take $M = G_0$ with the action of $H = G_0$ by left multiplication and we take the 2-form $\tilde{phi}$ on G_0. the construction gives a Lie group with the Lie algebra $\mathfrak{g}$.

Example
The Heisenberg group is $S^1 \times \mathbf{R}^{2n}$ with multiplication

$$(\lambda, v)(\lambda', v') = (e^{i\Omega(v,v')}\lambda\lambda', v + v')$$

where Ω is the standard skew-symmetric form on $\mathbf{R}^{2n}$. The Lie algebra gives the Heisenberg commutation relations.

2 Homogeneous spaces

If G is a Lie group and H is a Lie subgroup the set of cosets G/H is a manifold with a transitive action of G. Conversely, if M is a manifold with a transitive G action then $M = G/H$ where H is the stabiliser of some base point $m_0 \in M$. The tangent space of M at m_0 is naturally identified with $\mathfrak{g}/\mathfrak{h}$.

We will discuss two special classes: *symmetric spaces* and *co-adjoint orbits.*

2.1 Riemannian symmetric spaces

In general suppose $H \subset G$ as above and we have a positive definite quadratic form on $\mathfrak{g}/\mathfrak{h}$ invariant under the restriction of the adjoint action of $G$to H. This induces a Riemannian metric on G/H such that G acts by isometries. Symmetric spaces are a class of examples of this kind where the group structure and Riemannian structure interact in a specially simple way.

First, consider a Lie group G itself. Suppose we have an $\mathrm{ad} - G$-invariant positive definite quadratic form on $\mathfrak{g}$.

Proposition 2 *If G is compact these always exist.*

Then we get a *bi-invariant* Riemannian metric on G, preserved by left and right translations. (In the framework above, we can think of $G = (G \times G)/G$.

Exercise A bi-invariant metric is preserved by the map $g \mapsto g^{-1}$

Recall that the Levi-Civita connection of a Riemannian manifold is characterised by the conditions:

$$\nabla_X Y - \nabla_Y X = [X, Y],$$

$$\nabla_X (Y, Z) = (\nabla_X Y, Z) + (Y, \nabla_X X),$$

for vector fields X, Y, Z. Then the Riemann curvature tensor is

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

and the sectional curvature in a plane spanned by orthogonal vectors X, Y is

$$K(X, Y) = -(R(X, Y)X, Y).$$

In our case we restrict to left-invariant vector fields and we find that

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

Corollary 1 *The geodesics through* $1 \in G$ *are the* 1*-parameter subgroups.*

Corollary 2 *The exponential map of a compact Lie group is surjective.*

The curvature tensor is

$$R(X, Y)Z = \frac{1}{4}\left([X, [Y, Z] - [[Y, [X, Z] - [[X, Y], Z] + [Z, [X, Y]]\right),$$

which reduces to

$$R(X, Y)Z = -\frac{1}{4}[[X, Y], Z].$$

Hence

$$K(X, Y) = \frac{1}{4}|[X, Y]|^2.$$

In particular $K(X, Y) \geq 0$.

Now suppose we have a compact, connected Lie group G with a bi-invariant metric and an *involution* $\sigma : G \to G$: an automorphism with $\sigma^2 = 1$. Suppose that σ preserves the metric.

Set

$$K = \mathrm{Fix}\sigma = \{g \in G : \sigma(g) = g\}.$$

Then K is a Lie subgroup of G.

Also let $\tau : G \to G$ be defined by $\tau(g) = \sigma(g^{-1})$. The map τ is not a group homomorphism but we do have $\tau^2 = 1$. Set

$$M = \mathrm{Fix}(\tau) = \{g \in G : \tau(g) = g\}.$$

Then M is a submanifold of G. We define an action of G on M by

$$g(m) = gm\sigma(g)^{-1}.$$

Then the stabiliser of the point $1 \in M$ is K, so we can identify the G-orbit of 1 in M with G/K.

Now σ induces an involution of the Lie algebra $\mathfrak{g}$, which we also denote by σ. Since $\sigma^2 = 1$ we have a decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

into the ± 1 eigenspaces of σ. The -1 eigenspace $\mathfrak{p}$ is the tangent space of M at 1 and the derivative of the G action at the identity is twice the projection of $\mathfrak{g}$ onto $\mathfrak{p}$. So the G- orbit of 1 is an open subset of M. Since G is compact this orbit is also closed, so we see that the orbit is a whole connected component, M_0 say, of M. Such a manifold M_0 is a *compact Riemannian symmetric space.*

In general, suppose X is any Riemannian manifold and $f : X \to X$ is an isometry with $f^2 = 1$. Let F be a connected component of the fixed-point set of f. Then F is a *totally geodesic submanifold* of X; which is to say that any geodesic which starts in F with velocity vector tangent to F remains in F for all time. In this case the Riemann curvature tensor of the induced metric on F is simply given by the restriction of the curvature tensor of X. In particular if x, y are two tangent vectors to F at a point $p \in F$ the sectional curvature $K(x, y)$ is the same whether computed in F or in X.

We apply this to the isometry τ and we see that $M_0 = G/K$ is represented as a totally geodesic submanifold of G and the curvature is given by the same formula $K(x, y) = \frac{1}{4}|[x, y]|^2$, where now we restrict to $x, y \in \mathfrak{p}$. Up to a factor of 4, we get the same metric on G/K by regarding it as a submanifold of G as we do by using the general procedure and the identification $\mathfrak{g}/\mathfrak{k} = \mathfrak{p}$.

The conclusion is that we have a simple formula for the curvature of these compact symmetric spaces.

Examples

- $G = SO(n)$, $K = S(O(p) \times O(q))$ then G/K is the Grassmannian of p-dimensional subspaces of $\mathbf{R}^n$.
- the same but using complex or quaternionic co-efficients.
- $G = U(n)$, $K = O(n)$ then G/K is the manifold of Lagrangian subspaces of $\mathbf{R}^{2n}$.
- $G = SO(2n)$, $K = U(n)$ then G/K is the manifold of compatible complex structures $J : \mathbf{R}^{2n} \to \mathbf{R}^{2n} \quad J^2 = -1$.

Now consider the Lie algebra situation. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and the map given by 1 on $\mathfrak{k}$ and -1 on $\mathfrak{p}$ is a Lie algebra automorphism then the component of the bracket mapping $\mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ must vanish. We have

$$\mathfrak{k} \times \mathfrak{k} \to \mathfrak{k}, \mathfrak{k} \times \mathfrak{p} \to \mathfrak{p}, \mathfrak{p} \times \mathfrak{p} \to \mathfrak{k}.$$

(In fact the third component is the adjoint of the second defined by the metrics, so everything is determined by the Lie algebra $\mathfrak{k}$ and its orthogonal action on $\mathfrak{p}$.)

For example, if $G = SO(n), K = SO(n-1)$ then $\mathfrak{p} = \mathbf{R}^{n-1}$ is the standard representation of $SO(n-1)$.

We define a new bracket $[\ ,\ ]_*$ on $\mathfrak{k} \oplus \mathfrak{p}$ by reversing the sign of the component $\mathfrak{p} \times \mathfrak{p} \to \mathfrak{k}$. It is easy to check that $[\ ,\ ]_*$ satisfies the Jacobi identity. This is clearest if one works with $\mathfrak{k} \oplus i\mathfrak{p}$ inside the complexified Lie algebra. We define a new quadratic form Q^* by reversing the sign on the factor $\mathfrak{k}$. Then we get a new group, G^* say, containing K and the same discussion as before applies to the homogeneous space G^*/K. The difference is that G^* and G^*/K will not be compact. We have a bi-invariant pseudo-Riemannian metric on G^* but this induces a genuine Riemannian metric on G^*/K. When we compute the curvature $K(x,y)$ for $x, y \in \mathfrak{p}$ we get $Q^*([x,y])$. But $[x,y]$ lies in $\mathfrak{k}$ so

$$K(x,y) = -\frac{1}{4}|[x,y]|^2,$$

in terms of the positive definite form on $\mathfrak{k}$.

Conclusion: for any symmetric space of compact type, with (weakly) positive sectional curvature there is a dual space, of non-compact type, with (weakly) negative sectional curvature. The procedure can be reversed, so these symmetric spaces come in "dual pairs".

All this is a little imprecise since we could take products of compact and non-compact types, and products with Euclidean spaces or tori: it would be better to talk about *irreducible symmetric spaces.*

Examples

- The dual of the sphere $S^{n-1} = SO(n)/SO(n-1)$ is the hyperbolic space $SO_+(n-1,1)/SO(n-1)$.
- The dual of the complex projective space $\mathbf{CP}^{n-1} = SU(n)/U(n-1)$ is the complex hyperbolic space $\mathbf{CH}^{n-1} = SU(n-1,1)/U(n-1)$.
- The dual of $SU(n)/SO(n)$ is $SL(n,\mathbf{R})/SO(n)$, the space of positive definite symmetric matrices with determinant 1.
- The dual of $SU(n)$, regarded as $(SU(n) \times SU(n))/SU(n)$, is the space $SL(n,\mathbf{C})/SU(n)$ of positive definite Hermitian matrices of determinant 1. More generally the dual of a compact group G is G^c/G where G^c is the complexified group.

- The dual of $Sp(n)/U(n)$ is $Sp(n, \mathbf{R})/U(n)$, the space of compatible complex structures on $\mathbf{R}^{2n}$ with its standard symplectic form.

Precise definitions are:

Definition 1 *A Riemannian symmetric pair (G, K) consists of*

- *a Lie group G and a compact subgroup K,*
- *an involution σ of G such that K is contained in the fixed set $\mathrm{Fix}(\sigma)$ and contains the identity component of $\mathrm{Fix}(\sigma)$*
- *an $\mathrm{ad}G$-invariant, σ-invariant, quadratic form on $\mathfrak{g}$ which is positive definite on the -1 eigenspace $\mathfrak{p}$ of σ.*

A Riemannian globally symmetric space is a manifold of the form G/K, where (G, K) is a Riemannian symmetric pair as above, with the Riemannian metric induced from the the $\mathrm{ad}G$-invariant form on $\mathfrak{g}$.

These Riemannian manifolds can essentially be characterised by local differential geometric properties.

Proposition 3 *Let (M, g) be a Riemannian manifold. The following two conditions are equivalent*

- *For each point $x \in M$ there is a neighbourhood U of x and an isometry of U which fixes x and acts as -1 on TM_x.*
- *The covariant derivative ∇ Riem of the Riemann curvature tensor vanishes throughout M.*

In the above situation we call (M, g) a Riemannian locally symmetric space.

Proposition 4

- *A globally symmetric space is locally symmetric.*
- *If (M, g) is a locally symmetric space which is a complete Riemannian manifold then its universal cover is a Riemannian globally symmetric space.*
- *In any case if (M, g) is locally symmetric and x is any point in M then there is a neighbourhood U of x in M which is isometric to a neighbourhood in a Riemannian globally symmetric space.*

There is yet another point of view on these symmetric spaces. Suppose we have any homogeneous space $M = G/K$. Then G can be regarded as a principal K-bundle over M. The tangent bundle of M can be identified with the vector bundle associated to the action of K on $\mathfrak{g}/\mathfrak{k}$. Suppose we have an invariant form on $\mathfrak{g}$ giving a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then the translates of $\mathfrak{p}$ give a connection on this principal K- bundle, hence a connection on the tangent bundle of M. In general this will *not* be the same as the Levi-Civita connection, but for symmetric spaces it is. (In fact the two are equal precisely when the component $\mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ of the bracket vanishes.) This can be expressed by saying that the Riemannian holonomy group of a symmetric space G/K is contained in K.

The comprehensive reference for all this is the book of Helgason *Differential Geometry, Lie groups and symmetric spaces*. But most books on Riemannian geometry discuss parts of the theory (for example Cheeger and Ebin *Comparison Theorems in Riemannian geometry*).

2.2 Co-adjoint orbits

2.2.1 The symplectic picture

A Lie group G acts on its Lie algebra $\mathfrak{g}$ by the adjoint action. Thus it acts on the dual space $\mathfrak{g}^*$: the co-adjoint action. We consider the orbit $M \subset \mathfrak{g}^*$ of some $\xi \in \mathfrak{g}^*$. So $M = G/H$ where H is the stabiliser of ξ.

Example. Let Ω be the standard nondegenerate skew-symmetric form on $\mathbf{R}^{2n}$. The Heisenberg group is the set $\mathbf{R}^{2n} \times S^1$ with the multiplication

$$(x, \lambda).(y, \mu) = (x + y, \lambda\mu e^{i\Omega(x,y)}).$$

The adjoint orbits are either points or lines. The co-adjoint orbits are either points or copies of $\mathbf{R}^{2n}$.

Example. Take the group $U(n)$. We use the standard invariant form (and multiplication by i) to identify the Lie algebra and its dual with the space of Hermitian $n \times n$-matrices. Any such matrix is conjugate by an element of $U(n)$ to a diagonal matrix. A *maximal flag* in $\mathbf{C}^n$ is a chain of subspaces $V_1 \subset V_2 \ldots \subset V_{n-1} \subset V_n = \mathbf{C}^n$ with $\dim V_i = i$. Let $\mathbf{F}$ denote the set of maximal flags. Clearly $U(n)$ acts on $\mathbf{F}$ and $\mathbf{F} = U(n)/T$ where T is the n-dimensional torus consisting of diagonal unitary matrices. Fix real $\lambda_1, \ldots, \lambda_n$ with $\lambda_1 > \lambda_2 \ldots > \lambda_n$. Given a maximal flag $\{V_i\}$ let U_i be the orthogonal complement of V_{i-1} in V_i. So $\mathbf{C}^n = \bigoplus U_i$ and there is a unique hermitian matrix with eigenspaces U_i and corresponding eigenvalues λ_i. This gives a map

$$f_{\underline{\lambda}} : \mathbf{F} \to \mathfrak{g}^*,$$

whose image is a coadjoint orbit. Moreover the generic co-adjoint orbit has this form. The other co-adjoint orbits correspond, in a straightforward way, to flag manifolds with different sequences of dimensions; including Grassmannians and the projective space $\mathbf{CP}^{n-1}$.

Write $r_\xi : \mathfrak{g} \to \mathfrak{g}^*$ for the derivative of the co-adjoint action at ξ. Thus

$$\langle r_\xi(x), y\rangle = -\langle \xi, [x, y]\rangle,$$

for $x, y \in \mathfrak{g}$. So the tangent space of the orbit M at ξ is the image of r_ξ and the Lie algebra of H is the kernel of r_ξ.

Proposition 5 • *The linear map $\xi : \mathfrak{g} \to \mathbf{R}$ restricts to a Lie algebra homomorphism $\mathfrak{h} \to \mathbf{R}$.*

- *The recipe*

$$\omega(r_\xi(x), r_\xi(y)) = \langle \xi, [x, y]\rangle,$$

defines a nondegenerate skew-symmetric form on TM_ξ.

The proofs are rather trivial linear algebra exercises using the definitions.

By applying this at each point of the orbit we see that we have a G-invariant, nondegenerate, exterior 2-form ω on M. Clearly it is preserved by the action of G on M.

Proposition 6 *The form ω is closed, thus (M, ω) is a symplectic manifold.*

One way to see this is to use the formula

$$L_v\omega = di_v\omega + i_v d\omega,$$

for any vector field v, where L_v is the Lie derivative and i_v is the algebraic operation of contraction with v. Take v to be a vector field induced by the action of G on M, corresponding to some x in $\mathfrak{g}$. Then $L_v\omega = 0$ since G preserves ω. At the point ξ we have $v = r_\xi(x)$ and

$$\omega(v, r_\xi(y)) = \omega(r_\xi(x), r_\xi(y)) = \langle r_\xi(y), x\rangle,$$

Hence the 1-form $i_v\omega$ on M can be written as df where f is the restriction to M of linear function $f(\eta) = \langle \eta, x\rangle$. Thus $di_v\omega = ddf = 0$. It follows then that for any such vector field v we have $i_v d\omega = 0$. Since G acts transitively on M, these vector fields generate the tangent space at each point and we must have $d\omega = 0$.

The two structures appearing in the proposition above are related. To simplify the language and notation suppose that H is connected. (This is actually irrelevant for the local differential geometric discussion.) Call M an *integral orbit* if $\xi : \mathfrak{h} \to \mathbf{R}$ is induced by a Lie group homomorphism ρ from H to S^1.

Example For a suitable choice of normalisation of the invariant quadratic form on the Lie algebra, the co-adjoint orbit $f_{\underline{\lambda}}(\mathbf{F})$ of $U(n)$ discussed above is an integral orbit if and only if the λ_i are integers.

In the case of an integral orbit, we can form a principal S^1 bundle $E \to M$ associated to the principal H-bundle $G \to G/H$ and ρ, thought of as an action of H on S^1. This is an example of an important notion.

Definition 2 *Let G and Γ be Lie groups. A G-equivariant principal Γ-bundle is a manifold P with commuting actions of G (on the left) and Γ (on the right) such that the Γ-action defines a principal Γ-bundle.*

The definition implies that G acts on the base $M = P/\Gamma$ of the Γ bundle.
Examples

- For any Lie group G and subgroup K we can regard G as a G-equivariant K- bundle over $M = G/K$.
- Let M be any manifold with a G-action. Then the "frame bundle" P of the tangent bundle of M is a G-equivariant principle $GL(n, \mathbf{R}$- bundle over M, where $n = \dim M$.

In our situation the bundle $E \to M$ is a G-equivariant principle S^1-bundle over M. There is a natural connection on this bundle. To see this consider a point $z \in E$. The action of G gives a linear map $r : \mathfrak{g} \to TE_z$. We restrict r to the kernel of $\xi : \mathfrak{g} \to \mathbf{R}$ and the image of this in TE_z is (one checks) a subspace transverse to the S^1-fibre. One can also see that the curvature of this connection is the 2-form ω.

Recall the basic Hamiltonian construction in symplectic geometry. Let (V, Ω) be a symplectic manifold and $x \in TV_p$. Then

$$y \mapsto \Omega(x, y)$$

is an element of the dual space T^*V_p. So Ω is thought of as a linear map $TV_p \to T^*V_p$ and the condition that Ω is nondegenerate means that this is invertible so we have $\Omega^{-1} : T^*V \to TV$. Now if H is a function on V we take the derivative dH and apply Ω^{-1}, to get a vector field X_H. The vector fields appearing this way are characterised by the fact that, at least locally, they are exactly those which generate 1-parameter subgroups of "symplectomorphisms" (i.e. preserving ω).

What we get in this way is a homomorphism of *infinite dimensional* Lie algebras

$$C^\infty(V) \to \mathrm{SDiff}(V),$$

where the Lie algebra structure on $C^\infty(V)$ is the *Poisson bracket.* Now suppose a Lie group G acts on V, preserving Ω. So we have a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathrm{SDiff}(V)$. We say that the action is *Poisson* if this can be lifted to a Lie algebra homomorphism

$$\mu^* : \mathfrak{g} \to C^\infty(V).$$

Giving this is equivalent to giving a *moment map* $\mu : V \to \mathfrak{g}^*$. In fact the condition can be expressed directly in terms of μ by the requirements that

- μ is an *equivariant* map with respect to the given G-action on V and the co-adjoint action on $\mathfrak{g}^*$;
- Fix a basis e_i of $\mathfrak{g}$ and dual basis ϵ_i of $\mathfrak{g}^*$. Then μ has components μ_i which are functions on V. We require that these are Hamiltonian functions for the vector fields on V corresponding to the generators e_i.

Now consider the case when $(V, \Omega) = (M, \omega)$. We find that the action is Poisson and the moment map $\mu : M \to \mathfrak{g}^*$ is just the inclusion map. If (V, Ω) has a transitive Poisson action then the image of μ is a co-adjoint orbit M and the map $\mu : V \to M$ is a covering map. Thus, up to coverings, the co-adjoint orbits are the only symplectic manifolds with transitive Poisson actions.

Poisson actions are related to equivariant circle bundles. Suppose $E \to V$ is a principal S^1-bundle having a connection with curvature ω, a symplectic form on V. Then if E is an G-equivariant S^1-bundle the G action on V is Poisson. The constants in $C^\infty(V)$ act on E by constant rotation of the fibres. The corresponding co-adjoint orbit is integral.

Example Consider $V = \mathbf{R}^{2n}$ with its standard symplectic form Ω. There is an S^1-bundle $E \to V$ having a connection with curvature Ω. We can consider the action of $G_0 = \mathbf{R}^{2n}$ acting on V by translations, preserving Ω. This is not a Poisson action and it does not lift to the S^1-bundle. Now take G to be the Heisenberg group so we have homomorphisms

$$S^1 \to G \to G_0.$$

Then G acts on V (via the homomorphism to G_0) and this action *is* Poisson and does lift to E. In fact we can take E to be the Heisenberg group itself, and the action to be left multiplication.

General references for this subsection are the books of Arnold *Mathematical Methods in Classical Mechanics* and Guillemin and Sternberg *Symplectic techniques in physics*. Also the lecture course of Dominic Joyce.

2.2.2 The complex picture

Now suppose our Lie group G is compact, with a fixed invariant Euclidean form on $\mathfrak{g}$. So we can identify $\mathfrak{g}$ with its dual. Given $\xi \in \mathfrak{g}$ we have

$$\mathfrak{g} = \mathfrak{h} \oplus W$$

where H is the stabiliser of ξ and W is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. Then W is naturally identified with the tangent space of M. In this set-up, H is just the set of $\eta \in \mathfrak{g}$ with $[\eta, \xi] = 0$, so we certainly have $\xi \in \mathfrak{h}$. We consider the linear map

$$\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}.$$

So the kernel of ad_ξ is exactly $\mathfrak{h}$ and ad_ξ preserves W. Now ad_ξ lies in the Lie algebra $\mathfrak{so}(W)$ so by the standard classification of orthogonal matrices modulo conjugation we can decompose W into a sum of two dimensional subspaces on which $\exp(t\,\mathrm{ad}_\xi)$ acts as a rotation. By fixing the sign of the rotation we get a natural *complex structure* on the vector space W. This is easier to see after complexification. We write $W \otimes \mathbf{C} = W_- \oplus W_+$ where W_- is spanned by eigenvectors of $i\,\mathrm{ad}_\xi$ with eigenvalue < 0 and W_+ with eigenvalue > 0. Then W_-, W_+ are complex conjugates and we have an isomorphism $W_+ \to W$ of real vector spaces given by

$$w_+ \mapsto \overline{w_+} \oplus w_+.$$

The complex structure on W is inherited from that on W_+ via this isomorphism.

Now we want to see that this "almost complex" structure on M is in fact a complex structure.

Proposition 7 *The vector subspaces W_+ and $W_+ \oplus \mathfrak{h} \otimes \mathbf{C}$ are Lie subalgebras of $\mathfrak{g} \otimes \mathbf{C}$.*

This follows from THE KEY CALCULATION OF LIE ALGEBRA THEORY.

If w_a, w_b are eigenvectors of $i\,\mathrm{ad}_\xi$ with eigenvalues a, b then $[w_a, w_b]$ is either 0 or an eigenvector with eigenvalue $a + b$.

Let G^c be the complexification of G and $P \subset G^c$ the complex subgroup corresponding to $W_+ \oplus \mathfrak{h} \otimes \mathbf{C}$. Actually we only need neighbourhoods of the identity in these groups, so we do not need to worry about global questions of the existence of these groups. The implicit function theorem implies that there is a neighbourhood U of the identity in G^c such that any element $g \in U$ can be written as a product of an element in P and an element in G, uniquely up to multiplication by H. This means that a neighbourhood of ξ in $M = G/H$ is identified with a neighbourhood in G^c/P. The latter is obviously a complex manifold.

For example, the flag manifolds, complex Grassmannians and projective spaces which arise as co-adjoint orbits for $U(n)$ are clearly complex manifolds. We do have actions of the complexification $GL(n, \mathbf{C})$. The subgroups P are groups of "block lower triangular matrices".

A less obvious example is the Grassmann manifold $Gr_2(\mathbf{R}^n)$ of oriented planes in $\mathbf{R}^n$. Given such a plane choose an oriented orthonormal basis v_1, v_2. So v_i are vectors in $\mathbf{R}^n$ with

$$v_1.v_1 = v_2.v_2 = 1 \quad , \quad v_1.v_2 = 0.$$

Then $w = v_1 + iv_2$ is a vector in $\mathbf{C}^n$ with $w.w = 0$, for the obvious complex bilinear extension of the inner product. Changing the choice of v_i in the same plane changes w by multiplication by a complex scalar. Thus we see that $Gr_2(\mathbf{R}^n)$ can be identified with a complex quadric hypersurface $Q \subset \mathbf{CP}^{n-1}$. The complexified group $SO(n, \mathbf{C})$ obviously acts on Q.

The discussion in this subsection applies in some situations to non-compact groups. For example the co-adjoint orbit of $SL(2, \mathbf{R})$ which is one component of a two-sheeted hyperbola has an invariant complex structure (and can be thought of as the upper half-plane in $\mathbf{C}$). But the orbit which is a one-sheeted hyperbola does not have a natural complex structure.

3 Compact real forms

A vector subspace I in a Lie algebra $\mathfrak{g}$ is an *ideal* if $[I, \mathfrak{g}] \subset I$. This corresponds to the notion of a normal subgroup. A Lie algebra is called *simple* if it has no proper ideals, and is not abelian.

Examples

- The Lie algebras of $SL(n, \mathbf{C}), SO(n, \mathbf{C}), Sp(n, \mathbf{C}) \quad (n > 1)$ are simple except in the cases of $SO(2, \mathbf{C}), SO(4, \mathbf{C})$.
- The Lie algebra of the group of oriented isometries of $\mathbf{R}^n$ is not simple; it fits into an exact sequence

$$0 \to \mathbf{R}^n \to \mathfrak{g} \to \mathfrak{so}(n) \to 0.$$

- The Lie algebra of the Heisenberg group is not simple; it fits into an exact sequence

$$0 \to \mathbf{R} \to \mathfrak{g} \to \mathbf{R}^{2n} \to 0.$$

- The Lie algebra of upper triangular $n \times n$ matrices (in a field $k = \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$) is not simple; it fits into an exact sequence

$$0 \to \mathfrak{n} \to \mathfrak{g} \to k^n \to 0,$$

  where $\mathfrak{n}$ is the ideal of matrices with zeros on the diagonal.

There are general theorems, similar to finite group theory, which assert that any Lie algebra can be "built up" from simple and abelian ones; in exact sequences. Another notion is that of a "semisimple" Lie algebra. there are various definitions which turn out to be equivalent. The easiest is to say that a semisimple Lie algebra is a direct sum of simple algebras.

Theorem 2 *Let $\mathfrak{g}$ be a simple complex Lie algebra. Then there is a complex Lie group G with Lie algebra $\mathfrak{g}$ and a compact subgroup $K \subset G$ such that the Lie algebra $\mathfrak{g}$ is the complexification of the real Lie algebra $\mathfrak{k}$.*

For example we have $SU(n) \subset SL(n, \mathbf{C}), SO(n) \subset SO(n, \mathbf{C}), Sp(n) \subset Sp(n, \mathbf{C})$. The upshot is that the study of the structure and representations of simple (and semi-simple) complex Lie groups is essentially equivalent to the study of compact groups.

To prove the Theorem we consider a Lie bracket on $V = \mathbf{C}^n$ as a point w in the vector space

$$W = \Lambda^2 V^* \otimes V.$$

More generally we could consider any representation W of $SL(V)$. So we have a map $\rho : SL(V) \to SL(W)$. For brevity we will refer to a positive definite Hermitian form on a complex vector space as a "metric". A metric on V induces a metric on W in a standard way. One can express this abstractly as follows. We fix some metric H_0 on V, thus we have a $SU(V) \subset SL(V)$. Since $SU(V)$ is compact we can find a metric on W which is preserved by $SU(V)$ so ρ maps $SU(V)$ to $SU(W)$. Then ρ induces a map $SL(V)/SU(V) \to SL(W)/SU(W)$. Now $SL(V)/SU(V)$ can be identified with the metrics on V of a fixed determinant, and likewise for W. So each metric H on V, with fixed determinant, we have a metric $\rho(H)$ on W.

Now write $\mathcal{H}$ for the space of metrics on V, that is $\mathcal{H} = SL(V)/SU(V) = SL(n, \mathbf{C})/SU(n)$. We define a function F on $\mathcal{H}$ by

$$F(H) = |w|^2_{\rho(H)}.$$

Suppose we have found a point in $\mathcal{H}$ which is a critical point of F (in fact this will be a minimum as we shall see in a moment). We may as well suppose that this metric is H_0, so $SU(V) \subset SL(V)$ is the subgroup which preserves the metric. We let $G \subset SL(V)$ be the subgroup which fixes $w \in W$, under the action ρ, and let $K = G \cap SU(V)$. Then we have

Proposition 8 *In this situation, G is the complexification of K.*

The proof is very easy. We can think of the function F in an equivalent way as follows. We fix the metric $\rho(H_0)$ on W and for each $g \in SL(V)$ we define

$$\tilde{F}(g) = |\rho(g)(w)|^2_{\rho(H_0)}.$$

Then $\tilde{F}$ is a function on $SL(V)$ which is invariant under left multiplication by $SU(V)$ so descends to a function on $\mathcal{H}$, and this is exactly F. The derivative $d\rho$ is complex linear and maps the Lie algebra $\mathfrak{sl}(V)$ to $\mathfrak{sl}(W)$. It maps the real subspace $\mathfrak{su}(V)$ to $\mathfrak{su}(W)$. It follows that $d\rho$ must take the operation of forming the adjoint in V to that in W; i.e.

$$d\rho(\xi^*) = (d\rho(\xi))^*.$$

The condition that $\tilde{F}$ has a critical point is that

$$\langle w, d\rho(\xi) w \rangle = 0,$$

for all $\xi \in \mathfrak{sl}(V)$. Here $\langle\ ,\ \rangle$ denotes the real part of the Hermitian form $\rho(H_0)$ on W. In particular take $\xi = [\eta, \eta^*]$ for some η in the Lie algebra of G. Then if $A = d\rho(\eta)$ we have

$$d\rho([\eta, \eta^*) = [A, A^*].$$

So

$$\langle w, [A, A^*]w\rangle = 0$$

But this gives

$$|Aw|^2 - |A^*w|^2 = 0.$$

Thus $Aw = 0$ if and only if $A^*w = 0$. The condition that η is in the Lie algebra of G is precisely that $Aw = 0$. So we have $A^*w = 0$ and η^* also lies in the Lie algebra of G. Thus $\eta \mapsto -\eta^*$ is an antilinear isomorphism from Lie(G) to itself and the result follows immediately from the fact that multiplication by i interchanges the ± 1 eigenspaces of this map.

In the case we are interested in, when w is the bracket on $V = \mathfrak{g}$, the Lie algebra Lie(G) is the algebra of *derivations* of $\mathfrak{g}$, that is maps $\alpha : \mathfrak{g} \to \mathfrak{g}$ with

$$\alpha([x, y]) = [\alpha x, y] + [x, \alpha y].$$

Now suppose we have a critical point of F, as above. To prove the Theorem it suffices to see that the Lie algebra of G is the Lie algebra $\mathfrak{g}$ we started with. In general the adjoint action gives a Lie algebra homomorphism

$$\mathrm{ad}\ : \mathfrak{g} \to \mathrm{Lie}G.$$

The image is an ideal since for any $\alpha \in$ Lie(G) and $\xi \in \mathfrak{g}$

$$[\mathrm{ad}\ _\xi, \alpha] = \mathrm{ad}\ _{\alpha(\xi)}.$$

Since $\mathfrak{g}$ is simple the map must be an injection. The restriction of the standard from on $\mathfrak{su}(V)$ gives an invariant inner product on Lie(G) and we have an orthogonal direct sum

$$\mathrm{Lie}(G) = \mathfrak{g} \oplus I,$$

where I is also an ideal. But this means that $[I, \mathfrak{g}] = 0$, which is the same as saying that I acts trivially on $\mathfrak{g}$ so, by its definition as a set of operators on $\mathfrak{g}$, I must be trivial.

Thus our problem comes down to showing that F has a critical point, when $\mathfrak{g}$ is simple. More generally, for an arbitrary representation W as above we have

Theorem 3 *Either F has a critical point or there is a nontrivial subspace of V invariant under the stabiliser G of w.*

In the case at hand, a non-trivial subspace of $V = \mathfrak{g}$ invariant under the action of G would have to be invariant under the adjoint action, hence an ideal in $\mathfrak{g}$.

The key to proving the theorem is

Proposition 9 *F is convex along geodesics in $\mathcal{H}$*

In fact one finds that for any geodesic γ the function is a finite sum

$$F(\gamma(t)) = \sum a_i e^{n_i t},$$

with $a_i > 0$.

The group $SL(V)$ acts on $\mathcal{H}$ by isometries and the subgroup G preserves F. As an exercise in Riemannian geometry we can consider the more general situation of

- A simply-connected Riemannian manifold M of non-positive sectional curvature.
- A function f on M which is convex along geodesics.
- A group Γ of isometries of M, preserving f.

In this situation one can define the "sphere at infinity" $S_\infty(M)$ and Γ acts continuously on $S_\infty(M)$. Then we have

Theorem 4 *Either f achieves a minimum in M or there is a point in $S_\infty(M)$ fixed by Γ.*

To complete the proof of the main result we just have to interpret what it means for G to have a fixed point in $S_\infty(\mathcal{H})$. In terms of a base point $H_0 \in \mathcal{H}$ we can identify $S_\infty(\mathcal{H})$ with the unit sphere in the Lie algebra $\mathfrak{su}(V)$. This is written as a union of co-adjoint orbits (flag manifolds of different kinds) and the action of $SL(V)$ is given by the standard action on these orbits. So a fixed point in $S_\infty(\mathcal{H})$ is the same as a fixed flag, and in particular gives a non-trivial G-invariant subspace.

Full details of the proofs sketched above can be found in the preprint *Lie algebra theory without algebra* arXiv:math.DG/0702016v2. This is written in a way that does not require any technical background. The arguments are done there for the "real" case, but the main steps are identical. The corresponding result in the real case can be expressed in terms of symmetric spaces. For any simple real Lie algebra $\mathfrak{g}$ which is not the Lie algebra of a compact group there is a Lie group G with Lie algebra $\mathfrak{g}$ and a *maximal compact subgroup* $K \subset G$, such that G/K is a symmetric space of non-compact type. Moreover, K is unique up to conjugation. In this way we get a 2-1 correspondence between

- simply-connected, irreducible globally symmetric spaces,
- simple real Lie algebras which are not the Lie algebras of compact groups.

(For each Lie algebra $\mathfrak{g}$ in the latter category, we get a pair of dual symmetric spaces. If the Lie algebra is actually complex the pair is $K, K^c/K$ where K is a compact group and K^c its complexification.)

Expressed in terms of Lie algebras, the upshot is that the classification of simple real Lie algebras is reduced to either one of

- The classification of simple Lie algebras of compact groups, together with involutions;
- The classification of simple complex Lie algebras, together with **C**-linear involutions.

4 Representation Theory

4.1 The statement of the main theorem, and examples.

We consider a compact connected Lie group G, with a fixed invariant form on its Lie algebra. We want to describe the irreducible (finite-dimensional) complex representations of G. (By the result of Section 3, it would be the same to ask about the representations of a semi-simple complex Lie group.)

Theorem 5 *There is a one-to-one correspondence between the irreducible representations and integral co-adjoint orbits in $\mathfrak{g}^*$.*

More precisely, of course, the correspondence is described by a definite procedure.

Recall that a co-adjoint orbit M is a complex manifold. It is also a symplectic manifold and these structures are compatible in that the symplectic form arises from a Hermitian metric on the tangent space. (We forgot to mention this in Section 2, but it follows immediately from the definitions.) Thus (by definition), M is a *Kahler manifold.* If M is integral we have an S^1-bundle $U \to M$ and we can form the associated vector bundle L. This is naturally a holomorphic line bundle and (with the right choice of signs) it has a connection with curvature the Kahler form ω.

We make two constructions

- Starting with an integral orbit M we let $V = V_M$ be the space of holomorphic sections of $L \to M$. This is a vector space on which G acts.
- Starting with an irreducible representation V we consider the action of G on the projective space $\mathbf{P} = \mathbf{P}(V^*)$. This is a symplectic manifold and the action is Poisson, with a moment map $\mu : \mathbf{P} \to \mathfrak{g}^*$ (which we will write down explicitly later). We take the orbit $M = M(V)$ of a point where $|\mu|$ is maximal.

The first construction is essentially what is known as the *Borel-Weil* construction of the irreducible representations. One can compare it with the formation of induced representations for finite groups. That is, if $A \subset B$ are finite groups we consider the (zero-dimensional) manifold B/A and the group B as a principal A bundle over B/A. Then if U is a representation of A we form the vector bundle E over B/A associated to the principal bundle. The space of sections of $E \to B/A$ is a representation of B denoted $Ind_A^B U$.

To prove our theorem we will have to check/establish a number of things.

1. Any co-adjoint orbit M is simply connected (so we do not have to worry about coverings).
2. The vector space V_M is finite-dimensional.
3. The vector space V_M is non-trivial.
4. The vector space V_M is an irreducible representation of G.
5. Given an irreducible representation there is a *unique* G-orbit in the projective space where $|\mu|$ is maximal.
6. The orbit $M(V)$ is integral.
7. $V_{M(V)} = V$ for any irreducible representation V.
8. $M(V_M) = M$ for any integral orbit M.

But before getting on with this we discuss examples.

Example 1 $G = S^1$. Then $\mathfrak{g}^* = \mathbf{R}$ and the co-adjoint orbits are points. The integral co-adjoint orbits correspond (with a suitable normalisation of the inner product) to $\mathbf{Z} \subset \mathbf{R}$. Following the recipe we find the representation corresponding to $n \in \mathbf{Z}$ is the n-fold tensor power of the defining representation $S^1 \subset \mathbf{C}$.

Example 2 $G = SU(2)$. With a suitable normalisation, the integral orbits are the sets $|x| = d$ in $\mathbf{R}^3$, for integer $d \geq 0$. When $d = 0$ we get the trivial 1-dimensional representation. Otherwise, the orbits are 2-spheres and the line bundle is the one which we denoted by L_d before. As we (more-or-less) saw, the holomorphic sections correspond to holomorphic functions on $\mathbf{C}^2 \setminus \{0\}$ which are homogeneous of degree d. The only such are polynomials of degree d. The representation we get is S^d, the d-fold symmetric power of the standard 2-dimensional representation. If we take $SO(3)$ in place of $SU(2)$ the co-adjoint orbits are the same but the integrality condition is different: the integral orbits for $SO(3)$ are those corresponding to even values of d.

The fact that the S^d are the only irreducible representations of $SU(2)$ can be proved quite easily by an algebraic method. We extend to $SL(2, \mathbf{C})$ (by complex linearity) and consider standard generators H, X, Y of $\mathfrak{sl}_2$ which satisfy

$$[H, X] = 2X, [H, Y] = -2Y, [X, Y] = 2H.$$

Now consider the decomposition of a representation V into eigenspaces for H. A manipulation similar to that in the KEY CALCULATION from Section 2.2.2 shows that if $v \in V$ is an eigenvector of H with eigenvalue λ then Xv is either

0 or an eigenvector with eigenvalue $\lambda + 2$, and Yv is either 0 or an eigenvector with eigenvalue $\lambda - 2$. Take an eigenvector v with largest H-eigenvalue d say. Then one finds that $Xv = 0$ and V must be generated by

$$v, Yv, Y^2 v, \ldots, Y^r v,$$

where we go on until $Y^{r+1}v = 0$. Since the trace of the H action must be 0 we find that $r = d$ and the H-eigenvalues are

$$d, d-2, \ldots, -d;$$

and the dimension of V is $d + 1$. Since the Lie algebra action is entirely determined by the relations we see that $V = S^d$.

This algebraic approach can be applied to other groups (see Fulton and Harris, for example), but becomes considerably more complicated.

There is another, more geometric, way of thinking about a line bundle L over a complex manifold M and the space V of holomorphic sections. Suppose that at each point of M there is a section which does not vanish. The ratios of the sections at a point are well-defined complex numbers and we get a map

$$M \to \mathbf{P}(V^*),$$

defined in terms of a basis $s_1, \ldots, s_n$ of V by mapping $x \in M$ to $[s_1(x), \ldots, s_n(x)] \in \mathbf{P}^{n-1}$. Another way of saying this is that for each point $x \in M$ we have an evaluation map

$$e_x : V \to L_x,$$

which can be identified with an element of V^*, up to a scalar multiple. Conversely, given a complex submanifold (say) $M \subset \mathbf{P}^{n-1}$ which does not lie in any linear subspace we restrict the Hopf bundle $L_1 \to \mathbf{P}^{n-1}$ to M. The sections of L_1 over the projective space restrict to sections over M and we recover the previous set-up. In the case when $M = \mathbf{CP}^1$ is the Riemann sphere with the line bundle L_d we get an embedding

$$\mathbf{CP}^1 \to \mathbf{CP}^d$$

whose image is the "rational normal curve". Then $SL(2, \mathbf{C})$ acts on $\mathbf{CP}^d$ and the rational normal curve is the unique *closed orbit.*

Example Take $G = SU(3) = SU(W)$. The integral co-adjoint orbits correspond to integers $(\lambda_1, \lambda_2, \lambda_3)$ modulo the action $\lambda_i \mapsto \lambda_i + n$ (since we are working with $SU(3)$ rather than $U(3)$, and permutations of the λ_i. We can choose a representative $\lambda_1 = a, \lambda_2 = 0, \lambda_3 = -b$ with integers $a, b \geq 0$. The corresponding orbits are:

- The point 0, when $a = b = 0$.
- A copy of $\mathbf{P} = \mathbf{P}(W)$ when $a = 0, b > 0$,
- A copy of $\mathbf{P}^* = \mathbf{P}(W^*) = \mathbf{P}(\Lambda^2 W)$ when $b = 0, a > 0$.
- A copy of the flag manifold $\mathbf{F}$ when $a > 0, b > 0$.

The first case gives the trivial representation. The second gives the symmetric power $S^b W^*$. The third gives $S^a W$. The fourth case gives the kernel of the contraction map

$$S^a W \otimes S^b W^* \to S^{a-1} W \otimes S^{b-1} W^*.$$

Notice that we have holomorphic fibrations $\mathbf{F} \to \mathbf{P}$ and $\mathbf{F} \to \mathbf{P}^*$. The line bundles over $\mathbf{P}, \mathbf{P}^*$ can be pulled back to $\mathbf{F}$ and, if we prefer, everything can be expressed in terms of sections of line bundles over $\mathbf{F}$. (The analogue of this applies in general and is the more usual way of setting up the Borel-Weil construction.) The identification of the representation in the fourth case above is fairly straightforward, using the fact that the flag manifold $\mathbf{F}$ is cut out by a single equation in $\mathbf{P} \times \mathbf{P}^*$.

Example Take $G = SU(n) = SU(W)$. This goes much as before, but becomes more complicated. When the sequence λ_i takes just two distinct values $0, 1$ we get a co-adjoint orbit which is a Grassmannian $Gr_k(W)$. The sections of the corresponding line bundle give the Plücker embedding

$$Gr_k(W) \to \mathbf{P}(\Lambda^k).$$

The representation we get is $\Lambda^k W^*$ which is isomorphic to $\Lambda^{n-k} W$. For the general case we get subspaces (or, if you prefer, quotients) of the standard representations

$$S^{\mu_1} \Lambda^1 W \otimes S^{\mu_2} \Lambda^2 W \otimes \dots .$$

There is an intricate theory of Young diagrams etc. which describes these irreducible representations explicitlyin terms of tensors satisfying certain symmetry conditions. All this is related to the irreducible representations of the symmetric groups Σ_p. See Fulton and Harris, for example.

4.2 Proof of the main theorem

The main idea in the proof is this. Let V be a representation of G. Suppose we can find a G-orbit $\tilde{M}$ in $\mathbf{P}(V^*)$ which is a *complex submanifold.* (It follows that $\tilde{M}$ is a closed orbit for the action of the complexified group: the basic example is the rational normal curve in the case when $G = SU(2)$.) Then the symplectic form on $\mathbf{P}(V^*)$ restricts to symplectic form on $\tilde{M}$ and the moment map $\mu : \mathbf{P}(V^*) \to \mathfrak{g}^*$ restricts to a moment map on $\tilde{M}$. Thus we have a transitive Poisson action of G on $\tilde{M}$ and $\tilde{M}$ is a covering of a co-adjoint orbit. Assuming that we have established item 1, we have $\tilde{M} = M$. Consider the line bundle $H \to \mathbf{P}(V^*)$, so the space of holomorphic sections of H is V. Let p be a point of $\tilde{M}$. The stabiliser $H \subset G$ of p is the same as the stabiliser of $\mu(p)$ in the co-adjoint action. Checking the definitions we see that M is an integral co-adjoint orbit (item 6) and that H is identified with the line bundle $L \to M$ which we constructed. So restriction to $\tilde{M}$ gives a non-zero G-map from V to V_M. If V is irreducible this must be injective and, if we have established item 7, it must be an isomorphism.

For most of the proof we can think about the following general picture. We suppose we have a Kahler manifold (X, Ω) and a vector field Iv on X which generates a 1-parameter group of holomorphic isometries, hence symplectomorphisms. We suppose that $H : X \to \mathbf{R}$ is a Hamiltonian for Iv. The critical points of H are the zeros of v. The vector field v generates another flow $\alpha_t : X \to X$: these maps are holomorphic but are *not isometries.* The critical points of H are the zeros of v. At a zero we have a derivative ∇v which is a complex-linear endomorphism of the tangent space. This can be identified with the Hessian of H (matrix of second derivatives). In particular if p is a maximum of H the eigenvalues of ∇v are ≤ 0.

Lemma 1 *Suppose p is a point in X where H is maximal.*

- *Let N be another vector field on X and suppose that*

  $$[v, N] = \lambda N$$

  for some $\lambda < 0$. Then N vanishes at p.

- *Suppose F is a function which Poisson commutes with H and generates a vector field Z. Then $IZ(p)$ lies in the kernel of the Hessian of H. If $Z(p)$ does not vanish then the derivative of F at p in the direction $IZ(p)$ is non-zero.*

For the first part, it is a general fact that for any two vector fields $[V, N] = -(\nabla V)(N)$ at a zero of V. So if $N(p)$ is not zero it is an eigenvector for ∇v with positive eigenvalue which is impossible.

For the second part, any time we have a Hamiltonian F generating a vector field Z on a Kahler manifold we have $\nabla_{IZ} F = |Z|^2$. So all we need to show is that IZ lies in the kernel of the Hessian. But since F and H Poisson commute the vector Z lies in the kernel of the Hessian (as in the first part) and the kernel is a complex linear subspace.

Corollary 3 *Suppose a compact Lie group G acts on a compact Kahler manifold X, with moment map $\mu : X \to \mathfrak{g}^*$. If $p \in X$ is a point where $|\mu$ is maximal then the G-orbit of p is a complex submaifold of X.*

To see this, let $\xi = \mu(p)$ and let $H = \langle \xi, \mu \rangle$. This is the Hamiltonian for the action of the 1-parameter subgroup generated by ξ. (We identify $\mathfrak{g}$ with its dual.) Consider the decomposition of the complexified Lie algebra

$$\mathfrak{g} \otimes \mathbf{C} = W^- \oplus (\mathfrak{h} \otimes \mathbf{C}) \oplus W^+.$$

The complexification of the derivative of the action at p gives a $\mathbf{C}$-linear map from $\mathfrak{g} \otimes \mathbf{C}$ to TX_p. We want to see that this vanishes on the summands $\mathfrak{h} \otimes \mathbf{C}$ and W^-. For this implies that the tangent space to the orbit is a complex subspace of TX_p, which means that the orbit is a complex submanifold of X. So let N be a vector field on X corresponding to some eigenvector w in W^- for the action of ad $_{i\xi}$. Thus $[i\xi, w] = \lambda w$ for $\lambda < 0$ and the fact that the group action yields a Lie algebra homomorphism implies that $[v, N] = \lambda N$. The first statement in the proposition above shows that $N(p) = 0$. Now suppose there is a unit vector η in $\mathfrak{h}$ orthogonal to ξ. We set $F = \langle \eta, \mu \rangle$. This Poisson commutes with H, since $[\xi, \eta] = 0$, and vanishes at p. Suppose the corresponding vector field Z does not vanish at p. The second part of the proposition implies that the Hessian of $|\mu|^2$ is strictly positive in the direction iZ, which contradicts the fact that p is a maximum point for $|\mu|$.

We return to the picture of X, H, v, α_t, as above. We assume now that p is the unique point where H is maximised. Suppose that we have a holomorphic line bundle $L \to X$ and a lift $\tilde{\alpha}_t$ of the flow α_t to L. Then $\tilde{\alpha}_t$ acts on the fibre L_p over p with some weight λ, i.e.

$$\tilde{\alpha}_t(z) = e^{\lambda t}(z),$$

for $z \in L_p$. We say that a non-trivial holomorphic section s of L is a *highest weight vector* if it is an eigenvector for the induced action of $\tilde{\alpha}_t$, with the same weight: i.e.

$$\tilde{\alpha}_t s = e^{\lambda t} s.$$

Proposition 10 *If a highest weight vector exists it is unique up to constant scalar multiple.*

We show first that a highest weight vector cannot vanish at p. For if a section s vanishes at p it has an intrinsically defined derivative (∇s) which lies in $L_p \otimes_{\mathbf{C}}$

T^*X_p. This derivative is either zero or an eigenvector for the action of $\tilde{alpha}_t$. But the eigenvalues for the action on this space are all strictly less than λ. So the first derivative vanishes. This means that there is an intrinsically defined second derivative which lies in $L_p \otimes_{\mathbf{C}} s^2 T^* X_p$ but again there is no λ-eigenvalue for teh action here, so the second derivative vanishes, and so on. Thus we conclude that s is identically zero.

Now if s_1, s_2 are two highest weight vectors the ratio s_1/s_2 is a holomorphic function on a neighbourhood of p which is invariant under the flow α_t. It is clear that the only such are constants and by analytic continuation s_1 is a constant multiple of s_2 over the whole of X.

Let U be the set of points x in X such that $\lim_{t\to\infty} \alpha_t x = p$. Clearly U contains a neighbourhood of p, and it follows easily that U is open in X.

Proposition 11 *There is a highest weight vector over $U \subset X$.*

(By this we mean a holomorphic section over $U \subset X$ satisfying the equation $\tilde{\alpha}_t(s) = e^{\lambda t} s$).

There is no loss in supposing that the weight λ is zero (because we can change the lift of the action by multiplying by $e^{-\lambda t}$). Fix a non-zero point $z \in L_p$. Suppose $x \in U$ and let z' be a non-zero point in the fibre L_x. As $t \to \infty$ the flow $\tilde{\alpha}_t(z')$ converges to some non-zero point in L_p and there is a unique choice of z' such that this limit is z. Then we define $s(x) = z'$ and this is clearly a holomorphic $\tilde{\alpha}_t$-invariant section of L over U.

For the final steps we use a slightly more difficult fact– but one whose truth seems fairly plain:

(*) The complement of U is a complex subvariety of X.

Assuming this we have next:

Proposition 12 *Suppose the line bundle L arises from an S^1 bundle having a connection with curvature the Kahler form ω. Then there is a highest weight vector over all of X for the action of $\tilde{\alpha}_t$ on the holomorphic sections of L.*

We have to show that the section s we have constructed over U extends holomorphically to X. The hypotheses give us a Hermitian norm on the fibres of L. Given a point q in U set

$$f(t) = -\log(|s(e^{tv}q)|).$$

A calculation, using the fact that the curvature is ω, shows that this is a *convex* function of t. (This is related to the convexity phenomenon in Section 3.) Since $f(t)$ is bounded as $t \to +\infty$ it follows that $f(t)$ tends to ∞ as $t \to -\infty$. In particular f is bounded below. It follows easily that the holomorphic section s of L over U is bounded. Then a version of the Riemann extension theorem shows that the section extends holomorphically to X. (In fact we see that the extension of the section vanishes outside U. So *a posteriori* we see that (*) is true because the complement of U is the zero set of s.)

With all of this place we can quickly finish the proof.

Item 2. *the vector spaces V_M are finite-dimensional.* Indeed, the space of holomorphic sections of a line bundle over any compact complex manifold is finite-dimensional. This follows from the fact that a sequence of uniformly bounded holomorphic functions over a ball in $\mathbf{C}^n$ has a subsequence converging on compact subsets of the interior.

Let M be the co-adjoint orbit of $\xi \in \mathfrak{g}^* = \mathfrak{g}$. Then ξ generates an action by holomorphic isometries on M with Hamiltionian the linear function $x \to < x, \xi >$ which clearly has a unique maximum at ξ. If M is an integral orbit we have a holomorphic line bundle L and a lift of the action, so we are in the situation considered in Proposition 3 above (with $X = M$). So we have a highest weight vector and in particular the space V_M is non-trivial: this gives Item 3.

Suppose M is an integral co-adjoint orbit and that the space of sections V_M has a non-trivial decomposition $V_1 \oplus V_2$ as a G-representation. For each point $x \in M$ we have an evaluation map $V_1 \to L_x$. If this vanishes for one x it must do for all, by the transitive G-action, which is impossible if V_1 is non-trivial. So $e_\xi : V_1 \to L_\xi$ is non-trivial which means that there must be a highest weight vector in V_1. But similarly there must be a highest weight vector in V_2 and this contradicts the uniqueness. So we have Item 4 (the V_M are irreducible representations).

If M is a co-adjoint orbit it follows from (*) that M is simply-connected, since the complement of U has real codimension 2, so $\pi_1(M) = \pi_1(U)$ and any loop in U can be contracted into a small ball about p using the flow α_t. This gives item 1.

Now start with an irreducible representation V. We take $X = \mathbf{P}(V^*)$ and apply Corollary 1 to find a G-orbit $\tilde{M}$ which is a complex submanifold. Then the argument explained at the beginning of this sub-section shows that M is a copy of a integral co-adjoint orbit M and there is a natural restriction map from V to V_M which must be an isomorphism. This gives item 7 and item 6 (except for the fact that $M(V)$ is not strictly well-defined until we have established item 5).

The moment map for the action on a projective space $\mathbf{CP}^n$can be written down explicitly. First for the standard action of $U(n+1)$ we have

$$\mu(z) = \frac{i}{|z|^2} z z^*,$$

for a column vector z representing a point in $\mathbf{CP}^n$. Then for another group G acting we take the transpose of $\mathfrak{g} \to \mathfrak{u}(n)$ to get $\mathfrak{u}(n)^* \to \mathfrak{g}^*$ and compose with this. In particular suppose that $\xi \in \mathfrak{g}$ acts by iA on $\mathbf{C}^{n+1}$, for a self-adjoint matrix A. We can suppose A is diagonal with eigenvalues λ_a, ordered so that $\lambda_1 \geq \lambda_2 \geq \dots$. Then the function H is

$$H(\underline{z}) = \frac{1}{|\underline{z}|^2} \sum \lambda_a |z_a|^2.$$

Taking the point p where H is maximal corresponds to taking an eigenvector of A with largest eigenvalue. Also this eigenvalue is just the weight of the action on the fibre of H over p.

Now, since we can identify V with the sections of $L \to M$, we know that the largest eigenvalue of A has multiplicity 1 because these eigenvectors correspond to highest weight vectors. Let e_i be the basis for V^* of eigenvector for A. Consider a point $P = [\sum s_i e_i]$ in $\mathbf{P}(V^*)$. If $s_1 \neq 0$ then the limit of $[e^{At} P]$ as $t$tends to infinity is the point $P_0 = [1, 0 \dots, 0]$. So any closed set $Z \subset \mathbf{P}(V^*)$ either lies in the hyperplane $s_1 = 0$ or contains the point P_0. Thus there is at most one closed G^c-orbit not contained in any hyperplane. This gives the uniqueness of the maximising point for μ (item 5). It also proves item 8 (because certainly the embedding $\xi \mapsto e_\xi$ maps M to a closed G^c-invariant set in V_M which does not lie in any hyperplane, so this must be the set given by the maximum of $|\mu|$, by uniqueness).

There are many other ways of going about things. In particular, the *Kodaira embedding theorem* implies that for any bundle L with "positive" curvature some power L^k has a non-trivial holomorphic section. If we assume this we can avoid using (*) by arguing that s^k is holomorphic on X and working back from that. Notice that we have to use the positivity of the curvature somewhere since otherwise everything we say applies to the dual bundle L^*, which certainly does not have holomorphic sections. In the case of our application, when $X = M$ is a co-adjoint orbit, we can avoid using (*) by looking a bit more carefully at the structure we have. Then we can identify the complement of U explicitly.

The open set U is the first stage in a *stratification* of the co-adjoint orbit M by complex subvarieties, which is important for other purposes. In the case of the Grassmannians we get the stratification by "Schubert cells"; see Griffiths and Harris *Principles of algebraic geometry*, for example.

5 Structure theory for compact Lie groups

We consider a compact connected Lie group G. As in the previous Section it would be much the same to discuss semisimple complex Lie groups, using the result of Section 3. We first introduce the following concepts

- A *maximal torus* in G;
- The *Weyl group*;
- The *roots*;
- The *Weyl chambers.*

A torus is a Lie group isomorphic to $(S^1)^n$. Any compact connected Abelian Lie group is a torus. A maximal torus in G is a subgroup which is a torus and which is not contained in any strictly larger torus in G.

Proposition 13 *Any two maximal tori are conjugate*

We need a Lemma, in which we prove a little more than we need immediately.

Lemma 2 *Let ξ_0, ξ, ξ' be elements of $\mathfrak{g}$ with $[\xi_0, \xi] = [\xi_0, \xi'] = 0$. Then we can find $g \in G$ such that the adjoint action of g fixes ξ_0 and $[g(\xi), \xi'] = 0$.*

To see this let $H \subset G$ be the subgroup which fixes ξ_0 in the adjoint action. We maximise the function $\langle g(\xi), \xi' \rangle$ over $g \in H$. Without loss of generality the maximum occurs when $g = 1$. This means that for all η with $[\eta, \xi_0] = 0$ we have $\langle [\eta, \xi], \xi' \rangle = 0$, which is the same as saying $\langle \eta, [\xi, \xi'] \rangle = 0$. This means that $[\xi, \xi']$ is in the orthogonal complement of the Lie algebra $\mathfrak{h}$ of H. But ξ and ξ' each lie in $\mathfrak{h}$ so their bracket does too. Thus $[\xi, \xi'] = 0$, as required.

Now for any torus T a generic element ξ of Lie (T) will generate T in the sense that the closure of the 1-parameter subgroup $e^{\xi t}$ will be the whole of T. Given a pair of maximal tori $T, T' \subset G$ choose such generators ξ, ξ'. By applying the Lemma (with $\xi_0 = 0$) we see that after conjugation we may as well suppose that $[\xi, \xi'] = 0$. If ξ' does not lie in Lie (T) the two generate a strictly larger torus, contrary to maximality of T. Thus ξ' lies in Lie (T) which implies that T' is contained in T and $T = T'$ be maximality of T'.

Notice that this argument shows that, given one maximal torus T, any other torus is conjugate to a subgroup of T.

The *rank* of a G is the dimension of a maximal torus.

Examples

- A maximal torus in $U(n)$ is given by the diagonal matrices

$$\text{diag}(e^{i\lambda_1}, \dots, e^{i\lambda_n}),$$

  so $U(n)$ has rank n.

- A maximal torus in $SU(n)$ is given by the diagonal matrices as above, with $\sum \lambda_a = 0$. So $SU(n)$ has rank $n-1$.

- A maximal torus in $SO(2n)$ is given by the taking the maximal torus in $U(n)$ and the standard embedding $U(n) \subset SO(2n)$. So $SO(2n)$ has rank n.

- A maximal torus in $SO(2n+1)$ is given by taking the maximal torus in $SO(2n)$ and the standard embedding $SO(2n) \subset SO(2n+1)$. So $SO(2n+1)$ has rank n.

- A maximal torus in $Sp(n)$ is given by taking the maximal torus in $U(n)$ and the standard inclusion $U(n) \subset Sp(n)$, so $Sp(n)$ has rank n.

Fix a maximal torus $T \subset G$. The *Weyl group* is $W = N(T)/T$, where $N(T)$ is the normaliser of T. If $g \in G$ normalises T the adjoint action maps T to T, and T acts trivially on itself. So W acts on T. (In fact we could easily show that the action is effective so that W can also be defined as the group of automorphisms of T which are induced by inner automorphism in G.)

For example, the Weyl group of $U(n)$ is isomorphic to the symmetric group on n objects, acting by permutations of the eigenvalues $e^{i\lambda_a}$. Similarly for $SU(n)$. In particular the Weyl group of $SU(2)$ is $\{\pm 1\}$, acting by a reflection on $T = S^1$.

The Weyl group acts on Lie (T). Suppose two elements $\xi_1, \xi_2 \in$ Lie (T) are conjugate in G. Then it follows easily from Lemma 1 that there is an element of the Weyl group mapping ξ_1 to ξ_2. So we have the important fact that the adjoint orbits in G are in $1-1$ correspondence with the orbits of the Weyl group acting on Lie (T). (A variant of this is the fact that the conjugacy classes in G are in $1-1$ correspondence with the orbits of the Weyl group acting on T.)

Let V be any complex representation of G. By restriction we get a representation of T. This decomposes into a sum of *weight spaces*. The *weight lattice* Λ is the lattice in Lie $(T)^*$ consisting of linear maps $L(T) \to \mathbf{R}$ which take integer value on the kernel of exp : Lie $(T) \to T$. So a representation of T is specified by giving a collection of integer multiplicities $n_w > 0$ associated to weights $w \in \Lambda$. Using our fixed invariant inner product on $\mathfrak{g}$ we can identify Lie (T) with Lie $(T)^*$, so we can think of $\Lambda \subset$ Lie (T). The Weyl group acts on Lie (T) and Lie $(T)^*$, preserving Λ.

Examples

- If V is the complexification of a real representation, $V = A \otimes_{\mathbf{R}} \mathbf{C}$, then the non-zero weights come in pairs $\pm w$, with the same multiplicities. If A is already complex the weights of $A \otimes_{\mathbf{R}} \mathbf{C} = A \oplus \overline{A}$ are given by taking $\pm$ the weights of A.

- With standard co-ordinates $(\lambda_1, \ldots, \lambda_n)$ on Lie (T) for the maximal torus $T \subset U(n)$ the weights of the standard representation $\mathbf{C}^n$ are λ_i.

- For the representation $\Lambda^2 \mathbf{C}^n$ of $U(n)$ the weights are $\lambda_i + \lambda_j$ $(i \neq j)$. For $\Lambda^3 \mathbf{C}^n$ the weights are $\lambda_i + \lambda_j + \lambda_k$ $(i, j, k$ all different), and so on.

- For the representation $s^2 \mathbf{C}^n$ of $U(n)$ the weights are $\lambda_i + \lambda_j$ $(i = j$ allowed). Similarly for the higher symmetric powers.

Now consider the complexification of the adjoint representation $\mathfrak{g} \otimes \mathbf{C}$ of G. The weight 0 subspace just corresponds to Lie $(T) \otimes \mathbf{C} \subset \mathfrak{g} \otimes \mathbf{C}$. The nonzero weights of this representation are called the *roots* of G. They are elements of $\Lambda \subset$ Lie $(T)^*$ and occur in pairs $\pm\alpha$ (since the representation is the complexification of a real representation of G). But using our fixed inner product we will generally regard the roots as elements of Lie $(T) \subset \mathfrak{g}$. The number of roots is the difference $\dim G -$ rank G.

Examples

- With $G = U(n)$ and standard co-ordinates $(\lambda_1, \ldots, \lambda_n)$ on Lie (T) as above, the roots are $\lambda_i - \lambda_j$ for $i \neq j$. There are $n(n-1)$ of these and dim $G = n^2$, rank $(G) = n$.

- For $SU(n)$ everything is the same as for $U(n)$, except that the dimension and rank both drop by 1.

- With the same standard co-ordinates the roots of $SO(2n)$ are $\pm\lambda_i \pm \lambda_j$, for $i \neq j$. There are $2n(n-1)$ of these and dim $G = n(2n-1)$, rank $G = n$.

- In the same way, the roots of $SO(2n+1)$ are $\pm\lambda_i \pm \lambda_j$ for $i \neq j$ and $\pm\lambda_i$. There are $2n(n-1) + 2n = 2n^2$ roots and dim $G = n(2n+1)$, rank $G = n$.

- Also in the same way, the roots of $Sp(n)$ are $\lambda_i - \lambda_j$ for $i \neq j$ together with $\pm(\lambda_i + \lambda_j)$ $(i = j$ allowed). There are $n(n-1) + n(n+1) = 2n^2$ roots and dim $G = n(2n+1)$, rank $(G) = n$.

Remark

Given a pair of compact groups $H \subset G$ such that the maximal torus of H is also maximal in G we can find the roots of G by writing $\mathfrak{g} = \mathfrak{h} \oplus W$, say, where

W is a real representation of G. The roots of G are made up of the roots of H and the weights of $W \otimes \mathbf{C}$. In the examples above

$$\mathfrak{so}(2n) = \mathfrak{u}(n) \oplus \Lambda^2 \mathbf{C}^n,$$

$$\mathfrak{sp}(n) = \mathfrak{u}(n) \oplus s^2 \mathbf{C}^n,$$

$$\mathfrak{so}(2n+1) = \mathfrak{so}(2n) \oplus \mathbf{R}^{2n}.$$

So we can use this technique to identify the roots. Notice, by the way, that these decompositions, all correspond to symmetric spaces: $SO(2n)/U(n), Sp(n)/U(n), SO(2n+1)/SO(2n)$ as we discussed in Section 2.1. We will use this technique more in Section 7.

Now we write

$$\mathfrak{g} \otimes \mathbf{C} = (\text{Lie } (T) \otimes \mathbf{C}) \oplus \bigoplus R_\alpha$$

where α runs over the roots. To get back to the real Lie algebra we use the fact that the roots come in pairs $\pm\alpha$ with $R_{-\alpha} = \overline{R_\alpha}$ and the real part is spanned by sums $r_\alpha + \overline{r_\alpha}$ for $r_\alpha \in R_\alpha$. By definition, for $\xi \in \text{Lie } (T) \otimes \mathbf{C}$ and $r_\alpha \in R_\alpha$ we have

$$[\xi, r_\alpha] = \langle \alpha, \xi \rangle r_\alpha,$$

where $\langle \, , \, \rangle$ is the complex bilinear extension of the positive definite inner product on Lie $(T) \subset \mathfrak{g}$.

The same KEY CALCULATION as we used in Section 2.2.2 shows that if $r_\alpha \in R_\alpha, r_\beta \in R_\beta$ then one of the following occurs

- $\beta = -\alpha$ and $[r_\alpha, r_\beta] \in \text{Lie } (T)$;
- $\alpha + \beta$ is a root and $[r_\alpha, r_\beta] \in R_{\alpha+\beta}$;
- $[r_\alpha, r_\beta] = 0$..

We know that any adjoint orbit of G is the orbit of some $\xi \in \text{Lie } (T)$. Going back to the discussion in Section 2.2, we let H be the stabiliser of ξ, so H contains T. From the formula above we see that

$$\mathfrak{h} \otimes \mathbf{C} = \text{Lie } (T) \otimes \mathbf{C} \oplus \bigoplus_{\langle \alpha, \xi \rangle = 0} R_\alpha.$$

For each root $\alpha \in \text{Lie } (T)$ the corresponding *root plane* L_α is simply the orthogonal complement of α in Lie (T). These make up a finite number of hyperplanes in Lie (T). We see that if ξ is not in any root plane then $H = T$. (Here we use the fact from Section 4 that H is connected, since the co-adjoint orbit is simply connected.) Thus the generic co-adjoint orbit has the form G/T

but when ξ lie in one or more root planes we get a group H which strictly contains T, so the orbit is lower dimensional. This is just like the picture we have seen for $G = SU(n)$ where the generic orbit is the flag manifold $\mathbf{F}$ but for special choices of ξ we get smaller flag manifolds, Grassmannians, projective spaces etc.

The complement of the union of all the root planes in Lie (T) is an open set with a finite number of connected components. The closure of any one of these components is called a *Weyl chamber* in Lie (T).

Example
For $G = SU(2)$ there are two Weyl chambers each of which is a closed half-line in $\mathbf{R}$. For $G = SU(3)$ there are six Weyl chambers each of which is a wedge of angle $\pi/3$.

Imagine that we are setting out to classify compact Lie groups. A sensible strategy would be to proceed by the rank of the group. The first case would be

Proposition 14 *A compact connected Lie group G of rank 1 is isomorphic to S^1, $SU(2)$ or $SO(3)$.*

Suppose there are $2n$ roots. If $n = 0$ we clearly have $G = S^1$. If $n > 0$ then the non-trivial co-adjoint orbits are $2n$-dimensional submanifolds of S^{2n}, hence equal to S^{2n}. But we know that a co-adjoint orbit is a symplectic manifold and, by elementary de Rham theory, the only sphere which is symplectic is S^2 (since we need H^2 to be non-trivial). Thus $n = 1$ and the adjoint action maps G to $SO(3)$. It is clear that this is a local isomorphism, so the universal cover of G is $SU(2)$ and $G = SU(2)/N$ where N is a finite normal subgroup of $SU(2)$. But then it is clear that N must be in the centre of $SU(2)$ and since this centre is ± 1 we have the two possibilities $G = SU(2), G = SO(3)$.

Remark Notice that in the above argument we do not really need the compactness of G, once $n > 0$. All we need is that the Lie algebra has the given structure and that there is an invariant Euclidean form. Then the compactness of the group comes as part of the conclusion.

Now let G be any compact group and α a root.

Lemma 3 *Let β run over the roots which are non-zero real multiples of α and form the vector space*

$$\mathfrak{g}_0^c = \mathbf{C}\alpha \oplus \bigoplus_{\beta} R_\beta \subset \mathfrak{g} \otimes \mathbf{C}.$$

Then $\mathfrak{g}_0^c$ is a subalgebra of $\mathfrak{g} \otimes \mathbf{C}$ and its real part $\mathfrak{g}_0 = \mathfrak{g}_0^c \cap \mathfrak{g}$ is a subalgebra of $\mathfrak{g}$.

We only need to show that if $r \in R_{k\alpha}$ and $r' \in R_{-k\alpha}$ then $[r, r']$ is a multiple of α. But if $\xi \in \text{Lie }(T) \otimes \mathbf{C}$ is orthogonal to α then

$$\langle \xi, [r, r'] \rangle = -\langle [r, \xi], r' \rangle = [0, r'] = 0.$$

Now we have

Proposition 15 *If α is any root then $k\alpha$ is a root if and only if $k = \pm 1$, the dimension of R_α is 1 and the subalgebra $\mathfrak{g}_0$ is isomorphic to $\mathfrak{su}(2)$.*

To show this we apply the proposition above. We know that there is an abstract group G_0 corresponding to $\mathfrak{g}_0$ and by the remark following Proposition 2 we deduce that this is $SU(2)$ or $SO(3)$ (since by hypothesis there are at least two roots).

Remark. We can also show, easily, that $\mathfrak{g}_0$ corresponds to an $SU(2)$ or $SO(3)$ subgroup of G.

For a root α let $\rho_\alpha : \text{Lie }(T) \to \text{Lie }(T)$ be the reflection in the root plane L_α i.e.

$$\rho_\alpha(\xi) = \xi - \frac{2}{|\alpha|^2} \langle \xi, \alpha \rangle \alpha.$$

Proposition 16 *For each root α there is an element of the Weyl group W which acts as ρ_α on* Lie (T)*.*

(Since the Weyl group acts effectively on Lie (T), we could more simply say that ρ_α *is* an element of the Weyl group.)

This is clearly true for $SU(2), SO(3)$. In general let $\eta = r_\alpha + \overline{r_\alpha} \in \mathfrak{g}$ and consider the adjoint action of the 1-parameter subgroup $\exp(t\eta)$. This fixes elements ξ in the plane L_α, since then $[\xi, \eta] = 0$. By what we know for $SU(2)$, and the fact that ξ, α lie in a copy $\mathfrak{g}_0$ of $\mathfrak{su}(2)$, we can choose t such that the adjoint action maps α to $-\alpha$.

Now we take our classification programme one step further by considering the rank 2 case. In fact we express things in terms of a pair of roots in $\mathfrak{g}$.

Proposition 17 *Suppose α, β are roots in* Lie (T) *with $\alpha \neq \pm\beta$. Let $\Pi \subset$* Lie (T) *be the plane spanned by α, β. There are exactly $2k$ roots in Π, where $k = 2, 3, 4$ or 6. Moreover the normalised vectors $\alpha_i / |\alpha_i|$, for roots $\alpha_i \in \Pi$, are equally spaced around the unit circle.*

The reflections defined by the roots in Π act on Π and generate a finite group (since the Weyl group is finite). Identify Π, and its Euclidean form restricted from $\mathfrak{g}$, with the standard $\mathbf{R}^2$. Then we get a finite subgroup $A \subset O(2)$. Since A is not contained in $SO(2)$ it has an index 2 subgroup $A_0 = A \cap SO(2)$ which must be cyclic of order k for some $k \geq 2$. Let M be a generator of A_0: a rotation through an angle $\theta = 2\pi/k$.

Let Λ_Π be the intersection of the weight lattice with Π. Then Λ_Π contains α, β so it must have rank 2. The action of A preserves Λ_Π so we see that M is conjugate in $GL(2, \mathbf{R})$ to an element of $SL(2, \mathbf{Z})$, i.e. a matrix with integer entries. In particular the trace of M is an integer. So $2\cos(\theta)$ is an integer and the only possibilities are $\theta = \pm\pi/3, \pm\pi/2, \pm 2\pi/3, \pi$. The proposition follows immediately: the group A is generated by a pair of reflections defined by roots at an angle of $\theta/2$.

What we have here is essentially the familiar fact that the only lattices in the plane with a rotational symmetry are the square lattice and the hexagonal lattice.

There is a refinement of the Proposition which describes the ratios of the lengths of the roots in the cases when $k = 3, 4, 6$.

- When $k = 3$ all the roots in Π have the same length.
- When $k = 4$ the ratios of the lengths of successive roots as we go around the circle are alternately $\sqrt{2}, 1/\sqrt{2}$.
- When $k = 6$ the ratios are alternately $\sqrt{3}, 1/\sqrt{3}$.

This is a simple consequence of

Fact 1 *For any two roots α, β in* Lie (T) *the quantity*

$$\frac{2\langle\alpha, \beta\rangle}{|\alpha|^2}$$

is an integer.

We will not prove this (although there is no special difficulty in doing so).

Going back to our classification programme, we see that there are four possibilities for the pattern of the roots in the rank 2 case, and in fact these are all realised.

- $k = 2$, occurs for the groups $SO(4)$ or $SU(2) \times SU(2)$. (These groups are locally isomorphic, see Section 7.)
- $k = 3$, occurs for the group $SU(3)$.

- $k = 4$ occurs for the groups $SO(5)$ or $Sp(2)$ (again, locally isomorphic).
- $k = 6$ occurs for the exceptional group G_2, see Section 7.

We do not take the classification programme further here: it has all been done for us by Killing and Cartan. A *root system* in a Euclidean vector space V is defined to be finite set of non-zero vectors $\mathcal{R}$ such that

- The elements of $\mathcal{R}$ span V;
- If α is in $\mathcal{R}$ then $k\alpha$ is in $\mathcal{R}$ if and only $k = \pm 1$.
- For each $\alpha \in \mathcal{R}$ the reflection in the hyperplane orthogonal to α maps $\mathcal{R}$ to $\mathcal{R}$.
- For any $\alpha, \beta \in \mathcal{R}$ the quantity

  $$\frac{2\langle \alpha, \beta \rangle}{|\alpha|^2}$$

  is an integer.

Then the roots of any compact Lie group form a root system (in some subspace of Lie (T)). On the other hand, purely as a matter of Euclidean geometry, the root systems have been completely classified.

We now have a very detailed picture of the structure of our Lie algebra $\mathfrak{g}$. We explain this for the complexified algebra, where the notation is a little easier, but everything can be restated in terms of $\mathfrak{g}$ if we prefer. Each space R_α is one dimensional so we choose a basis element r_α. We know that $[r_\alpha, r_{-\alpha}]$ is a nonzero multiple of α and we can multiply by a factor so that without loss of generality

$$[r_\alpha, r_{-\alpha}] = \alpha.$$

If α, β are roots with $\alpha \neq \pm\beta$ then either $\alpha + \beta$ is a root in which case

$$[r_\alpha, r_\beta] = \epsilon_{\alpha\beta} r_{\alpha+\beta},$$

for some $\epsilon_{\alpha\beta} \in \mathbf{C}$ or $\alpha + \beta$ is not a root in which case $[r_\alpha, r_\beta] = 0$. And of course we know that for $\xi \in$ Lie $(T)^c$ we have

$$[\xi, r_\alpha] = \langle \xi, \alpha \rangle r_\alpha.$$

So we have a complete description of the Lie algebra in terms of the root system and the collection of numbers $\epsilon_{\alpha\beta}$. These are not uniquely defined. If we change the basis elements r_α by scalar multiples μ_α, with $\mu_{-\alpha} = \mu_\alpha^{-1}$, we get an equivalent collection

$$\tilde{\epsilon}_{\alpha\beta} = \mu_\alpha \mu_\beta \mu_{\alpha+\beta}^{-1} \, \epsilon_{\alpha\beta}.$$

If we take any $\epsilon_{\alpha\beta}$ we can define a bracket by the formulae above. The Jacobi identity becomes a system of equations for the numbers $\epsilon_{\alpha\beta}$. If we can solve these we get a Lie algebra.

Fact 2 *Given a root system there is a way to choose $\epsilon_{\alpha\beta}$ to solve these equations, and the solution is unique up to equivalence.*

In this way, the classification of root systems leads to the renowned Cartan-Killing classification of compact Lie groups, or semisimple complex Lie groups.

Now we go back to look at the geometry of the Weyl chambers in Lie (T). Clearly the Weyl group acts by permuting the chambers.

Proposition 18 *The action of the Weyl group on the set of chambers is simply transitive.*

To see that the action is transitive we join two Weyl chambers by a path which crosses one root plane at a time, in a transverse fashion. Then the corresponding product of reflections does the job.

To see that the action is simply transitive, suppose p is an element of the Weyl group and B is a chamber with $pB = B$. Then p has finite order n say, and all powers p^r map B to B. Choose any point η in the interior of B and set

$$\xi = \sum_1^n p^r \eta.$$

Then η is an interior point in B (by convexity) which is fixed by p. Let $g \in G$ be an element whose adjoint action realises p. Then g is in the stabiliser H_ξ of ξ. But since ξ is an interior point of B it does not lie in any root plane, so $H_\xi = T$ and we see that g is in T and so p is the identity.

For $\xi \in$ Lie (T) let W_ξ be the subgroup of the Weyl group which fixes ξ. A similar argument to that above shows that W_ξ acts transitively on the set of Weyl chambers which contain ξ. From this it is elementary to deduce.

Corollary 4 *No two distinct points in the same Weyl chamber are in the same orbit of W acting on* Lie (T).

IMPORTANT CONCLUSION

We fix a Weyl chamber B_0 which we call the *fundamental chamber*. Then the adjoint (or equivalently co-adjoint) orbits of G in $\mathfrak{g}$ can be identified with the points of B_0. The integral co-adjoint orbits can be identified with the elements of the weight lattice in B_0.

Remark There is a useful way to visualise the relation between the orbits of G in $\mathfrak{g}$ and of W in Lie (T). Let ξ be a point in Lie (T). The orbit of ξ under the action of the Weyl group is a finite set $W\xi \in$ Lie (T). Let P be the convex hull of this set. Thus P is a convex polytope in the Euclidean space Lie (T). Consider the adjoint orbit $M_\xi \subset \mathfrak{g}$ of ξ. It is a fact that under orthogonal projection from $\mathfrak{g}$ to Lie (T) the manifiold M_ξ maps onto P. For example if M is an adjoint orbit of $SU(n)$ which is a copy of $\mathbf{CP}^{n-1}$ the set P is an $n-1$ simplex and the projection map, in suitable co-ordinates, is given by

$$[z_1, \ldots, z_n] \mapsto \frac{1}{\sum |z_i|^2}(|z_1|^2, \ldots, |z_n|^2).$$

Given a choice of fundamental Weyl chamber B_0, we say that a root α is *positive* if $\langle \alpha, \xi \rangle \geq 0$ for all $\xi \in B_0$.

The fundamental Weyl chamber B_0 can be defined by finite set of inequalities $\langle \alpha, \xi \rangle \geq 0$ for roots α. There is a unique minimal set of such inequalities. The roots appearing in this set are called the "simple roots". (To be more precise, a root α is simple if $\langle \alpha, \xi \rangle \geq 0$ on B_0 and if equality occurs when ξ is in the interior of some codimension-1 face of the boundary.)

Fact 3 *The angle between two simple roots is* $\geq \pi/2$.

Thus the angle between two simple roots is one of $\pi/2, 2\pi/3, 3\pi/4, 5\pi/6$. The *Dynkin diagram* is defined by taking a node for each simple root and joining nodes by $0, 1, 2, 3$ bonds in the four cases respectively.

Fact 4 *The root system can be recovered from the Dynkin diagram*

Fact 5 *The Lie algebra is simple if and only if the Dynkin diagram is connected*

Fact 6 *If the Lie algebra* $\mathfrak{g}$ *has trivial centre then the number of simple roots is equal to the rank. We can choose* $\omega_1, \dots \omega_r$ *in* B_0 *such that*

$$B_0 = \{\sum a_i \omega_i : a_i \geq 0\}.$$

Fact 7 *If* G *is simply connected (which implies that* $\mathfrak{g}$ *has trivial centre) then we can choose the* ω_i *to be weights and such that the intersection of the weight lattice with* B_0 *is given by vectors* $\sum a_i \omega_i$ *for integers* $a_i \geq 0$.

6 More on representations

6.1 The general picture

We consider a compact, connected Lie group G with maximal torus T, as in the previous Section. Recall that we have a weight lattice $\Lambda \subset \text{Lie}\ (T)$ and a complex representation of T is specified by a a finite collection of weights $\mu \in \Lambda$ and multiplicities n_μ. If we have any representation of G, restriction to T gives a set of weights and multiplicities and these are *invariant under the action of the Weyl group*. We also have a *root lattice* $\Lambda_R \subset \Lambda$, the subgroup generated by the roots of G.

Fix a fundamental Weyl chamber B_0. From what we know in Section 5, the integral co-adjoint orbits are in 1-1 correspondence with points in $\Lambda \cap B_0$. So the main theorem of Section 4 takes the following form.

The irreducible representations are in 1-1 correspondence with the points in $\Lambda \cap B_0$.

Given $\xi \in \Lambda \cap B_0$, let V_ξ be the corresponding irreducible representation. Let $P_\xi \subset \text{Lie }(T)$ be the convex hull of the finite set $W\xi$, as considered in Section 5. Then we have

Proposition 19

- ξ *is a weight of* V_ξ *and the multiplicity* n_ξ *is* 1.
- *The weights of* V_ξ *are contained in the intersection of* P_ξ *with the coset* $\xi + \Lambda_R \subset \Lambda$

The weight ξ of V_ξ is called the "highest weight", and the corresponding eigenvector in V_ξ (which is unique up to scalars, since $n_\xi = 1$) is called the "highest weight vector".

The proof of the proposition is (in our approach) an exercise in the techniques used in Section 4. The first item we have already established, when we constructed a particular section s of $L \to M$. Let $Q_\xi \subset \text{Lie }(T)$ be the convex hull of the weights, so $P_\xi \subset Q_\xi$. To see that $Q_\xi = P_\xi$ it suffices to show that the only extreme point of Q_ξ in B_0 is ξ (for then, by the Weyl group invariance, the only extreme points of Q_ξ are those in the orbit $W\xi$). This statement follows by considering the action of a suitable 1-parameter subgroup on M, and using one of the results from Section 4. To see that the weights are contained in $\xi + \Lambda_R$ we can consider the derivatives of a section of L at the point $p \in M$ fixed by $T \subset H$. The first non-vanishing derivative is well-defined (without using a connection), as in Section 4. If σ were a section belonging to a weight not in $\xi + \Lambda_R$ one sees that all derivatives of σ must vanish at p, so σ is identically zero by analytic continuation.

Remark

We stated as a "fact" in Section 5 that for a simply connected group G of rank r there are r fundamental weights ω_i such that the weights in B_0 are

$\sum a_i\omega_i$ for integers $a_i \geq 0$. For each ω_i there is a unique simple root α_i such that $\langle \alpha_i, \omega_i \rangle \neq 0$. On the other hand, each ω_i is associated to an irreducible representation, V_i say. So we can label the vertices of the Dynkin diagram with these fundamental irreducible representations.

We have described the irreducible representations as sections of line bundles over the different co-adjoint orbits. Just as we saw in the case of $SU(3)$ there is another, slightly different, approach. Consider any point ξ in the interior of the Weyl chamber B_0. The co-adjoint orbit is identified with G/T and as ξ varies we get the same complex structure. So we can think of a fixed complex manifold G/T (the flag manifold in the case when $G = SU(n)$). From this point of view we think of a weight ξ as defining a homomorphism from T to S^1 and hence a line bundle $L_\xi \to G/T$. Then our representation V_ξ, when ξ is a weight in the interior of B_0, is the space of holomorphic sections of L_ξ over G/T. Now suppose ξ is a weight on the boundary of B_0. Then we have a different co-adjoint orbit $M_\xi = G/H$ but $T \subset H$ so we have a fibration $G/T \to M_\xi$. Checking the definitions, one sees that this is a holomorphic fibration and the line bundle L_ξ over G/T is the lift of the line bundle $L \to M_\xi$. So the holomorphic sections of $L_\xi \to G/T$ can be identified with sections of $L \to M_\xi$. Thus we see that for all ξ in the (closed) Weyl chamber B_0 we can describe V_ξ as the space of holomorphic sections of $L_\xi \to G/T$. This the Borel-Weil Theorem, in the form usually stated.

We will now take a more algebraic point of view. For each weight μ we have a vector space U_μ of dimension n_μ and $V_\xi = \bigoplus_\mu U_\mu$. We consider the representation of the complex Lie algebra $\mathfrak{g}^c = \mathfrak{g} \otimes \mathbf{C} = \text{Lie } (T)^c \oplus \bigoplus_\alpha \mathbf{C} r_\alpha$, where α runs over the roots. Then, by definition of the weights, an $\eta \in \text{Lie } (T)^c$ acts on U_μ as scalar multiplication by $\langle \eta, \mu \rangle$. Our KEY CALCULATION shows that a root α acts as

$$g_\alpha : U_\mu \to U_{\mu+\alpha}$$

interpreted as 0 if $\mu + \alpha$ is not a weight. The algebraic approach to proving the main theorem (irreducible representations$\leftrightarrow$ highest weight vectors) extends what we did for $\mathfrak{sl}_2(\mathbf{C})$ in Chapter 4. If we have an irreducible representation V we take a highest weight vector e (with a suitable definition of "highest")and repeatedly apply the operators g_α for *negative* roots α. The vectors we get in this way generate V. See Fulton and Harris (for example) for all this. (Just as in the approach of Chapter 4, the most difficult part is the construction of a representation with a given highest weight.)

THIS COMPLETES THE CORE OF THE COURSE

SHORT BREAK.

Creators of the theory:

- S. Lie 1842-1900
- W. Killing 1847-1923
- E. Cartan 1869-1951
- H. Weyl 1885-1955

WHAT YOU SHOULD REMEMBER FROM THIS COURSE

1. Symmetric spaces: generalise Euclidean/spherical/hyperbolic geometries.
2. Irreducible representations$\leftrightarrow$ integral coadjoint orbits $\leftrightarrow$ orbits of weights under Weyl group $\leftrightarrow$ weights in Weyl chamber.
3. The "key calculation" for eigenvectors in Lie algebra actions: passage back and forth between compact/complex semisimple groups.

6.2 Spin representations

We have seen that $\pi_1(SO(3)) = \mathbf{Z}/2$ (since the universal cover is $SU(2)$). It follows easily from the fibration

$$SO(n-1) \to SO(n) \to S^{n-1},$$

that $\pi_1(SO(n)) = \mathbf{Z}/2$ for all $n \geq 3$. Thus there are compact, simply connected, groups Spin(n) which are double covers of the $SO(n)$. (And Spin$(3) = SU(2)$.)

In general if $\tilde{G} \to G$ is a finite covering of compact groups then we can identify Lie (T), the Weyl chambers and the roots for the two cases but the weight lattices will be different. There are representations of $\tilde{G}$ which do not factor through G. We saw this in the case of $SU(2) \to SO(3)$.

Background (not essential) According to the fact stated in Section 5, the weights in the fundamental Weyl chamber for the simply connected group Spin$(2m+1)$ are linear combinations of m fundamental weights ω_i, each corresponding to an irreducible representation. It turns out that $m-1$ of these representations are just the exterior powers $\Lambda^i \mathbf{C}^{2m+1}$ for $1 \leq i \leq m-1$, so these factor through $SO(2m+1)$. The other representation does not factor and is the *spin representation* S of Spin$(2m+1)$. Similarly, Spin$(2m)$ has rank m and $m-2$ of these fundamental representations come from the exterior powers $\Lambda^i \mathbf{C}^{2m}$ for $1 \leq i \leq m-2$ but we have two more representations S^+, S^- which do not factor through $SO(2m)$.

Our task is to construct these spin representations explicitly. The even and odd cases are a little different, so we begin with the even case. In one sense, our general theory constructs these representations. The relevant co-adjoint orbit of $SO(2m)$ is $M = SO(2m)/U(m)$–which can be viewed as the adjoint orbit of the standard $I_0 : \mathbf{R}^{2m} \to \mathbf{R}^{2m}$. With the right scale factor, working with the group $SO(2m)$, we get the homomorphism $\det : U(n) \to S^1$ and the corresponding line bundle $L \to M$. This can be described as follows. A point of M is a complex structure $I : \mathbf{R}^{2m} \to \mathbf{R}^{2m}$ which allows us to think of $\mathbf{R}^{2m}$ as a complex vector space. The we get a 1-dimensional complex vector space by taking the dual of the top exterior power, and this is the fibre of L over I.

Now define

$$\tilde{U}(m) = \{(g, z) \in U(m) \times S^1 : \det g = z^2\}.$$

This is a group which double covers $U(m)$. (We can think of working in $\tilde{U}(m)$ as the same as working in $U(m)$, but being allowed a choice of square root of the determinant.) It is easy to see that in the double cover Spin$(2m) \to SO(2m)$ the preimage of $U(m)$ is a copy of $\tilde{U}(m) \subset$ Spin$(2m)$. We can regard the same coadjoint orbit M as a coadjoint orbit of Spin$(2m)$ but now

$$M = \text{Spin}(2m)/\tilde{U}(m).$$

With this description we get a square root of the line bundle L, i.e. a line bundle $L^{1/2}$ with

$$L = L^{1/2} \otimes L^{1/2}.$$

The representation S^+ of Spin$(2m)$ is given by the space of holomorphic sections of $L^{1/2}$ over M. The representation S^- which is obtained similarly from the orbit of a complex structure compatible with the *opposite orientation* of $\mathbf{R}^{2m}$. (Thus a choice of one of the representations $S^\pm$ is the same as the choice of an orientation of $\mathbf{R}^{2m}$.)

But we want a more explicit description. Recall that the Lie algebra of $SO(n)$ can be identified with $\Lambda^2\mathbf{R}^n$. If w_i is a standard basis of $\mathbf{R}^n$ we have standard elements $w_{ij} = -w_{ji}$ spanning $\mathfrak{so}(n)$ and the brackets $[w_{ij}, w_{kl}]$ are specified by

- 0 if $|\{i,j\} \cap \{k,l\}| = 2$;
- 0 if $|\{i,j\} \cap \{k,l\}| = 0$;
- $[w_{ij}, w_{jl}] = w_{il}$ if i, j, l distinct.

Lemma 4 *Suppose we have a complex vector space V and linear maps $\Gamma_i : V \to V$ which satisfy the relations*

$$\Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 0 \quad (i \neq j)$$

$$\Gamma_i^2 = 1.$$

Then the map $w_{ij} \mapsto \Gamma_i\Gamma_j$ defines a representation of $\mathfrak{so}(n)$ on V.

This is easy to check, using the relations defining $\mathfrak{so}(n)$. More invariantly we can thing of the input as a family of linear map $\Gamma_w : V \to V$ for $w \in \mathbf{R}^n$ such that

$$\Gamma_w^2 = |w|^2 1_V$$

Now start with an m-dimensional Hermitian vector space E. The wedge product gives a map $E \otimes \Lambda^* E \to \Lambda^* E$. Explicitly in terms of a standard basis we have wedge products

$$e_\alpha : \Lambda^p E \to \lambda^{p+1} E.$$

Using the metric we get a complex linear map defined by contraction

$$\overline{E} \otimes \Lambda^* E \to \Lambda^* E.$$

Explicitly, we have the contraction operators

$$e_\alpha^* : \Lambda^p E \to \Lambda^{p+1} E.$$

Lemma 5

$$e_\alpha e_\beta^* + e_\beta^* e_\alpha = \delta_{\alpha\beta} 1.$$

This is straightforward to check.

Now given $v \in E$ with complex co-ordinates v_α in the standard basis set

$$\Gamma_v = \sum v_\alpha e_\alpha + \sum \overline{v}_\beta e_\beta^*.$$

Then expanding out we find that

$$\Gamma_v^2 = 1.$$

This uses the preceding Lemma and the obvious relations $e_\alpha e_\beta = -e_\beta e_\alpha, e_\alpha^* e_\beta^* = -e_\beta^* e_\alpha^*$. So we conclude that we get a representation of $\mathfrak{so}(2m)$ on $\Lambda^* E$. By general theory, this Lie algebra representation corresponds to a representation of the simply connected group Spin$(2m)$. This is the total spin representation S. However S obviously decomposes into $S = S^+ \oplus S^-$ corresponding to the even and odd parts of the exterior algebra, and these are the irreducible representations that we want. Our maps Γ_i give maps of representations of Spin$(2m)$:

$$\mathbf{R}^{2m} \otimes_{\mathbf{R}} S^+ \to S^- \ , \ \mathbf{R}^{2m} \otimes_{\mathbf{R}} S^- \to S^+.$$

These are called the *Clifford multiplication maps.*

Now consider $SO(2m-1) \subset SO(2m)$. That is we fix a unit vector in $\mathbf{R}^{2m}$. Then from the above we get a map $S^+ \to S^-$ which is an isomorphism. In other words when restricted to $SO(2m-1)$ the representations $S^\pm$ become isomorphic. This defines the spin representation S of $SO(2m-1)$.

Another approach, in place of complex structures, is to use induction on dimension. Suppose that for $m = 2n$ we have Γ_i as above where $S = S^+ \oplus S^-$ and $\Gamma_i = \gamma_i + \gamma_i^*$ where $\gamma_i : S^+ \to S^-$. Then we get the same structure for $m = 2n+1$ by taking

$$\Gamma_{2m+1} = (1) \oplus (-1) : S^+ \oplus S^- \to S^+ \oplus S^-$$

Suppose that we have S for $m = 2n+1$ and $\Gamma_i^* = \Gamma_i$. Then we set $S^+ = S, S^- = S$ and $\gamma_i = \sqrt{-1}\Gamma_i$. We also let $\gamma_{2n+2} = 1$.

Thus we construct the spin representations inductively.

The embedding $M = SO(2n)/U(n) \subset \mathbf{P}(S^+)$ can be described more explicitly as follows. Fix a decomposition $\mathbf{C}^{2n} = L_0 \oplus L_0^*$ by isotropic subspaces. Then a generic n-dimensional subspace can be written as the graph of a linear map from L_0 to L_0^* i.e an element of $L_0^* \otimes L_0^*$. The condition that the graph is isotropic is that this map is skew symmetric. So we parametrise a dense open set in M by $\Lambda^2 L_0^*$. Identifying S^+ with $\Lambda^{\text{even}} L_0^*$ the embedding is defined by

$$A \mapsto [\exp(A/2)],$$

where exp is computed in the commutative algebra Λ^{even}.

Extra facts

- When we restrict to $SU(m) \subset \mathrm{Spin}(2m)$ the spin representations are identified with the even and odd exterior powers. When we restrict to the larger group $\tilde{U}(m)$ we get the representations

$$\Lambda^{\text{even}}\mathbf{C}^m \otimes (\det \mathbf{C}^m)^{1/2} \quad , \Lambda^{\text{odd}}\mathbf{C}^m \otimes (\det \mathbf{C}^m)^{1/2}.$$

- If we identify the Lie (T) for $\mathrm{Spin}(2m)$ with that of $SO(2m)$ and use the same co-ordinates as before the weights of S are

$$\frac{1}{2}\left(\pm\lambda_1 \pm \lambda_2 \pm \lambda_3 \ldots \pm \lambda_m\right).$$

  For S^+ we take those terms with an even number of $+$ signs and for S^- those with an odd number of $-$ signs. Notice that the weight lattice of $\mathrm{Spin}(2m)$, in these co-ordinates, is given by $\sum a_i\lambda_i$ where $a_i \in \mathbf{Z}/2$ and $a_i = a_j$ modulo 1, for all i, j.

- If we consider $Spin(2m) \subset Spin(2m+1)$ then the spin representation S in $2m+1$ dimensions decomposes as $S^+ \oplus S^-$ where $S^\pm$ are the spin representations in $2m$ dimensions.

- The spin representations $S^\pm$ of $\mathrm{Spin}(2m)$ start life as complex vector spaces of dimension 2^{m-1}. But they can be considered in a variety of ways, depending on the dimension. We have a complex *antilinear* map

$$* : \Lambda^p E \to \Lambda^{m-p} E,$$

  defined by

$$\theta \wedge (*\theta) = |\theta|^2 \text{ vol}.$$

  When $m = 0$ modulo 4, $*$ maps Λ^{even} to Λ^{even} and has $** = 1$. This implies that S^+ is the complexification of a real representation of $\mathrm{Spin}(2m)$, and similarly for S^-.

  When $m = 2$ modulo 4, $*$ again maps Λ^{even} to Λ^{even} but $** = -1$. This implies that S^+ is naturally a quaternionic vector space, and similarly for S^-.

  When $m = \pm 1$ modulo 4, $*$ maps Λ^{even} to Λ^{odd} and this implies that the representations $S^\pm$ are duals of each other.

  Likewise, in the odd case, the spin representation of $\mathrm{Spin}(n)$ is real if $n = \pm 1$ modulo 8 and quaternionic if $n = \pm 3$ modulo 8.

- Write $S(\mathbf{R}^{2m})$ for the total spin space associated to a Euclidean vector space. (More precisely the "association" is only up to a sign.) Then we have
$$S(\mathbf{R}^{2p} \oplus \mathbf{R}^{2q}) = S(\mathbf{R}^{2p}) \otimes S(\mathbf{R}^{2q})$$
and the decomposition into $\pm$ parts works in the obvious way.

- There is an analogue of the spin representation associated to the real symplectic group $Sp(n, \mathbf{R})$. This is a noncompact group and the representation, called the *metaplectic representation* is an infinite-dimensional unitary representation of a double cover of $Sp(n, \mathbf{R})$. The infinite dimensionality brings in analytical issues but, ignoring these, we proceed as follows. In the orthogonal case the relations we need for our maps Γ_v can be written as
$$\Gamma_v \Gamma_w + \Gamma_w \Gamma_v = \langle v, w \rangle 1.$$
Now we take v to lie in $\mathbf{R}^{2m}$ endowed with the standard *symplectic* form Ω and consider instead the relations
$$\Gamma_v \Gamma_w - \Gamma_w \Gamma_v = \Omega(v, w) 1.$$
Given such a family we get a representation of the Lie algebra of $Sp(n, \mathbf{R})$. Take a Hermitian space E as a before and consider the symmmetric (i.e. polynomial) algebra $s^*(E)$. We have maps e_α, defined by multiplication, and e^*_α, defined by differentiation, and we proceed as before using the Heisenberg relations
$$[e_\alpha, e^*_\beta] = \delta_{\alpha\beta}.$$

6.3 The Weyl character formula.

The data of the weights and multiplicities of a representation can equivalently be encoded in the character $\chi_V : T \to \mathbf{C}$. This is the character of V restricted to T. Since each conjugacy class in G contains a representative in T there is no loss of information. The character is obviously invariant under the action of the Weyl group. In the case of the representation s^k of $SU(2)$ we have

$$\chi(x) = e^{ikx} + e^{i(k-2)x} + \ldots + e^{-ikx}.$$

This can be written as

$$\chi(x) = \frac{e^{i(k+1)x} - e^{-i(k+1)x}}{e^{ix} - e^{-ix}}.$$

(Here x is a co-ordinate on Lie $(T) = \mathbf{R}$ and we are regarding the character as a periodic function on Lie (T).)

For the general case: given a weight λ, let e_λ be the corresponding complex-valued function. This can be thought of as a function on Lie (T) or T (by periodicity). We have $e_{\lambda+\mu} = e_\lambda e_\mu$. The character is

$$\chi_V = \sum_\lambda n_{\lambda,V} e_\lambda.$$

Fix a fundamental Weyl chamber and let ρ be one half the sum of the positive roots. Define a homomorphism sgn : $W \to \{\pm 1\}$ by the determinant of the action on Lie (T). Given a highest weight vector $\xi \in B_0$ set

$$A_{\rho+\xi}(x) = \sum_{w \in W} \mathrm{sgn}(w) e_{w(\xi+\rho)}(x).$$

In general $\xi + \rho$ may not be a weight, so we have to interpret this on Lie (T) for the moment. Let

$$D(x) = \prod_\alpha (e_{\alpha/2}(x) - e_{-\alpha/2}(x)),$$

where the product is taken over the positive roots α.

The Weyl character formula is

Theorem 6 *For a weight ξ in B_0 the character of the corresponding irreducible representation is*

$$\chi_{V_\xi} = \frac{A_{\rho+\xi}}{D}.$$

The statement should be taken as including the assertion that the right hand side actually is a finite sum of multiples of e_λ for weights λ.

There are various ways of proving this Theorem. One, as in Adams (following Weyl), uses a careful discussion of the orthogonality of characters. (This also leads to another proof—the original proof— of the main theorem about representations, but not in a very explicit form.) Another approach, covered in Fulton and Harris, is entirely algebraic. Both these proofs require some rather detailed arguments. We discuss a proof which involves more background but leads straight to the formula.

Preliminaries

Note first that the Weyl formula is interesting and non-trivial for "familiar" groups such as $SU(3), SU(4), \ldots$. So, if you prefer, it makes good sense not to worry too much about the arguments involving the Weyl group etc., in the general case, but just verify the assertions in these familiar cases.

Lemma 6 *Let α_0 be a simple root. Then the reflection defined by α_0 permutes all the positive roots not equal to α_0*

By definition α_0 corresponds to a codimension-1 face of the boundary of B_0. Let ξ be a generic point on this face and choose nearby points ξ_+, ξ_- with ξ_+ in the interior of B_0 and ξ_- the reflection of ξ_- in the root plane L_{α_0}. So the line segment from ξ_+ to ξ_- does not cross any root planes apart from L_{α_0}. So, for any positive root α, the inner products $\langle \alpha, \xi_\pm \rangle$ have the same sign and this sign is positive, since ξ_+ is in the interior of B_0. But if the relection ρdefined by α_0 took α to a negative root we would have

$$\langle \alpha, \xi_- \rangle = \langle \alpha, \rho(\xi_+) \rangle = \langle \rho(\alpha), \xi_+ \rangle < 0,$$

in contradiction to the above.

A function f on Lie (T) is called *alternating* if $f(wx) = \mathrm{sgn}(w) f(x)$ for $w \in W$. Clearly $A_{\rho+\xi}$ is alternating. Two elementary observations are

Corollary 5 *1. The function $D(x)$ is alternating.*

2. The element ρ lies in the interior of the fundamental Weyl chamber B_0.

These facts are easy to show, given the preceding lemma. For (1) we see that each time we apply a reflection we change the sign of just one term in the product defining D. For (2), we see that for a simple root α_0

$$2\langle \alpha_0, \rho \rangle = \langle \alpha_0, \alpha_0 \rangle + \langle S, \alpha_0 \rangle,$$

where S is the the sum is of the positive roots not equal to α_0. By the lemma, S is fixed by the reflection defined by α_0, so $\langle S, \alpha_0 \rangle = 0$.

There is no loss of generality in supposing that ρ is a weight of G. For the truth of the Weyl formula is not affected by taking finite coverings. Consider the adjoint representation of G, which maps to some orthogonal group $SO(d)$. This may not lift to Spin(d) but we can construct a double cover $\tilde{G} \to G$ which does have this property. So we can suppose that the adjoint representation lifts to Spin and then ρ is a weight, by what we know about the weight lattice of Spin. This is not really essential but will simplify language. Geometrically, the line bundle L_ρ over G/T, corresponding to ρ, is a square root $K^{-1/2}$ of the dual "anti-canonical" line bundle of the manifold G/T.

Two extra facts which we do not need to use in the proof of the Theorem, but which are useful to know for orientation are:

- The denominator D can be written in another way:

$$D(x) = A_\rho(x) = \sum_{w \in W} \text{sgn}(w) e_{w(\rho)}(x).$$

  This follows from the main theorem by taking the trivial representation, but it is not hard to prove directly using the alternating property of D.

- If ρ is a weight then the map $\sigma \mapsto \sigma + \rho$ gives a 1-1 correspondence between weights σ in B_0 and weights $\sigma + \rho$ in in the interior of B_0.

The basic theorem about representations of a finite group A can be stated as follows. Consider the space $C(A)$ of complex-valued functions on A. This is a representation of $A \times A$, with A acting on itself by left and right translation. The statement is that, as representations of $A \times A$

$$C(A) = \bigoplus V^* \otimes V,$$

where the sum runs over the irreducible representations V of A and we let A act on the left on V^* and on the right on V. All the properties of orthogonality of characters etc. can be read off from this.

The same theorem holds, ignoring some technicalities, for a compact Lie group. We let $C(G)$ be the space of complex-valued "functions" on G and we have

$$C(G) = \bigoplus V^* \otimes V.$$

The technicalities involve what kind of functions we consider and precisely what we mean by the direct sum, running over the infinitely many irreducible representations. But these issues appear already in the case when $G = S^1$

$$C(S^1) = \bigoplus_{r \in \mathbf{Z}} \mathbf{C}\ e^{irx},$$

which is the theory of Fourier series. There are a variety of different precise interpretations we can put on the formula e.g.

- The space formed by the finite sums on the right hand side is dense in $C^\infty(S^1)$,
- If we take l^2 sums on the right hand side we get an isomorphism with $L^2(S^1)$,
- If we take sums on the right hand side which do not grow too fast we get distributions on S^1

and so on. In fact these issues will be irrelevant for our discussion so we just ignore them. (The essential thing one needs is to break up the functions on G into a sum of finite dimensional representations, and this can be seen by using the eigenspaces of the Laplace operator on G, or using integral operators as in Adams.)

Now consider $T \subset G$. Given a weight μ we can consider the functions on G which transform by the weight $-\mu$ under the right action of T on G. These are just the same as the *smooth* sections $\Gamma(L_\mu)$ of the line bundle $L_\mu \to G/T$. But now G acts on $\Gamma(L_\mu)$ and we can decompose into irreducibles

$$\Gamma(L_\mu) = \bigoplus_V m_{V,\mu} V,$$

for certain multiplicities $m_{V,\mu}$. Now the formula above implies that $m_{V,\mu} = n_{V,\mu}$: i.e.*the multiplicity with which the weight μ occurs in a representation V is the same as the multiplicity with which the representation V occurs in the space of smooth sections of L_μ.*

The analogue of this for finite groups and induced representations is the *Frobenius reciprocity theorem.*

Input from complex geometry: the Dolbeault complex and vanishing theorems

Suppose in general that we have a complex manifold M and a holomorphic line bundle $L \to M$. The holomorphic sections of L can be viewed as the smooth sections which satisfy a linear partial differential equation, a version of the Cauchy-Riemann equations. This equation can be written as $\overline{\partial} s = 0$ where $\overline{\partial}$ is a linear differential operator

$$\overline{\partial} : \Gamma(L) \to \Gamma(L \otimes \tau),$$

where τ is the conjugate dual of the tangent bundle of M. This extends to the Dolbeault complex

$$\overline{\partial} : \Gamma(L \otimes \Lambda^p \tau) \to \Gamma(L \otimes \Lambda^{p+1} \tau),$$

with $\overline{\partial}^2 = 0$. We form the corresponding Dolbeault cohomology groups $H^p(M, L)$, so $H^0(M, L) \subset \Gamma(L)$ is the same as the holomorphic sections. The basic fact we need is that when $M = G/T$ and $L = L_{\lambda-\rho}$ for a weight $\lambda \in B_0$ the cohomology groups vanish for $p \geq 1$. Further, if λ is not in the interior of B_0 then the 0-dimensional cohomology vanishes as well. These statements follows from the *Kodaira Vanishing Theorem.* This may seem more familiar if we write $L_{\lambda-\rho} = L_\lambda \otimes K^{1/2}$ and note that L_λ is a "semi-positive" line bundle and $K^{-1/2}$ is a strictly "positive" line bundle. This last follows from the second part of the corollary above (that ρ is in the interior of B_0.)

(**Remark** Those who have encountered the Kodaira Vanishing theorem will recognise similarities between the proof and the identities used in 6.2 above.)

Now we are ready to derive the Weyl formula. Let μ be a weight and consider the Dolbeault complex of the line bundle L_μ over G/T. The group G acts on everything and each space of sections $\Gamma(L \otimes \Lambda^p \tau)$ can be expressed as a sum over irreducibles. The $\overline{\partial}$-operators can only map between pieces corresponding to the same irreducible. That is, we have

$$\Gamma(L \otimes \Lambda^p \tau) = \bigoplus_W N_{p,W} \otimes W$$

say, where W runs over the irreducible representations of G. Then the $\overline{\partial}$-operator is defined by linear maps $N_{p,W} \to N_{p+1,W}$.so If we fix the representation V and look at the corresponding sub-complex we get a finite dimensional complex

$$N_0 \to N_1 \to N_2 \to \dots$$

where we have written $N_p = N_{p,V}$. Using the well-known relation relation between the Euler characteristics of a complex and its cohomology we see that the alternating sum

$$\sum_p (-1)^p \mathrm{dimN_p}$$

is given by the multiplicity of V in the cohomology of L_μ, taken with suitable signs. Now suppose that V has weight ξ and that $\mu + \rho$ lies in the fundamental Weyl chamber B_0. Then by what we know about the cohomology we have

$$\sum_p (-1)^p \dim N_p = \delta_{\xi\mu}$$

(i.e. equal to zero unless $\mu = \xi$ when it is 1.)

Using a Hermitian metric, we can identify the bundle τ with the tangent bundle of the complex manifold. But the decomposition into root spaces allows us to write this as

$$\tau = \bigoplus L_\alpha$$

where α runs over the positive roots. (This is not a holomorphic isomorphism, but that does not matter here.) Thus the exterior power $\Lambda^p \tau$ can be identified with the direct sum of line bundles $L_{\alpha_1 + \dots \alpha_p}$ where the sum runs over distinct positive roots $\alpha_1, \dots, \alpha_p$. Using our "reciprocity principle',' this means that

$$\mathrm{dim} N_p = \sum n_{\mu+\alpha_1+\dots+\alpha_p, V}.$$

Let $E(x) = \prod(1 - e_{-\alpha}(x))$, where the product runs over the positive roots. We see that $\sum(-1)^p \mathrm{dim} N_p$ is the coefficient of e_μ in the product

$$(\sum_\lambda n_{\lambda,V} e_\lambda(x))E(x),$$

that is, in $\chi_V(x)E(x)$. So our formula above says that the only term in $\chi_V(x)E(x)$ corresponding to a weight in $B_0 - \rho$ is e_ξ. Clearly $D(x) = E(x)e_\rho(x)$, so the only term in the product $\chi_V(x)D(x)$ corresponding to a weight in B_0 is $e_{\xi+\rho}$. But we know that $\chi_V(x)D(x)$ is alternating, by (1) of the Corollary, under the action of the Weyl group, so we must have

$$\chi_V(x)D(x) = \sum \mathrm{sgn}(w) e_{w(\xi+\rho)}(x),$$

which is the Weyl formula.

Remark From our argument we can derive (most of) the generalisation due to Bott of the Borel-Weil theorem. Consider any holomorphic line bundle L_μ

over G/T. The cohomology groups are representations of G. Then $H^*(G/T, L_\mu)$ is zero if $\mu + \rho$ lies in a root plane. Otherwise, $\mu + \rho$ lies in the interior of some Weyl chamber wB_0 and the cohomology is zero except in a certain dimension $p(w)$. In this dimension we get a copy of V_ξ where ξ is the weight $w^{-1}(\mu+\rho)-\rho$ which is in B_0. What we are missing is an precise description of $p(w)$; we only see that $p(w)$ is odd or even as $\operatorname{sgn}(w)$ is ± 1. The whole discussion, and the role of ρ, is clearer if one works with the *Dirac operator* in place of the Dolbeault complex.

7 Exceptional groups and special isomorphisms

7.1 Low dimensions:special isomorphisms

The dimension of the spin representation of $SO(n)$ grows very rapidly as a function of n, but for low values of n the spin representation can be "smaller" than the usual fundamental representation on $\mathbf{R}^n$. This leads to *special isomorphisms* within the families of classical Lie groups. We write $\sim$ for local isomorphism (i.e. isomorphism up to coverings.)

- $\mathrm{Spin}(3) \sim SU(2) \cong Sp(1)$, and $\mathbf{C}^2 = \mathbf{H}$ is the spin representation.
- $\mathrm{Spin}(4) \sim SU(2) \times SU(2)$ and the $\mathbf{C}^2$ representations of the two factors are the representations $S^{\pm}$.
- $\mathrm{Spin}(5) \sim Sp(2)$ and $\mathbf{H}^2$ is the spin representation.
- $\mathrm{Spin}(6) \sim SU(4)$ and $\mathbf{C}^4$ is the positive spin representation (the negative spin representation is the dual).

And there we stop: apart from these *special isomorphisms*, the groups $SU(n), \mathrm{Spin}(n), Sp(n)$ are all distinct. These special isomorphisms can all be seen in a variety of other ways.

- $SO(3) \sim SU(2)$: we have seen this many times. The adjoint representation gives a double covering $SU(2) \to SO(3)$.
- $SO(4) \sim SU(2) \times SU(2) \sim SO(3) \times SO(3)$: We consider the exterior power $\Lambda^2 \mathbf{R}^4$. In the presence of a metric and orientation there is a $*$-operation $* : \Lambda^2 \to \Lambda^2$ with $*^2 = 1$. Then $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, the ± 1 eigenspaces of $*$. These are 3-dimensional. The symmetry group $SO(4)$ acts on $\Lambda^2_{\pm}$ and this gives a homomorphism $SO(4) \to SO(3) \times SO(3)$.
- $SO(5) \sim Sp(2)$. We think of $Sp(2)$ acting on the quaternionic projective line $\mathbf{HP}^1$ i.e.
  $$\mathbf{HP}^1 = Sp(2)/Sp(1) \times Sp(1).$$
  Now use the fact that $\mathbf{HP}^1 = S^4$ and the previous isomorphism to recognise this, up to coverings, as
  $$S^4 = SO(5)/SO(4).$$
- $SO(6) \sim SU(4)$. Start with $SU(4)$ acting on $\mathbf{C}^4$. Similar to the real case there is a $*$ operator $* : \Lambda^2 \to \Lambda^2$, with $*^2 = 1$, but this is now complex antilinear (since it uses the Hermitian form on $\mathbf{C}^4$). We encountered this in Section 6.2. The eigenspaces $\Lambda^2_{\pm}$ are 6-dimensional real vector spaces (since $*$ is antilinear). The action on one of these gives a homomorphism $SU(4) \to SO(6)$.

In the above we have discussed compact groups. We get similar isomorphisms between non-compact forms with the same complexification: for example $SL(2, \mathbf{R}) \sim SO(2,1)$. Another example is $SL(4, \mathbf{R}) \sim SO(3,3)$ which comes from the fact that, given a volume form, the wedge product defines a natural quadratic form on $\Lambda^2 \mathbf{R}^4$. Then we can get $SU(4) \sim SO(6)$ by a slightly different route: first complexifying and then taking maximal compact subgroups.

7.2 Dimensions 7 and 8: Triality and G_2.

What happens when we go further? The special isomorphisms stop but in dimensions 7 and 8 the spin representations are roughly the same size as the fundamental representation and this leads to exceptional phenomena, and in turn the existence of the exceptional Lie groups. Since we defined the spin representation in dimension $2n-1$ by passing to dimension $2n$ we skip 7 dimensions for the moment and go straight to Spin(8).

The two spin representations $S^{\pm}$ of Spin(8) are each 8 dimensional and have natural real structures. So we get two homomorphisms Spin(8) $\to SO(S^{\pm})$ and it is easy to see that these are local isomorphisms. It follows that there are inner automorphisms of Spin(8) which take the standard representation to either of the spin representations. In fact the inner automorphisms permute the three 8 dimensional representations $\mathbf{R}^8, S^+, S^-$ in all possible ways. This is the phenomenon of "triality". The symmetry is also evident in the Dynkin diagram. The Clifford multiplication

$$\mathbf{R}^8 \times S^+ \to S^-,$$

gives, after transposition, a trilinear map

$$\mathbf{R}^8 \times S^+ \times S^- \to \mathbf{R},$$

which is preserved, up to sign, by the automorphisms interchanging the three representations.

Now go back to 7 dimensions. The spin representation gives an action of Spin(7) on the 8-dimensional real vector space S and hence on the unit sphere in this space, which we denote by $\Sigma(S)$. So $\Sigma(S)$ is a copy of S^7. We claim that this action is transitive. Indeed we can think of Spin(7) $\subset$ Spin(8) as the stabiliser of a unit vector in the standard representation on $\mathbf{R}^8$. By triality it is the same to show that if $H \subset$ Spin(8) is the stabiliser of a unit spinor in S^+ then H acts transitively on the unit sphere in the standard representation $\mathbf{R}^8$. Fix a complex structure $\mathbf{R}^8 = \mathbf{C}^4$. Then the group $SU(4)$ preserves the form $1 + *1 \in \Lambda^0 \oplus \Lambda^4$,and this is a real element in the sense of the real structure on S^+. So H contains $SU(4)$ and this already acts transitively on the unit sphere in $\mathbf{C}^4$.

Given the above, we can define a Lie group $G_2 \subset \mathrm{Spin}(7)$ to be the stabiliser of a unit spinor ψ. So we have

$$S^7 = \Sigma(S) = \mathrm{Spin}(7)/G_2,$$

and G_2 has dimension $\frac{1}{2}7.6 - 7 = 14$. We claim now that $Spin(7)$ acts transitively on the unit sphere bundle of $\Sigma(S)$, i.e. on pairs of orthogonal vectors in S. By the same principle as before it is the same to show that H acts transitively on pairs of orthonormal vectors in $\mathbf{R}^8$. Since we already know it acts transitively on unit vectors we can consider pairs of the form $(e_1, e_2), (e_1, e_2')$. Then we choose a complex structure on $\mathbf{R}^8$ so that e_2, e_2' are in the same 1-dimensional complex subspace. Then, with this complex structure, there is an element of $SU(4)$ which fixes e_1 and takes e_2 to e_2'.

From the above, the group G_2 acts transitively on the unit sphere in the tangent bundle of $\Sigma(S)$ at the point ψ, i.e. on S^6. We look at the stabiliser G of a point in this unit sphere bundle, which has dimension 8. By the same principle this is the same as looking at the stabiliser of a unit spinor ψ and a pair of orthonormal vectors e_1, e_2 in $\mathbf{R}^8$. Choose a complex structure on $\mathbf{R}^8$ with $e_2 = Ie_1$. Then the standard embedding $SU(3) \subset SU(4)$ maps into the stabiliser and since $SU(3)$ has dimension 8 we see that it must be the whole group. Thus

$$S^6 = G_2/SU(3).$$

It follows that we can build the Lie algebra of G_2 as

$$\mathfrak{g}_2 = \mathfrak{su}(3) \oplus V$$

for some representation V of $SU(3)$. It is not hard to identify this as the standard representation on $\mathbf{C}^3$. It follows that G_2 has rank 2, with same maximal torus as $SU(3)$. The roots of G_2 are given by adjoining $\pm$ the weights of the standard representation to the roots of $SU(3)$. This gives a configuration of two concentric hexagons in the plane and realises the case $k = 6$ discussed in Section 5.

: Further facts about G_2.

The decomposition $\mathfrak{g}_2 = \mathfrak{su}(3) \oplus V$ is *not* that of a symmetric space. That is, there is a non-zero component of the bracket mapping $\mathbf{C}^3 \times \mathbf{C}^3$ to $\mathbf{C}^3$. This is the complex analogue of the cross product in $\mathbf{R}^3$ and in standard co-ordinates is given by

$$(z \times w)_i = \frac{1}{2} \sum \epsilon_{ijk} \overline{z}_j \overline{w}_k.$$

Take $\mathbf{R}^7 = \mathbf{R} \oplus \mathbf{C}^3$. We can define a skew-symmetric cross product $\times : \mathbf{R}^7 \times \mathbf{R}^7 \to \mathbf{R}^7$ by

- The component $\mathbf{C}^3 \times \mathbf{C}^3 \to \mathbf{C}^3$ is the cross product above;
- The component $\mathbf{C}^3 \times \mathbf{R} \to \mathbf{C}^3$ is scalar multiplication, and $\mathbf{R} \times \mathbf{C}^3 \to \mathbf{C}^3$ defined by skew symmetry.
- The component $\mathbf{C}^3 \times \mathbf{C}^3 \to \mathbf{R}$ is the imaginary part of the Hermitian form on $\mathbf{C}^3$.

This is visibly invariant under the action of $SU(3)$ on $\mathbf{R}^7$. It is also invariant under the action of $G_2 \subset SO(7)$. We can alternatively define G_2 to be the subgroup of $GL(7, \mathbf{R})$ which preserves this cross product.

The cross product on $\mathbf{R}^7$ can be used to define the *octonion* product on $\mathbf{R}^8 = \mathbf{R}1 \oplus \mathbf{R}^7$, with the same formulae as for the quaternions. The octonions are not associative. A 3-dimensional subspace of $\mathbf{R}^7$ which is closed under the cross-product and on which the product is isomorphic to that on $\mathrm{Im}\mathbf{H}$ is called an *associative subspace*. Let M be the set of associative subspaces. Then G_2 acts transitively on M and it turns out that $M = G_2/SO(4)$. Furthermore this is a symmetric space (and the only compact symmetric space associated to G_2).

7.3 Lie algebra constructions

7.3.1 Construction of F_4, E_8.

Now we go back to our discussion of the Lie algebras associated to symmetric spaces in Section 2.1. Suppose we have a Lie algebra $\mathfrak{g}$, with a nondegenerate invariant quadratic form, and a Euclidean representation on a vector space V (i.e. a homomorphism $\mathfrak{g} \to \mathfrak{so}(V)$: all of this can be done in either the real or complex cases). Then we have $\mathfrak{g} \otimes V \to V$ which we can transpose using the inner products to get $V \otimes V \to \mathfrak{g}$. Using these we can build a bracket $[,] : W \times W \to W$ where $W = \mathfrak{g} \oplus V$. Thus the component of $[,]$ mapping $V \times V$ to V is defined to be 0. Now $[,]$ will *not* always satisfy the Jacobi identity (usually it will not), but the only problem comes from the component of $J(x, y, z) = [x, [y, z]] + [y, [z, x]] + [z, [x, y]]$ which maps $V \times V \times V$ to V. In fact we can think of the equivalent data

$$\Omega(x, y, z, w) = \langle J(x, y, z), w \rangle$$

as an element of $\Lambda^4 V^*$, canonically determined by the input $(\mathfrak{g}, V)$. Call V a *good representation* of $\mathfrak{g}$ if J (or equivalently Ω) vanishes. In this case we get a new Lie algebra $(W, [,])$. If we had a good technique to calculate Ω we would have a good method for constructing interesting Lie algebras, but direct calculations can be complicated.

Now suppose we have a Lie algebra $\mathfrak{g}$, with invariant form, and three Euclidean representations V_1, V_2, V_2. Suppose we have a $\mathfrak{g}$-invariant trilinear map

$$B : V_1 \times V_2 \times V_3 \to \mathbf{R}.$$

Suppose also that we have automorphisms of $\mathfrak{g}$ which permute the V_i cyclically and which preserve B up to sign. Now take

$$U = \mathfrak{g} \oplus V_1 \oplus V_2 \oplus V_2,$$

and define a skew-symmetric bracket $[,]$ on U as follows.

- The component $\mathfrak{g} \otimes V_i \to V_i$ as before.
- The component $V_i \otimes V_i \to \mathfrak{g}$ as before.
- Define $V_1 \otimes V_2 \to V_3$ by transposing B, and similarly for cyclic permutations, and the terms required by skew-symmetry.
- All other components are set to 0.

Now suppose that V_1 is a good representation of $\mathfrak{g}$. (By the symmetry we assume, V_2 and V_3 are also good representations.) Then we have a pair of Lie algebras $\mathfrak{g} \subset \mathfrak{g}^+$ as above, with $\mathfrak{g}^+ = \mathfrak{g} \oplus V_1$. Suitable components of $[,]$ on U give a map

$$\mathfrak{g}^+ \otimes (V_2 \oplus V_3) \to V_2 \oplus V_3.$$

Suppose we can show that this defines an action of $\mathfrak{g}^+$ on $V = V_2 \oplus V_3$. Examining the definitions we see that this is Euclidean and that $[,]$ on $U = \mathfrak{g}^+ \oplus V$ is defined by the same procedure as before. Now we claim that in this situation V is a good representation of $\mathfrak{g}^+$. For, by what we have said, the only possible difficulty comes from $J(x, y, z)$ where x, y, z all lie in $V_2 \oplus V_3$. That is to say, Ω lies in $\Lambda^4(V_2 \oplus V_3)$. But by symmetry we have

$$\Omega \in \Lambda^4(V_2 \oplus V_3) \cap \Lambda^4(V_3 \oplus V_1) \cap \Lambda^4(V_1 \oplus V_1 \oplus V_2,$$

and this intersection is 0. So the Jacobi identity is satisfied. (Of course one does need to think carefully about the signs here, but the argument holds up.)

We apply this in two cases. First we take $\mathfrak{g} = \mathfrak{so}(8)$ and the three representations $\mathbf{R}^8, S^+, S^-$. The initial hypotheses express what we know about triality. We recognise $\mathfrak{so}(8) \oplus \mathbf{R}^8$ as $\mathfrak{so}(9)$ (corresponding to the description of S^8 as as symmetric space), so $\mathbf{R}^8$ is a good representation. Also we can recognise $S^+ \oplus S^-$ as the spin representation S of $\mathfrak{so}(9)$, and check that the recipe above does define the usual action. So we conclude that there is a Lie algebra, called $\mathfrak{f}_4$ with

$$\mathfrak{f}_4 = \mathfrak{so}(8) \oplus \mathbf{R}^8 \oplus S^+ \oplus S^-,$$

and $\mathfrak{so}(8) \subset \mathfrak{so}(9) \subset \mathfrak{f}_4$. We get a corresponding group F_4, either by general theory or more concretely by taking the automorphisms of the Lie algebra, and then the universal cover. With the right choice of signs F_4 is compact. It has dimension $\frac{1}{2}8.7 + 8 + 8 + 8 = 52$ and rank 4; the maximal torus is the same as that of $\mathfrak{so}(8)$. We also get a new symmetric space $F_4/\mathrm{Spin}(9)$ which is called the Cayley, or Moufang, plane. It is the analogue, for the octonions, of the real, complex and quaternionic projective planes. (But the analogy is not straightforward and there is no analogue of the higher dimensional projective spaces.)

For the second application we start with $\mathfrak{g} = \mathfrak{so}(8) \oplus \mathfrak{so}(8)$ and the representations

$$V_1 = \mathbf{R}^8_1 \otimes \mathbf{R}^8_2 \; , V_2 = \; S^+_1 \otimes S^+_2 \; , \; V_3 = S^-_1 \otimes S^-_2 .$$

Here the lower indices ($\mathbf{R}^8_1$ etc.) mean that either the first or second copy of $\mathfrak{so}(8)$ acts. Triality on each copy of $\mathfrak{so}(8)$ puts us in the setting considered above (and using the tensor product of the two maps B). Now we recognise $\mathfrak{so}(8) \oplus \mathfrak{so}(8) \oplus \mathbf{R}^8_1 \otimes \mathbf{R}^8_2$ as $\mathfrak{so}(16)$, corresponding to the symmetric space description of the Grassmannian of 8-planes in $\mathbf{R}^{16}$. So the V_i are good representations of $\mathfrak{g}$. Then we recognise $V_2 \oplus V_3$ as the positive spin representation $S^+(\mathbf{R}^{16})$, (and check that our rules define the usual action). So we conclude that we have a Lie algebra

$$\mathfrak{e}_8 = \mathfrak{so}(8) \oplus \mathfrak{so}(8) \oplus V_1 \oplus V_2 \oplus V_3,$$

and a chain

$$\mathfrak{so}(8) \oplus \mathfrak{so}(8) \subset \mathfrak{so}(16) \subset \mathfrak{e}_8.$$

All the same remarks as before apply. We get a compact simply connected Lie group E_8 of rank 8 and dimension $7.8 + 3.8.8 = 248$. The maximal torus in $\mathrm{Spin}(16)$ is still maximal in E_8. We get a new symmetric space $E_8/\mathrm{Spin}(16)$.

7.3.2 Configurations of roots: constructions of E6 and E7

We will now get another view of E_8. This will make it obvious that it contains subgroups $E_6 \subset E_7 \subset E_8$ and these are related to two classical topics: the 27 lines in a cubic surface and the 28 bitangents of a quartic curve.

For $r \geq 1$ consider the standard Lorentzian form $x_0^2 - x_1^2 - \ldots - x_r^2$ on $\mathbf{R}^{1,r}$ and the integer lattice $\mathbf{Z}^{1,r} \subset \mathbf{R}^{1,r}$. Set $K = (-3; 1, \ldots, 1)$, an element of $bZ^{1,r}$. If $r \leq 8$ then $K.K \geq 0$ and the form is negative definite on the orthogonal complement of K. From now on we assume $r \leq 8$. Let Λ be the intersection of the integer lattice with this orthogonal complement and let $\mathcal{C}$ be the set of vectors of length -2 in Λ, i.e.

$$\mathcal{C} = \{C \in Z^{r+1} : C.C = -2, K.C = 0\}.$$

Let $\mathcal{L}$ be the set of vectors L in $\mathbf{Z}^{r+1}$ with

$$L.L = -1 \quad , K.L = -1.$$

Digression for those interested These definitions will look familiar to algebraic geometers. Suppose S is a complex cubic surface in $\mathbf{CP}^3$. Then the integral homology group $H_2(S)$ is $\mathbf{Z}^7$ and the "intersection form"is the standard Lorentzian form above. Tbe element K represents the canonical class of S. It is shown in surface theory that the conditions defining $\mathcal{L}$ characterise homology classes which are represented by *exceptional curves* in S; copies of $\mathbf{CP}^1$ with self-intersection -1. So $\mathcal{L}$ can be identified with the set of exceptional curves. Likewise $\mathcal{C}$ can be identified with the classes of "vanishing cycles", or -2 curves in resolutions of singular cubic surfaces. The same discussion applied to the other values of r. When $r = 7$ we consider the surface S formed as a double cover of the plane branched over a smooth quartic curve $\Sigma \subset \mathbf{CP}^2$. For each bitangent line to Σ we get a pair of exceptional curves and all arise in this way.

The sets $\mathcal{C}, \mathcal{L}$ are finite and it is easy to enumerate them. For example when $r = 6$ consider $(a; b_1, \dots, b_6) \in \mathbf{Z}^7$. This lies in $\mathcal{L}$ if

$$3a + \sum b_i = 1 \quad , \quad a^2 + 1 = \sum b_i^2.$$

Now the Cauchy-Schwartz inequality in this case gives

$$\left(\sum b_i\right)^2 \leq 6. \sum b_i^2,$$

so $(3a-1)^2 \leq (a^2+1)$ or $3a^2 - 6a - 5 \leq 0$. Since a is an integer this leads to $a = 0, 1, 2$. We get

1. When $a = 0$ six elements of $\mathcal{L}$ like $(0, 1, 0, \dots, 0)$;
2. When $a = 1$ fifteen elements of $\mathcal{L}$ like $(1, -1, -1, 0, \dots, 0)$;
3. When $a = 2$ six elements of $\mathcal{L}$ like $(2, -1, -1, -1, -1, -1, 0)$.

So $\mathcal{L}$ has 27 elements when $r = 6$, corresponding to the classical fact that there are 27 lines on a cubic surface. Indeed, continuing the digression for a moment, we can represent a cubic surface S as the "blow-up" of the plane at 6 points. The six classes of the first type above can be taken to be the exceptional curves in this blow-up, the fifteen of the second type to be the proper transforms of lines through 2 of the points and the six of the third class to be the proper transforms of conics through 5 of the points. When $r = 7$ we perform a similar analysis and find that $\mathcal{L}$ has $56 = 2.28$ elements. These correspond to the 28 bitangents of a quartic curve.

It is important to realise that the division into the three caes above is not really intrinsic to the set-up. Let Γ be the group of linear automorphisms of $\mathbf{Z}^{1,r}$

which preserve the element K and the quadratic form. This is a finite group which acts on $\mathcal{C}, \mathcal{L}$ and it is the natural symmetry group of the situation. But Γ does not preserve the division into the three classes; in fact it acts transitively on $\mathcal{L}$.

We leave it as an exercise to show that for any distinct $L, L' \in \mathcal{L}$ the inner product $L.L'$ is either 0 or 1. Thus $\mathcal{L}$ is a finite set equipped with a simple combinatorial structure $\mathcal{L} \times \mathcal{L} \to \{0, 1\}$ given by the inner product, and this is preserved by the finite group Γ. In the case when $r = 6$ this is the classical study of the incidence realtions of the "double six" configuration of lines on a cubic surface (see Hilbert and Cohn Vossen "Geometry and the imagination" for example).

If $L, L' \in \mathcal{L}$ then $L.L' = 0$ if and only if $L - L' \in \mathcal{C}$. Similarly for $C, C' \in \mathcal{C}$ with $C \neq \pm C'$ we have $C.C' = 0, \pm 1$ and $C + C' \in \mathcal{C}$ if and only if $C.C' = 1$. Also of course we have a map $\mathcal{C} \to \mathcal{C}$ taking C to $-C$. Thus we have combinatorial structures both on $\mathcal{C}, \mathcal{L}$ individually and connecting the two sets, everything invariant under Γ. To see that Γ is large note that for any element C of $\mathcal{C}$ defines a reflection map

$$x \mapsto x + (x.C)C,$$

which is in Γ.

It is clear from the definitions that the set $\mathcal{C}$ is a root system (in its span). Now we claim that for $1 \leq r \leq 8$ the set $\mathcal{C}$ is the root system associated to a complex Lie algebra $\mathfrak{h}_r^c$. That is we let $\mathfrak{h}^c$ as a vector space be the direct sum of $T^c = T \otimes \mathbf{C}$ and root spaces $\mathbf{C} w_C$ for each $C \in \mathcal{C}$. Suppose given $\epsilon(C, C') = \pm 1$ for each pair $C, C' \in cC$ with $C'.C = 1$ satisfying $\epsilon(C, C') = -\epsilon(C, C')$. Then we define a bracket by

- For $\alpha, \beta \in T^c$ we set $[\alpha, \beta] = 0$.
- For α in T^c and $C \in \mathcal{C}$ we set $[\alpha, w_C] = (\alpha.C) W_C$.
- For each $C \in \mathcal{C}$ we set $[w_C, w_{-C}] = C$.
- For $C, C' \in \mathcal{C}$ with $C' \neq \pm C$ we set $[w_C, w_{C'}] = \epsilon(C, C') w_{C+C'}$ if $C.C' = 1$ and $[w_C, w_{C'}] = 0$ if $C.C' = 0, -1$.

Then we have

Proposition 20 *There is a way to define ϵ so that $[,]$ satisfies the Jacobi identity.*

Of course this is essentially a special case of the fact we stated in Section 5, but we do not want to appeal to that here.

It is elementary to verify Proposition 2 from the combinatorics of the situation, *up to a sign*—that is, if we take any choice of ϵ. The difficulty is to choose ϵ so that all the signs work out correctly. The point here is that while the problem is invariant under Γ, so that if we have one solution ϵ we get another by applying any element of Γ, there is no *solution* invariant under Γ, although all solutions are equivalent in the sense that we can change our basis vectors w_C by a sign $\mu(C) = \pm 1$, with $\mu(-C) = \mu(C)$, and given one solution ϵ we get an equivalent one

$$\tilde{\epsilon}(C, C') = \mu(C)\mu(C')\mu(C + C')\epsilon(C, C').$$

Assume the above Proposition for the moment. Form a vector space V with basis v_L for $L \in \mathcal{L}$. Suppose we have signs $g(C, L) = \pm 1$ for each pair $L \in \mathcal{L}, C \in \mathcal{C}$ with $C.L = 1$.

Proposition 21 *There is a way to choose g such that the recipe*

$$\alpha(v_L) = (\alpha.L)v_L \quad , w_C(v_L) = g(C, L)v_{C+L}$$

defines an action of $\mathfrak{g}$ on V.

Again, this is easy to check up to sign. Of course the choice of g will depend on the choice of ϵ.

Now we fix attention on the case $r = 8$. In this case Proposition 3 is rather vacuous, assuming we have established Proposition 2. Since $K.K = 1$ the map $L \mapsto L + K$ is a bijection from $\mathcal{L}_8$ to $\mathcal{C}_8$ and the representation in question is just the adjoint representation. Of course the Lie algebra $\mathfrak{h}_8$ will be $\mathfrak{e}_8$, so what we have to do is to match up the description here with our previous construction.

Take $(y_1, \dots, y_8)$ as the standard co-ordinates on the Lie algebra of the maximal torus in $\mathfrak{so}(16)$. The roots of $SO(16)$ are of the form

$$(\dots, \pm 1, \dots, \pm 1 \dots)$$

with all other entries 0, and there are 112 of these. The weights of the positive spin representation are

$$\frac{1}{2}(\pm 1, \pm 1, \dots, \pm 1)$$

where we take an even number of minus signs, and there are 128 of these. The roots of E_8 are the union of these two sets, since $\mathfrak{e}_8 = \mathfrak{so}(16) \oplus S^+$. We take our form to be negative definite, $-\sum y_i^2$. Then all these roots y have $y.y = -2$. Let Λ' be the lattice in $\mathbf{R}^8$ consisting of vectors $(y_1, \dots, y_8)$ in $\frac{1}{2}\mathbf{Z}^8$ with $y_i = y_j$ mod 1 and $\sum y_i = 0$ mod 2. Then it is easy to check that the roots of E_8 are precisely the vectors in Λ' with $y.y = -2$. So all we have to check is that the lattices Λ and Λ', with their negative definite forms, are isomorphic. For then our general structure theory shows that the bracket on $\mathfrak{e}_8$, when transferred to the other setting, has the given form. (Actually our general theory will tell us that we get a bracket by taking some $\epsilon(C, C') \in \mathbf{R}$ but it is not hard to see that we can arrange $\epsilon(C, C') = \pm 1$ in this case.

To check that Λ and Λ' are isomorphic we observe that, since $K.K = 1$, we have $\mathbf{Z} \oplus \Lambda = \mathbf{Z}K \oplus \Lambda = \mathbf{Z}^{1,8}$ with the standard Lorentzian form. So we consider $\mathbf{Z} \oplus \Lambda' \oplus \mathbf{Z} \subset \mathbf{R} \oplus \mathbf{R}^8$. Write a vector in $\mathbf{R} \oplus \mathbf{R}^8$ as $(y_0; y_1, \dots, y_8)$ and use the quadratic form $y_0^2 - y_1^2 \dots - y_8^2$. Set

$$\kappa = (1; 0, \dots, 0),$$

$$q_1 = (1, \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}),$$
$$q_2 = (1, 1, 1, 0, \dots, 0),$$
$$q_3 = (1, 1, 0, 1, 0, \dots, 0),$$

and similarly down to

$$q_8 = (1; 0, 0, \dots, 1).$$

Then

$$\kappa.\kappa = 1, q_i.q_i = -1, \kappa.q_i = 1, q_i.q_j = 0.$$

On the other hand, if L_i $(i = 1, \dots, 8)$ is the standard basis vector for $\mathbf{R}^{1,8}$ with $L_i^2 = -1$, we have

$$K.K = 1, L_i.L_i = -1, K.L_i = 1, L_i.L_j = 0$$

So there is a unique isometry of $\mathbf{R}^{1,8}$ taking κ to K and q_i to l_i. Thus this isometry takes $\Lambda' \otimes \mathbf{R}$ to $\Lambda \otimes \mathbf{R}$ and it is straightforward to check that it actually takes Λ' to Λ.

We have now established our results for the case when $r = 8$. Change notation slightly to write $\mathcal{C}_r, \mathcal{L}_r$ for the different sets $\mathcal{C}, \mathcal{L}$. There is are obvious inclusions $\mathcal{C}_{r-1} \subset \mathcal{C}_r$; just taking vectors with last entry zero. Given a choice of ϵ for $\mathcal{C}_r$ we get a choice for $\mathcal{C}_{r-1}$ by this embedding and the definitions show immediately that if the Jacobi identity holds in $\mathfrak{h}_r$ it does also in $\mathfrak{h}_{r-1}$. This proves Proposition 2, and we get a chain of Lie algebras

$$\mathfrak{h}_1 \subset \mathfrak{h}_2 \subset \dots \subset \mathfrak{h}_7 \subset \mathfrak{h}_8 = \mathfrak{e}_8.$$

Similarly, there is an obvious embedding of $\mathcal{L}_{r-1}$ in $\mathcal{L}_r$ and the Proposition 2 for $(\mathfrak{h}_r, V_r)$ implies the same statement for $(\mathfrak{h}_{r-1}, V_{r-1})$. So we have representations V_r of $\mathfrak{h}_r$ for all $1 \leq r \leq 8$.

The groups H_6, H_7 are the remaining exceptional groups E_6, E_7 and we have constructed them along with representations of dimension $27, 56$ respectively.

Now consider the relation between $\mathcal{C}_{r-1}$ and $\mathcal{C}_r$ in more detail. Let L_r be the standard basis vector $(0; 0, \dots, 1)$, as above. We define a map

$$i^+ : \mathcal{L}_{r-1} \to \mathcal{C}_r$$

by $i^+(L) = L - L_r$, where we regard $\mathbf{R}^{1,r-1} \subset \mathbf{R}^{1,r}$ as usual. We define $i^- : \mathcal{L}_{r-1} \to \mathcal{C}_r$ by $i^-(L) = -i^+(L) = L_r - L$. When $r \leq 7$ one finds that the elements of $\mathcal{C}_r$ are

1. Vectors of the kind $\pm(0; 1, -1, 0, \dots, 0)$: there are $r(r-1)$ of these.
2. Vectors of the form $\pm(1; -1, -1, -1, 0, \dots, 0)$. These only occur when $r \geq 3$ and there are $r(r-1)(r-2)/3$ of these.
3. Vectors of the form $\pm(2; -1, -1, -1, -1, -1, -1, 0, \dots, 0)$. These only occur when $r \geq 6$ and there are 2 of there are $2C_6^r$ of these.

In particular we have $L_r.C = 0, \pm 1$ for all $C \in \mathcal{C}_r$. If $L_r.C = 0$ then C lies in the copy of $\mathcal{C}_{r-1} \subset \mathcal{C}_r$. If $L_r.C = \pm 1$ then C is an element of $i^{\pm}(\mathcal{L}_{r-1})$. So for $r \leq 7$ we have a disjoint union

$$\mathcal{C}_r = \mathcal{C}_{r-1} \cup i^+(\mathcal{L}_{r-1}) \cup i^-(\mathcal{L}_{r-1}).$$

This translates into a vector space isomorphism

$$\mathfrak{h}_r^c = (\mathfrak{h}_{r-1}^c \oplus \mathbf{C}) \oplus V_{r-1} \oplus V_{r-1}^*,$$

or for the real forms

$$\mathfrak{h}_r = (\mathfrak{h}_{r-1} \oplus \mathbf{R}) \oplus V_{r-1}.$$

It is clear from the definitions that $\mathfrak{g}_{r-1} \oplus \mathbf{R}$ is a Lie subalgebra and the component of the bracket mapping $V_{r-1} \times V_{r-1}$ to V_{r-1} vanishes, so we have a symmetric pair. In other words, for $r \leq 7$ there is a group $\hat{H}_{r-1} \subset H_r$, locally isomorphic to $H_{r-1} \times S^1$, and $X_r = H_r / \hat{H}_{r-1}$ is a symmetric space.

When $r = 8$ a new feature arises. There are 16 vectors in $\mathcal{C}_8$ of the form $\pm(3; -2, -1, \dots, -1)$. Thus there are two vectors $\pm c$ say having inner product ± 2 with L_8. So now we have

$$\mathcal{C}_8 = \mathcal{C}_7 \cup \{c, -c\} \cup i^+\mathcal{L}_7 \cup i^-\mathcal{L}_7,$$

and this translates into a symmetric pair decomposition

$$\mathfrak{e}_8 = (\mathfrak{e}_7 \oplus \mathfrak{su}(2)) \oplus V_7,$$

and a symmetric space $X_8 = E_8/\hat{H}_7$, with $\hat{H}_7$ locally isomorphic to $H_7 \times SU(2)$.

To sum up, we have another way of building up E_8 via a chain of symmetric pairs. The groups and represention which occur are

1. $H_1 = S^1$ and V_1 is the standard 1-dimensional representation.
2. $H_2 = U(2)$ and $X_2 = U(2)/S^1 \times S^1$ is the Riemann sphere $\mathbf{CP}^1$.
3. $H_3 = SU(3) \times SU(2)$ and $X_3 = (U(3) \times SU(2))/(U(2) \times S^1)$ is $\mathbf{CP}^2 \times \mathbf{CP}^1$.
4. $H_4 = SU(5)$ and $X_4 = SU(5)/(U(3) \times SU(2))$ is the Grassmannian of 2-planes in $\mathbf{C}^5$.
5. $H_5 = \mathrm{Spin}(10)$ and $X_5 = \mathrm{Spin}(10)/\tilde{U}(5)$.
6. $H_6 = E_6$ and we have an exceptional symmetric space $X_6 = E_6/(\mathrm{Spin}(10) \times S^1)$, associated to the representation S^+ of Spin(10).
7. $H_7 = E_7$ and we have an exceptional symmetric space $X_7 = E_7/(E_6 \times S^1)$ associated to the 27-dimensional representation of E_6.
8. $H_8 = E_8$ and we have the exceptional symmetric space $X_8 = E_8/(E_7 \times SU(2))$ associated to the 56-dimensional representation of E_7.

(The identification of the groups above is only meant up to local isomorphism.)

8 Lie groups and topology

8.1 The cohomology ring of a group

Throughout, G will be a compact Lie group.

We start by considering the real cohomology $H^*(G; \mathbf{R})$. It is a *graded-commutative ring.*

The multiplication $m : G \times G \to G$ induces a co-product

$$\Delta : H^*(G) \to H^*(G) \otimes H^*(G).$$

This is a homomorphism of algebras. The fact that $m(1, g) = m(g, 1) = g$ implies that

$$\Delta\alpha = 1 \otimes \alpha + \alpha \otimes 1 + \underline{\Delta}(\alpha),$$

where, for $\alpha \in H^p$,

$$\underline{\Delta}(\alpha) \in \bigoplus_{i=1}^{p-1} H^i \otimes H^{p-i}.$$

More generally we consider a *Hopf algebra A* which is just an abstraction of this algebraic structure.

Hopf's Theorem A finite dimensional Hopf algebra over a field of characteristic 0 is the exterior algebra generated by certain homogeneous elements of odd degree.

Outline proof

Let $I \subset A$ be the ideal of elements with vanishing A^0 component.

Then $I^2 \subset I$ is also an ideal in A.
Elementary arguments give a finite set of homogeneous elements $e_i \in I$ which

- generate I,
- map to a basis for I/I^2.

We claim that these do the job. Need to show

- e_i have odd degree,
- there is no multilinear relation between the e_i.

Suppose $e = e_1$ has even degree. Some power e^p must vanish (finite dimensionality). Suppose for simplicity that $e^2 = 0$. Then

$$0 = \Delta(e^2) = e^2 \otimes 1 + 1 \otimes e^2 + 2e \otimes e + S,$$

where $S \in I \otimes I^2 + I^2 \otimes I$. This implies that $e = 0$. The same argument applies for all p.

Suppose there is a relation and without loss of generality that e_1 is the term of highest degree appearing in the relation. This means that e_1 cannot arise in any term Δe_i for any e_i appearing in the relation. Write the relation as $e_1 P + Q = 0$. Applying Δ one finds that $e_1 \otimes P$ lies in the sum of $e_i \otimes I$ for $i > 1$ and $I^2 \otimes I$ which contradicts the choice of e_i.

In general an element $\alpha \in A$ is called primitive if

$$\Delta\alpha = 1 \otimes \alpha + \alpha \otimes 1.$$

Taking duals we have the $\Delta^* A^* \otimes A^* \to A^*$. This is a product operation on A^* which will not in general be associative. It can be shown that Δ^* is associative if and only if A is an exterior algebra on its primitive elements. The whole structure is determined by the degrees of the primitive elements.

In our topological situation we have the *Pontrayagin product*

$$\Delta^* = m_* : H_*(G) \otimes H_*(G) \to H_*(G),$$

and associativity follows from the group law.

The Leray-Serre spectral sequence

We want to use the *spectral sequence* of a fibration

$$F \to E \to B.$$

Assume for simplicity that B is 1-connected. We start with

$$E_2^{p,q} = H^p(B) \otimes H^q(F).$$

There are differentials

$$d_2 : E_2^{p,q} \to E_2^{p+2,q-1},$$

with $d_2^2 = 0$ and we form cohomology groups $E_3^{p,q}$.

There are differentials

$$d_3 : E_3^{p,q} \to E_3^{p+3,q-2},$$

and we take cohomology to get $E_4^{p,q}$ and so on. There is a "limit" $E_\infty^{p,q}$ and

$$H^k(E) = \bigoplus_{p+q=k} E_\infty^{p,q}.$$

It is better to say that there is a filtration of $H^k(E)$ whose successive quotients are the $E_\infty^{p,q}$. there are also product structures at each stage, compatible with the cup products on $H^*(F), H^*(B), H^*(E)$.

Sketch proof (see Griffiths and Harris *Principles of algebraic geometry* 3.5 for example).

In algebra we get a spectral sequence any time we have a filtered complex.

Take the complex Ω^* of differential forms on E filtered by saying that $\mathcal{F}^p\Omega^*$ consists of forms with "at least p terms in the base direction".

Suppose for example that we want to compute $H^1(E)$. There are classes coming from the base, which can be represented in $\mathcal{F}^1$. We work modulo these, so we seek a closed 1-form α on E which is not identically zero on the fibres.

- The first condition is that α is closed on the fibres so defines a class in H^1 of each fibre.
- The second condition is that this cohomology class is constant as the fibre varies.
- Now, fixing a class in $H^1)(F)$, we can choose α so that $d\alpha = \tilde{\omega}$ lies in $\mathcal{F}^2(\Omega^2)$. The fact that $d\tilde{\omega} = 0$ implies that $\tilde{\omega}$ is the lift of a closed 2-form ω on B. This defines $d_2 : H^1(F) \to H^2(B)$.

The first two steps correspond to the E_0, E_1 terms of the spectral sequence. Once we reach E_2 everything is expressed in terms of cohomology.

Examples

- We have
$$U(n-1) \to U(n) \to S^{2n-1}.$$
Using the spectral sequence and induction we find that $H^*(U(n))$ is the exterior algebra on generators in dimensions $1, 3, \ldots, 2n-1$.

- Similarly for
$$Sp(n-1) \to Sp(n) \to S^{4n-1}.$$
We find that $H^*(Sp(n))$ has generators in dimensions $3, 7, \ldots 4n-1$.

- The orthogonal groups are more complicated. We find $H^*(SO(2n+1))$ has generators in dimensions $3, 7, \ldots, 4n-1$ while $H^*(SO(2n))$ has generators in dimensions $3, 7, \ldots, 4n-5, 2n-1$. For example $H^*(SO(4))$ has two generators in dimension 3.

Spin(m) and $SO(m)$ have the same rational cohomology.

More generally the same is true for any finite coverings.

For the exceptional group G_2 we can use either
$$SU(3) \to G_2 \to S^6$$
or
$$G_2 \to \text{Spin}(7) \to S^7,$$
to see that the cohomology has generators in dimensions $3, 11$.

For the exceptional group F_4 we use the symmetric space $P = F_4/Spin(9)$.

Clifford multiplication gives a map $s^2(S(\mathbf{R}^9) \to \mathbf{R}^9$ and so a quadratic map $S^{15} \to S^8$. A study of the Jacobi equation (for example) shows that P is obtained by attaching a 16-ball to S^8 using this map on the boundary. So $H^*(P)$ has a generator u in dimension 8 and one other class $u^2 \in H^{16}$.

One finds that the$H^*(F_4)$ has generators in dimensions $3, 11, 15, 23$.

Consider the map $S : G \to G$ defined by $S(g) = g^2$. Then $S^* : H^*(G) \to H^*(G)$ is the composite of Δ and the cup-product. It follows that $S^* e_i = 2e_i + \sigma$ where $\sigma \in I^2$. The top dimensional class of G is represented by the product $\Pi = e_1 e_2 \dots e_s$. It follows that $S^*(\Pi) = 2^s \Pi$. Thus S has degree 2^s. If h is a generic element of a maximal torus $T \subset G$ then all solutions of $g^2 = h$ lie in T and the number of these is 2^r, where r is the rank of G. It follows (after checking signs) that $s = r$.

8.2 Classifying spaces

In algebraic topology one considers the *classifying space* BG of a topological group G. By definition this is the quotient of a contractible space EG on which G acts freely. Thus there is a fibration $G \to EG \to BG$. For a Lie group G we can approximate EG by finite dimensional manifolds. Isomorphism classes of G bundles over a space X are in 1-1 correspondence with homotopy classes of maps from X to BG.

Example If $G = U(n)$ then BG is the Grassmannian of n-planes in $\mathbf{C}^\infty$, which for our purposes can be studied by taking n-planes in $\mathbf{C}^N$ for sufficiently large N, in any given problem or calculation.

Borel's Theorem Suppose we have a spectral sequence (of vector spaces over a field k of characteristic zero) with

$$E_2^{p,q} = A^q \otimes B^p,$$

where A is an exterior algebra on generators $e_i \in A^{2l_i - 1}$, B is an algebra with $B^0 = k$, and the sequence is compatible with products. Suppose $E_\infty^{p,q} = 0$ for $p + q > 0$. Then B is a polynomial algebra on generators b_i in dimensions $2l_i$.
The b_i appear in the spectral sequence as $b_i = d_{2l_i} e_i$.

It follows that for a compact Lie group G, $H^*(BG)$ is a polynomial algebra, as above.

For the classical groups G we see that $H^*(BG)$ has generators as follows

- $U(n)$: generators $c_i \in H^{2i}$ for $i = 1, \dots, n$.
- $Sp(n)$: generators $p_i \in H^{4i}$ for $i = 1, \dots, n$
- $SO(2n+1)$ generators $p_i \in H^{4i}$ for $i = 1, \dots, n$.
- $SO(2n)$ generators p_i as above and $e \in H^{2n}$.

8.3 Differential forms approach

General fact: If a compact connected group G acts on a manifold M we can compute the cohomology of M from the complex of G-invariant forms.

This follows because d commutes with the operation of averaging over the G-action and G acts trivially on the cohomology.

We can apply this to the left action of G on itself. Then we see that $H^*(G)$ can be computed from a complex $D : \Lambda^p\mathfrak{g}^* \to \Lambda^{p+1}\mathfrak{g}^*$ defined by the product with the bracket in $\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ and contraction $\mathfrak{g} \otimes \Lambda^p\mathfrak{g}^* \to \Lambda^{p-1}\mathfrak{g}^*$.

Note. This complex is defined for any Lie algebra and leads to the notion of Lie algebra cohomology. For example we saw in Section one that classes in H^2 correspond to central extensions. For non-compact groups it is not directly related to the topological cohomology.

We can apply the same principle to the right action and see that $H^*(G)$ can be computed from the G-invariants in $\Lambda^*(\mathfrak{g})$.

Let $\iota : G \to G$ be $\iota(g) = g^{-1}$. Then ι acts on the bi-invariant p forms as multiplication by $(-1)^p$. It follows that d vanishes on these forms. So we see that $H^p(G)$ can be identified with the G-invariants in $\Lambda^p\mathfrak{g}^*$.

Example

If $\langle\ ,\ \rangle$ is an invariant inner product then

$$(x, y, z) \mapsto \langle x, [y, z]\rangle,$$

defines an invariant 3-form, hence a class in $H^3(G)$. This is non-zero if G is not abelian.

Chern-Weil Theory: First treatment

Let $P \to M$ be a principal G-bundle. Choose a connection A. the curvature is a section of the vector bundle $\lambda^2 T^*M \otimes \mathrm{ad}P$ over M, where $\mathrm{ad}P$ is the bundle assoaciated to teh adjoint representation.

Let $\phi \in s^p(\mathfrak{g}^*)$ be an ad-invariant polynomial. We can regard it as a function on $\mathfrak{g}$. Then $\phi(F)$ is a well-defined $2p$-form on M.

Main fact This form is cloaed.

Proof Compute in a trivialisation of P at a point in M such that the connection form vanishes.

Corollary The cohomology class of $\phi(F)$ is independent of the choice of connection.

Proof. Any two connections can be joined by a path. We lift P to $M \times [0,1]$ and apply the homotopy invariance of de Rham cohomology.

The conclusion is that we construct a "characteristic class" in $H^{2p}(M)$, which is an invariant of the bundle. Applying the construction to the universal bundle one sees that what we have constructed is a homomorphism of rings

$$s^*_G \mathfrak{g}^* \to H^*(BG),$$

where s^*_G denotes the G-invariant polynomials.

We will see presently that this map is an isomorphism.

Note that $S^*_G(\mathfrak{g}^*)$ can be identified with the polynomials on the Lie algebra of a maximal torus invariant under the Weyl group.

- $U(n)$: the Weyl group acts as permutations of $\lambda_1, \dots, \lambda_n$. The invariant polynomials are generated by the standard elementary symmetric functions
$$\sigma_1 = \sum \lambda_i, \sigma_2 = \sum \lambda_i \lambda_j.$$
Up to factors of 2π these correspond to the Chern classes.
- For $Sp(n)$ or $SO(2n+1)$ the Weyl group permits us to change any number of signs. The invariants are generated by symmetric functions of λ_i^2. We get the Pontrayagin classes $p_i \in H^{4i}$.
- For $SO(2n)$ we can only change an even number of signs. We get another invariant $e = \lambda_1 \dots \lambda_n$, the Euler class.

The invariant polynomial corresponding to the Euler class is the Pfaffian. It can be written more directly as follows. Identify the Lie algebra of $SO(2n)$ with $\Lambda^2 \mathbf{R}^{2n}$ and map $\Omega \in \Lambda^2$ to $\Omega^n \in \Lambda^{2n} = \mathbf{R}$. The Pfaffian is a square root of the determinant. Notice that it needs an orientation: the Euler class is not defined for $O(2n)$ bundles.

Go back to $T \subset \mathrm{Spin}(9) \subset F_4$. One sees that the Pontrayagin class p_2 of Spin(9) does not come from a Weyl group-invriant polynomial on the torus. Rather there are three such degree 4 polynomials ϕ_1, ϕ_2, ϕ_3 which are interchanged by the Weyl group. The product $\phi_1\phi_2\phi_3$ is an invariant degree 12 polynomial and this gives a characteristic class in H^{24}, matching up with our previous calculation.

8.4 Equivariant cohomology

If G acts on X we set $X_G = X \times_G EG$ and define the equivariant cohomology $H^*_G(X) = H^*(X_G)$.

There is a fibration $X \to X_G \to BG$ and hence a spectral sequence in cohomology. The pull back gives a map $H^*(BG) \to H^*_G(X)$ so $H^*_G(X)$ can be regarded as a module over the "co-efficient" ring $H^*(BG)$.

The inclusion of a fibre gives a restriction map $H^*_G(X) \to H^*(X)$. An equivariant cohomology class can be regarded as an ordinary cohomology class with extra data which defines the extension over X_G. More generally if $P \to B$ is any principal G-bundle and $\mathcal{X} \to B$ is the associated bundle with fibre X then an equivariant cohomology class automatically extends over $\mathcal{X}$.

Motivation

Isomorphisms classes of S^1 bundles over M correspond to classes in $H^2(M; \mathbf{Z})$. Thus if G acts on M, to give a class in $H^2_G(M)$ should be the same as defining a G-equivariant line bundle. A connection on a line bundle gives a curvature form ω which represents the ordinary cohomology class. Lifting the action is the same as giving a map $\mu : M \to \mathfrak{g}^*$ which is equivariant and satisfies the "Hamiltonian" property: if we fix a basis for $\mathfrak{g}$ so $\mu = (\mu_a)$ then

$$d\mu_a = i(v_a)(\omega),$$

where v_a are the corresponding vector fields on M. This gives a candidate for a de Rham definition of $H^2_G(M)$ whose cochains we can write as sums $\omega + \mu$ in $\Omega^2 \oplus \mathfrak{g}^* \otimes \Omega^0$.

Now we want to have products in our theory so we should be able to multiply such objects. This suggests that 4-cochains should lie in $\Omega^4 \oplus \mathfrak{g}^* \otimes \Omega^2 \oplus s^2(\mathfrak{g}^*) \otimes \Omega^0$.

Suppose we have vector spaces $\mathfrak{g}, T$ and a linear map $\mathfrak{g} \to T$. Then from $1 \in \mathfrak{g} \otimes \mathfrak{g}^*$ we get an element in $T \otimes \mathfrak{g}^*$ and the tensor product of contraction and multiplication gives

$$I : \Lambda^q(T^*) \otimes s^p(\mathfrak{g}^*) \to \Lambda^{q-1}(T^*) \otimes s^{p+1}(\mathfrak{g}^*).$$

Clearly $I^2 = 0$. Apply this in the tangent spaces of a manifold M with G action. We get

$$I : \Omega^q(M) \otimes s^p \to \Omega^{q-1} \otimes s^{p+1},$$

where we write $s^p = s^p(\mathfrak{g}^*)$.

Now let $C_G(M)$ be the G-invariants in $\Omega^* \otimes s^*$.

Lemma $dI + Id = 0$ in $C_G(M)$.

For simplicity work with an invariant metric and identify $\mathfrak{g}, \mathfrak{g}^*$. The proof boils down to the fact that if $[e_i, e_j] = \sum c_{ijk} e_k$, in an orthonormal basis, then c_{ijk} is totally skew symmetric.

Thus $D = d + I$ is a differential in $C_G(M)$. We grade by $q + 2p$. We get cohomology groups $H^*_{dR,G}(M)$.

This is the *Cartan model* for equivariant cohomology.

Biographic

Sophus Lie 1842-1899
Eli Cartan 1869-1951
Henri Cartan 1904-2008

We can also think of elements of $C_G(M)$ as equivariant polynomial maps

$$f : \mathfrak{g} \to \Omega^*(M).$$

The differential can then be defined by

$$(Df)(\xi) = d[f(\xi)] + i_\xi f(\xi),$$

where i_ξ is the contraction with the action of ξ.

The complex is filtered by p. We get a spectral sequence with

$$E_2^{2p,q} = H^q(M) \otimes s^p_G,$$

and $E_2^{2p+1,q} = 0$. The sequence converges to $H^*_{G,dR}(M)$.

Here s^p_G denotes the G-invriants in s^p. This identification of the E_2 term uses the fact that G acts trivially on the ordinary cohomology.

Next we show that if G acts freely then $H^*_{G,dR}$ is the ordinary cohomology of M/G. Consider

$$I : s^p(\mathfrak{g}^*) \otimes \Lambda^q(\mathfrak{g}^*) \to s^{p+1}(\mathfrak{g}^*) \otimes \Lambda^{q-1}(\mathfrak{g}^*.$$

If we regard the objects as differential forms on $\mathfrak{g}$ with polynomial co-efficients then I is the contraction with $v = \sum x_i \frac{\partial}{\partial x_i}$. We also have a map

$$J : s^p(\mathfrak{g}^*) \otimes \Lambda^q(\mathfrak{g}^*) \to s^{p-1}(\mathfrak{g}^*) \otimes \Lambda^{q+1}(\mathfrak{g}^*,$$

defined by the exterior derivative. So $IJ - JI = (p+q)$. This means that we can invert I on the kernel of I. More generally if $\mathfrak{g} \to T$ is injective, so $T = \mathfrak{g} \oplus W$ say, the same argument applies.

Now if a class in $H^*_{G,dR}(M)$ is represented in $\bigoplus_{p \leq p_0}$ we can use the preceding to show that it is also represented in $\bigoplus_{p \leq p_0 - 1}$. When $p_0 = 0$ the forms in question are precisely those which lift from M/G.

Now suppose that G acts on a contractible space Y. We claim that

$$H^*_{G,dR}(X \times Y) = H^*_{G,dR}(X).$$

For the pull-back gives a map of filtered complexes

$$C_{G,dR}(X) \to C_{G,dR}(X \times Y),$$

which induces an isomorphism on the E_2 term of the spectral sequences and thus an isomorphism on the cohomology.

Thus

$$H^*_{G,dR}(M) = H^*_{G,dR}(M \times E_G) = H^*(M_G) = H^*_G(M).$$

Taking $M = pt.$ we get $H^*(BG) = s^*_G$. The Chern-Weil construction for a principle bundle $P \to X$ becomes a particular case of the map

$$H^*(BG) \to H^*_G(P),$$

in the case of a free action.

Go back to a bundle $\mathcal{X} \to B$ induced from a principal bundle $P \to B$ by an action of G on X. Choose a connection so we have horizontal and vertical subspaces

$$T\mathcal{X} = V \oplus H,$$

and a curvature F. We define a map from $C_G(X)$ to differential forms on $\mathcal{X}$. This is induced by a map which takes a polynomial $f \in s^p(\mathfrak{g}^*)$ to $f(F) \in \Lambda^{2p} H^*$. The basic fact is that this is a chain map so a representative for an equivariant cohomology class on X defines a specific closed form on $\mathcal{X}$.

The Matthai-Quillen form

This represents the Thom class in $H^{2n}_{comp}(\mathbf{R}^{2n})$. Actually it is neater to use rapidly decaying rather than compactly supported forms. Start with $n = 1$.

Consider the 2-form $\omega = e^{-r^2/2} r dr d\theta$, in polar co-ordinates. The Hamiltonian is $e^{-r^2/2}$. So

$$e^{-r^2/2}(\omega + \sigma),$$

is a closed equivariant form where σ denotes the standard generator for the dual of Lie($SO(2)$).

Next take co-ordinates $x_i y_i$ on $\mathbf{R}^{2n}$ and forms $\omega_i = dx_i dy_i$. Let T be the corresponding maximal torus in $SO(2n)$ and σ_i the standard basis for Lie(T). Then by taking products we see that

$$(\omega_1 + \sigma_1)(\omega_2 + \sigma_2) \dots (\omega_n + \sigma_n) e^{-r^2/2}$$

is a closed T-equivariant form.

For the general case it is easier to work with maps $f : \mathfrak{g} \to \Omega^*$, with $G = SO(2n)$. We define

$$f(\xi) = * \exp(\xi) e^{-r^2/2},$$

where we identify $\mathfrak{g}$ with Λ^2, compute the exponential in the exterior algebra and $*$ is the Hodge $*$-operator.

The proof that this is equivariantly closed comes down to the identity

$$i(v)(\exp(\xi)) = w_\xi \wedge \exp(\xi),$$

where $v = \sum x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}$ is the "radial" vector field and $w_\xi = i(v)(\xi)$. It suffices to check this when $\xi = \sum a_i \omega_i$ which is easy.

Notice that the s^n component of our Thom form is the Pfaffian. Now if we have a vector bundle $E \to B$ with fibre $\mathbf{R}^{2n}$ and structure group $SO(2n)$, and a connection, we get a closed $2n$-form τ on E by our general construction. This has integral 1 over each fibre and restricts on the zero section to the Pfaffian of the curvature. By considering a smooth section we get

$$\int_B \mathbf{Pfaff}(F) = \sharp(\text{Zeros}).$$

In particular if $E = TB$ we get the generalised Gauss-Bonnet formula

$$\int_B \mathbf{Pfaff}(F) = \text{Euler Number}.$$

Transgression and Chern-Simons invariants

Take the action of G on itself by right multiplication. This is free and the quotient is a point so the equivariant cohomology is trivial. The Cartan complex is a purely finite-dimensional gadget. Let b be an invariant polynomial of degree

$p > 0$. This defines a cochain in the complex so we can apply the construction above to write $b = (D + I)Tb$. The equivariant class has a component $(Tb)_0$ say in $\Lambda^{2p-1}\mathfrak{g}_G^*$. If $b \in s^p$ is a polynomial generator then $(Tb)_0 \in \Lambda^{2p-1}$ is the corresponding exterior generator.

An explicit formula for going between invariants in s^* and Λ^* is

$$(Tb)_0 = P(\theta, [\theta, \theta], [\theta, \theta], \ldots, [\theta, \theta]),$$

where we write $P(\ , \ , \ , \ )$ for the corresponding multilinear form and θ is the identity in $\mathfrak{g} \otimes \mathfrak{g}^*$.

Now let $\pi : P \to X$ be a principle G-bundle with connection. The polynomial b defines a Chern-Weil form $b(F) \in \Omega^{2p}(X)$. The equivariant cochain Tb defines a $(2p-1)$-form ϕ say on the total space of P such that

$$d\phi = \pi^*(b(F)).$$

In a case when $b(F) = 0$ then ϕ yields a cohomology class on P which is the *Chern-Simons invariant* of the connection.

Localisation

This is one of the most important applications of equivariant cohomology. If $E \to B$ is a fibration with fibre a compact oriented n-manifold M there is an "integration over the fibre" map

$$H^{n+r}(E) \to H^r(B).$$

In particular this gives us a map $H_G^{n+2p}(M) \to H^{2p}(BG)$. In the de Rham approach this is simply given by integration

$$\Omega^n(M) \otimes s^{2p} \to s^{2p}.$$

If G acts freely then the cohomology vanishes in dimensions $\geq n$ so this map is zero.

Suppose for simplicity that $G = S^1$. If $\Omega \otimes \sigma$ is the tope term in a equivariantly closed form we can use the local formulae to write $\Omega = d\chi$ away from the fixed points of the action, where χ is given explicitly by the lower terms. This leads to a formula for the integral of Ω in terms of local data at these fixed points. For example if $n = 2n$, ω is a sympectic form and H is the Hamiltonian for a circle action this applies to integrals

$$\int_M H^p \omega^m,$$

which are given by *Duistermaat-Heckmann formulae* at the fixed points.

8.5 Other topics

Homogeneous spaces

We can also study the cohomology of homogeneous spaces. Suppose $M = G/K$ with G compact and choose a K-invariant complement so

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Then $H^*(M)$ can be computed from a complex consisting of the k-invariants in $\Lambda^*\mathfrak{p}$ with differential defined by the component of the bracket $\mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$.

When M is a symmetric space this differential is trivial so the K-invariant forms give the cohomology.

Let $T \subset G$ be a maximal torus. The T bundle $G \to G/T$ gives a map $H^*(BT) \to H^*(G/T)$. Now $H^*(BT)$ can be identified with the polynomial functions on Lie(T) and $H^*(BG) \subset H^*(BT)$ with the polynomials invariant under the Weyl group W. The general result is that $H^*(G/T)$ is the quotient of $H^*(BT)$ by the ideal generated by $H^*(BG)$.

To see this one can use the fact that G/T has no cohomology in odd dimensions (this is true for any symplectic manifold with a torus action, by elementary Morse theory).

There is a fibration $G/T \to BT \to BG$ and all the terms in E_2^{pq} are in even dimensions so the spectral sequence collapses and the result follows by staring at this.

Example $G = U(3)$. Then $H^*(BT)$ is polyomials in h_1, h_2, h_3 say and $H^*(G/T)$ is generated by the $h_i \in H^2$ with relations

$$h_1 + h_2 + h_3 = h_1h_2 + h_2h_3 + h_3h_1 = h_1h_2h_3 = 0.$$

K-theory

One can also consider *equivariant K-theory.* For a G-space X, this is the Grothendieck group of g-equivariant complex vector bundles over X. When X is a point we get the representation ring $R(G)$ of the group, analogous to $H^*(BG)$ in equivariant cohomology. It follows from Weyl's Theorem that, if G is simply connected, $R(G)$ is a polynomial algebra generated by the fundamental representations.

9 Problems

Qn. 1.

Show that the exponential map does not map onto $SL(2,\mathbf{R})$. Can you describe explicitly those matrices in $SL(2,\mathbf{R})$ which have a "logarithm"?

Qn. 2. Let G be a connected Lie group with a simple Lie algebra. Show that any normal subgroup of G is contained in the centre of G.

Qn. 3. Let V be a Euclidean vector space and write Λ^p for the exterior powers $\Lambda^p V$. Show that there is a non-trivial $SO(V)$-invariant contraction map

$$\Lambda^p \times \Lambda^q \to \Lambda^{p+q-2}$$

which we write as $(\alpha, \beta) \mapsto \alpha \circ \beta$.

1. Show that Lie algebra structures on V, compatible with the Euclidean structure, correspond to elements $B \in \Lambda^3 V$ with $B \circ B = 0$.
2. Given a B as above let $b : \Lambda^p \to \Lambda^{p+1}$ be the map $b(\alpha) = B \circ \alpha$. Show that $b^2 : \Lambda^p \to \Lambda^{p+2}$ is zero,so we have a cochain complex with cohomology groups H^p. Can you interpret the meaning of these, for small p, in terms of the Lie algebra? Show that if V is 6 dimensional and B corresponds to the Lie algebra $SO(4)$ the cohomology group H^3 is 2-dimensional.
3. In the case when V is the Lie algebra of a compact group G, with a bi-invariant metric, can you see any relation between the complex above and the de Rham complex of differential forms on G?

(The general topic which this question leads into is *Lie algebra cohomology.*)

Qn. 4.

Show that, with its standard invariant Riemannian metric suitably scaled, the sectional curvatures of $\mathbf{CP}^2$ lie between 1 and 1/4. Then show the same for $\mathbf{CP}^n, \mathbf{HP}^n$ $(n \geq 2)$.

(The sectional curvature in a plane $\Pi \subset TM_p$ is the quantity $K(X,Y)$ defined in the lectures, where X, Y form an orthonormal basis for Π.)

Qn.5.

Let $Z \subset \mathbf{C}^3$ be the set defined by the equation

$$|z_0|^2 - (|z_1|^2 + |z_2|^2) = 1.$$

Show that the symmetric space dual to $\mathbf{CP}^2$ can be identified with the quotient of the Z by S^1. Find another model of this space as the unit ball B in

$\mathbf{C}^2$, with a suitable Riemannian metric. (This is the analogue of the disc model for the real hyperbolic plane.)

Qn. 6.

Consider the set-up discussed in Section 3 for the case of $\mathfrak{sl}_2(\mathbf{C})$. We have a function F defined on the space of Hermitian metrics on $\mathbf{C}^3$. Verify the assertion that on any geodesic γ through H_0 the function $F(\gamma(t))$ is a finite sum $\sum a_r e^{rt}$. What can you say about the values of the exponents r which occur?

Qn. 7.

let $\mathfrak{g}$ be a real of complex Lie algebra. One standard definition of what it means for $\mathfrak{g}$ to be semisimple is that that the Killing form of $\mathfrak{g}$ is nondegenerate. Using the main result of Section 3, show that this is equivalent to the definition stated in Section 3 (i.e. semisimple if and only if a direct sum of simple algebras).

Qn. 8.

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^*$ its dual. Write $C^\infty(\mathfrak{g}^*)$ for the space of smooth functions on $\mathfrak{g}^*$. Show that there is a way to define a "bracket" $\{f, g\}$ making $C^\infty(\mathfrak{g}^*)$ an (infinite dimensional) Lie algebra. How is this related to the symplectic structure on the co-adjoint orbits?

Qn. 9.

Let M be an integral co-adjoint of a Lie group G. Verify the assertion made in lectures that there is a connection on the corresponding S^1 bundle $E \to M$ with curvature the symplectic form on M.

Qn. 10.

Give a description of the co-adjoint orbits of $SO(n), Sp(n)$ which makes it apparent that they are complex manifolds.

Qn. 11.

Consider the standard symplectic form on $\mathbf{R}^{2n}$. Let W be the space of polynomial functions on $\mathbf{R}^{2n}$ of degree less than or equal to 2. Show that W is closed under the Poisson bracket and so becomes a Lie algebra. Find a Lie group with this Lie algebra.

Qn. 12.

Find the weights and multiplicities of the representations $s^p \otimes s^q$ of $SU(2)$. Hence, or otherwise, decompose this as a sum of irreducibles. Find the decompositions of $\Lambda^2(s^p)$ and $s^2(s^p)$.

Qn.13.

Let P, Q be complementary subspaces in $\mathbf{C}^n$ (i.e. $\mathbf{C}^n = P \oplus Q$), with dim $P = p$. By considering the graphs of linear maps, show that there is an open dense subset V of the Grassmannian of p-planes in $\mathbf{C}^n$ which can be identified with $P^* \otimes Q$. Let D be the complement of V. Show that if we consider the Grassmannian as a co-adjoint orbit of $U(n)$ and take Q to be the orthogonal complement of P this set V coincides with the set U considered in Section 4 (with $P = p$). Now let P' be another subspace, so we get another subset V' of the Grassmannian. Find an equation, in terms of the "co-ordinates" on V' as above, defining the intersection $D \cap V'$.

Qn. 14. Give an algebraic proof of the result from Section 5; that if α is a root then the only roots $k\alpha$ occur when $k = \pm 1$, and that the dimension of the root-space R_α is 1.

Qn. 15. (For those who like Riemannian geometry.)

A fibre bundle $\pi : X \to Y$ is called a *Riemannian submersion* if X, Y are Riemannian manifolds and the derivative of π maps the orthogonal complement in each TX_x of the tangent space to the fibres isometrically to the tangent space $TY_{\pi(x)}$.

1. Show that the sectional curvature of X in a plane orthogonal to the fibre is no greater than the sectional curvature to Y in the corresponding plane. (This can be done by direct, but rather lengthy, calculation. But there is also a more conceptual argument using the relation between sectional curvature and geodesics.)

2. Suppose $M = G/H$ is a co-adjoint orbit of a compact group G endowed with its Kahler metric. Show that if v is a tangent vector at a point in M then the sectional curvature in the plane spanned by v, Iv is strictly positive.

3. Hence show, using the second variation formula for length, that there are no length-minimising closed geodesics in M and deduce that M is simply connected.

Qn. 16.

Find a set of simple roots for $Sp(3)$ and verify the form of the Dynkin diagram. Find the corresponding "fundamental weights" ω_i and try to identify the corresponding representations. (You probably want to consider $U(3) \subset Sp(3)$ and start with the roots of $U(3)$.)

Qn 17.

Same question for $SO(7)$ (or Spin(7).)

(It would probably be very helpful for your understanding to attempt these questions: if you get stuck you could look in Fulton and Harris, Part III, for example. But it would best if you do so *after* trying the questions by yourself. Beware that there are different standard notations for the "n" in the symplectic group $Sp(n)$.)

Qn.18.

Show that the centre of Spin(n) is isomorphic to

- $\mathbf{Z}/2$ if n is odd.
- $\mathbf{Z}/2 \times \mathbf{Z}/2$ if n is divisible by 4,
- $\mathbf{Z}/4$ if $n = 2$ modulo 4.

Qn. 19.

Using the "reciprocity principal" from Section 6, or otherwise, find a decomposition of the functions on S^2 into irreducible representations of $SO(3)$. How is this related to the study of "harmonic polynomials" on $\mathbf{R}^3$, and the Legendre polynomials?

Qn. 20.

Show that, in the case of $U(n)$, the equality of the two ways of writing the denominator D in the Weyl character formula becomes the formula for the Vandermonde determinant $\det(z_i^j)$.

Qn 21.

Using our description of the irreducible representations $V_{a,b}$ of $SU(3)$, find a formula for the dimension $\dim V_{a,b}$. Then obtain this in another way using the Weyl character formula.

Qn. 22.

Let G be a compact Lie group and T a maximal torus. Show that there is a non-trivial action of the Weyl group of G on G/T. (The action does *not* preserve the standard complex structure.)

Qn. 23.

For $n \geq 1$ let $SL(n, \mathbf{H})$ be the group of $n \times n$ quaternion matrices whose determinant, regarded as a real $4n \times 4n$ matrix, is 1. Show that the the complexification of $SL(n, \mathbf{H})$ is $SL(2n, \mathbf{C})$.

Show that that $SL(2, \mathbf{H})$ is locally isomorphic to $S0(5,1)$ and that the natural action of $SL(2, \mathbf{H})$ on $S^4 = \mathbf{HP}^1$ coincides with the action of $SO(5,1)$ on a quadric hypersurface in $\mathbf{RP}^5$.

Qn. 24. (Longer)

Find a subgroup $SO(4) \subset G_2$ and identify the roots of $SO(4)$ inside the set of roots of G_2. What is the Euclidean form on the Lie algebra of $SO(4)$ induced from that of G_2? Show that $G_2/SO(4)$ is a symmetric space and identify the corresponding Euclidean representation of $SO(4)$.

(There are many ways of going about this. One way is to follow the line indicated in Section 7 and consider associative subspaces in the imaginary Cayley numbers. Another way is to start with an oriented Euclidean $\mathbf{R}^4$ and construct an $SO(4)$ invariant cross -product on

$$\mathbf{R}^7 = \Lambda^2_+(\mathbf{R}^4) \oplus \mathbf{R}^4.$$

Then show that, with a suitable choice of constants, this coincides with the cross-product derived from $\mathbf{R}^7 = \mathbf{R} \oplus \mathbf{C}^3$. A third way is to spot the representation of $SO(4)$ asked for in the last part of the question, then show that you can build a Lie algebra using that.)

Qn 25.(Longer)

1. Suppose $M = G/H$ is a compact symmetric space and $\mathfrak{p}$ is the corresponding Euclidean representation of H, which can be identified with the tangent space to M at the base point. Let x, y be orthogonal vectors in $\mathfrak{p}$. Show that the the sectional curvature of M is zero in the plane spanned by x, y if and only if y is orthogonal to tangent space of the H-orbit at x.

2. Consider the action of Spin(9) on the spin space S, regarded as a real vector space of dimension 16. Fix some $\psi \in S$. For any unit vector $n \in S^9$ we can take the orthogonal complement $n^{\text{perp}} \cong \mathbf{R}^8$ and decompose $S = S^+(\mathbf{R}^8) \oplus S^-(\mathbf{R}^8)$, and hence $\psi = \psi^+ \oplus \psi^-$, say. Show that we can choose n so that $\psi^- = 0$.

3. Hence, or otherwise, show that Spin(9) acts transitively on the unit sphere in S.

4. Now consider the Cayley/Moufang plane $X = F_4/\text{Spin}(16)$, with its symmetric-space Riemannian metric, inducing a "distance function" $d(p, q)$. Show that the sectional curvature of X is strictly positive and that X is "2-point homogeneous" in the sense that if p, q, p', q' are points in X with $d(p, q) = d(p', q')$ then there is a $g \in F_4$ such that $g(p) = p', g(q) = q'$.

You may be interested to go further. For example, you could show that $S^{15} = \text{Spin}(9)/\text{Spin}(7)$, for a non-standard embedding $\text{Spin}(7) \subset \text{Spin}(9)$. Or you could find a fibration

$$S^7 \to S^{15} \to S^8,$$

analogous to

$$S^3 \to S^7 \to S^4,$$

and

$$S^1 \to S^3 \to S^2.$$

(Topologically, the space X can be constructed by attaching a 16-dimensional ball to the 8-sphere using the map above on the boundary of the ball.) You could look for the appropriate definition of a "line" in X, and show that any two distinct points lie on a unique line.

Q, 26. Let $G = SO(2n)$ and $M = S^{2n-1}$ with the standard action. Find an extension of the volume form in $\Omega^{2n-1}(M)$ to an equivariant form (i.e. a closed element of the Cartan complex.)

References

[1] ...

[2] ...