2.11. A general formula involving $\zeta(s)$. It was observed by Müntz† that several of the formulae for $\zeta(s)$ which we have obtained are particular cases of a formula containing an arbitrary function.

We have formally

$$\int_0^\infty x^{s-1}\sum_{n=1}^\infty F(nx)\,dx = \sum_{n=1}^\infty \int_0^\infty x^{s-1}F(nx)\,dx$$
$$= \sum_{n=1}^\infty \frac{1}{n^s}\int_0^\infty y^{s-1}F(y)\,dy$$
$$= \zeta(s)\int_0^\infty y^{s-1}F(y)\,dy,$$

where $F(x)$ is arbitrary; and the process is justifiable if $F(x)$ is bounded in any finite interval, and $O(x^{-\alpha})$, where $\alpha > 1$, as $x \to \infty$. For then

$$\sum_{n=1}^\infty \left|\frac{1}{n^s}\right| \int_0^\infty |y^{s-1}F(y)|\,dy$$

exists if $1 < \sigma < \alpha$, and the inversion is justified.

Suppose next that $F'(x)$ is continuous, bounded in any finite interval, and $O(x^{-\beta})$, where $\beta > 1$, as $x \to \infty$. Then as $x \to 0$

$$\sum_{n=1}^\infty F(nx) - \int_0^\infty F(ux)\,du = x\int_0^\infty F'(ux)(u-[u])\,du$$
$$= x\int_0^{1/x} O(1)\,du + x\int_{1/x}^\infty O\{(ux)^{-\beta}\}\,du = O(1),$$

i.e.
$$\sum_{n=1}^\infty F(nx) = \frac{1}{x}\int_0^\infty F(v)\,dv + O(1) = \frac{c}{x} + O(1),$$

say. Hence

$$\int_0^\infty x^{s-1}\sum_{n=1}^\infty F(nx)\,dx$$
$$= \int_0^1 x^{s-1}\left\{\sum_{n=1}^\infty F(nx) - \frac{c}{x}\right\}dx + \frac{c}{s-1} + \int_1^\infty x^{s-1}\sum_{n=1}^\infty F(nx)\,dx,$$

and the right-hand side is regular for $\sigma > 0$ (except at $s = 1$). Also for $\sigma < 1$

$$\frac{c}{s-1} = -c\int_1^\infty x^{s-2}\,dx.$$

† Müntz (1).

Hence we have Müntz's formula

$$\zeta(s)\int_0^\infty y^{s-1}F(y)\,dy = \int_0^\infty x^{s-1}\left\{\sum_{n=1}^\infty F(nx) - \frac{1}{x}\int_0^\infty F(v)\,dv\right\}dx, \quad (2.11.1)$$

valid for $0 < \sigma < 1$ if $F(x)$ satisfies the above conditions.

If $F(x) = e^{-x}$ we obtain (2.7.1); if $F(x) = e^{-\pi x^2}$ we obtain a formula equivalent to those of § 2.6; if $F(x) = 1/(1+x^2)$ we obtain a formula which is also obtained by combining (2.4.1) with the functional equation. If $F(x) = x^{-1}\sin \pi x$ we obtain a formula equivalent to (2.1.6), though this $F(x)$ does not satisfy our general conditions.

If $F(x) = 1/(1+x)^2$, we have

$$\sum_{n=1}^\infty F(nx) - \frac{1}{x}\int_0^\infty F(v)\,dv = \sum_{n=1}^\infty \frac{1}{(1+nx)^2} - \frac{1}{x}$$
$$= \frac{1}{x^2}\left[\frac{d^2}{d\xi^2}\log\Gamma(\xi+1)\right]_{\xi=1/x} - \frac{1}{x}.$$

Hence
$$\frac{(1-s)\pi}{\sin \pi s}\zeta(s) = \int_0^\infty \xi^{1-s}\left\{\frac{d^2}{d\xi^2}\log\Gamma(\xi+1) - \frac{1}{\xi}\right\}d\xi,$$

and on integrating by parts we obtain (2.9.2).