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Galerkin-Finite Element Methods for Parabolic Equations

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The purpose of this paper is to present a survey of error estimates for Galerkin-finite element methods applied to parabolic initial-boundary value problems. In doing so we shall depend on known results pertaining to the corresponding elliptic problem. We shall concentrate on the error originating from the discretization in the space variables and only quote at the end some work related to the discretization in time.

Before we state our parabolic problem we consider briefly the elliptic problem

$$Au \equiv -\sum_{j,k=1}^{N} \frac{\partial}{\partial x_j}\left(a_{jk}\frac{\partial u}{\partial x_k}\right)+a_0 u = f \quad \text{in } \Omega,\ u=0 \text{ on } \partial\Omega,$$

where Ω is a bounded smooth domain in R^N and where the coefficients are smooth with (a_{jk}) positive definite and a_0 nonnegative in $\bar{\Omega}$. This problem may also be stated in weak form: Find $u\in H_0^1(\Omega)$ such that

$$A(u,\varphi)=(f,\varphi) \quad \text{for} \quad \varphi\in H_0^1(\Omega),$$

where

$$A(u,v)=\int_\Omega\left(\sum_{j,k=1}^{N} a_{jk}\frac{\partial u}{\partial x_k}\frac{\partial v}{\partial x_j}+a_0 uv\right)dx, \quad (u,v)=\int_\Omega uv\,dx.$$

Let $\{S_h\}$ denote a family of finite dimensional subspaces of $H_0^1(\Omega)$, depending on the "small" parameter h, with the property that for some integer $r\geqslant 2$,

$$\inf_{\chi\in S_h}\{\|w-\chi\|+h\|w-\chi\|_1\}\leqslant Ch^r\|w\|_r \quad \text{for} \quad w\in H_0^1(\Omega)\cap H^r(\Omega),$$

where $\|\cdot\|_s$ denotes the norm in $H^s(\Omega)$ and $\|\cdot\|=\|\cdot\|_0$. A simple example (with $r=2$) of such a family is obtained by approximating the domain Ω from the interior by a union Ω_h of triangles with diameter at most h, and considering continuous functions which are linear on each triangle and vanish outside Ω_h. More generally, one may consider continuous functions which reduce to polynomials of degree $r-1$ on triangles. Nontrivial modifications near the boundary are then often necessary.

The "standard" Galerkin-finite element method for our boundary value problem is then to find $u_h \in S_h$ such that

$$A(u_h, \chi) = (f, \chi) \quad \text{for} \quad \chi \in S_h.$$

Setting $e = u_h - u$ we have at once $A(e, \chi) = 0$ for $\chi \in S_h$ and hence

$$C^{-1}\|e\|_1^2 \leq A(e, e) = A(e, \chi - u) \leq C\|e\|_1 \inf_{\chi \in S_h} \|u - \chi\|_1,$$

so that by our assumptions on $\{S_h\}$,

$$\|e\|_1 \leq Ch^{r-1}\|u\|_r \quad \text{for} \quad u \in H_0^1(\Omega) \cap H^r(\Omega).$$

A famous duality argument by Aubin [1], Nitsche [18] and Oganesjan and Ruchovetz [23] shows the L_2 estimate needed to conclude the optimal order error estimate

$$\|e\| + h\|e\|_1 \leq Ch^r\|u\|_r \quad \text{for} \quad u \in H_0^1(\Omega) \cap H^r(\Omega).$$

We now turn to our main target, the initial boundary value problem ($u_t = \partial u/\partial t$, $R_+ = \{t \geq 0\}$).

(1) $u_t + Au = f$ in $\Omega \times R_+$, $u = 0$ on $\partial\Omega \times R_+$, $u(x, 0) = v(x)$ in Ω,

which we write in weak form, with $u(\cdot, t) \in H_0^1(\Omega)$,

$$(u_t, \varphi) + A(u, \varphi) = (f, \varphi) \quad \text{for} \quad \varphi \in H_0^1(\Omega).$$

The corresponding "standard" Galerkin-finite element semidiscrete problem is then to find $u_h(t) \in S_h$ such that

(2) $$(u_{h,t}, \chi) + A(u_h, \chi) = (f, \chi) \quad \text{for} \quad \chi \in S_h, \quad t \geq 0, \quad u_h(0) = v_h,$$

where v_h is a suitable approximation of v in S_h. This may be considered as an initialvalue problem for a system of ordinary differential equations in the coefficients of u_h with respect to some basis in S_h.

Error estimates for (2) were given in e.g. Price and Varga [24], Douglas and Dupont [11], Fix and Nassif [15], Wheeler [32] and Dupont [14]. We show the following for $e = u_h - u$.

THEOREM 0. *For u sufficiently smooth in $\Omega \times [0, t_0]$ and with a suitable choice of v_h we have*

$$\|e(t)\| + h\|e(t)\|_1 \leq C(u)h^r \quad \textit{for} \quad 0 \leq t \leq t_0.$$

PROOF. Following Wheeler [30] we define the elliptic projection $P_1: H_0^1(\Omega) \to S_h$ by $A(P_1u - u, \chi) = 0$ for $\chi \in S_h$. By above we then have

(3) $$\|P_1u - u\| + h\|P_1u - u\|_1 \leq Ch^r\|u\|_r \quad \text{for} \quad u \in H_0^1(\Omega) \cap H^r(\Omega).$$

In the parabolic case, set $\theta=u_h-P_1u$ and $\varrho=P_1u-u$. By our definitions we have

$$(\theta_t,\chi)+A(\theta,\chi)=-(\varrho_t,\chi)\quad\text{for}\quad \chi\in S_h.$$

Choosing in particular $\chi=\theta_t$ we find

$$\|\theta_t\|^2+\frac{1}{2}\frac{d}{dt}A(\theta,\theta)=-(\varrho_t,\theta_t)\le\frac{1}{2}\|\varrho_t\|^2+\frac{1}{2}\|\theta_t\|^2.$$

Hence after integration, with $v_h=P_1v$ so that $\theta(0)=0$, in view of (3),

$$C^{-1}\|\theta\|_1^2\le A(\theta,\theta)(t)\le A(\theta,\theta)(0)+\int_0^t\|\varrho_t\|^2\,d\tau\le C(u)h^{2r}.$$

This completes the proof since $e=\theta+\varrho$ so that, using (3) once more,

$$\|e\|+h\|e\|_1\le\|\varrho\|+h\|\varrho\|_1+\|\theta\|_1\le C(u)h^r.$$

In the rest of this paper we shall, following Bramble, Schatz, Thomée and Wahlbin [8], write the semidiscrete equation in a somewhat different form. Thus let $T_h\colon L_2(\Omega)\to S_h$ denote the solution operator of the discrete elliptic problem, defined by

(4) $$A(T_hf,\chi)=(f,\chi)\quad\text{for}\quad\chi\in S_h.$$

The semidiscrete problem (2) may then be written

(5) $$T_hu_{h,t}+u_h=T_hf\quad\text{for}\quad t\ge 0,\quad u_h(0)=v_h.$$

The continuous problem (1) may analogously be put into the form

(6) $$Tu_t+u=Tf\quad\text{for}\quad t\ge 0,\quad u(0)=v,$$

where $T=A^{-1}$. The operator T_h has the properties:

(i) T_h is selfadjoint, positive semidefinite on $L_2(\Omega)$ and positive definite on S_h,

(ii) There is an integer $r\ge 2$ such that

$$\|T_hf-Tf\|\le Ch^s\|f\|_{s-2}\quad\text{for}\quad 2\le s\le r.$$

We may now consider the discrete problem (5) assuming only that T_h is an approximate solution operator of the elliptic problem satisfying (i) and (ii). In this fashion we also include into our considerations methods other than the standard Galerkin method described above. For instance, one may cover situations when the functions in S_h cannot easily be made to satisfy the homogeneous boundary conditions. One way of dealing with such a situation, which is contained in the above framework, is to use in the discrete problem rather than the bilinear form $A(\cdot,\cdot)$ a form with boundary terms, such as the following form proposed by Nitsche [19],

$$B_h(v,w)=A(v,w)-\left\langle v,\frac{\partial w}{\partial \nu}\right\rangle-\left\langle\frac{\partial v}{\partial \nu},w\right\rangle+\beta h^{-1}\langle v,w\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L_2(\partial\Omega)$, $\partial/\partial\nu$ the conormal derivative on $\partial\Omega$ and β a positive constant. Another method included is the Lagrange multiplier method of Babuška [2] which employs a separate family of approximating functions on $\partial\Omega$.

Subtracting (6) from (5) we find for the error

(7) $$T_h e_t + e = \varrho \equiv (T_h - T)Au = (P_1 - I)u,$$

where the elliptic projection is now defined by $P_1 = T_h A$. Using the properties (i) and (ii) one proves easily by the energy method (cf. [8]):

THEOREM 1. *We have for* $t \geq 0$,

$$\|e(t)\| \leq C\Big\{\|e(0)\| + h^r\Big[\|v\|_r + \int_0^t \|u_t\|_r \, d\tau\Big]\Big\}.$$

In particular, for the homogeneous equation ($f=0$) with $v_h = P_1 v$ or $P_0 v$ (P_0 denotes the L_2-projection onto S_h), we find under the appropriate compatibility conditions on v at $\partial\Omega$,

$$\|e(t)\| \leq C_\varepsilon h^r \|v\|_{r+\varepsilon};$$

a somewhat more precise argument yields this inequality with $\varepsilon = 0$. In this case, it is in fact possible to show convergence of order r, even for time derivatives, under much weaker regularity assumptions than above, when t is bounded away from zero (cf. [8]).

THEOREM 2. *Let* $j \geq 0$ *and* $v_h = P_0 v$. *Then for the homogeneous equation,*

$$\|D_t^j e(t)\| \leq C h^r t^{-r/2-j} \|v\| \quad (D_t = \partial/\partial t).$$

Results of this nature were also discussed by spectral representations in Blair [4], Thomée [28], Helfrich [17], Fujita and Mizutani [16] and recently by the energy method in Sammon [25].

The estimate of Theorem 2 for the homogeneous equation may be combined with Theorem 1 to derive error estimates for the nonhomogeneous equation for t bounded away from zero, which require smoothness of the solution only near t (cf. [30]).

THEOREM 3. *Let* $j \geq 0$ *and* $v_h = P_0 v$. *Then for the general nonhomogeneous equation, for* $t \geq \delta > 0$,

$$\|D_t^j e(t)\| \leq C h^r \Big\{\sum_{l=0}^{j} \|D_t^l u(t)\|_r + \int_{t-\delta}^{t} \|D_t^{j+1} u\|_r \, d\tau + \|v\| + \int_0^t \|f\| \, d\tau\Big\}.$$

In one application below, we shall need an error estimate in H^1 (cf. [31]).

THEOREM 4. *Consider the standard Galerkin method* (2) *for the nonhomogeneous equation and let* $v_h = P_0 v$. *Then for* $t \geq \delta > 0$,

$$\|D_t^j e(t)\|_1 \leq C h^{r-1} \Big\{\sum_{l=0}^{j} \sup_{t-\delta \leq \tau \leq t} \|D_t^l u(\tau)\|_r + \Big(\int_{t-\delta}^{t} \|D_t^{j+1} u\|_{r-1}^2 \, d\tau\Big)^{1/2}\Big\}$$

$$+ C h^r \Big\{\|v\| + \int_0^t \|f\| \, d\tau\Big\}.$$

The above estimate in H^1 shows an order of approximation in the gradient of the solution which is one order less than that for u itself. We shall now present a result from [30] which shows that if the finite element spaces are based on uniform partitions in a specific sense (which we shall here only refer to as uniform) in the interior domain Ω_0, then difference quotients of u_h may be used to approximate any derivative of u in the interior of Ω_0 to order $O(h^r)$. This generalizes a result in the elliptic case by Bramble, Nitsche and Schatz [6]. In addition to the global norms used above we use for $\tilde{\Omega}\subset\Omega$ the norms $|\cdot|_{\tilde{\Omega}}$ and $\|\cdot\|_{s,\tilde{\Omega}}$ in $L^\infty(\tilde{\Omega})$ and $H^s(\tilde{\Omega})$, respectively, and set $N_0=[N/2]+1$.

THEOREM 5. *Let S_h be uniform on $\Omega_0\subset\Omega$ and assume that T_h is such that*

$$A(T_hf,\chi)=(f,\chi)\quad for\quad \chi\in S_h\quad with\quad \operatorname{supp}\chi\subset\Omega_0.$$

Let $v_h=P_0v$ and let Q_h be a finite difference operator approximating D^α with order of accuracy r. Then for $t\geqq\delta>0$, $\Omega_2\subset\subset\Omega_1\subset\subset\Omega_0$,

$$|Q_hu_h(t)-D^\alpha u(t)|_{\Omega_2}\leqq Ch^r\Big\{\sup_{t-\delta\leqq\tau\leqq t}\|u(\tau)\|_{r+|\alpha|+N_0,\Omega_1}$$
$$+\Big(\int_{t-\delta}^{t}(\|u_t\|^2_{r+|\alpha|+N_0-1,\Omega_1}+\|u\|_r^2+\|f\|^2)\,d\tau\Big)^{1/2}+\|v\|+\int_0^t\|f\|\,d\tau\Big\}.$$

Notice the local character of the stringent regularity assumptions.

We shall now turn to global estimates in the maximum-norm and denote by $|\cdot|$ and $|\cdot|_r$ the norms in $L_\infty(\Omega)$ and $W^r_\infty(\Omega)$, respectively. The following result was proved in [8] (for $N=1$, cf. [33]).

THEOREM 6. *Assume that T_h satisfies*

$$|T_hw|\leqq C|Tw|_1,\quad \|T_hw\|\leqq C\|Tw\|_1.$$

Then for $t\geqq0$ we have

$$|e(t)|\leqq C\Big\{\sum_{j=0}^{N_0-1}|(I-P_1)D_t^ju(t)|+\|D_t^{N_0}e(t)\|\Big\}.$$

The proof consists of a simple iteration argument using the error equation (7) and noticing that T_h is a bounded operator from $L_q(\Omega)$ to $L_p(\Omega)$ if $0<q^{-1}-p^{-1}<N^{-1}$.

Combining this with a property such as

$$|(I-P_1)v|\leqq Ch^r(\log h^{-1})^{\delta_{2,r}}|v|_r$$

(for a survey of such estimates, see Nitsche [20]) and the above estimates for time derivatives we have under the appropriate assumptions, for $t\geqq\delta>0$,

$$|e(t)|\leqq C(u)h^r(\log h^{-1})^{\delta_{2,r}}.$$

Using weighted norms, Nitsche [21] (cf. also Dobrowolski [10]) proved the following result which is uniform for small t and in which the number of derivatives entering is independent of N. Here we are concerned with the standard Galerkin method with $A=-\Delta$ and the subspaces are assumed to consist of C^0 piecewise polynomials of degree $r-1$ on a quasiuniform partition into simplices, or isoparametric modifications.

THEOREM 7. *For T_h and S_h as stated and with $v_h=P_1 v$, $r\geq 3$, we have for any N,*

$$|e(t)|\leq Ch^r\Big\{|u(t)|_r+|u_t(t)|_r+\Big(\int_0^t |u_{tt}|_r^2\,d\tau\Big)^{1/2}\Big\}.$$

Recent work [26] shows that under the present types of assumptions the following discrete a priori estimate holds for solutions of the homogeneous semidiscrete equation (for $N\geq 5$ under an additional assumption about the discrete elliptic problem), namely

$$|u_h(t)|\leq C(\log h^{-1})^{p_N}|v_h|.$$

This has as a consequence for the error in the nonhomogeneous problem:

THEOREM 8. *Under the above assumptions, and with $v_h=P_1 v$, we have*

$$|e(t)|\leq Ch^r\Big(\log\frac{1}{h}\Big)^{p_N+\delta_{2,r}}\Big\{|u(t)|_r+\int_0^t |u_t|_r\,d\tau\Big\}.$$

In the analysis of different finite element methods for elliptic problems the duality argument quoted above for showing L_2 error estimates from the basic H^1 estimates, also yields error estimates in negative norms. To state such an estimate, set for $s\geq 0$, $\|v\|_{-s}=(T^s v, v)^{1/2}$. This norm can be shown equivalent to

$$\sup\Big\{\frac{(v,\varphi)}{\|\varphi\|_s};\ \varphi\in C^\infty(\bar\Omega),\ A^j\varphi=0\ \text{ on }\ \partial\Omega\ \text{ for }\ j<s/2\Big\}.$$

The negative norm estimate for the elliptic problem can then be expressed as

$$\|(I-P_1)u\|_{-(r-2)}\leq Ch^{2r-2}\|u\|_r\quad\text{for}\quad u\in H_0^1(\Omega)\cap H^r(\Omega),$$

and thus shows convergence in $\|\cdot\|_{-(r-2)}$ which is of higher order than that in the L_2-norm if $r>2$. In the rest of the paper we now assume this estimate to hold in addition to (ii) or that now

(ii′) $$\|T_h f-Tf\|_{-p}\leq Ch^{p+q+2}\|f\|_q,\ 0\leq p,\ q\leq r-2.$$

One may show similar estimates for the parabolic problem (cf. [31]).

THEOREM 9. *With $v_h=P_0 v$ or $P_1 v$ we have for $t\geq 0$,*

$$\|e(t)\|_{-(r-2)}\leq Ch^{2r-2}\Big\{\|v\|_r+\int_0^t \|u_t\|_r\,d\tau\Big\}.$$

For the purpose of proof, one introduces the semi-inner product $(v,w)_{-s,h}=(T_h^s v, w)$ and the corresponding seminorm $\|\cdot\|_{-s,h}$. It can easily be seen by (ii) that

$$\begin{aligned}\|v\|_{-s,h}&\leq C\{\|v\|_{-s}+h^s\|v\|\},\\ \|v\|_{-s}&\leq C\{\|v\|_{-s,h}+h^s\|v\|\},\end{aligned}\qquad 0\leq s\leq r-2.$$

It is therefore sufficient to show the desired estimates in $\|\cdot\|_{-(r-2),h}$. This is done by the energy method similarly to the proof for $s=0$, using the fact that T_h is selfadjoint and positive semidefinite with respect to $(\cdot,\cdot)_{-(r-2),h}$.

One may also derive negative norm estimates for time derivatives. These require additional smoothness only near t.

THEOREM 10. *Let $j\geq 0$ and $v_h=P_0v$ or P_1v. Then for $t\geq\delta>0$,*

$$\|D_t^j e(t)\|_{-(r-2)}\leq Ch^{2r-2}\left\{\sum_{l=0}^{j}\|D_t^l u(t)\|_r+\int_{t-\delta}^{t}\|D_t^{j+1}u\|_r\,d\tau+\int_0^t\|u_t\|_r\,d\tau\right\}.$$

We shall now give two examples from [31] utilizing the above negative norm error estimates to show pointwise convergence of order $O(h^{2r-2})$ for certain approximation procedures. Following Douglas and Dupont [12] such procedures are referred to as superconvergent, in as much as they are of higher order than the optimal order basic error estimates in L_2 or L_∞. The first estimate in the literature of this nature for Galerkin methods for parabolic equations (cf. [27]) concerned the pure initial value problem in one space dimension, with S_h consisting of smooth splines on a uniform mesh, and shows that if v_h is taken as the interpolant of v, then an associated finite difference operator has accuracy of order $2r-2$ and used known results from finite difference theory. For collocation methods for ordinary differential equations a similar phenomenon was observed by de Boor and Swartz [5].

Our first example here concerns superconvergence at knots for C^0 elements in one space dimension. This was proved first by Douglas, Dupont and Wheeler [13] using as a comparison function a so called quasi-projection of the exact solution into the subspace. Their approach required a more special choice of discrete initial data and somewhat higher regularity of the exact solution than the one described here.

Recall first the following simple fact for the solution of the two-point boundary value problem

$$Au=f \quad \text{on} \quad (0,1), \quad u(0)=u(1)=0,$$

and the corresponding semidiscrete solution $u_h=T_hf\in S_h$ where T_h is defined by (4) and where S_h consists of piecewise polynomials of degree $r-1$, with $\chi(0)=\chi(1)=0$ and with only continuity required at the knot $x=x_0$. With $g=g_{x_0}$ the Green's function of A with boundary values zero, we have for $e=u_h-u$,

$$e(x_0)=A(e,g)=A(e,g-\chi) \quad \text{for} \quad \chi\in S_h.$$

Since $g\in H^r(0,x_0)\cap H^r(x_0,1)\cap C^0(0,1)$ one finds easily

$$|e(x_0)|\leq Ch^{r-1}\|e\|_1\leq C(u)h^{2r-2}.$$

We now state a corresponding result for the parabolic equation.

THEOREM 11. *With the above assumptions on T_h and S_h, we have in the parabolic case for any $n \geqslant 0$,*

$$|e(x_0, t)| \leqslant C\left\{h^{r-1} \sum_{j=0}^{n} \|D_t^j e(t)\|_1 + h^r \|D_t^{n+1} e(t)\| + \|D_t^{n+1} e(t)\|_{-2n}\right\}.$$

It follows by our previous estimates that under suitable regularity assumptions,

$$|e(x_0, t)| \leqslant C(u) h^{2r-2} \quad \text{for} \quad t > 0.$$

The proof uses the representation

$$e(x_0, t) = \sum_{j=0}^{n} (-1)^j L(D_t^j e, T^j g) + (-1)^{n+1} (D_t^{n+1} e, T^n g),$$

where $g = g_{x_0}$ is as above, and

$$L(e, v) = (e_t, v) + A(e, v) = L(e, v - \chi) \quad \text{for} \quad \chi \in S_h,$$

and depends on the fact that $T^j g$ may be well approximated by an element of S_h

In the case that the finite element spaces are based on uniform partitions in the way quoted in connection with Theorem 8 in the interior domain $\Omega_0 \subset\subset \Omega \subset R^N$, it is possible to show that for any derivative D^α one may find a local approximation of $D^\alpha u$ from u_h in Ω_0. To see this we first quote the following lemma (Theorem 3 in [29]) which generalizes to the case of derivatives a construction due to Bramble and Schatz [7].

LEMMA. *Let ∂_h^α denote the forward difference quotient corresponding to D^α and ψ the B-spline in R^N of order $r-2$. Then there exists a function K_h of the form*

$$K_h(x) = h^{-N} \sum_\gamma k_\gamma \psi(h^{-1}x - \gamma),$$

with $k_\gamma = 0$ when $|\gamma_j| \geqslant r-1$ such that for $\Omega_1 \subset\subset \Omega_0 \subset\subset \Omega$, $e = u_h - u$,

$$|K_h * \partial_h^\alpha u_h - D^\alpha u|_{\Omega_1} \leqslant C\Big\{h^{2r-2} |u|_{2r-2+|\alpha|, \Omega_0} + \sum_{|\beta| \leqq r-2+N_0} \|\partial_h^{\alpha+\beta} e\|_{-(r-2), \Omega_0} + h^{r-2} \sum_{|\beta| \leqq r-2} |\partial_h^{\alpha+\beta} e|_{\Omega_0}\Big\},$$

with $\|\cdot\|_{-(r-2), \Omega_0}$ the dual norm to that in $H_0^{r-2}(\Omega_0)$.

In order to use this estimate we need to have at our disposal the appropriate estimates for $\partial_h^\beta e$. Such estimates may be derived by the techniques developed for elliptic problems in Nitsche and Schatz [22], Bramble, Nitsche and Schatz [6] and yield:

THEOREM 12. *Under the above assumptions, we have for the parabolic problem*

$$|K_h * \partial_h^\alpha u_h(t) - D^\alpha u(t)|_{\Omega_1} \leqslant C(u) h^{2r-2} + C \sum_{l=0}^{m} \{\|D_t^l e(t)\|_{-(r-2), \Omega_0} + h^{2r-2} \|D_t^l e(t)\|_{\Omega_0}\},$$

where $C(u)$ depends on u and certain of its derivatives on Ω_0 at time t and m is a positive integer.

Combining this with our above global estimates we have e.g. with $v_h = P_0 v$, for $t \geq \delta > 0$,

$$|K_h * \partial_h^\alpha u_h(t) - D^\alpha u(t)|_{\Omega_1} = O(h^{2r-2}) \quad \text{as} \quad h \to 0.$$

Several of the papers referred to above complete the error analysis by also discussing the error introduced by discretizing in the time variable, particularly by means of the backward Euler or Crank–Nicolson methods. For more general time discretizations than these we quote in particular Crouzeix [9] for Runge–Kutta type methods in the nonhomogeneous case and Baker, Bramble and Thomée [3] for estimates for homogeneous equations with smooth and nonsmooth data.

References

1. J. P. Aubin, *Behaviour of the error of the approximate solutions of boundary value problems for linear elliptic operators by Galerkin's and finite difference methods,* Ann. Scuola Norm. Sup. Pisa **21** (1967), 599—637.

2. I. Babuška, *The finite element method with Lagrangian multipliers,* Numer. Math. **20** (1973), 179—192.

3. G. A. Baker, J. H. Bramble and V. Thomée, *Single step Galerkin approximations for parabolic problems,* Math. Comp. **31** (1977), 819—847.

4. J. Blair, *Approximate solution of elliptic and parabolic boundary value problems,* thesis, Univ. of California, Berkeley, 1970.

5. C. de Boor and B. Swartz, *Collocation at Gaussian points,* SIAM J. Numer. Anal. **10** (1973), 582—606.

6. J. H. Bramble, J. A. Nitsche and A. H. Schatz, *Maximum norm interior estimates for Ritz—Galerkin methods,* Math. Comp. **29** (1975), 677—688.

7. J. H. Bramble and A. H. Schatz, *Higher order local accuracy by averaging in the finite element method,* Math. Comp. **31** (1977), 94—111.

8. J. H. Bramble, A. H. Schatz, V. Thomée and L. B. Wahlbin, *Some convergence estimates for semidiscrete Galerkin type approximations for parabolic equations,* SIAM J. Numer. Anal. **14** (1977), 218—241.

9. M. Crouzeix, *Sur l'approximation des équations différentielles opérationnelles linéaires par des méthodes de Runge-Kutta,* thesis, University of Paris VI, 1975.

10. M. Dobrowolski, *L^∞-convergence of finite element approximations to quasilinear initial boundary value problems,* Preprint no. 148, Sonderforschungsbereich 72, Universität Bonn 1977.

11. J. Douglas, Jr. and T. Dupont, *Galerkin methods for parabolic equations,* SIAM J. Numer. Anal. **7** (1970), 575—626.

12. *Some superconvergence results for Galerkin methods for the approximate solution of two-point boundary value problems,* Topics in Numerical Analysis (J. J. Miller, ed.) Academic Press, New York, 1973, pp. 89—92.

13. J. Douglas, Jr., T. Dupont and M. F. Wheeler, *A quasiprojection analysis of Galerkin methods for parabolic and hyperbolic equations,* Math. Comp. **32** (1978), 345—362.

14. T. Dupont, *Some L^2 error estimates for parabolic Galerkin methods,* The Mathematical Foundations of the Finite Element Methods with Applications to Partial Differential Equations, A. K. Aziz, ed., Academic Press, New York, 1972, pp. 491—501.

15. G. Fix and N. Nassif, *On finite element approximations to time dependent problems*, Numer. Math. **19** (1972), 127—135.

16. H. Fujita and A. Mizutani, *On the finite element method for parabolic equations* I, J. Math. Soc. Japan **28** (1976), 749—771.

17. H.-P. Helfrich, *Fehlerabschätzungen für das Galerkinverfahren zur Lösung von Evolutionsgleichungen*, Manuscripta Math. **13** (1974), 219—235.

18. J. A. Nitsche, *Ein Kriterium für die Quasi-Optimalität des Ritzschen Verfahrens*, Numer. Math. **11** (1968), 346—348.

19. ———, *Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind*, Abh. Math. Sem. Univ. Hamburg **36** (1971), 9—15.

20. ———, *L_∞-error analysis for finite elements*, The 3rd Conference on the Mathematics of Finite Elements and Applications, Brunel University, 1978.

21. ———, *L_∞-convergence of finite element Galerkin approximations on parabolic problems*, Rev. Française Automat. Informat. Recherche Opérationelle Sér. Rouge **13** (1979), 31—54.

22. J. A. Nitsche and A. H. Schatz, *Interior estimates for Ritz-Galerkin methods*, Math. Comp. **28** (1974), 937—958.

23. L. A. Oganesjan and P. A. Rukhovets, *Investigation of the convergence rate of variational-difference schemes for elliptic second order equations in a two-dimensional domain with a smooth boundary*. Ž. Vyčisl. Mat. i Mat. Fiz. **9** (1969), 1102—1120. (Russian) (Translation: USSR Comput. Math. and Math. Phys.).

24. H. S. Price and R. S. Varga, *Error bounds for semi-discrete Galerkin approximations of parabolic problems with application to petroleum reservoir mechanics*, Numerical Solution of Field Problems in Continuum Physics, SIAM-AMS Proc., vol. II, Amer. Math. Soc., Providence, R. I. 1970, pp. 74—94.

25. P. H. Sammon, *Approximations for parabolic equations with time dependent coefficient*, thesis, Cornell University, 1978.

26. A. H. Schatz, V. Thomée and L. B. Wahlbin, *Maximum norm stability and error estimates in parabolic finite element equations* (to appear).

27. V. Thomée, *Spline approximation and difference schemes for the heat equation*, The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, edited by A. K. Aziz, Academic Press, New York, 1972, pp. 711—746.

28. ———, *Some convergence results for Galerkin methods for parabolic boundary value problems*, Mathematical Aspects of Finite Elements in Partial Differential Equations, C. de Boor, ed , Academic Press, New York, 1974, pp. 55—88.

29. ———, *High order local approximations to derivatives in the finite element method*, Math. Comp. **31** (1977), 652—660.

30. ———, *Some interior estimates for semidiscrete Galerkin approximations for parabolic equations*, Math. Comp. **33** (1979), 37—62.

31. ———, *Negative norm estimates and superconvergence in Galerkin methods for parabolic problems*, Technical Report No. 1978—6, Math. Dept., Chalmers Univ. of Technology and the University of Göteborg.

32. M. F. Wheeler, *A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations*, SIAM J. Numer. Anal. **10** (1973), 723—759.

33. ———, *L_∞ estimates of optimal order for Galerkin methods for one dimensional second order parabolic and hyperbolic equations*, SIAM J. Numer. Anal. **10** (1973), 908—913.

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