A NOTE ON THE OSEEN KERNELS

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ABSTRACT. The Oseen operators are $\Delta^{-1}\partial_{x_j}\partial_{x_k}e^{t\Delta}$, where Δ is the standard Laplace operator and $t \in \mathbb{R}_+$. We give an explicit expression for the kernels of these Fourier multipliers which involves the incomplete gamma function and the confluent hypergeometric functions of the first kind. This explicit expression provides directly the classical decay estimates with sharp bounds. Although the computations are elementary and the definition of the Oseen kernels goes back to the 1911 paper of this author, we were not able to find the simple explicit expression below in the literature.

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1. Introduction

The (Marcel) Riesz operators $(R_j)_{1\le j\le n}$ are the following Fourier multipliers (we use the notation $\hat u$ for the Fourier transform of u: our normalization is given in the formula (3.1) of our appendix)

$$(1.1)\qquad (\widehat{R_j u})(\xi) = \xi_j|\xi|^{-1}\hat u(\xi), \qquad R_j = D_j/|D| = (-\Delta)^{-1/2}\frac{\partial}{i\partial x_j}.$$

The R_j are selfadjoint bounded operators on $L^2(\mathbb{R}^n)$ with norm 1. The Riesz operators are the natural multidimensional generalization of the Hilbert transform, given by the convolution with pv $\frac{i}{\pi x}$ which is the one-dimensional Fourier multiplier by $\operatorname{sign}\xi$. These operators are the paradigmatic singular integrals, introduced by Calderón and Zygmund and are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and send L^1 into L^1_w. However they are not continuous on the Schwartz class, because of the singularity at the origin. The Leray-Hopf projector[1] is the following matrix valued Fourier multiplier, given by

$$(1.2)\qquad \mathbf{P}(\xi) = \operatorname{Id} - \frac{\xi\otimes\xi}{|\xi|^2} = \big(\delta_{jk} - |\xi|^{-2}\xi_j\xi_k\big)_{1\le j,k\le n}, \quad \mathbf{P} = \mathbf{P}(D) = \operatorname{Id} - R\otimes R.$$

We can also consider the $n\times n$ matrix of operators given by $\mathbf{Q} = R\otimes R = (R_jR_k)_{1\le j,k\le n}$ sending the vector space of $L^2(\mathbb{R}^n)$ vector fields into itself. The operator $\mathbf{Q}$ is selfadjoint and is a projection since $\sum_l R_l^2 = \operatorname{Id}$ so that $\mathbf{Q}^2 = (\sum_l R_jR_lR_lR_k)_{j,k} = \mathbf{Q}$. As a result the operator

$$(1.3)\qquad \mathbf{P} = \operatorname{Id} - R\otimes R = \operatorname{Id} - |D|^{-2}(D\otimes D) = \operatorname{Id} - \Delta^{-1}(\nabla\otimes\nabla)$$

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[1] That projector is also called the Helmholtz-Weyl projector by some authors.

is also an orthogonal projection, the Leray-Hopf projector (a.k.a. the Helmholtz-Weyl projector); the operator $\mathbf{P}$ is in fact the orthogonal projection onto the closed subspace of L^2 vector fields with null divergence. We have for a vector field $u = \sum_j u_j \partial_j$, the identity $\operatorname{grad}\operatorname{div} u = \nabla(\nabla \cdot u)$, and thus

$$\operatorname{grad}\operatorname{div} = \nabla \otimes \nabla = \Delta\ R \otimes R, \quad \text{so that} \tag{1.4}$$

$$\mathbf{Q} = R \otimes R = \Delta^{-1} \operatorname{grad}\operatorname{div}, \quad \operatorname{div} R \otimes R = \operatorname{div}, \tag{1.5}$$

which implies $\operatorname{div} \mathbf{P}u = \operatorname{div} u - \operatorname{div}(R \otimes R)u = 0$, and if $\operatorname{div} u = 0$, we have $\mathbf{Q}u = 0$ and $u = \mathbf{Q}u + \mathbf{P}u = \mathbf{P}u$. This operator plays an important role in fluid mechanics since the Navier-Stokes system ([7], [3], [6]) for incompressible fluids can be written as

$$\begin{cases} \partial_t v + \mathbf{P}((v \cdot \nabla)v) - \nu \triangle v = 0, \\ \mathbf{P}v = v, \\ v_{|t=0} = v_0. \end{cases} \tag{1.6}$$

As already said for the Riesz operators, $\mathbf{P}$ is not a classical pseudodifferential operator, because of the singularity at the origin: however it is indeed a Fourier multiplier with the same continuity properties as those of R, and in particular is bounded on L^p for $p \in (1, +\infty)$. In three dimensions the **curl** operator is given by the matrix

$$\mathbf{curl} = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} = \mathbf{curl}^* \tag{1.7}$$

so that $\mathbf{curl}^2 = -\Delta \operatorname{Id} + \operatorname{grad}\operatorname{div}$ and (the Biot-Savard law)

$$\operatorname{Id} = (-\Delta)^{-1} \mathbf{curl}^2 + \Delta^{-1} \operatorname{grad}\operatorname{div} = (-\Delta)^{-1} \mathbf{curl}^2 + \operatorname{Id} - \mathbf{P}, \tag{1.8}$$

which gives

$$\mathbf{curl}^2 = -\Delta \mathbf{P}, \tag{1.9}$$

so that $[\mathbf{P}, \mathbf{curl}] = 0$ and

$$\mathbf{P}\,\mathbf{curl} = \mathbf{curl}\,\mathbf{P} = \mathbf{curl}(-\Delta)^{-1} \mathbf{curl}^2 = \mathbf{curl}\big(\operatorname{Id} - \Delta^{-1} \operatorname{grad}\operatorname{div}\big) = \mathbf{curl} \tag{1.10}$$

since $\mathbf{curl} \operatorname{grad} = 0$ (note also that the transposition of the latter gives $\operatorname{div} \mathbf{curl} = 0$). The solutions of (1.6) are satisfying

$$v(t) = e^{t\nu\Delta} v_0 - \int_0^t e^{(t-s)\nu\Delta} \mathbf{P} \nabla\big(v(s) \otimes v(s)\big) ds.$$

2. The action of the Leray projector on Gaussian functions

We want now to compute the action of $\mathbf{P}$ on Gaussian functions.

Lemma 2.1. *Let $n \ge 1$ be an integer, $1 \le j,k \le n$ and $a > 0$. Then, with $u_a(x) = a^{n/2}e^{-\pi a|x|^2}$, we have*

(2.1)

for $j \ne k$, $(R_jR_ku_a)(x) = -x_jx_k|x|^{-n-2}\gamma(1+\frac{n}{2}, a\pi|x|^2)\pi^{-n/2}$,

$$(R_j^2u_a)(x) = -x_j^2|x|^{-n-2}\gamma(1+\frac{n}{2}, a\pi|x|^2)\pi^{-n/2} + \frac{1}{2}|x|^{-n}\gamma(\frac{n}{2}, a\pi|x|^2)\pi^{-n/2}, \tag{2.2}$$

where γ is the incomplete gamma function (see below a reminder).

A reminder. We recall the definition of the (lower) incomplete Gamma function (see e.g. [2], [1]),

$$\begin{aligned}\gamma(a,x) &= \int_0^x t^{a-1}e^{-t}dt \\ &= a^{-1}x^ae^{-x}{}_1F_1(1;1+a;x) = a^{-1}x^a{}_1F_1(a;1+a;-x),\end{aligned} \tag{2.3}$$

where ${}_1F_1$ is the confluent hypergeometric function of the first kind. Also for n positive integer, we have

$$\gamma(n,x) = (n-1)!\big(1 - e^{-x}\sum_{0\le k\le n-1}\frac{x^k}{k!}\big) = \Gamma(n)\big(1 - e^{-x}\sum_{0\le k<n}\frac{x^k}{k!}\big).$$

The confluent hypergeometric function has a hypergeometric series given by

$${}_1F_1(a;b;z) = 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \dots = \sum_{k\ge0}\frac{(a)_k}{(b)_k}\frac{z^k}{k!},$$

where $(x)_n$ stands for the Pochhammer symbol

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\dots(x+n-1).$$

We note also the following identity

$$\forall a \in \mathbb{C}\backslash\mathbb{Z}_-^*, \quad {}_1F_1(1;1+a;z) = \sum_{k\ge0}\frac{z^k}{(a+1)\dots(a+k)} \tag{2.4}$$

which is an entire function of the variable z for these values of a; as a result, we can write for $\operatorname{Re} a > 0, x \ge 0$,

$$\gamma(a,x) = a^{-1}x^ae^{-x}\sum_{k\ge0}\frac{x^k}{(a+1)\dots(a+k)}, \tag{2.5}$$

and this implies that

$$\forall a > 0, \forall x \ge 0, \quad a^{-1}x^ae^{-x} \le \gamma(a,x) \le \min\big(\Gamma(a), a^{-1}x^a\big). \tag{2.6}$$

We have also

$$\gamma(1+a,x) = a\gamma(a,x) - x^ae^{-x}. \tag{2.7}$$

Remark 2.2. *The above lemma can be generalized easily to the case where A is a complex-valued symmetric matrix with a positive definite real part, with $u_A(x) = (\det A)^{1/2} e^{-\pi\langle Ax,x\rangle}$, with $(\det A)^{1/2} = e^{\frac{1}{2}\operatorname{trace}\operatorname{Log} A}$ (see the appendix for the choice of the determination of* $\operatorname{Log} A$*).*

Proof of the lemma. We consider, for $t > 0$ the smooth function

$$F_{j,k}(t,x) = \int_{\mathbb{R}^n} e^{2i\pi x\xi} e^{-t4\pi^2|\xi|^2} \xi_j\xi_k|\xi|^{-2} d\xi, \tag{2.8}$$

and we note that

$$\begin{aligned}\frac{\partial F_{jk}}{\partial t}(t,x) &= -4\pi^2 \int e^{2i\pi x\xi} e^{-t4\pi^2|\xi|^2}\xi_j\xi_k d\xi \\ &= -4\pi^2 \frac{1}{(2i\pi)^2}\partial_{x_j}\partial_{x_k}\int e^{2i\pi x\xi}e^{-t4\pi^2|\xi|^2}d\xi = \partial_{x_j}\partial_{x_k}(e^{-\frac{|x|^2}{4t}})(4\pi t)^{-n/2},\end{aligned} \tag{2.9}$$

so that

$$\text{for } j\neq k,\ \frac{\partial F_{jk}}{\partial t}(t,x) = (4\pi t)^{-n/2}e^{-\frac{|x|^2}{4t}}(\frac{x_jx_k}{4t^2}), \tag{2.10}$$

$$\text{for } j = k,\ \frac{\partial F_{jj}}{\partial t}(t,x) = (4\pi t)^{-n/2}e^{-\frac{|x|^2}{4t}}(\frac{x_j^2}{4t^2} - \frac{1}{2t}). \tag{2.11}$$

Since we have also $F_{j,k}(+\infty, x) = 0$, we obtain for $j \neq k$, $x \neq 0$,

$$\begin{aligned}F_{jk}(\frac{1}{4\pi},x) &= \int_{+\infty}^{1/4\pi}(4\pi t)^{-n/2}e^{-\frac{|x|^2}{4t}}\frac{x_jx_k}{4t^2}dt \\ &= -x_jx_k\int_0^{\pi|x|^2} s^{n/2}|x|^{-n}e^{-s}4s^2|x|^{-4}|x|^2s^{-2}ds\pi^{-n/2} \\ &= -x_jx_k|x|^{-n-2}\int_0^{\pi|x|^2} s^{n/2}e^{-s}ds\pi^{-n/2},\end{aligned} \tag{2.12}$$

i.e. for $j \neq k$,

$$\int_{\mathbb{R}^n} e^{2i\pi x\xi}e^{-\pi|\xi|^2}\xi_j\xi_k|\xi|^{-2}d\xi = -x_jx_k|x|^{-n-2}\int_0^{\pi|x|^2}s^{n/2}e^{-s}ds\pi^{-n/2}. \tag{2.13}$$

For $j = k$, we have

$$\begin{aligned}F_{jj}(\frac{1}{4\pi},x) &= \int_{+\infty}^{1/4\pi}(4\pi t)^{-n/2}e^{-\frac{|x|^2}{4t}}(\frac{x_j^2}{4t^2}-\frac{1}{2t})dt \\ &= -x_j^2|x|^{-n-2}\int_0^{\pi|x|^2}s^{n/2}e^{-s}ds\pi^{-n/2} + \frac{1}{2}|x|^{-n}\int_0^{\pi|x|^2}s^{\frac{n}{2}-1}e^{-s}ds\pi^{-n/2},\end{aligned} \tag{2.14}$$

so that

$$\begin{aligned}&\int_{\mathbb{R}^n} e^{2i\pi x\xi}e^{-\pi|\xi|^2}\xi_j^2|\xi|^{-2}d\xi \\ &= -x_j^2|x|^{-n-2}\int_0^{\pi|x|^2}s^{n/2}e^{-s}ds\pi^{-n/2} + \frac{1}{2}|x|^{-n}\int_0^{\pi|x|^2}s^{\frac{n}{2}-1}e^{-s}ds\pi^{-n/2}.\end{aligned} \tag{2.15}$$

As a consequence, for $t>0, j\neq k$, we have

$$F_{jk}(t,x)=-\gamma(1+\frac{n}{2},\frac{|x|^2}{4t})\pi^{-n/2}\frac{x_jx_k}{|x|^{2+n}}, \tag{2.16}$$

and

$$F_{jj}(t,x)=-\gamma(1+\frac{n}{2},\frac{|x|^2}{4t})\pi^{-n/2}\frac{x_j^2}{|x|^{2+n}}+\frac{1}{2}\gamma(\frac{n}{2},\frac{|x|^2}{4t})\pi^{-n/2}\frac{1}{|x|^n}. \tag{2.17}$$

As a result, we have indeed, with $a>0$, $j\neq k$,

$$(R_jR_ku_a)(x)=\int_{\mathbb{R}^n}e^{2i\pi x\xi}e^{-\pi a^{-1}|\xi|^2}\xi_j\xi_k|\xi|^{-2}d\xi \tag{2.18}$$

$$=a^{n/2}\int_{\mathbb{R}^n}e^{2i\pi a^{1/2}x\xi}e^{-\pi|\xi|^2}\xi_j\xi_k|\xi|^{-2}d\xi \tag{2.19}$$

$$=a^{n/2}F_{jk}(\frac{1}{4\pi},a^{1/2}x) \tag{2.20}$$

$$=-x_jx_k|x|^{-n-2}\gamma(1+\frac{n}{2},a\pi|x|^2)\pi^{-n/2}, \tag{2.21}$$

and for $j=k$,

$$(R_j^2u_a)(x)=\int_{\mathbb{R}^n}e^{2i\pi x\xi}e^{-\pi a^{-1}|\xi|^2}\xi_j^2|\xi|^{-2}d\xi \tag{2.22}$$

$$=a^{n/2}\int_{\mathbb{R}^n}e^{2i\pi a^{1/2}x\xi}e^{-\pi|\xi|^2}\xi_j^2|\xi|^{-2}d\xi \tag{2.23}$$

$$=a^{n/2}F_{jj}(\frac{1}{4\pi},a^{1/2}x) \tag{2.24}$$

$$=-x_j^2|x|^{-n-2}\gamma(1+\frac{n}{2},a\pi|x|^2)\pi^{-n/2}+\frac{1}{2}|x|^{-n}\gamma(\frac{n}{2},a\pi|x|^2)\pi^{-n/2}. \square$$

Theorem 2.3. *Let $n\geq 1$ be an integer and $\Delta=\sum_{1\leq j\leq n}\partial_{x_j}^2$ be the standard Laplace operator on $\mathbb{R}^n$. For $t\geq 0$, we define the Oseen matrix operator*

$$\Omega(t)=\Delta^{-1}(\nabla\otimes\nabla)e^{t\Delta}=(I-\mathbf{P})e^{t\Delta}=\Delta^{-1}(\partial_{x_j}\otimes\partial_{x_k})_{1\leq j,k\leq n}e^{t\Delta}. \tag{2.25}$$

The operator $\Omega(t)$ is the Fourier multiplier by the matrix $\Omega(t,\xi)=|\xi|^{-2}(\xi\otimes\xi)e^{-4\pi t|\xi|^2}$ and is given by the convolution (w.r.t. the variable x) with the matrix $\big(F_{jk}(t,x)\big)_{1\leq j,k\leq n}$ where

$$\text{for } j\neq k,\ F_{jk}(t,x)=-x_jx_k|x|^{-n-2}\gamma(1+\frac{n}{2},\frac{|x|^2}{4t})\pi^{-n/2}, \tag{2.26}$$

$$F_{jj}(t,x)=-x_j^2|x|^{-n-2}\gamma(1+\frac{n}{2},\frac{|x|^2}{4t})\pi^{-n/2}+\frac{1}{2}|x|^{-n}\gamma(\frac{n}{2},\frac{|x|^2}{4t})\pi^{-n/2}, \tag{2.27}$$

$$F_{jj}(t,x)=\gamma(\frac{n}{2},\frac{|x|^2}{4t})\pi^{-n/2}\frac{1}{2}|x|^{-n-2}\big(|x|^2-nx_j^2\big)+x_j^2|x|^{-2}(4\pi t)^{-n/2}e^{-\frac{|x|^2}{4t}}. \tag{2.28}$$

On $t>0$, the functions F_{jk} are real analytic functions of the variable $t^{-1/2}x$ multiplied by $t^{-n/2}$. We have also

$$F_{jk}(t,x)=(4\pi t)^{-n/2}F_{jk}\big(\frac{1}{4\pi},x(4\pi t)^{-1/2}\big), \tag{2.29}$$

and with $|\mathbb{S}^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$,

$$|F_{jk}(t,x)| \le |x|^{-n}\frac{n+1}{|\mathbb{S}^{n-1}|}, \qquad |F_{jk}(t,x)| \le \big(\frac{|x|^2}{2(n+2)t}+\frac{1}{n}\big)(4\pi t)^{-n/2}. \tag{2.30}$$

Moreover we have

$$\text{for } j\ne k,\ F_{jk}(t,x) = -\frac{2}{n+2}\frac{x_jx_k}{4t}(4\pi t)^{-n/2}e^{-\frac{|x|^2}{4t}}{}_1F_1(1;2+\frac{n}{2};\frac{|x|^2}{4t}) \tag{2.31}$$

and

$$F_{jj}(t,x) = -\frac{2}{n+2}\frac{x_j^2}{4t}(4\pi t)^{-n/2}e^{-\frac{|x|^2}{4t}}{}_1F_1(1;2+\frac{n}{2};\frac{|x|^2}{4t}) + \frac{1}{n}(4\pi t)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4t}}{}_1F_1(1;1+\frac{n}{2};\frac{|x|^2}{4t}). \tag{2.32}$$

The proof is an immediate consequence of (2.16), (2.17), (2.6). □

Remark 2.4. *This theorem provides a direct proof, using special functions, of the estimates established in a more general context in* [4] *as well as those stated on page 27 of* [6].

Remark 2.5. *We get easily from the first part of the previous theorem that the kernel of the operator* $I-\mathbf{P}$, *which is the matrix Fourier multiplier* $|\xi|^{-2}(\xi\otimes\xi)$, *is the singular integral given by the (principal-value) convolution with the matrix* $(f_{jk}(x))$ *where*

$$\text{for } j\ne k,\ f_{jk}(x) = -x_jx_k|x|^{-n-2}\Gamma(1+\frac{n}{2})\pi^{-n/2} = -x_jx_k|x|^{-n-2}\frac{n}{|\mathbb{S}^{n-1}|}, \tag{2.33}$$

$$f_{jj}(x) = |x|^{-n-2}(|x|^2-nx_j^2)|\mathbb{S}^{n-1}|^{-1}+n^{-1}\delta_0(x). \tag{2.34}$$

We note also that the functions $g_{jk} = f_{jk}-n^{-1}\delta_{j,k}\delta_0$ *are homogeneous of degree* $-n$ *on* $\mathbb{R}^n\backslash\{0\}$ *with integral* 0 *on* $\mathbb{S}^{n-1}$ *so that the principal value*

$$\langle T_{jk},\varphi\rangle = \lim_{\epsilon\to 0_+}\int_{|x|\ge\epsilon} g_{jk}(x)\varphi(x)dx$$

actually defines a homogeneous distribution T_{jk} *of degree* $-n$ *on* $\mathbb{R}^n$ *(*[5]*).*

3. Appendix

The Fourier transformation. The Fourier transform of a function u in the Schwartz class $\mathscr{S}(\mathbb{R}^n)$ is defined by the formula

$$\hat u(\xi) = \int e^{-2i\pi x\xi}u(x)dx, \tag{3.1}$$

and it is an isomorphism of $\mathscr{S}(\mathbb{R}^n)$ so that $u(x) = \int e^{2i\pi x\xi}\hat u(\xi)d\xi$. That isomorphism extends to an isomorphism of the temperate distributions $\mathscr{S}'(\mathbb{R}^n)$ via the duality formula $\langle \widehat{T},\phi\rangle_{\mathscr{S}',\mathscr{S}} = \langle T,\hat\phi\rangle_{\mathscr{S}',\mathscr{S}}$. The Fourier transform is also a unitary transformation of $L^2(\mathbb{R}^n)$.

The logarithm of a nonsingular symmetric matrix. The set $\mathbb{C}\backslash\mathbb{R}_-$ is star-shaped with respect to 1, so that we can define the principal determination of the logarithm for $z \in \mathbb{C}\backslash\mathbb{R}_-$ by the formula

$$\operatorname{Log} z = \oint_{[1,z]} \frac{d\zeta}{\zeta}. \tag{3.2}$$

The function Log is holomorphic on $\mathbb{C}\backslash\mathbb{R}_-$ and we have $\operatorname{Log} z = \ln z$ for $z \in \mathbb{R}_+^*$ and by analytic continuation $e^{\operatorname{Log} z} = z$ for $z \in \mathbb{C}\backslash\mathbb{R}_-$. We get also by analytic continuation, that $\operatorname{Log} e^z = z$ for $|\operatorname{Im} z| < \pi$.

Let Υ_+ be the set of symmetric nonsingular $n \times n$ matrices with complex entries and nonnegative real part. The set Υ_+ is star-shaped with respect to the Id: for $A \in \Upsilon_+$, the segment $[1, A] = \big((1-t)\operatorname{Id} + tA\big)_{t\in[0,1]}$ is obviously made with symmetric matrices with nonnegative real part which are invertible, since for $0 \le t < 1$, $\operatorname{Re}\big((1-t)\operatorname{Id} + tA\big) \ge (1-t)\operatorname{Id} > 0$ and for $t = 1$, A is assumed to be invertible. We can now define for $A \in \Upsilon_+$

$$\operatorname{Log} A = \int_0^1 (A - I)\big(I + t(A - I)\big)^{-1} dt. \tag{3.3}$$

We note that A commutes with $(I + sA)$ (and thus with $\operatorname{Log} A$), so that, for $\theta > 0$,

$$\begin{aligned}\frac{d}{d\theta}\operatorname{Log}(A + \theta I) = &\int_0^1 \big(I + t(A + \theta I - I)\big)^{-1} dt \\ &- \int_0^1 \big(A + \theta I - I\big)t\big(I + t(A + \theta I - I)\big)^{-2} dt,\end{aligned}$$

and since $\frac{d}{dt}\Big\{\big(I+t(A+\theta I-I)\big)^{-1}\Big\} = -\big(I+t(A+\theta I-I)\big)^{-2}(A+\theta I-I)$, we obtain by integration by parts $\frac{d}{d\theta}\operatorname{Log}(A + \theta I) = (A + \theta I)^{-1}$. As a result, we find that for $\theta > 0, A \in \Upsilon_+$, since all the matrices involved are commuting,

$$\frac{d}{d\theta}\left((A + \theta I)^{-1} e^{\operatorname{Log}(A+\theta I)}\right) = 0,$$

so that, using the limit $\theta \to +\infty$, we get that $\forall A \in \Upsilon_+, \forall \theta > 0,\ e^{\operatorname{Log}(A+\theta I)} = (A+\theta I)$, and by continuity

$$\forall A \in \Upsilon_+, \quad e^{\operatorname{Log} A} = A, \quad \text{which implies} \quad \det A = e^{\operatorname{trace}\operatorname{Log} A}. \tag{3.4}$$

Using (3.4), we can define for $A \in \Upsilon_+$, using (3.3)

$$(\det A)^{-1/2} = e^{-\frac{1}{2}\operatorname{trace}\operatorname{Log} A} = |\det A|^{-1/2} e^{-\frac{i}{2}\operatorname{Im}(\operatorname{trace}\operatorname{Log} A)}. \tag{3.5}$$

- When A is a positive definite matrix, $\operatorname{Log} A$ is real-valued and $(\det A)^{-1/2} = |\det A|^{-1/2}$.
- When $A = -iB$ where B is a real nonsingular symmetric matrix, we note that $B = PD^tP$ with $P \in O(n)$ and D diagonal. We see directly on the formulas (3.3),(3.2) that

$$\operatorname{Log} A = \operatorname{Log}(-iB) = P(\operatorname{Log}(-iD))^tP, \quad \operatorname{trace}\operatorname{Log} A = \operatorname{trace}\operatorname{Log}(-iD)$$

and thus, with (μ_j) the (real) eigenvalues of B, we have $\operatorname{Im}(\operatorname{trace}\operatorname{Log} A) = \operatorname{Im}\sum_{1\le j\le n}\operatorname{Log}(-i\mu_j)$, where the last Log is given by (3.2). Finally we get,

$$\operatorname{Im}(\operatorname{trace}\operatorname{Log} A) = -\frac{\pi}{2}\sum_{1\le j\le n}\operatorname{sign}\mu_j = -\frac{\pi}{2}\operatorname{sign} B$$

where $\operatorname{sign} B$ is the signature of B. As a result, we have when $A = -iB$, B real symmetric nonsingular matrix

$$(\det A)^{-1/2} = |\det A|^{-1/2} e^{i\frac{\pi}{4}\operatorname{sign}(iA)} = |\det B|^{-1/2} e^{i\frac{\pi}{4}\operatorname{sign} B}. \tag{3.6}$$

Proposition 3.1. *Let A be a symmetric nonsingular $n \times n$ matrix with complex entries such that $\operatorname{Re} A \ge 0$. We define the Gaussian function v_A on $\mathbb{R}^n$ by $v_A(x) = e^{-\pi\langle Ax,x\rangle}$. The Fourier transform of v_A is*

$$\widehat{v_A}(\xi) = (\det A)^{-1/2} e^{-\pi\langle A^{-1}\xi,\xi\rangle}, \tag{3.7}$$

where $(\det A)^{-1/2}$ is defined according to the formula (3.5). In particular, when $A = -iB$ with a symmetric real nonsingular matrix B, we get

$$\operatorname{Fourier}(e^{i\pi\langle Bx,x\rangle})(\xi) = \widehat{v_{-iB}}(\xi) = |\det B|^{-1/2} e^{i\frac{\pi}{4}\operatorname{sign} B} e^{-i\pi\langle B^{-1}\xi,\xi\rangle}.$$

Proof. Let us define Υ_+^* as the set of symmetric $n \times n$ complex matrices with a positive definite real part (naturally these matrices are nonsingular since $Ax = 0$ for $x \in \mathbb{C}^n$ implies $0 = \operatorname{Re}\langle Ax, \bar{x}\rangle = \langle(\operatorname{Re} A)x, \bar{x}\rangle$, so that $\Upsilon_+^* \subset \Upsilon_+$).

Let us assume first that $A \in \Upsilon_+^*$; then the function v_A is in the Schwartz class (and so is its Fourier transform). The set Υ_+^* is an open convex subset of $\mathbb{C}^{n(n+1)/2}$ and the function $\Upsilon_+^* \ni A \mapsto \widehat{v_A}(\xi)$ is holomorphic and given on $\Upsilon_+^* \cap \mathbb{R}^{n(n+1)/2}$ by (3.7). On the other hand the function $\Upsilon_+^* \ni A \mapsto e^{-\frac{1}{2}\operatorname{trace}\operatorname{Log} A} e^{-\pi\langle A^{-1}\xi,\xi\rangle}$ is also holomorphic and coincides with previous one on $\mathbb{R}^{n(n+1)/2}$. By analytic continuation this proves (3.7) for $A \in \Upsilon_+^*$.

If $A \in \Upsilon_+$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we have $\langle\widehat{v_A}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \int v_A(x)\hat{\varphi}(x)dx$ so that $\Upsilon_+ \ni A \mapsto \langle\widehat{v_A}, \varphi\rangle$ is continuous and thus (note that the mapping $A \mapsto A^{-1}$ is an homeomorphism of Υ_+), using the previous result on Υ_+^*,

$$\langle\widehat{v_A}, \varphi\rangle = \lim_{\epsilon\to 0_+}\langle\widehat{v_{A+\epsilon I}}, \varphi\rangle = \lim_{\epsilon\to 0_+}\int e^{-\frac{1}{2}\operatorname{trace}\operatorname{Log}(A+\epsilon I)} e^{-\pi\langle(A+\epsilon I)^{-1}\xi,\xi\rangle}\varphi(\xi)d\xi$$

$$\text{(by continuity of Log on } \Upsilon_+ \text{ and domin. cv.)} \qquad = \int e^{-\frac{1}{2}\operatorname{trace}\operatorname{Log} A} e^{-\pi\langle A^{-1}\xi,\xi\rangle}\varphi(\xi)d\xi,$$

which is the sought result. □

Some standard examples of Fourier transform. Let us consider the Heaviside function defined on $\mathbb{R}$ by $H(x) = 1$ for $x > 0$, $H(x) = 0$ for $x \le 0$. With the notation of this section, we have, with δ_0 the Dirac mass at 0, $\check{H}(x) = H(-x)$,

$$\widehat{H} + \widehat{\check{H}} = \hat{1} = \delta_0, \quad \widehat{H} - \widehat{\check{H}} = \widehat{\operatorname{sign}}, \qquad \frac{1}{i\pi} = \frac{1}{2i\pi}2\widehat{\delta_0}(\xi) = \widehat{D\operatorname{sign}}(\xi) = \xi\widehat{\operatorname{sign}}\xi$$

so that $\xi\big(\widehat{\text{sign}}\xi - \frac{1}{i\pi}pv(1/\xi)\big) = 0$ and $\widehat{\text{sign}}\xi - \frac{1}{i\pi}pv(1/\xi) = c\delta_0$ with $c = 0$ since the lhs is odd. We get

$$\widehat{\text{sign}}(\xi) = \frac{1}{i\pi}pv\frac{1}{\xi}, \qquad \widehat{pv(\frac{1}{\pi x})} = -i\,\text{sign}\,\xi, \qquad \hat{H} = \frac{\delta_0}{2} + \frac{1}{2i\pi}pv(\frac{1}{\xi}). \tag{3.8}$$

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