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LIQUID DROPS IN A VISCOUS FLUID UNDER THE INFLUENCE OF GRAVITY AND SURFACE TENSION

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We consider the steady fall of a drop of fluid under its own weight in an infinite reservoir of another viscous fluid; the shape of the drop is determined by surface tension. For small data we prove existence and uniqueness of a classical solution to this problem.

1. INTRODUCTION

A drop of a viscous fluid in an infinite reservoir of another fluid will move under its own weight if the surrounding fluid has different density. This paper is concerned with steady motions, i.e. the motions of the two fluids are independent of time with respect to an observer attached to the interface between them. We assume that the shape of this bounding surface is governed by surface tension.

The problem then consists in determining the velocity field and the pressure in the drop as well as in the outer fluid (which is at rest at infinity), the capillary surface and its speed of falling, when the volume of the drop and the gravitational field are given.

The situation we investigate is realized in everyday[1] phenomena as raindrops. If the drops are small enough experiments indicate that the motion is practically independent of time. J.E. McDonald [7] has argued that the shape

[1] We refer to Bonn where the paper was written. The support of Sonderforschungsbereich 72 at the University of Bonn is gratefully acknowledged

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of falling drops can only be determined from dynamical considerations: the velocity field around the drop must be taken into account to get a shape that is in agreement with the experimental facts. This parallels the discussion about the shape of an axially symmetric body of least resistance moving with constant speed. I. Newton [8] who was the first to formulate this problem proceeded on the assumption that a solution could be found without knowing the flow around the body. This led him to a variational problem which he was able to solve. The solution however turned out not to be in agreement with the experiments[2].

The aim of this paper is to prove existence and uniqueness of a solution to the problem of a falling drop provided the data are small. This may be regarded as a first step of a mathematically rigorous treatment of McDonald's investigations. A typical example for the shape of a falling drop is given in figure 1.

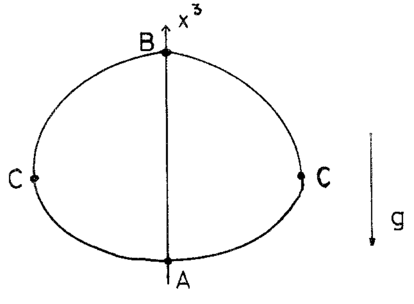

Fig. 1

In order to compare our solution with the experiments one must be able to locate the extrema of the curvature at the points (or lines resp.) A, B, C, cf. the discussion in McDonald's paper. The problem of a falling drop is closely related to the one of a falling rigid body; this was solved by H.F. Weinberger in [9], [10].

[2] For a discussion of Newton's variational problem (including its criticism for purely mathematical reasons) we refer to P. Funk [4] pp. 616-621, where also references to hydrodynamical papers can be found which are relevant for this problem

2. FORMULATION OF THE PROBLEM

Let Ω denote the region occupied by the drop of fluid whose density and viscosity are denoted by ρ_1 and μ_1; $\Sigma \equiv \partial\Omega$ is the interface between the two fluids, and the fluid in $\mathcal{E} := \mathbb{R}^3 \setminus (\Omega \cup \Sigma)$ is of density ρ_2 and viscosity μ_2. We assume that there is a uniform gravitational field $\underline{g} = (0,0,g)$, and that the drop Ω moves vertically downward with velocity $\underline{\gamma} = (0,0,\gamma)$. If the fluid in $\mathcal{E}$ is at rest at infinity then in a coordinate system that is attached to the drop the velocity at infinity is $-\underline{\gamma}$. Let $\underline{v}$ denote the difference between the velocity in $\mathcal{E}$ and the limiting velocity, and p the pressure in $\mathcal{E}$; then $\underline{v}$ and p satisfy the equations of motion

$$(2.1) \qquad \begin{aligned} -\mu_2 \Delta \underline{v} + \nabla p + \rho_2((\underline{v}-\underline{\gamma}) \cdot \nabla)\underline{v} &= \rho_2 \underline{g} \\ \nabla \cdot \underline{v} &= 0 \end{aligned} \qquad \text{in } \mathcal{E} .$$

At infinity there holds

$$(2.2) \qquad \underline{v}(x) \to 0 \quad \text{as} \quad |x| \to \infty.$$

If the corresponding variables in Ω are $\underline{u}$ and q, we get

$$(2.3) \qquad \begin{aligned} -\mu_1 \Delta \underline{u} + \nabla q + \rho_1((\underline{u}-\underline{\gamma}) \cdot \nabla)\underline{u} &= \rho_1 \underline{g} \\ \nabla \cdot \underline{u} &= 0 \end{aligned} \qquad \text{in } \Omega .$$

On the interface Σ the velocity and the tangential components of the stress vector must be continuous. So if $\underline{\tau}_k(x)$, $k = 1,2$, span the tangent plane at $x \in \Sigma$, if $\underline{n}$ is the normal and

$$(2.4) \qquad T_{ij}(\underline{v},p) = -p\delta_{ij} + 2\mu_2 D_{ij}(\underline{v}) \quad ; \quad D_{ij}(\underline{v}) = \frac{1}{2}\left(\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i}\right),$$

the boundary conditions on Σ are

$$(2.5) \qquad \underline{v}-\underline{u} \equiv 0, \ \underline{\tau}_k \cdot \{\underline{\underline{T}}(v)-\underline{\underline{T}}(u)\} \cdot \underline{n} \equiv 0 .$$

As Σ moves with uniform speed $\underline{\gamma}$ we have in addition to (2.5)

(2.6) $$\underline{v} \cdot \underline{n} = \underline{u} \cdot \underline{n} = \underline{\gamma} \cdot \underline{n} \quad \text{on} \quad \Sigma \quad .$$

The unknown speed of falling is determined by equating the viscous force to the net weight of the drop. As $\underline{g} = \nabla(\underline{g} \cdot x)$ we can write for the weight $\rho_1 |\Omega| \underline{g} = \oint_\Sigma \rho_1 (\underline{g} \cdot x) \underline{n} \, d\sigma$, and the equilibrium of these forces is expressed by

(2.7) $$\oint_\Sigma \underline{\underline{T}}(\underline{v},\tilde{p}) \cdot \underline{n} \, d\sigma = \oint_\Sigma \underline{\underline{T}}(\underline{u},\tilde{q}) \cdot \underline{n} \, d\sigma \quad ,$$

where

$$\tilde{p} = p + \rho_1 \underline{g} \cdot x \quad , \quad \tilde{q} = q + \rho_2 \underline{g} \cdot x \quad .$$

The surface tension produces a jump in the normal component of the stress which is proportional to the mean curvature H of Σ

(2.8) $$2\kappa H = \underline{n} \cdot \{\underline{\underline{T}}(\underline{v},\tilde{p}) - \underline{\underline{T}}(\underline{u},\tilde{q})\} \cdot \underline{n} \quad \text{on} \quad \Sigma \quad .$$

Finally, the volume of the drop is prescribed, too:

(2.9) $$\text{meas } \Omega = V.$$

For any closed surface Σ of mean curvature H there holds

$$2\oint H \underline{n} \, d\sigma = 0 \quad ,$$

which can be derived from the integration by parts formula:

$$-2\oint_\Sigma H \underline{n} \psi \, d\sigma = \oint_\Sigma \delta\psi \, d\sigma$$

for all $\psi \in C_c^1(U)$, where U is a (three-dimensional) neighborhood of Σ and δ denotes the tangential component of the gradient: $\delta\psi = \nabla\psi - (\nabla\psi \cdot \underline{n})\underline{n}$. From (2.5) and

(2.7) we infer that this necessary condition is satisfied by the prescribed mean curvature in (2.8).

The result of the paper is contained in the following

THEOREM

Let $|\rho_1-\rho_2|$ be small. Then there exists a unique solution $(\underline{v},p,\underline{u},q,\Sigma,\underline{\gamma})$ of (2.1) - (2.9) with the regularity properties:

$$\underline{v} \in C^{2+\alpha}(\mathcal{E} \cup \Sigma) \ , \ \underline{u} \in C^{2+\alpha}(\Omega \cup \Sigma)$$
$$p \in C^{1+\alpha}(\mathcal{E} \cup \Sigma) \ , \ q \in C^{1+\alpha}(\Omega \cup \Sigma)$$
$$\Sigma \in C^{3+\alpha} \ .$$

The solution is axially symmetric. The problem is also solvable for some exterior force densities.

REMARK

Once a regular solution is established we deduce higher regularity $\underline{v} \in C^{k+\alpha}$ etc. from the fact that the forces are regular; this follows immediately from classical regularity theorems for the Navier-Stokes equations and the equation for surfaces of prescribed mean curvature.

The existence proof consists of an approximation procedure similar to the one we used in [2] to study the flow of two viscous fluids which are separated by a capillary surface but which are subject to (otherwise arbitrary) exterior forces such that the viscous force on Σ vanishes. It means that in [2] $\underline{\gamma}$ is not an unknown of the problem but vanishes a priori. The method we use here to determine $\underline{\gamma}$ is due to Weinberger [10].

For $\rho_1-\rho_2 = 0$ the solution to (2.1) - (2.9) is obviously $\underline{v} \equiv 0$, $\underline{u} \equiv 0$, $p = c_1$, $q = c_2$, $\underline{\gamma} = 0$; Ω is the ball of radius $R = (3V/4\pi)^{1/3}$, and the constants c_1 and c_2 satisfy the relation (2.7): $2\kappa R^{-1} = c_1-c_2$. We call this solution $s_o \equiv (\underline{v}_o, p_o, \underline{u}_o, q_o, \Sigma_o, \underline{\gamma}_o)$ and construct s_{m+1} from s_m in the following way.

From $\underline{v}_m, p_m, \underline{u}_m, q_m$, the solution to the Navier-Stokes equations (2.1), (2.3) in $\Omega_{m-1}, \mathcal{E}_{m-1}$ with boundary conditions (2.2), (2.5), (2.6), (2.7), we get the new interface Σ_m exactly as in [2] §7 by the variational problem

$$(2.10) \qquad \oint_S \sqrt{g}\,\sqrt{1+|\mathcal{D}u|^2} + \frac{2}{3}\oint_S h^* u \sqrt{g}\, d\xi \to \min$$

in $\mathcal{E} = BV_R(S) \cap \left\{\frac{1}{3}\oint_S u\sqrt{g}\, d\xi = V\right\} \setminus \operatorname{span}[u_1, u_2, u_3]$; u_i are the eigenfunctions to the Laplace-Beltrami operator on the unit sphere S to the eigenvalue 2. Here Σ_m is described as a graph over the sphere S, $\Sigma_m = \{x = (\xi, u_m(\xi)) : u_m : S \to \mathbb{R}\}$ and for the mean curvature h_m^* we have to insert

$$(2.11) \qquad h_m^*(\xi) = \underline{n}(\xi, u_{m-1}(\xi)) \cdot \Big\{\underline{\underline{T}}\Big(\underline{v}_m(\xi, u_{m-1}(\xi)), p_m(\xi, u_{m-1}(\xi))\Big) - \underline{\underline{T}}\Big(\underline{u}_m(\xi, u_{m-1}(\xi)), q_m(\xi, u_{m-1}(\xi))\Big)\Big\} \cdot \underline{n}(\xi, u_{m-1}(\xi)).$$

To construct $\underline{v}_{m+1}, p_{m+1}, \underline{u}_{m+1}, q_{m+1}, \underline{\gamma}_{m+1}$ we regard the underlying domains Ω_{m+1} and $\mathcal{E}_{m+1}$ as fixed, and solve the Navier-Stokes equations there with boundary conditions (2.2), (2.5), (2.6), (2.7). As in (2.5) the difference of the boundary values is prescribed instead of the values for $\underline{v}_{m+1}, \underline{u}_{m+1}, \underline{\underline{T}}(\underline{v}_{m+1}), \underline{\underline{T}}(\underline{u}_{m+1})$ themselves we will use the following approximation procedure. To simplify the notation we let Δ be the interface that separates $\mathbb{R}^3$ into D and E. Then we condsider in D and E resp. the boundary value problems

$$(E;1) \quad \begin{cases} -\mu_2 \Delta \underline{V}_\ell + \nabla P_\ell + \rho_2((\underline{V}_\ell - \underline{\Gamma}_\ell)\cdot\nabla)\underline{V}_\ell = \rho_2 \underline{g} \\ \qquad\qquad\qquad\qquad \nabla\cdot\underline{V}_\ell = 0 \end{cases} \text{ in } E$$
$$\quad \underline{V}_\ell(x) \to 0 \quad \text{as} \quad |x| \to \infty$$
$$\quad \underline{V}_\ell\cdot\underline{n} = \underline{\Gamma}_\ell\cdot\underline{n};\ \underline{\tau}_k\cdot\underline{\underline{T}}(\underline{V}_\ell)\cdot\underline{n} = \alpha_{k\ell} \text{ on } \Gamma;\ \oint_\Delta(\underline{\underline{T}}(\underline{V}_\ell) - \underline{\underline{A}})\cdot\underline{n}\, d\sigma = \underline{G}$$

where $G = (\rho_1-\rho_2)V\underline{g}$, and $\underline{\tau}_k$ are the tangents at Δ.

$$(D;1)\begin{cases} -\mu_1\Delta\underline{U}_\ell + \nabla Q_\ell + \rho_1((\underline{U}_\ell-\underline{\Lambda}_\ell)\cdot\nabla)\underline{U}_\ell = \rho_1\underline{g},\ \nabla\cdot\underline{U}_\ell = 0 & \text{in } D \\ \underline{U}_\ell\cdot\underline{n} = \underline{\Lambda}_\ell\cdot\underline{n}\ ,\ \underline{U}_\ell\cdot\underline{\tau}_k = \beta_{k\ell} & \text{on } \Delta \end{cases}$$

(E;1) describes the steady fall of a body D under its own weight but with boundary conditions and an additional force $\oint_\Delta \underline{\underline{A}}\cdot\underline{n}\,d\sigma$ which correspond to the behavior of fluid drops rather than rigid bodies.

If we set

$$(2.12)\begin{cases} \alpha_{k0} = 0\ ,\ \underline{\underline{A}}_0 = 0\ ,\ \underline{\Lambda}_\ell = \underline{\Gamma}_\ell\ ,\ \beta_{k\ell} = \underline{V}_\ell\cdot\underline{\tau}_k \\ \alpha_{k\ell} = \underline{\tau}_k\cdot\underline{\underline{T}}(\underline{U}_{\ell-1})\cdot\underline{n}\ ,\ \underline{\underline{A}}_\ell = \underline{\underline{T}}(\underline{U}_{\ell-1},Q_{\ell-1}) \end{cases}$$

we get a sequence of functions $\{\underline{V}_\ell,P_\ell,\underline{U}_\ell,Q_\ell,\underline{\Gamma}_\ell\}$ which converges to a solution $(\underline{V},P,\underline{U},Q,\underline{\Gamma})$ of (2.1), (2.3) in E and D with (2.2), (2.5), (2.6), (2.7) on Δ. Choosing $\Delta = \Sigma_m$ and applying this procedure for every $m \in \mathbb{N}$, we get $(\underline{v}_{m+1},p_{m+1},\underline{u}_{m+1},q_{m+1},\Sigma_{m+1},\underline{\gamma}_{m+1})$, the new element s_{m+1} of the approximating sequence to our original problem. We show that $T: s_m \to s_{m+1}$ is a contraction by proving continuous dependence of the solutions $(\underline{v}_m,p_m)$, $(\underline{u}_m,q_m)$ and Σ_m on the data.

The method of proof parallels [2] §7 except for the boundary-value problem (E,1), which is essentially contained in H.F. Weinberger's contribution [10]. In this paper two different problems for the steady fall of bodies in a fluid are investigated: either the shape and a downward orientation of the body B are given and one seeks a position of the center of mass which will result in a steady falling motion with the given downward orientation (the body is assumed to be hollow such that masses can be moved inside of it); or the mass distribution and the shape of B are given and one seeks a steady motion with an orientation to be determined. In both cases the body will in general undergo both translation and rotation about the vertical axis. To the latter problem several types of symmetric bodies are considered, cf. [10] §6; so if B is axially symmetric with respect to the x^3-axis and if the center of gravity lies on this axis then there is a solution without rotation; the velocity field around B is axially symmetric, and the body moves in $(-x^3)$-direction. For physical reasons, cf. (2.5), one cannot expect that a drop will rotate, and therefore we will perform the successive approximations in the class of axially symmetric functions such that Weinberger's theorem can be applied. To do so one needs a minor modification of this theorem which is due to the inhomogeneous boundary conditions in (E;1).

3. EXISTENCE AND UNIQUENESS

To solve the boundary value problems (E;1) and (D;1) we first have to transform the boundary conditions into homogeneous ones such that all data remain in the class of axially symmetric functions.

LEMMA 1

Let Δ be an axially symmetric surface as in §2, D the compact domain with $\partial D = \Delta$, $E = \mathbb{R}^3 \setminus \bar{D}$. Let $A_\sigma(D)$ and $A_\sigma(E)$ denote the class of solenoidal functions $\underline{\phi}$ which are axially symmetric and which satisfy $\underline{\phi} \cdot \underline{n} = \underline{c} \cdot \underline{n}$ on Δ, $c = \text{const}$.

(i) To every $\underline{V} \in A_\sigma(E) \cap H_2^1(E)$ there exists a vector field $\underline{U}_0 \in A_\sigma(D) \cap H_2^1(D)$ with support in a neighborhood of Δ such that

$$(3.1) \quad \underline{U}_0 = \underline{V} \quad \text{on} \quad \Delta$$

$$(3.2) \quad \| \underline{U}_0 \|_{H_2^1(D)} \leq C(\Delta) \| \underline{V} \|_{H_2^1(E)} .$$

(ii) To every $\underline{U} \in A_\sigma(D) \cap H_2^1(D)$ there exists a vector field $\underline{V}_0 \in A_\sigma(E) \cap H_2^1(E)$ with support in a neighborhood of Δ such that

(3.3) $\underline{V}_0 \cdot \underline{n} = \underline{U} \cdot \underline{n}\ ,\ \underline{\tau}_k \cdot \underline{\underline{D}}(\underline{V}_0) \cdot \underline{n} = \underline{\tau}_k \cdot \underline{\underline{D}}(\underline{U}) \cdot \underline{n}$ on Δ

(3.4) $\| \underline{V}_0 \|_{H_2^1(E)} \leq C(\Delta) \| \underline{U} \|_{H_2^1(D)}$

PROOF

In a neighborhood of Δ we choose orthogonal coordinates α^1, α^2, α^3 such that Δ is given by $\alpha^3 = 0$ and the unit vector $(0,0,1)$ coincides with the exterior normal at Δ; α^2 is chosen to be the azimuth, hence we require the functions $\underline{U}_0$ and $\underline{V}_0$ to be independent of α^2.

If the α^2-components of the data $\underline{V}$ and $\underline{U}$ vanish the problem reduces to extending the boundary data in the plane, and the proof of (i) is well known. For (ii) we refer to [2] lemma 3. This condition will be fulfilled in the proof of the theorem because the data solve the Navier-Stokes system under symmetry conditions for which theorem 5 in Weinberger [10] applies.

If we allow however an exterior force $\underline{f}$ which depends only on the distance from the x^3-axis and is not necessarily symmetric with respect to reflection in the (x^2,x^3)-plane, $\underline{U}$ and $\underline{V}$ will not have this symmetry either. Then we proceed as follows. For $\alpha^3 < 0$ we define $\underline{U}_0$ to be

$$U_0^1(\alpha^1,\alpha^2,\alpha^3) = V^1(\alpha^1,\alpha^2,-\alpha^3\ c\ (\alpha^1,\alpha^2,\alpha^3))$$
$$U_0^2(\alpha^1,\alpha^2,\alpha^3) = V^2(\alpha^1,\alpha^2,-\alpha^3\ c\ (\alpha^1,\alpha^2,\alpha^3))$$
$$U_0^3(\alpha^1,\alpha^2,\alpha^3) = -V^3(\alpha^1,\alpha^2,-\alpha^3\ c\ (\alpha^1,\alpha^2,\alpha^3))$$

where we assume that $V^3 = \underline{V} \cdot \underline{n}$ vanishes on Δ.

V^i are the α^i-components of the vector $\underline{V}$, and the definition holds in a neighborhood of Δ only. Now we determine the function c to make $\underline{U}_0$ solenoidal; if Δ were a part of the plane $\{x^3 = 0\}$ one could choose $c \equiv 1$.

Let $A_i^2 = \sum_{j=1}^{3} (\partial x^j/\partial\alpha^i)^2$, then the divergence operator in the α-coordinates reads as

$$\mathrm{Div}\,\underline{V} = A_1A_2A_3\left\{\frac{\partial}{\partial\alpha^1}\left(\frac{V^1}{A_2A_3}\right) + \frac{\partial}{\partial\alpha^2}\left(\frac{V^2}{A_1A_3}\right) + \frac{\partial}{\partial\alpha^3}\left(\frac{V^3}{A_1A_2}\right)\right\}$$

which gives for $\underline{U}_0$

$$\begin{aligned}\mathrm{Div}\,\underline{U}_0 = A_1A_2A_3\Big\{&\left(\frac{\partial}{\partial\alpha^1}\,\frac{1}{A_2A_3}\right)V^1 + \frac{1}{A_2A_3}(V^1{}_{,1}-V^1{}_{,3}c_{,1})\\ &+\left(\frac{\partial}{\partial\alpha^2}\,\frac{1}{A_1A_3}\right)V^2 + \frac{1}{A_1A_3}(V^2{}_{,2}-V^2{}_{,3}c_{,2})\\ &-\left(\frac{\partial}{\partial\alpha^3}\,\frac{1}{A_1A_2}\right)V^3 + \frac{1}{A_1A_2}\,V^3{}_{,3}c_{,3}\Big\}\end{aligned}$$

The subscripts $,i$ denote differentiation with respect to α^i. This quasilinear equation has a unique solution c in $-2\varepsilon < \alpha^3 < 0$ with initial data $c \equiv 1$ on Δ because $\mathrm{Div}\,\underline{V} = 0$ for $\alpha^3 > 0$ and because $|A_i(\alpha^1,\alpha^2,\alpha^3)-A_i(\alpha^1,\alpha^2,-\alpha^3)|$ is small.

If the coefficients do not depend on α^2 the solution c will be independent, too. The vector field $\underline{U}_0$ will be cut off such that it vanishes for $\alpha^3 = -2\varepsilon$. Consider a scalar function $\eta \in C^\infty(\mathbb{R}^3) \cap A(\mathbb{R}^3)$ (where A denotes the set of axially symmetric functions) such that

$$\eta(\alpha) \equiv 1 \quad \forall\alpha \in N_\varepsilon = \{(\alpha^1,\alpha^2,\alpha^3): -\varepsilon < \alpha^3 < \varepsilon\}$$

$$\eta(\alpha) \equiv 0 \quad \forall\alpha \text{ outside of } N_{2\varepsilon}$$

Clearly $\eta\underline{U}_0 \in A$ and $\mathrm{supp}\,\eta\underline{U}_0 \subset N_{2\varepsilon}^- = N_{2\varepsilon} \cap \overline{D}$. Let $\underline{U}^*$ be the unique solution to

$$(3.5)\qquad \begin{cases}\mathrm{div}\,\underline{U}^* = \mathrm{div}(\eta\underline{U}_0) \equiv \underline{U}_0\cdot\nabla\eta & \text{in } N_{2\varepsilon}^-\setminus N_\varepsilon^-\\ \underline{U}^* = 0 & \text{on } \partial(N_{2\varepsilon}^-\setminus N_\varepsilon^-)\end{cases}$$

which satisfies the estimate

$$(3.6) \qquad \int_{N^-_{2\varepsilon} \setminus N^-_{\varepsilon}} |\nabla \underline{U}^*|^2 \, d\alpha \le C \int_{N^-_{2\varepsilon} \setminus N^-_{\varepsilon}} |\underline{U}_0 \cdot \nabla \eta|^2 \, d\alpha$$

cf. M. Giaquinta - G. Modica [5] theorem 0.3. Together with $\underline{U}^*$ its derivative $\underline{U}_2 = \frac{\partial}{\partial \alpha^2} \underline{U}^*$ is an element of $(\ker \operatorname{div})^{\perp}$; by definition of α^2 also $\underline{U}_2$ vanishes on Δ, and $\underline{U}_2$ is therefore the unique solution of (3.5) to the data $\frac{\partial}{\partial \alpha^2}(\underline{V} \cdot \nabla \eta)$. Hence $\underline{U}^* \in A$ and $\eta \underline{U}_0 - \underline{U}^*$ (which we call $\underline{U}_0$ again) is the vector field which satisfies (3.1), (3.2).

To show (ii) we proceed in the same way and define for $\alpha^3 > 0$

$$V_0^1(\alpha^1,\alpha^2,\alpha^3) = -U^1(\alpha^1,\alpha^2,-\alpha^3 d(\alpha^1,\alpha^2,\alpha^3))$$
$$V_0^2(\alpha^1,\alpha^2,\alpha^3) = -U^2(\alpha^1,\alpha^2,-\alpha^3 d(\alpha^1,\alpha^2,\alpha^3))$$
$$V_0^3(\alpha^1,\alpha^2,\alpha^3) = U^3(\alpha^1,\alpha^2,-\alpha^3 d(\alpha^1,\alpha^2,\alpha^3))$$

To solve (D;1) and (E;1) we introduce the following function spaces.

DEFINITION 1

Let $\kappa'(E)$ be the space of smooth, solenoidal vector fields $\underline{\varphi}$ in E which vanish outside $B_R(0)$, R large enough and have boundary values of the form $\underline{\varphi} \cdot \underline{n} = \underline{\phi} \cdot \underline{n}$, $\underline{\phi} = (0,0,\phi) = \text{const}$; $\kappa'_A(E)$ denotes the supspace of vector fields which are axially symmetric with respect to the x^3-axis. The closure of $\kappa'(E)$ and $\kappa'_A(E)$ with respect to the norm

$$(3.7) \qquad \mathcal{D}(\underline{\varphi},\underline{\varphi}) \equiv \|\underline{\varphi}\|^2 = \int_E \underline{\underline{D}}(\underline{\varphi}) : \underline{\underline{D}}(\underline{\varphi}) \, dx$$

is denoted by $\kappa(E)$ and $\kappa_A(E)$ resp.

$\mathcal{H}'(D)$ consists of all smooth, solenoidal vector fields $\underline{\psi}$ with compact support in D, and $\mathcal{H}'_A(D)$ contains the axially symmetric functions. $\mathcal{H}(D)$ and $\mathcal{H}_A(D)$ are their closures under the norm (3.7).

DEFINITION 2

Let $\underline{V}_0 \in A_\sigma(E) \cap H_2^1(E)$ be an extension of the boundary data:

$$\underline{V}_0 \cdot \underline{n} = 0 \quad , \quad \underline{\tau}_k \cdot \underline{\underline{T}}(\underline{V}_0) \cdot \underline{n} = \alpha_k \qquad \text{on} \quad \Delta \quad .$$

Then $\underline{W} = \underline{V} - \underline{V}_0$ is called a weak solution to $(E;\circ)$ with data α_k, $\underline{\underline{A}}$, $\underline{g}$ if $\underline{W} \in \kappa_A(E)$ and if for all $\underline{\varphi} \in \kappa'(E)$ there holds

$$4\mu_2 \mathcal{D}(\underline{W},\underline{\varphi}) - \rho_2 \int_E \underline{W} \cdot \nabla\underline{\varphi} \cdot (\underline{W}-\underline{\Gamma})\,dx = -4\mu_2 \mathcal{D}(\underline{V}_o,\underline{\varphi})$$
$$+ \rho_2 \int_E \underline{\varphi} \cdot [(\underline{W}\cdot\nabla)\underline{V}_0 + (\underline{V}_0\cdot\nabla)\underline{W}]\,dx$$
$$+ \underline{\phi} \cdot \underline{G}_0 + \oint_\Delta (\underline{\varphi}+\underline{\phi}) \cdot \underline{\tau}_k\, \alpha_k\, d\sigma$$

where $\underline{\phi}$ is the constant vector such that $\underline{\varphi} \cdot \underline{n} = \underline{\phi} \cdot \underline{n}$, and $\underline{G}_0 = -\rho_2 \underline{g}$ meas $D + \underline{G} + \oint_\Delta \underline{\underline{A}} \cdot \underline{n}\, d\sigma$.

DEFINITION 3

Let $\underline{U}_0 \in A_\sigma(D) \cap H_2^1(D)$ be an extension of the boundary data:

$$\underline{U}_0 \cdot \underline{n} = \underline{\Lambda} \cdot \underline{n} \quad , \quad \underline{U}_0 \cdot \underline{\tau}_k = \beta_k \qquad \text{on} \quad \Delta \ .$$

Then $\underline{Z} = \underline{U} - \underline{U}_0$ is called a weak solution to $(D;\circ)$ with data $\underline{\Lambda} =$ const, β_k, $\underline{g}$ if $\underline{Z} \in H_A(D)$ and if for all $\underline{\psi} \in H'(D)$ there holds

$$4\mu_1 \mathcal{D}(\underline{Z},\underline{\psi}) - \rho_1 \int_D \underline{Z} \cdot \nabla\underline{\psi} \cdot (\underline{Z}-\underline{\Lambda})\,dx - \rho_1 \int \underline{g} \cdot \underline{\psi}\, dx$$
(3.9)
$$= -4\mu_1 \mathcal{D}(\underline{U}_0,\underline{\psi}) + \rho_1 \int_D \underline{\psi} \cdot [(\underline{U}_0\cdot\nabla)\underline{Z} - (\underline{Z}\cdot\nabla)\underline{U}_0]\,dx \ .$$

The definition of a weak solution for the Navier-Stokes equations with a Dirichlet condition on the boundary is well known whereas definition 2 needs a short explanation. If $\underline{V}$ solves (E;$\cdot$) then $\underline{W}$ satisfies

$$(3.10)\quad \begin{aligned} -\mu_2\Delta\underline{W} + \nabla P + \rho_2((\underline{W}-\underline{\Gamma})\cdot\nabla)\underline{W} &= \rho_2\underline{g} + \mu_2\Delta\underline{V}_0 \\ &+ \rho_2(\underline{W}\cdot\nabla)\underline{V}_0 + \rho_2(\underline{V}_o\cdot\nabla)\underline{W} \end{aligned}$$

and on the boundary

$$\underline{W}\cdot\underline{n} = 0 \quad , \quad \underline{\tau}_k\cdot\underline{\underline{T}}(\underline{W})\cdot\underline{n} = 0$$

If we multiply (3.10) by $\underline{\varphi}\in\kappa'(E)$ and integrate over E we get

$$\begin{aligned} 4\mu_2\mathcal{D}(\underline{W},\underline{\varphi}) + \rho_2\int_E \underline{W}\cdot\nabla\underline{\varphi}\cdot(\underline{W}-\underline{\Gamma})dx &= \int_E \rho_2\underline{g}\cdot\underline{\varphi}\,dx \\ &- 4\mu_2\mathcal{D}(\underline{V}_0,\underline{\varphi}) + \rho_2\int_E \underline{\varphi}\cdot[(\underline{W}\cdot\nabla)\underline{V}_0 + (\underline{V}_0\cdot\nabla)\underline{W}]\,dx \\ &+ \oint_\Delta \underline{\varphi}\cdot\underline{\underline{T}}(\underline{W},P)\cdot\underline{n}\,d\sigma + \oint_\Delta \underline{\varphi}\cdot\underline{\underline{T}}(\underline{V}_0,0)\cdot\underline{n}\,d\sigma \end{aligned}$$

We have

$$\begin{aligned} \rho_2\int_E \underline{\varphi}\cdot\underline{g}\,dx &= \rho_2\int_E \underline{\varphi}\cdot\nabla(\underline{g}\cdot x)\,dx \\ &= \rho_2\oint_\Delta (\underline{g}\cdot x)(\underline{\varphi}\cdot\underline{n})\,d\sigma \\ &= \rho_2\oint_\Delta (\underline{g}\cdot x)(\underline{\phi}\cdot\underline{n})\,d\sigma \\ &= -\rho_2\,\underline{\phi}\cdot\int_D \underline{g}\,dx \\ &= -\rho_2\,\underline{\phi}\cdot\underline{g}\text{ meas } D \end{aligned}$$

$$\begin{aligned} \oint_\Delta \underline{\varphi}\cdot\underline{\underline{T}}(\underline{W},P)\cdot\underline{n}\,d\sigma &+ \oint_\Delta \underline{\varphi}\cdot\underline{\underline{T}}(\underline{V}_0,0)\cdot\underline{n}\,d\sigma \\ &= \oint_\Delta \underline{\varphi}\cdot\underline{\underline{T}}(\underline{V},P)\cdot\underline{n}\,d\sigma \\ &= \oint_\Delta (\underline{\varphi}+\underline{\varphi}_0)\cdot\underline{\underline{T}}(\underline{V},P)\cdot\underline{n}\,d\sigma - \oint_\Delta \underline{\varphi}_0\cdot\underline{\underline{T}}(\underline{V},P)\cdot\underline{n}\,d\sigma \end{aligned}$$

$$(3.11) \quad = \underline{\phi} \cdot \oint_{\Delta} \underline{\underline{T}}(\underline{V},P) \cdot \underline{n} \, d\sigma - \oint_{\Delta} \sum_{k=1}^{2} \underline{\varphi}_0 \cdot \underline{\tau}_k \, \alpha_k \, d\sigma$$

because $\underline{\varphi}_0 = \underline{\phi} - \underline{\varphi}$ is a vector field which is tangential to Δ: Using (2.12) we get the definition of weak solution of $(E;\cdot)$.

For $\underline{\varphi} \in \kappa(E)$ its boundary values are of class $L_2(\Delta)$, such that for regular α_k the last integral in (3.11) is well defined. For the constant $\underline{\phi}$ there holds

LEMMA 2

Let $\underline{\varphi} \in \kappa'(E)$ and let $\underline{\phi}$ be the constant such that $\underline{\phi} \cdot \underline{n} = \underline{\varphi} \cdot \underline{n}$. Then

$$(3.12) \quad |\underline{\phi}|^2 \quad C(\Delta) \int_E |\nabla \underline{\varphi}|^2 \, dx$$

where C is independent of $\underline{\varphi}$.

The lemma shows that also for functions in $\kappa(E)$ their normal component is the inner product of a constant vector and the normal itself. That boundary values of the form $\underline{\phi} + \underline{\Psi} \wedge x$ are preserved under completion has been shown by Weinberger [10]. Here a similar, but simpler proposition follows along the same lines.

PROOF

Let $h: E \to \mathbb{R}$ be the solution of

$$\Delta h = 0 \quad \text{in } E, \qquad h(x) \to 0 \quad \text{as } |x| \to \infty$$
$$h = \underline{\phi} \cdot \underline{n} \quad \text{on } \Delta .$$

Clearly h has finite Dirichlet integral $D(h) = \int_E |\nabla h|^2 dx$, and the $L_2(\Delta)$-norm of h can be bounded in terms of $D(h)$

$$\oint_{\Delta} |\underline{\Phi} \cdot \underline{n}|^2 \, d\sigma \leq C_1 \, D(h)$$

With a constant depending only on Δ we get

$$|\underline{\Phi}|^2 \leq C_2 \oint_\Delta |\underline{\Phi} \cdot \underline{n}|^2 \, d\sigma \quad .$$

If we choose a smooth vector field $\underline{N}$ that coincides with the normal on Δ and vanishes outside a large sphere, the function $\underline{N} \cdot \underline{\varphi}$ has the same boundary values on Δ than h. Therefore Dirichlet's principle leads to

$$|\underline{\Phi}|^2 \leq C_1 C_2 \, D(h) \leq C_1 C_2 \, D(\underline{N} \cdot \underline{\varphi}) \leq C \, D(\underline{\varphi}) \quad ,$$

and the lemma is proved.

LEMMA 3

The boundary value problem (E;•) has a unique solution $\underline{V} \in \kappa_A(E) \cap C^{2+\alpha}(\overline{E})$, $P \in C^{1+\alpha}(\overline{E})$ provided the data in (2.12) are derived from a regular vector field $\underline{U} \in H_A(D)$, and if $|\underline{g}|$, $\| \underline{U} \|_{C^{2+\alpha}(\overline{D})}$, $\| Q \|_{C^{1+\alpha}(\overline{D})}$ are small.

PROOF

The existence of a solution $\underline{V}$ follows immediately from [10] Theorem 4 and the symmetry consideration on p. 439. As in [10] (6.1) we show that there is a $\underline{V} \in \kappa_A(E)$ such that (3.8) is satisfied for all $\underline{\varphi} \in \kappa_A'(E)$ because for smooth α_k the boundary integral is a bounded functional. If R denotes rotation through some angle θ, S reflection in the (x^2,x^3)-plane, and $T\underline{V}(x) = R^{-1}\underline{V}(Rx)$, $S\underline{V}(x) = S^{-1}\underline{V}(Sx)$, then all expressions in (3.8) which are of the same type than the ones in [10] (6.1) do not change if $\underline{\varphi}$ is replaced by $T\underline{\varphi}$ or $S\underline{\varphi}$. Choosing instead of [10] (6.2) the vector field $\frac{1}{2}(\underline{\psi} + S\underline{\psi})$ we find that (3.8) is satisfied for all $\underline{\varphi} \in \kappa'(E)$.

The prescribed boundary values are of mixed type, and H_2^2-regularity up to the boundary is well known. As Δ does not rotate a fundamental solution to the Oseen linearization of (E;·) can be given explicitely; therefore K.I. Babenko's result [1] can be applied, and the solution is known to belong to the class of PR-solutions:

$$(3.13)\qquad |\underline{V}(x)-\underline{\Gamma}| < c|x|^{-\frac{1}{2}-\varepsilon}$$

This class was introduced by R. Finn, and as the H^2_2-estimate gives a bound for the $C^{0+\frac{1}{2}}$-norm of $\underline{V}$ on the boundary we can use Finn's representation formulas for the Dirichlet problem, cf. [3]. Higher regularity as well as uniqueness of the solution are then well known.

REMARK

(D;·) is a Dirichlet problem in a bounded domain, and therefore the existence and uniqueness of a solution $\underline{U} \in C^{2+\alpha}(\overline{D}) \cap A \quad Q \in C^{1+\alpha}(\overline{D}) \cap A$ follows as for (E;·).

LEMMA 4

For given Δ, $\underline{g}$, ρ_1, ρ_2, μ_1, μ_2 there exists a sequence of functions $\{\underline{V}_\ell, P_\ell, \underline{U}_\ell, Q_\ell\}$ which are solutions to (E;l), (D;l), (2.12). If the data are small this sequence converges to an axially symmetric solution of (2.1) - (2.7) in given domains D and E.

The proof follows as in [2] §7 because the difference of two solutions in E or D resp. can be estimated by the difference of their boundary values.

PROOF OF THE THEOREM

First we note that the approximation procedure that was defined in §2 can be carried out in the class of axially symmetric functions. As far as the velocity fields and the pressures are concerned this is contained in lemma 3. That Σ_m is axially symmetric for $h^*_m \in A$ follows from the minimum property of $\Sigma_m = \{(\xi, u_m(\xi)) : u_m : S \to \mathbb{R}\}$. Choose coordinates $\xi = (\xi^1, \xi^2)$ on S such that ξ^2 is the azimuth and associate to every (smooth) function $u: S \to \mathbb{R}$ its symmetrization u_0 by

$$\int_0^{2\pi} u^3(\xi^1,\xi^2)\,d\xi^2 = \int_0^{2\pi} u_0^3(\xi^1)\,d\xi^2$$

By construction u_0 encludes the same volume and

$$\oint_S h^*(\xi^1)u^3(\xi)\sqrt{g^*}\,d\xi = \oint_S h^*(\xi^1)u_0^3(\xi^1)\sqrt{g^*}\,d\xi$$

for axially symmetric functions h^*. As $A(u)$ is the perimeter of $\Omega = \{(\xi,\rho): \xi \in S,\ 0 \leq \rho < u(\xi)\}$, we get $A(u) \geq A(u_0)$ hence u_0 is in $BV_R(S)$ again. So we finally get for the functional

$$I(u) = A(u) + \frac{2}{3}\oint h^* u^3 \sqrt{g^*}\,d\xi, \quad A(u) = \oint_S \sqrt{g}\sqrt{1+|Du|^2}$$

that its value decreases if u is replaced by u_0. As u is the uniquely determined minimum it must coincide with u_0, i.e. u is axially symmetric.

Hence there exists a sequence $\{\underline{v}_m, p_m, \underline{u}_m, q_m, \Sigma_m, \underline{\gamma}_m\}$, where $(\underline{v}_m, p_m, \underline{u}_m, q_m, \underline{\gamma}_m)$ is the unique solution to (2.1) - (2.7) in $\mathcal{E}_{m-1}$ and Ω_{m-1}, and Σ_m is the interface given by (2.8), (2.9) with $\underline{u} = \underline{u}_m$, $\underline{v} = \underline{v}_m$, $p = p_m$, $q = q_m$. The convergence of this series follows as in [2]. It is perhaps worth to mention that the continuous dependence of $\underline{v}_m$, p_m (and similarly of $\underline{u}_m, q_m$) on the underlying domain $_{m-1}$ can be shown without referring to propositions on general elliptic systems. Once it is shown that this can be done in the H_2^2-norm we can use the representations of Finn [3]

$$\underline{v}_m(x) = \oint_{\Sigma_{m-1}} \underline{\underline{g}}(x,y)\cdot\underline{v}_m(y)\,d\sigma_y + \int_{\mathcal{E}_{m-1}} \nabla\underline{\underline{g}}(x,y)\cdot\underline{v}_m(y)\cdot\underline{v}_m(y)\,dy$$

and apply to it the results of Lichtenstein [6], who studied in several papers the behavior of such integrals when Σ_{m-1}

and $\mathcal{E}_{m-1}$ vary. Depending on the singularity of $\nabla \underline{\underline{g}}$ one gets a bound in the $C^{1+\alpha}$-norm which can be used to estimate

$$\underline{v}_m(x) = \oint_{\Sigma_{m-1}} \underline{\underline{g}}(x,y)\, \underline{v}_m(y)\, d\sigma_y + \int_{m-1} \underline{\underline{g}}(x,y) \cdot [(\underline{v}(y) \cdot \nabla)\underline{v}(y)]dy$$

in the $C^{2+\alpha}$-norm finally.

REMARK

Let $\underline{f}$ be an exterior force density which is axially symmetric and depends only on the distance r from the x^3-axis. We assume $\underline{f}$ to be smooth and to influence the flow inside the drop only (e.g. a drop of a charged fluid in a magnetic field). We then have to solve $(D;\ell)$ with $\rho_1 \underline{g} + \underline{f}$ on the right hand side. The problem is no longer invariant with respect to reflection in the (x^2, x^3)-plane, hence the solution satisfies only $\underline{U}(Rx) = R\underline{U}(x)$. This follows again from theorem 4 in [10]. As also such functions can be continued across Δ, cf. lemma 1, the rest of the proof follows as before.

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