On the Navier–Stokes equation with boundary conditions based on vorticity

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Dedicated to the memory of Professor Dr. Erhard Meister.

We treat the Stokes and the Navier–Stokes equation with the conditions $\mathbf{curl}^k\boldsymbol{u}\cdot\boldsymbol{n} = 0$ $(k = 0, 1, 2)$ on the boundary of the flow field. The approach is based on a spectral analysis and properties of operator **curl**.

1 Introduction

Suppose that Ω is a bounded simply connected domain in $\mathbb{R}^3$ which is either a convex polyhedron or its boundary $\partial\Omega$ is of the class $C^{2,1}$. Put $Q_T = \Omega \times (0, T)$.

There exists an extensive theory of the Navier–Stokes equation with the homogeneous Dirichlet boundary condition (= the non–slip boundary condition). In this paper, we consider the Navier–Stokes initial–boundary value problem

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} = -\nabla(p + \tfrac{1}{2}\,|\boldsymbol{u}|^2) + \nu\Delta\boldsymbol{u} + \boldsymbol{f} \quad (\text{in } Q_T), \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad (\text{in } Q_T), \tag{2}$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}) \quad (\text{for } \boldsymbol{x} \in \Omega) \tag{3}$$

with the boundary conditions

$$\mathbf{curl}^k\boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad (k = 0, 1, 2) \quad \text{a.e. on } \partial\Omega \times (0, T). \tag{4}$$

$\boldsymbol{u}$ represents the unknown velocity, p is an associated pressure and the positive constant ν is the coefficient of viscosity.

It is not usual in the theory of boundary value problems to prescribe a condition on the highest derivative of the unknown function on the boundary as it is done in (4). In our case, it is enabled by a special structure of divergence–free vector functions which preserve more information on the normal component than on the tangential component on the boundary. (See R. Temam [28], p. 9, V. Girault and P.–A. Raviart [10], pp. 22–30 or G. P. Galdi [7], pp. 112–115 for more details.) Another interesting question is a physical sense of the boundary conditions (4). Obviously, the first two conditions, corresponding to $k = 0$, 1, coincide with the usual Dirichlet non–slip boundary condition $\boldsymbol{u} = \mathbf{0}$ on $\partial\Omega \times (0, T)$. If we accept that the term $\nu\Delta\boldsymbol{u}$ $(= -\nu\,\mathbf{curl}^2\boldsymbol{u})$ expresses the force due to viscosity then the third boundary condition in (4), i.e. $\mathbf{curl}^2\boldsymbol{u}\cdot\boldsymbol{n}\,|_{\partial\Omega} = 0$, represents the requirement that the normal

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component of the viscous force on the boundary is zero. On the other hand, the boundary conditions (4) do not exclude a non–zero tangential velocity on $\partial\Omega$, however it follows from Lemma 3.3 that the velocity on $\partial\Omega$ represents a potential vector field.

We show that under some assumption on $\boldsymbol{f}$ the boundary conditions (4) naturally induce the same boundary conditions for vorticity in Section 2. We deal with operator **curl** in Section 3. We derive (or re–derive) some properties which we apply in the next sections. In Sections 4 and 5, we show that the basic theory of the Stokes and the Navier–Stokes equations can be extended to the case of the boundary conditions (4). Finally, we treat some problems whose solution is different with the boundary conditions (4) than with the classical non–slip boundary condition in Section 6.

We shall use the following notation:

- $\boldsymbol{n}$ is the outer normal vector on $\partial\Omega$.
- $\|\,.\,\|_r$, respectively $\|\,.\,\|_{m,r}$, is the norm of a scalar– or vector– or tensor–valued function with its components in $L^r(\Omega)$, respectively in $W^{m,r}(\Omega)$.
- $(\,.\,,\,.)_2$, respectively $(\,.\,,\,.)_{1,2}$, is the scalar product of scalar– or vector–valued functions with the components in $L^2(\Omega)$, respectively in $W^{1,2}(\Omega)$.
- $L^2_\sigma(\Omega)^3$ is the closure of $\{\boldsymbol{u} \in C_0^\infty(\Omega)^3;\ \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega\}$ in $L^2(\Omega)^3$. $L^2_\sigma(\Omega)^3$ coincides with $\{\boldsymbol{u} \in L^2(\Omega)^3;\ \operatorname{div}\boldsymbol{u} = 0$ in Ω in the sense of distributions and $(\boldsymbol{u}\cdot\boldsymbol{n})|_{\partial\Omega} = 0$ in the sense of traces$\}$. (See R. Temam [28], p. 15, V. Girault and P.–A. Raviart [10], p. 29 or G. P. Galdi [7], p. 115.)
- P_σ is the orthogonal projection of $L^2(\Omega)^3$ onto $L^2_\sigma(\Omega)^3$.
- $D^1 = \{\boldsymbol{u} \in W^{1,2}(\Omega)^3 \cap L^2_\sigma(\Omega)^3;\ (\mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{n})|_{\partial\Omega} = 0$ in the sense of traces$\}$. D^1 is a closed subspace of $W^{1,2}(\Omega)^3$.
- D^{-1} is the dual to D^1. The duality between the elements of D^{-1} and D^1 is denoted by $\langle\,.\,,\,.\rangle_\Omega$. The norm in D^{-1} is denoted by $\|\,.\,\|_{-1,2}$.
- $A = \mathbf{curl}\,|_{D^1}$ (Thus, $D^1 = D(A)$.)
- $D^2 = \{\boldsymbol{u} \in W^{2,2}(\Omega)^3;\ \nabla\cdot\boldsymbol{u} = 0 \text{ in } \Omega,\ (\mathbf{curl}^k\boldsymbol{u}\cdot\boldsymbol{n})|_{\partial\Omega} = 0\ (k = 0,1,2)$ in the sense of traces$\}$
- $\sigma(A)$ (respectively $\rho(A)$) is the spectrum (respectively the resolvent set) of operator A, as an operator in $L^2_\sigma(\Omega)^3$.

$L^2_\sigma(\Omega)^3$ is a closed subspace of $L^2(\Omega)^3$. The orthogonal complement of $L^2_\sigma(\Omega)^3$ in $L^2(\Omega)^3$ is the space $\{\boldsymbol{v} = \nabla\varphi;\ \varphi \in W^{1,2}(\Omega)\}$ (see R. Temam [28], p. 15, V. Girault and P.–A. Raviart [10], p. 27 or G. P. Galdi [7], p. 99). D^1 is dense in $L^2_\sigma(\Omega)^3$ and $A : D^1 \longrightarrow L^2_\sigma(\Omega)^3$.

2 Natural boundary conditions for vorticity

Denote $\boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{u}$. Applying operator **curl** to the Navier–Stokes equation (1), we obtain the well known equation

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{f} + \nu\Delta\boldsymbol{\omega} \quad (\text{in } Q_T). \tag{5}$$

Moreover, $\boldsymbol{\omega}$ is also divergence–free, i.e.

$$\nabla\cdot\boldsymbol{\omega} = 0 \quad (\text{in } Q_T). \tag{6}$$

If the Navier–Stokes system is considered with the no–slip boundary condition for velocity then we only obtain that $\mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{n} = \boldsymbol{\omega}\cdot\boldsymbol{n} = 0$ on $\partial\Omega\times(0,T)$, but this information is not sufficient in order to formulate a well–posed boundary–value problem for function $\boldsymbol{\omega}$, based on equations (5) and (6).

In this section, we are going to show that if $\boldsymbol{u}$ satisfies the boundary conditions (4) and $\mathbf{curl}\,\boldsymbol{f}\cdot\boldsymbol{n} = 0$ on $\partial\Omega\times(0,T)$ then $\boldsymbol{\omega}$ (if it is smooth enough) also satisfies the boundary conditions (4), i.e.

$$\mathbf{curl}^k\boldsymbol{\omega}\cdot\boldsymbol{n} = 0 \quad (k = 0,1,2) \quad \text{a.e. on } \partial\Omega\times(0,T). \tag{7}$$

The validity of (7) for $k = 0, 1$ directly follows from (4). Thus, we only need to show that $\boldsymbol{\omega}$ satisfies (7) with $k = 2$, i.e. $\mathbf{curl}^2\boldsymbol{\omega}\cdot\boldsymbol{n} = 0$ for $\boldsymbol{x}\in\partial\Omega$. Since $\nu\Delta\boldsymbol{\omega} = -\nu\,\mathbf{curl}^2\boldsymbol{\omega}$ in equation (5), it is sufficient to show that

$$[\,(\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}\,]\cdot\boldsymbol{n} \;=\; 0 \quad \text{a.e. on } \partial\Omega\times(0,T). \tag{8}$$

Let $\boldsymbol{x}_0\in\partial\Omega$. Suppose that $\boldsymbol{x}_0$ does not lie on an edge or that it is not a vertex if Ω is a convex polyhedron. We can assume without loss of generality that $\boldsymbol{x}_0 = [0,0,0]$, the boundary of Ω coincides with the graph of a twice continuously differentiable function $x_3 = f(x_1,x_2)$ which satisfies $f_{,1}(0,0) = f_{,2}(0,0) = 0$ and the points $\boldsymbol{x} = [x_1,x_2,x_3]\in\Omega$ locally satisfy $x_3 < f(x_1,x_2)$ in the neighborhood of $\boldsymbol{x}_0$. Let us show that

$$\omega_{3,1}(\boldsymbol{x}_0,t) \;=\; -\omega_1(\boldsymbol{x}_0,t)\,f_{,11}(0,0) - \omega_2(\boldsymbol{x}_0,t)\,f_{,12}(0,0) \tag{9}$$

at first. Indeed, using the information that $\boldsymbol{u}\cdot\boldsymbol{n} = \boldsymbol{\omega}\cdot\boldsymbol{n} = 0$ for $\boldsymbol{x}\in\partial\Omega$, we obtain:

$$\begin{aligned}
\omega_{3,1}(\boldsymbol{x}_0,t) \;&=\; \lim_{x_3\to 0-}\,\omega_{3,1}(0,0,x_3,t) \;=\; \lim_{x_3\to 0-}\,\frac{d}{dx_1}\,\omega_3(x_1,0,x_3+f(x_1,0),t)\Big|_{x_1=0}\\
&=\; \frac{d}{dx_1}\,\omega_3(x_1,0,f(x_1,0,),t)\Big|_{x_1=0} \;=\; \lim_{x_1\to 0}\,\frac{1}{x_1}\,\Big[\omega_3(x_1,0,f(x_1,0),t) - \omega_3(0,0,0,t)\Big]\\
&=\; \lim_{x_1\to 0}\,\frac{1}{x_1}\,\boldsymbol{\omega}(x_1,0,f(x_1,0),t)\cdot\boldsymbol{n}(0,0,0)\\
&=\; \lim_{x_1\to 0}\,\boldsymbol{\omega}(x_1,0,f(x_1,0),t)\,\frac{\boldsymbol{n}(0,0,0)-\boldsymbol{n}(x_1,0,f(x_1,0))}{x_1}\\
&=\; \boldsymbol{\omega}(\boldsymbol{x}_0,t)\,\frac{d}{dx_1}\,\boldsymbol{n}(x_1,0,f(x_1,0))\Big|_{x_1=0} \;=\; \boldsymbol{\omega}(\boldsymbol{x}_0,t)\cdot\boldsymbol{n}_{,1}(\boldsymbol{x}_0).
\end{aligned} \tag{10}$$

However, $\boldsymbol{n} = (-f_{,1},\,-f_{,2},\,1)/R$ where $R = [f_{,1}^2 + f_{,2}^2 + 1]^{1/2}$ and so it can be calculated that

$$\boldsymbol{n}_{,1}(\boldsymbol{x}_0) \;=\; \Big(-f_{,11}(0,0),\,-f_{12}(0,0),\,0\Big).$$

This, together with (10), gives (9). We can analogously prove that

$$\omega_{3,2}(\boldsymbol{x}_0,t) \;=\; -\omega_1(\boldsymbol{x}_0,t)\,f_{,12}(0,0) - \omega_2(\boldsymbol{x}_0,t)\,f_{,22}(0,0), \tag{11}$$

$$u_{3,j}(\boldsymbol{x}_0,t) \;=\; -u_1(\boldsymbol{x}_0,t)\,f_{,1j}(0,0) - u_2(\boldsymbol{x}_0,t)\,f_{,2j}(0,0) \quad (j=1,2). \tag{12}$$

Substituting (9), (11) and (12) into (8) and using the identities $u_3 = \omega_3 = 0$ and $\boldsymbol{n} = (0,0,1)$ for $\boldsymbol{x} = \boldsymbol{x}_0$, we can show that (8) is satisfied at point $\boldsymbol{x}_0$. The freedom of choice of $\boldsymbol{x}_0$ implies that it holds a.e. on $\partial\Omega\times(0,T)$.

Since equation (5) is essentially of the same nature as equation (1), the theory explained in Sections 4 and 5 provides a possibility to solve the system (5), (6) with the boundary conditions (7).

3 Some properties of operator A

Many authors have already studied or quoted properties of operator **curl** and have dealt with the solution of the equation $\mathbf{curl}\,\boldsymbol{u} = \boldsymbol{g}$. Let us mention e.g. O. A. Ladyzhenskaya and V. A. Solonnikov [15], V. Girault and P.-A. Raviart [10], W. von Wahl [30], W. Borchers and H. Sohr [3], Z. Yosida and Y. Giga [31], A. Mahalov and B. Nicolaenko [21]. In this section, we show that operator A ($= \mathbf{curl}\,|_{D^1}$) is a selfadjoint operator in space $L^2_\sigma(\Omega)^3$ and we discuss the form of its eigenvalues and eigenfunctions. Note that the selfadjointness of the same operator in the case of a smooth bounded simply connected domain was proved by Z. Yosida and Y. Giga in [31], too, and a possibility of a selfadjoint realization of operator **curl** with stress to exterior domains is studied by R. Picard in [24][1].

[1] The authors discovered the results in [31] and [24] after having this paper almost finished.

Lemma 3.1 *There exists $c_1 > 0$ such that if $\boldsymbol{g} \in L^2_\sigma(\Omega)^3$ then the equation*

$$A\boldsymbol{u} \,=\, \boldsymbol{g} \tag{13}$$

has a unique solution $\boldsymbol{u} \in D^1$ such that

$$\|\boldsymbol{u}\|_{1,2} \,\leq\, c_1\, \|\boldsymbol{g}\|_2\,. \tag{14}$$

This lemma follows from Theorem 4.4 quoted in A. Mahalov and B. Nicolaenko [21]. The authors refer to a series of works of O. A. Ladyzhenskaya, V. A. Solonnikov and their co–workers (see e.g. [15]) for the proof and they also mention that the same result was later re–discovered by C. Bardos. Nevertheless, since the proof is instructive and we shall also use some of its ideas later, we include it here, too.

P r o o f. If Ω is a smooth simply connected domain and $\boldsymbol{g} \in L^2_\sigma(\Omega)^3$ is given then due to W. Borchers and H. Sohr [3], Theorem 2.1, the equation $\mathbf{curl}\, \boldsymbol{u}_0 = \boldsymbol{g}$ is solvable in $W^{1,2}_0(\Omega)^3$. Moreover, there exists $c_2 > 0$, independent of $\boldsymbol{g}$, such that

$$\|\boldsymbol{u}_0\|_{1,2} \,\leq\, c_2\, \|\boldsymbol{g}\|_2\,. \tag{15}$$

The Neumann problem

$$\Delta\varphi \,=\, -\nabla\cdot\boldsymbol{u}_0 \quad \text{in } \Omega, \qquad \frac{\partial\varphi}{\partial\boldsymbol{n}}\bigg|_{\partial\Omega} = 0 \tag{16}$$

has a solution $\varphi \in W^{2,2}(\Omega)$ which can be chosen so that

$$\|\varphi\|_{2,2} \,\leq\, c_3\, \|\nabla\boldsymbol{u}_0\|_2\,. \tag{17}$$

The constant c_3 is independent of $\boldsymbol{u}_0$. (See e.g. [10], p. 15.) The function $\boldsymbol{u} = \boldsymbol{u}_0 + \nabla\varphi$ is the solution of equation (1). Estimates (15) and (17) imply (14).

Assume that Ω is a convex polyhedron now. Put $H(\mathbf{curl};\Omega) = \{\boldsymbol{w} \in L^2(\Omega)^3;\ \mathbf{curl}\,\boldsymbol{w} \in L^2(\Omega)^3\}$. Theorem 3.5 in V. Girault and P.–A. Raviart [10], p. 47, says that the problem $\mathbf{curl}\,\boldsymbol{u} = \boldsymbol{g}$, $\operatorname{div}\boldsymbol{u} = 0$, $(\boldsymbol{u}\cdot\boldsymbol{n})|_{\partial\Omega} = 0$ has a solution $\boldsymbol{u}$ in $H(\mathbf{curl};\Omega) \cap L^2_\sigma(\Omega)^3$. However, $H(\mathbf{curl};\Omega) \cap L^2_\sigma(\Omega)^3$ is continuously embedded into $W^{1,2}(\Omega)^3$ ([10], pp. 54–55). Hence $\boldsymbol{u} \in D^1$ and $\boldsymbol{u}$ satisfies equation (1). Theorem 3.9 in [10], p. 55, implies that

$$\|\nabla\boldsymbol{u}\|_2 \,\leq\, c_4\, \|\boldsymbol{g}\|_2 \tag{18}$$

where c_4 is independent of $\boldsymbol{g}$. This estimate, together with the fact that the normal component of $\boldsymbol{u}$ is zero a.e. on $\partial\Omega$, implies (14).

The uniqueness of $\boldsymbol{u}$ can be easily proved by contradiction. □

Corollary 3.2 *There exist constants c_5, $c_6 > 0$ such that*

$$c_5\, \|A\boldsymbol{u}\|_2 \,\leq\, \|\boldsymbol{u}\|_{1,2} \,\leq\, c_6\, \|A\boldsymbol{u}\|_2 \quad \text{for all } \boldsymbol{u} \in D^1. \tag{19}$$

$(A\boldsymbol{u}, A\boldsymbol{v})_2$ is a scalar product, equivalent to $(\boldsymbol{u}, \boldsymbol{v})_{1,2}$, in D^1.

The first estimate in (19) is obvious. The second estimate follows from Lemma 3.1 if we use it with $\boldsymbol{g} = \mathbf{curl}\,\boldsymbol{u}$. The assertion about the scalar product $(A\boldsymbol{u}, A\boldsymbol{v})_2$ is the consequence of the equivalence of the norms $\|A\boldsymbol{u}\|_2$ and $\|\boldsymbol{u}\|_{1,2}$ in D^1.

If the boundary of Ω is smooth then $D^2 = D(A^2)$. (The inclusion $D^2 \subset D(A^2)$ is obvious. In order to prove the inclusion $D(A^2) \subset D^2$, it is sufficient to show that if $\boldsymbol{f} \in D^1$ then $A^{-1}\boldsymbol{f} \in W^{2,2}(\Omega)^3$. However, it is an easy consequence of the identity $W^{2,2}(\Omega)^3 = \{\boldsymbol{v} \in L^2(\Omega)^3;\ \nabla\cdot\boldsymbol{v} \in W^{1,2}(\Omega),\ \mathbf{curl}\,\boldsymbol{v} \in W^{1,2}(\Omega)^3,\ \boldsymbol{v}\cdot\boldsymbol{n} \in W^{3/2,2}(\partial\Omega)\}$, see [10], p. 56.) Moreover, since D^2 is a closed subspace of $W^{2,2}(\Omega)^3$ and A^{-2} maps $L^2_\sigma(\Omega)^3$ onto D^2, A^{-2} is a bounded operator from $L^2_\sigma(\Omega)^3$ into $W^{2,2}(\Omega)^3$. This implies that

$$c_7\, \|A^2\boldsymbol{u}\|_2 \,\leq\, \|\boldsymbol{u}\|_{2,2} \,\leq\, c_8\, \|A^2\boldsymbol{u}\|_2 \quad \text{for all } \boldsymbol{u} \in D^2. \tag{20}$$

Lemma 3.3 *D^1 can be characterized by the identities*

$$D^1 = \Big\{\boldsymbol{u} = \boldsymbol{u}_0 + \nabla\varphi;\ \boldsymbol{u}_0 \in W_0^{1,2}(\Omega)^3,\ \Delta\varphi = -\nabla\cdot\boldsymbol{u}_0 \text{ in } \Omega \text{ and } \partial\varphi/\partial\boldsymbol{n}\Big|_{\partial\Omega} = 0\Big\}, \quad (21)$$

$$D^1 = P_\sigma\, W_0^{1,2}(\Omega)^3. \quad (22)$$

Proof. It is obvious that the right hand side of (21) is included in the left hand side. Let us prove the opposite inclusion. Suppose that $\boldsymbol{u} \in D^1$. The representation of $\boldsymbol{u}$ in the desired form $\boldsymbol{u}_0 + \nabla\varphi$, in the case when Ω is a smooth simply connected domain, was already derived in the first part of the proof of Lemma 3.1. If Ω is a bounded convex polyhedron, we can put $\boldsymbol{g} = \mathbf{curl}\,\boldsymbol{u}$. Then $\boldsymbol{g} \in L^2_\sigma(\Omega)^3$ and the equation $\mathbf{curl}\,\boldsymbol{u}_0 = \boldsymbol{g}$ (for the unknown $\boldsymbol{u}_0$) is solvable in $W_0^{1,2}(\Omega)^3$. (See e.g. [10], pp. 37–39.) Since $\mathbf{curl}(\boldsymbol{u} - \boldsymbol{u}_0) = \mathbf{0}$ in Ω and Ω is simply connected, $\boldsymbol{u} - \boldsymbol{u}_0$ can be expressed in the form of $\nabla\varphi$ where φ has the properties stated in (21).

Projection P_σ can be expressed in the form $P_\sigma = I + \nabla(-\Delta)^{-1}\mathrm{div}$ where the operator $(-\Delta)$ is inverted with the homogeneous Neumann boundary condition. (See e.g. P. L. Lions [19], p. 84.) Identity (22) is an immediate consequence of this representation of P_σ and (21) because $\boldsymbol{u}$ on the right hand side of (21) corresponds to $P_\sigma\boldsymbol{u}_0$. □

Lemma 3.4 *A is a closed and symmetric operator in $L^2_\sigma(\Omega)^3$.*

Proof. The closedness of operator A is an easy consequence of inequalities (19) and the closedness of space D^1 in $W^{1,2}(\Omega)^3$.

Let us prove the symmetry of A. Suppose that $\boldsymbol{u}, \boldsymbol{v} \in D^1$. Then

$$\begin{aligned} \int_\Omega A\boldsymbol{u}\cdot\boldsymbol{v}\,d\boldsymbol{x} &= \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{v}\,d\boldsymbol{x} = \int_\Omega \mathrm{div}(\boldsymbol{u}\times\boldsymbol{v})\,d\boldsymbol{x} + \int_\Omega \boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}\,d\boldsymbol{x} \\ &= \int_{\partial\Omega} \boldsymbol{n}\cdot(\boldsymbol{u}\times\boldsymbol{v})\,dS + \int_\Omega \boldsymbol{u}\cdot A\boldsymbol{v}\,d\boldsymbol{x}. \end{aligned} \quad (23)$$

Using Lemma 3.3, we can express $\boldsymbol{u}$ in the form $\boldsymbol{u}_0 + \nabla\varphi$, where $\boldsymbol{u}_0 \in W_0^{1,2}(\Omega)^3$ and $\nabla\varphi \in W^{1,2}(\Omega)^3$. Similarly, $\boldsymbol{v} = \boldsymbol{v}_0 + \nabla\psi$ for some $\boldsymbol{v}_0 \in W_0^{1,2}(\Omega)^3$ and $\nabla\psi \in W^{1,2}(\Omega)^3$. Then

$$\begin{aligned} \int_{\partial\Omega} \boldsymbol{n}\cdot(\boldsymbol{u}\times\boldsymbol{v})\,dS &= \int_{\partial\Omega} \boldsymbol{n}\cdot(\nabla\varphi\times\nabla\psi)\,dS = \int_\Omega \mathrm{div}(\nabla\varphi\times\nabla\psi)\,d\boldsymbol{x} \\ &= \int_\Omega \Big(\mathbf{curl}\,\nabla\varphi\cdot\nabla\psi - \nabla\varphi\cdot\mathbf{curl}\,\nabla\psi\Big)\,d\boldsymbol{x} = 0 \end{aligned}$$

and the symmetry of operator A follows from (23). □

Lemma 3.5 *$0 \in \rho(A)$, A is a selfadjoint operator in $L^2_\sigma(\Omega)^3$ and the resolvent operator $(\lambda I - A)^{-1}$ is compact in $L^2_\sigma(\Omega)^3$ for all $\lambda \in \rho(A)$.*

Proof. The fact that $0 \in \rho(A)$ follows from Lemma 3.1.

$\rho(A)$ is an open set and so there exists a neighborhood $U_R(0)$ in the complex domain which is a part of $\rho(A)$ and the defect of $(A - \zeta I)$ is zero for all $\zeta \in U_R(0)$. However, the defect is constant in the half–planes $Im\,\zeta > 0$ and $Im\,\zeta < 0$ (T. Kato [12], p. 270) and so A has the deficiency index $(0, 0)$. Since it is also closed and symmetric, it is selfadjoint. (T. Kato [12], p. 271.)

The compactness of the resolvent operator follows from Lemma 3.1 and inequalities (19). □

Lemma 3.6 *$\sigma(A) = \{\lambda_i;\ i \in \mathbf{Z}^*\}$ $(\mathbf{Z}^* = \mathbf{Z} - \{0\})$ where*

a) λ_i are isolated real eigenvalues, each of them has the same finite algebraic and geometric multiplicity.

b) $\lambda_i < 0$ if $i < 0$, $\lambda_i > 0$ if $i > 0$ and $\lambda_i \le \lambda_j$ if $i < j$.

c) The corresponding eigenfunctions $\boldsymbol{e}^i$ can be chosen so that they form a complete orthonormal system in $L^2_\sigma(\Omega)^3$.

d) $D^1 = \Big\{ \boldsymbol{u} \in L^2_\sigma(\Omega)^3;\ \sum_{i\in\mathbf{Z}^*} |\,(\boldsymbol{u},\,\boldsymbol{e}^i)_2\,|^2\,\lambda_i^2 < +\infty \Big\}.$

e) $A\boldsymbol{u} = \sum_{i\in\mathbf{Z}^*} \lambda_i\,(\boldsymbol{u},\,\boldsymbol{e}^i)_2\,\boldsymbol{e}^i \quad$ *for* $\boldsymbol{u} \in D^1$.

P r o o f. The fact that $\sigma(A)$ consists of countably many real isolated eigenvalues, whose algebraic and geometric multiplicities are finite and equal, follows e.g. from T. Kato [12], pp. 187 and 270–271. $\sigma(A)$ cannot have any finite cluster point because the resolvent operator of A is compact. On the other hand, let us show that both $+\infty$ and $-\infty$ are the cluster points of $\sigma(A)$.

Let B_R be a ball with radius R in Ω. We can assume without loss of generality that the center of B_R is at the origin. Let g be a C^∞ non–zero function of one variable, defined on the interval $[0,+\infty)$, with its support in $(0,R)$. Denote by $\boldsymbol{i}_3$ the unit vector $(0,0,1)$. Put $\boldsymbol{v}(\boldsymbol{x}) = (\boldsymbol{i}_3 \times \boldsymbol{x})\,g(|\boldsymbol{x}|) = (-x_2,x_1,0)\,g(|\boldsymbol{x}|)$ for $\boldsymbol{x} \in \Omega$, Then $\nabla\cdot\boldsymbol{v} = 0$ in Ω and

$$\begin{aligned}\mathbf{curl}\,\boldsymbol{v}(\boldsymbol{x}) &= (0,\,0,\,2)\,g(|\boldsymbol{x}|) + \frac{g'(|\boldsymbol{x}|)}{|\boldsymbol{x}|}\,(-x_1x_3,\,-x_2x_3,\,x_1^2+x_2^2),\\ \mathbf{curl}^2\boldsymbol{v}(\boldsymbol{x}) &= (x_2,\,-x_1,\,0)\left[2\,\frac{g'(|\boldsymbol{x}|)}{|\boldsymbol{x}|} + \Delta g(|\boldsymbol{x}|)\right]\end{aligned}$$

for $\boldsymbol{x} \in \Omega$, $|\boldsymbol{x}| > 0$. Moreover, $\boldsymbol{v}(\boldsymbol{x})\cdot\mathbf{curl}\,\boldsymbol{v}(\boldsymbol{x}) = 0$ and $\mathbf{curl}\,\boldsymbol{v}(\boldsymbol{x})\cdot\mathbf{curl}^2\boldsymbol{v}(\boldsymbol{x}) = 0$ a.e. in Ω. Put

$$\begin{aligned}\boldsymbol{u}_1(\boldsymbol{x}) &= \alpha\,[\boldsymbol{v}(\boldsymbol{x}) + \mathbf{curl}\,\boldsymbol{v}(\boldsymbol{x})],\\ \boldsymbol{u}_2(\boldsymbol{x}) &= \alpha\,[\boldsymbol{v}(\boldsymbol{x}) - \mathbf{curl}\,\boldsymbol{v}(\boldsymbol{x})]\end{aligned}$$

(α is a real parameter.) Then both $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ belong to D^1 and

$$\begin{aligned}(A\boldsymbol{u}_1,\,\boldsymbol{u}_1)_2 &= \int_\Omega \mathbf{curl}\,\boldsymbol{u}_1\cdot\boldsymbol{u}_1\,d\boldsymbol{x} \;=\; 2\alpha^2\int_{B_R}|\mathbf{curl}\,\boldsymbol{v}|^2\,d\boldsymbol{x},\\ (A\boldsymbol{u}_2,\,\boldsymbol{u}_2)_2 &= \int_\Omega \mathbf{curl}\,\boldsymbol{u}_2\cdot\boldsymbol{u}_2\,d\boldsymbol{x} \;=\; -2\alpha^2\int_{B_R}|\mathbf{curl}\,\boldsymbol{v}|^2\,d\boldsymbol{x}.\end{aligned}$$

Choosing parameter α sufficiently large, we can make the scalar products $(A\boldsymbol{u}_1,\boldsymbol{u}_1)_2$ and $(A\boldsymbol{u}_2,\boldsymbol{u}_2)_2$ arbitrarily large with the positive or the negative sign. This shows that the numerical range of operator A is unbounded from below as well as from above and consequently, $\sigma(A)$ is also unbounded from below and from above. Thus, the eigenvalues of A can be ordered and numbered so that they satisfy statements a) – b).

Items c) – e) follow e.g. from [12], pp. 356–360. □

Example – the eigenvalue and the eigenfunction of operator A on a cube. We assume that $\Omega = (-\pi,\pi)^3$ in this example. The eigenfunction $\boldsymbol{u}$ of operator A can be expressed in the form of a Fourier expansion

$$\boldsymbol{u}(\boldsymbol{x}) \;=\; \sum_{k\in\mathbf{Z}^3} \boldsymbol{c}_k\exp(\mathrm{i}\,\boldsymbol{x}\cdot\boldsymbol{k}).$$

If $\boldsymbol{u}$ is an eigenfunction of A corresponding to an eigenvalue λ then $A\boldsymbol{u} = \lambda\boldsymbol{u}$ and consequently $A^2\boldsymbol{u} = \mathbf{curl}^2\boldsymbol{u} = -\Delta\boldsymbol{u} = \lambda^2\boldsymbol{u}$. From this, we can easily deduce that each eigenvalue λ must have the form $\lambda = \pm\sqrt{k_1^2+k_2^2+k_3^3}$ where $\boldsymbol{k} = (k_1,k_2,k_3) \in \mathbf{Z}^3$ and any eigenfunction $\boldsymbol{u}$ associated with eigenvalue λ is of the form

$$\boldsymbol{u}(\boldsymbol{x}) \;=\; \sum_{|k|^2=\lambda^2} \boldsymbol{c}_k\exp(\mathrm{i}\,\boldsymbol{x}\cdot\boldsymbol{k}). \tag{24}$$

We have done the calculation with $\lambda = \sqrt{1^2+2^2+3^2} = \sqrt{14}$. Then the sum on the right hand side of (24) contains only the terms with $\boldsymbol{k} = (k_1, k_2, k_3)$ being permutations of numbers 1, 2, 3 with positive or negative signs. One of the possible solutions is

$$\boldsymbol{u}(\boldsymbol{x}) \;=\; \sum_{l=1}^{4} \Big([\boldsymbol{a}(l) + \mathrm{i}\,\boldsymbol{b}(l)] \exp(\mathrm{i}\,\boldsymbol{x}\cdot\boldsymbol{k}(l)) + [\boldsymbol{a}(l) - \mathrm{i}\,\boldsymbol{b}(l)] \exp(-\mathrm{i}\,\boldsymbol{x}\cdot\boldsymbol{k}(l)) \Big) \tag{25}$$

where $\boldsymbol{k}(1) = (2,3,1)$, $\boldsymbol{k}(2) = (2,3,-1)$, $\boldsymbol{k}(3) = (2,-3,1)$, $\boldsymbol{k}(4) = (-2,3,1)$ and

$$\begin{aligned}
\boldsymbol{a}(1) &= (0.0113,\, 0.0000,\, -0.0219), & \boldsymbol{b}(1) &= (0.0175,\, -0.0089,\, 0.0111),\\
\boldsymbol{a}(2) &= (-0.0004,\, 0.0000,\, 0.0219), & \boldsymbol{b}(2) &= (-0.0175,\, 0.0320,\, -0.0111),\\
\boldsymbol{a}(3) &= (0.0004,\, 0.0000,\, -0.0219), & \boldsymbol{b}(3) &= (-0.0175,\, 0.0089,\, -0.0034),\\
\boldsymbol{a}(4) &= (-0.0113,\, 0.0000,\, 0.0219), & \boldsymbol{b}(4) &= (-0.0175,\, 0.0320,\, -0.0034).
\end{aligned}$$

4 The steady Stokes problem with the boundary conditions (4)

The operator $S = -P_\sigma \Delta$ is usually called the Stokes operator. (Obviously, $S = P_\sigma \mathbf{curl}^2$ on divergence–free vector fields.) Let us mention three possible domains of this operator (we shall successively denote them $D_1(S)$, $D_2(S)$ and $D_3(S)$) on which S is selfadjoint, as an operator in $L^2_\sigma(\Omega)^3$, with a compact inverse.

1. $D_1(S) = W^{2,2}(\Omega)^3 \cap W^{1,2}_0(\Omega)^3 \cap L^2_\sigma(\Omega)^3$. The operator $S_1 = S|_{D_1(S)}$ and the corresponding Stokes problem $S_1\boldsymbol{u} = P_\sigma \boldsymbol{f}$ (for $\boldsymbol{f} \in L^2(\Omega)^3$) is widely studied and discussed in literature. (See e.g. R. Temam [28], V. Girault and P.–A. Raviart [10], M. Feistauer [5] and G. P. Galdi [7].)

2. $D_2(S) = \{\boldsymbol{u} \in L^2_\sigma(\Omega)^3;\ \mathbf{curl}\,\boldsymbol{u} \in W^{1,2}(\Omega)^3,\ \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} \text{ on } \partial\Omega\}$. The operator $S_2 = S|_{D_2(S)}$ can be represented as $\mathbf{curl}_1 \circ \mathbf{curl}_2$ where $\mathbf{curl}_2$ $(= \mathbf{curl}\,|_{D_2(S)})$ maps $D_2(S)$ onto $R(\mathbf{curl}_2) = \{\boldsymbol{u} \in W^{1,2}(\Omega)^3;\ \nabla\cdot\boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} \text{ on } \partial\Omega\}$ and $\mathbf{curl}_1$ $(= \mathbf{curl}\,|_{R(\mathbf{curl}_2)})$ maps $R(\mathbf{curl}_2) = D(\mathbf{curl}_1)$ onto $L^2_\sigma(\Omega)^3$. Operator S_2 and its properties are studied in O. A. Ladyzhenskaya and V. A. Solonnikov [15] and B. Nicolaenko and A. Mahalov [21]. The authors discuss, except others, the spectral properties of operator S_2 and the relation between eigenvalues and eigenfunctions of the operators S_2 and $\mathbf{curl}$.

3. $D_3(S) = D(A^2)$, $S_3 = S|_{D^2} = A^2$. (P_σ can be omitted in front of A^2 because it commutes with A.) We are going to deal with this operator in more details in this section. The steady Stokes problem can be in our case defined by the equation

$$A^2\boldsymbol{u} \;=\; P_\sigma \boldsymbol{f} \qquad (\text{for } \boldsymbol{f} \in L^2(\Omega)^3) \tag{26}$$

together with the requirement that $\nabla\cdot\boldsymbol{u} = 0$ and $\boldsymbol{u}$ satisfies the boundary conditions (4). Equation (26) can also be written in the form

$$A^2\boldsymbol{u} \;=\; \boldsymbol{f} \;-\; \nabla p \tag{27}$$

where p is an appropriate scalar function such that $\nabla p = (I - P_\sigma)\boldsymbol{f}$. ($p$ corresponds to the pressure in the Navier–Stokes equation.)

The corresponding weak formulation is: Given $\boldsymbol{f} \in D^{-1}$. We look for $\boldsymbol{u} \in D^1$ (a *weak solution of (26) and (4)*) such that

$$\int_\Omega A\boldsymbol{u}\cdot A\boldsymbol{\phi}\, d\boldsymbol{x} \;=\; \langle \boldsymbol{f}, \boldsymbol{\phi}\rangle_\Omega \tag{28}$$

for all $\boldsymbol{\phi} \in D^1$.

It is obvious that each strong solution of (26), (4) is a weak solution. The next lemma shows that under the assumptions on appropriate smoothness of the weak solution the opposite assertion is also true.

Lemma 4.1 *If $\boldsymbol{u}$ is a weak solution of (26), (4) that is in $W^{2,2}(\Omega)^3$ and if $\boldsymbol{f} \in L^2(\Omega)^3$ then there exists $p \in W^{1,2}(\Omega)$ such that $\boldsymbol{u}$, p is a strong solution of (27), (4).*

Proof. Using at first the test functions $\boldsymbol{\phi}$ that have compact supports in Ω in (28) and integrating by parts, we can show that there exists $p \in W^{1,2}(\Omega)$ such that the pair $\boldsymbol{u}$, p satisfies equation

$$\mathbf{curl}\, A\boldsymbol{u} = \boldsymbol{f} - \nabla p$$

a.e. in Ω. It remains to show that $\boldsymbol{u}$ satisfies the boundary condition $\mathbf{curl}^2\boldsymbol{u} \cdot \boldsymbol{n} = 0$ in the sense of traces on Ω. Choosing a general test function $\boldsymbol{\phi}$, integrating by parts in (28) and using also the fact that $\boldsymbol{u}$ is a solution of (27), we obtain the identity

$$\int_{\partial\Omega} \mathbf{curl}\, \boldsymbol{u} \cdot (\boldsymbol{\phi} \times \boldsymbol{n})\, dS \;=\; 0. \tag{29}$$

Function $\boldsymbol{\phi}$ can be, in accordance with Lemma 3.3, expressed in the form $\boldsymbol{\phi} = \boldsymbol{\phi}_0 + \nabla\varphi$ where $\boldsymbol{\phi}_0 \in W_0^{1,2}(\Omega)^3$ and φ is the solution of the Neumann problem (16) (with $\boldsymbol{\phi}_0$ instead of $\boldsymbol{u}_0$). Substituting this form of $\boldsymbol{\phi}$ into (29), we obtain:

$$\begin{aligned} 0 \;&=\; \int_{\partial\Omega} \mathbf{curl}\, \boldsymbol{u} \cdot (\nabla\varphi \times \boldsymbol{n})\, dS = -\int_{\partial\Omega} \boldsymbol{n} \cdot (\nabla\varphi \times \mathbf{curl}\, \boldsymbol{u})\, dS \\ &=\; -\int_{\Omega} \nabla \cdot (\nabla\varphi \times \mathbf{curl}\, \boldsymbol{u})\, d\boldsymbol{x} \;=\; \int_{\Omega} \nabla\varphi \cdot \mathbf{curl}^2\boldsymbol{u}\, d\boldsymbol{x} \;=\; \langle(\mathbf{curl}^2\boldsymbol{u} \cdot \boldsymbol{n}),\, \varphi\rangle_{\partial\Omega}. \end{aligned} \tag{30}$$

$\mathbf{curl}^2\boldsymbol{u}$ is a divergence–free function in $L^2(\Omega)^3$ and so its normal component on the boundary belongs to $W^{-1/2,2}(\partial\Omega)$ (see e.g. [10], p. 27). The right hand side of (30) therefore expresses the duality between the elements of $W^{-1/2,2}(\partial\Omega)$ and $W^{1/2,2}(\partial\Omega)$. The set of traces on $\partial\Omega$ of all possible functions φ is dense in $W^{1/2,2}(\partial\Omega)$. Thus, (30) implies that $\mathbf{curl}^2\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ in the sense of traces. □

Theorem 4.2 (Existence and uniqueness of a solution to the steady Stokes problem.)
1. There exists a positive constant c_9 such that the weak Stokes problem (28) has for every $\boldsymbol{f} \in D^{-1}$ a unique solution $\boldsymbol{u}$. It satisfies the estimate

$$\|\boldsymbol{u}\|_{1,2} \;\leq\; c_9\, \|\boldsymbol{f}\|_{-1,2}\, . \tag{31}$$

2. If the boundary of Ω is smooth then there exists a positive constant c_{10} such that if $\boldsymbol{f} \in L^2(\Omega)^3$ then the solution $\boldsymbol{u}$ belongs to D^2 and it satisfies

$$\|\boldsymbol{u}\|_{2,2} \;\leq\; c_{10}\, \|\boldsymbol{f}\|_{2}\, . \tag{32}$$

Proof. 1. Given $\boldsymbol{f} \in D^{-1}$, the right hand side of (28) is a linear continuous functional on D^1. Thus, due to the Riesz theorem, there exists a unique $\boldsymbol{u} \in D^1$ such that the mentioned functional can be represented by a scalar product of $\boldsymbol{u}$ and $\boldsymbol{\phi}$ in D^1. Due to Corollary 3.2, the scalar product can be expressed as $(A\boldsymbol{u}, A\boldsymbol{\phi})_2$. Estimate (31) is an easy consequence of the Schwarz inequality and inequalities (19).

2. Part 2 of the theorem follows from the Stokes equation (26) and Lemma 3.1. One can solve equation (26) in two steps: $A\boldsymbol{v} = P_\sigma \boldsymbol{f}$ and $A\boldsymbol{u} = \boldsymbol{v}$. Both equations have, due to Lemma 3.1, solutions in D^1. Obviously, $\boldsymbol{u} \in D^2$. Estimate (32) is a consequence of (20). □

It is also possible to formulate a corresponding non–steady strong or weak Stokes problem with the boundary conditions (4) and to prove the existence and uniqueness of solutions.

5 The Navier–Stokes problem with the boundary conditions (4) – elements of the basic theory

Using operator A, the Navier–Stokes equation (1) can be written in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + A\boldsymbol{u} \times \boldsymbol{u} \;=\; -\nabla(p + \tfrac{1}{2}\,|\boldsymbol{u}|^2) - \nu A^2 \boldsymbol{u} + \boldsymbol{f}. \tag{33}$$

The equation of continuity (2) and the boundary conditions (4) can be involved in the requirement that $\boldsymbol{u}(.\,,t) \in D^2$ for $t \in (0,T)$. If we formally expand function $\boldsymbol{u}$ in the series

$$\boldsymbol{u}(\boldsymbol{x},t) \;=\; \sum_{i \in \mathbf{Z}^*} a_i(t)\, \boldsymbol{e}^i(\boldsymbol{x}), \tag{34}$$

substitute it to equation (33), multiply by $\boldsymbol{e}^i$ and integrate on Ω, we obtain the equivalent infinite system of ordinary differential equations

$$\frac{d}{dt}\, a_i(t) \;+\; \sum_{j,k \in \mathbf{Z}^*} \lambda_j\, a_j(t)\, a_k(t)\, b_{ijk} \;+\; \nu\, \lambda_i^2\, a_i(t) \;=\; f_i(t) \quad (i = -\infty, \dots, +\infty) \tag{35}$$

where

$$b_{ijk} = \int_\Omega \boldsymbol{e}^i \cdot (\boldsymbol{e}^j \times \boldsymbol{e}^k)\, d\boldsymbol{x}, \qquad f_i(t) = \int_\Omega \boldsymbol{f}(\boldsymbol{x},t) \cdot \boldsymbol{e}^i(\boldsymbol{x})\, d\boldsymbol{x}.$$

Note that the analogous system was also obtained by A. Mahalov and B. Nicolaenko in paper [21], Section 4, in a cylindrical domain in context with flows which have a weakly aligned uniformly large vorticity. (This property, as it is shown by the authors, excludes the appearance of singularities.)

System (35) enables the formal derivation of the energy estimate: If we multiply the i–th equation in (35) by a_i and sum over i from $-\infty$ to $+\infty$, we obtain

$$\frac{d}{dt}\, \frac{1}{2} \sum_{i \in \mathbf{Z}^*} a_i^2(t) \;+\; \nu \sum_{i \in \mathbf{Z}^*} \lambda_i^2\, a_i^2(t) \;=\; \sum_{i \in \mathbf{Z}^*} f_i(t)\, a_i(t) \;\leq\; \frac{\nu}{2} \sum_{i \in \mathbf{Z}^*} \lambda_i^2 a_i^2(t) \;+\; \frac{1}{2\nu} \sum_{i \in \mathbf{Z}^*} \frac{f_i^2(t)}{\lambda_i^2}$$

etc. The nonlinear term in (35) leads to zero. (This can be easily shown by interchanging i with k and using the skew–symmetry of b_{ijk}, i.e. the identity $b_{kji} = -b_{ijk}$.)

System (35) also enables to deduce a class of globally in time strong solutions of the Navier–Stokes initial–boundary value problem (1)–(4): If the initial value $\boldsymbol{u}_0$ is chosen so that it is a multiple of one of the eigenfunctions of operator A (i.e. $\boldsymbol{u}_0 = \alpha\, \boldsymbol{e}^i$; $\alpha \in \mathbb{R}$) and if $\boldsymbol{f}(\boldsymbol{x},t) = f(t)\, \boldsymbol{e}^i(\boldsymbol{x}) + \nabla\varphi$ where $f \in L^1_{loc}([0,+\infty))$ and $\varphi \in L^1_{loc}([0,+\infty);\, W^{1,2}(\Omega))$ then the corresponding strong solution is

$$\boldsymbol{u}(\boldsymbol{x},t) \;=\; \mathrm{e}^{-\nu\lambda_i^2 t}\, \alpha\, \boldsymbol{e}^i(\boldsymbol{x}) + \int_0^t \mathrm{e}^{-\nu\lambda_i^2 (t-\tau)}\, f(\tau)\, \boldsymbol{e}^i(\boldsymbol{x})\, d\tau \tag{36}$$

because the nonlinear terms in all equations in (35) are identically equal to zero. This is the reason why this function $\boldsymbol{u}$ is a solution of the non–steady Stokes problem, too. Obviously, if $\nu = 0$ then formula (36) provides a solution of the Euler equation. Formula (36) represents one of few cases when solution of the Navier–Stokes equation or the Euler equation can be expressed explicitly.

The problem (1)–(4) can be weakly formulated in this way: Let $\boldsymbol{f} \in L^2(0,T;\, D^{-1})$ and $\boldsymbol{u}_0 \in L^2_\sigma(\Omega)^3$. We look for $\boldsymbol{u} \in L^\infty(0,T;\, L^2_\sigma(\Omega)^3) \cap L^2(0,T;\, D^1)$ (so called *weak solution of (1)–(4)*) such that

$$\begin{aligned} &\int_{Q_T} \Big(-\boldsymbol{u} \cdot \frac{\partial \boldsymbol{\phi}}{\partial t} + (A\boldsymbol{u} \times \boldsymbol{u}) \cdot \boldsymbol{\phi} + A\boldsymbol{u} \cdot A\boldsymbol{\phi} \Big)\, d\boldsymbol{x}\, dt - \int_\Omega \boldsymbol{u}_0 \cdot \boldsymbol{\phi}(.\,,0)\, d\boldsymbol{x} \\ &\quad = \int_0^T \langle \boldsymbol{f}(.\,,t), \boldsymbol{\phi}(.\,,t) \rangle_\Omega\, dt \end{aligned} \tag{37}$$

for all $\boldsymbol{\phi} \in C^\infty([0,T];\, D^1)$ such that $\boldsymbol{\phi}(.\,,T) = \mathbf{0}$.

It can be easily shown that if $\boldsymbol{u}$ (together with some pressure p) is a strong solution of (1)–(4) then $\boldsymbol{u}$ is a weak solution. In order to confirm the sense of the weak formulation, it is also necessary to show the opposite, i.e. that to a sufficiently smooth weak solution $\boldsymbol{u}$ there exists an associated pressure p so that $\boldsymbol{u}$, p is a strong solution. The most steps of the proof are quite standard. The crucial part is to show that $\boldsymbol{u}$ satisfies in the sense of traces the third boundary condition $\mathbf{curl}^2\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on $\partial\Omega\times(0,T)$. However, this can be done analogously as in the proof of Lemma 4.1, using the representation of the test function $\boldsymbol{\phi}$ in the form $\boldsymbol{\phi} = \boldsymbol{\phi}_0 + \nabla\varphi$ where $\boldsymbol{\phi}_0(.\,,t) \in W_0^{1,2}(\Omega)^3$ and $\varphi(.\,,t)$ is a solution of the Neumann problem (16) (with $\boldsymbol{\phi}_0(.\,,t)$ instead of $\boldsymbol{u}_0$).

The boundary conditions (4) enable to re–derive many results which are already known to hold for the Navier–Stokes equation with the homogeneous Dirichlet boundary condition. See Theorems 5.1–5.3. We are not going to show complete proofs because their ideas are mostly the same as the ideas of the corresponding theorems dealing with the no–slip boundary condition. (See e.g. O. A. Ladyzhenskaya [16], R. Temam [28], G. P. Galdi [8] and [9], M. Feistauer [5] and P. L. Lions [19].)

Theorem 5.1 (Global in time existence of a weak solution.) *There exists at least one weak solution $\boldsymbol{u}$ of the problem (1)–(4). This solution satisfies the energy inequality*

$$\|\boldsymbol{u}(.\,,t)\|_2^2 + 2\nu\int_0^t \|A\boldsymbol{u}(.\,,\sigma)\|_2^2\,d\sigma \;\leq\; 2\int_0^t \langle \boldsymbol{f}(.\,,\sigma), \boldsymbol{u}(.\,,\sigma)\rangle_\Omega\,d\sigma + \|\boldsymbol{u}_0\|_2^2 \tag{38}$$

for $t\in(0,T)$ and $\lim_{t\to 0+}\|\boldsymbol{u}(.\,,t) - \boldsymbol{u}_0\|_2 = 0$.

This theorem can be proved by the Galerkin method. The idea of the proof is due to J. Leray [18] ($\Omega = \mathbb{R}^3$) and it was extended by E. Hopf [11] to a bounded domain Ω with the non–slip boundary condition. (See e.g. the review article of G. P. Galdi [9].) The boundary conditions (4) do not cause remarkable differences. The approximations can be constructed e.g. as linear combinations of eigenfunctions $\boldsymbol{e}^i$ of operator A.

Theorem 5.2 (Uniqueness of a weak solution.) *Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two weak solutions of the problem (1)–(4). Let $\boldsymbol{u}$ satisfy the energy inequality (38) and let $\boldsymbol{v}$ satisfy at least one of the conditions*

a) $\boldsymbol{v}\in L^s(0,T;\, L^r(\Omega)^3)$ *for some* $s\in[2,+\infty]$, $r\in(3,+\infty]$ *such that* $2/s+3/r\leq 1$,

b) $\boldsymbol{v}\in L^\infty(0,T;\, L^3(\Omega)^3)$ *and* $\boldsymbol{v}(.\,,t)$ *is right–continuous in the* L^3*–norm in* $[0,T)$.

Then $\boldsymbol{v} = \boldsymbol{u}$ *in* Q_T.

This theorem can also be proved in the same way as the analogous Theorem 4.2 considering the Dirichlet boundary condition in the article of G. P. Galdi [9]. Let us mention e.g. J. L. Lions and G. Prodi [20], G. Prodi [25], O. A. Ladyzhenskaya [16] and H. Kozono and H. Sohr [14] of all authors who mostly contributed to the proof in the case when $\Omega = \mathbb{R}^3$ or when the Dirichlet boundary condition is used. The boundary conditions (4) also enable to perform all important steps, especially all necessary integrations by parts.

Theorem 5.3 (Local in time existence of a strong solution.) *Let the boundary of Ω be smooth, $u_0\in D^2$ and $\boldsymbol{f}\in L^2(Q_T)^3$. Then there exists $T_1\in(0,T]$ such that the problem (1)–(4) has a solution $\boldsymbol{u}$ whose restriction to $\Omega\times(0,T_1)$ belongs to $L^2(0,T_1;\, W^{2,2}(\Omega)^3)$ and its time derivative belongs to $L^2(Q_T)^3$.*

The analogous theorem in the case of the non–slip boundary condition is proved e.g. in the book [16] by O. A. Ladyzhenskaya, Section VI. The key role plays the a–priori estimate which can be formally obtained by multiplying the Navier–Stokes equation by $P_\sigma\Delta\boldsymbol{u}$ and integrating over Ω. However, the same estimate can also be obtained if we formally multiply equation (33) by $A^2\boldsymbol{u}$, integrate over Ω and use the boundary conditions (4).

One can easily write down the weak formulation of the steady problem defined by the equations (1) (without the time derivative of $\boldsymbol{u}$), (2) and by the boundary conditions (4). The sense of the weak formulation can be confirmed similarly as in Lemma 4.1 and the existence of a steady weak solution can also be proved e.g. by the Galerkin method. The important point is to show that there exists $c_{11} > 0$ such that

$$\int_\Omega \Big((A\boldsymbol{u} \times \boldsymbol{u}) \cdot \boldsymbol{u} + \nu A\boldsymbol{u} \cdot A\boldsymbol{u} \Big)\, d\boldsymbol{x} - \langle \boldsymbol{f}, \boldsymbol{u} \rangle_\Omega \;\geq\; \nu \, \|\boldsymbol{u}\|_{1,2}^2 - c_{11} \, \|\boldsymbol{f}\|_{-1,2} \, \|\boldsymbol{u}\|_{1,2} \tag{39}$$

for $\boldsymbol{u} \in D^1$. (See e.g. G. P. Galdi [9], p. 15–16, or R. Temam [28], p. 165, for details.) In our case, (39) follows from estimates (19). The uniqueness of the weak solution (for large ν or small $\|\boldsymbol{f}\|_{-1,2}$) can be proved in the same way as the analogous theorem in [28], p. 167.

6 The Navier–Stokes problem with the boundary conditions (4) – some finer results

Suppose for simplicity that the boundary of Ω is smooth and $\boldsymbol{f} = \boldsymbol{0}$ in this section.

If the Navier–Stokes equation is considered with the homogeneous Dirichlet boundary condition then the Laplace operator, defined on $W^{2,2}(\Omega)^3 \cap W_0^{1,2}(\Omega)^3 \cap L^2_\sigma(\Omega)^3$, does not commute with projector P_σ. Thus, applying P_σ to the Navier–Stokes equation, the viscous term becomes $\nu P_\sigma \Delta \boldsymbol{u}$. However, in the case of the boundary conditions (4), P_σ commutes with operator A and therefore $P_\sigma \Delta \boldsymbol{u} = -P_\sigma A^2 \boldsymbol{u} = -A^2 P_\sigma \boldsymbol{u} = -A^2 \boldsymbol{u}$ for $\boldsymbol{u} \in D^2$. This fact has interesting consequences e.g. for the regularity of $\partial \boldsymbol{u}/\partial t$ and ∇p. The difficulties encountered in the case of the Dirichlet boundary condition are explained in details in the book of P. L. Lions [19], pp. 83–86. If $\boldsymbol{u}$ is a weak solution of (1)–(3) then the regularity of $\partial \boldsymbol{u}/\partial t$ and ∇p can be derived from equation (1), considered as an equation satisfied in the sense of distributions. In the case of the Dirichlet boundary condition, one can obtain that

$$\frac{\partial \boldsymbol{u}}{\partial t} \in L^2(0,T;\, V^{-1,2}) \;+\; [\, L^s(0,T;\, V^{-1,3s/(3s-2)}) \cap L^q(0,T;\, L^r(\Omega)^3)\,] \tag{40}$$

for $1 \leq s < +\infty$, $1 \leq q < 2$, $r = 3q/(4q-2)$ and where $V^{-1,m}$ is the dual to $V^{1,m} = \{\boldsymbol{u} \in W_0^{1,m}(\Omega)^3;\ \nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega\}$. (This holds because $P_\sigma \Delta \boldsymbol{u} \in L^2(0,T;\, V^{-1,2})$ and $P_\sigma(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \in L^s(0,T;\, V^{-1.3s/(3s-2)}) \cap L^q(0,T;\, L^r(\Omega)^3)$ – see P. L. Lions [19], pp. 82, 87.) In the case of the boundary conditions (4), $P_\sigma \Delta \boldsymbol{u} = \Delta \boldsymbol{u} \in L^2(0,T;\, W^{-1,2}(\Omega)^3)$ and this provides a better information because $W^{-1,2}(\Omega)^3 \subset V^{-1,2}$. Thus, we have

$$\frac{\partial \boldsymbol{u}}{\partial t} \in L^2(0,T;\, W^{-1,2}(\Omega)^3) \;+\; [\, L^s(0,T;\, V^{-1,3s/(3s-2)}) \cap L^q(0,T;\, L^r(\Omega)^3)\,] \tag{41}$$

instead of (40). We can obtain the same information about ∇p, too.

The boundary conditions (4) also enable to improve the information on the interior regularity of $\partial \boldsymbol{u}/\partial t$ and ∇p. Suppose that $\Omega_1 \subset \Omega$, $(t_1, t_2) \subset (0,T)$ and the weak solution $\boldsymbol{u}$ of (1)–(3) belongs to $L^s(t_1, t_2;\, L^r(\Omega_1)^3)$ where $2/s + 3/r \leq 1$. Then, due to J. Serrin [26] (see also G. P. Galdi [9]), $\boldsymbol{u}$ has all space derivatives in $L^\infty(t_1+\zeta, t_2-\zeta;\, L^\infty(\Omega_2)^3)$ where $\Omega_2 \subset \overline{\Omega}_2 \subset \Omega_1$ and $\zeta > 0$ satisfies $t_1 + \zeta < t_2 - \zeta$. The same statement can also be proved about $\partial \boldsymbol{u}/\partial t$ and ∇p if $\Omega = \mathbb{R}^3$. However, it is only known that $\partial \boldsymbol{u}/\partial t$ and ∇p have all their space derivatives in $L^\alpha(t_1+\zeta, t_2-\zeta;\, L^\infty(\Omega_2)^3)$ with arbitrary $\alpha \in [1,2)$ in the case of $\Omega \neq \mathbb{R}^3$ and the Dirichlet boundary condition on $\partial\Omega$. (See e.g. J. Neustupa and P. Penel [23], Lemma 2 or P. Kučera and Z. Skalák [14], [27].) The next theorem shows that if the boundary conditions (4) are considered then we can derive the same result on the interior regularity of $\partial \boldsymbol{u}/\partial t$ and ∇p in the case of a bounded domain Ω (satisfying the assumptions from the beginning of the paper) as in the case of $\Omega = \mathbb{R}^3$.

Theorem 6.1 *Let $\boldsymbol{u}$ be a weak solution of the problem (1)–(4) (whose existence is guaranteed by Theorem 5.1). Let Ω_1 and Ω_2 be sub–domains of Ω such that $\overline{\Omega}_2 \subset \Omega_1$ and let ζ be a positive number such that $t_1 + \zeta < t_2 - \zeta$. Suppose that at least one of the conditions*

(i) $\boldsymbol{u} \in L^s(t_1, t_2;\, L^r(\Omega_1)^3)$ *for some* $s \in [2, +\infty]$, $r \in (3, +\infty]$ *such that* $2/s + 3/r = 1$,

(ii) $\boldsymbol{u} \in L^\infty(t_1, t_2;\, L^3(\Omega_1)^3)$ *and the norm of* $\boldsymbol{u}$ *in* $L^\infty(t_1, t_2;\, L^3(\Omega_1)^3)$ *is sufficiently small* *is satisfied. Then* $\partial \boldsymbol{v}/\partial t$, ∇p *and their space derivatives of an arbitrary order belong to* $L^\infty(t_1+\zeta, t_2-\zeta;\, L^\infty(\Omega_2)^3)$.

Proof. Applying the operator of divergence to equation (1), we obtain that p satisfies the equation

$$\Delta p \,=\, -\frac{\partial^2}{\partial x_i\,\partial x_j}\,(u_i\,u_j) \tag{42}$$

in the sense of distributions in Ω and in a classical sense in Ω_1 for a.a. $t \in (t_1+\zeta, t_2-\zeta)$. ($u_1$, u_2 and u_3 are the components of $\boldsymbol{u}$.) Let μ be a C^∞ cut–off function such that $\mu(\boldsymbol{x}) = 1$ for $\boldsymbol{x} \in \Omega_2$, $0 \le \mu(\boldsymbol{x}) \le 1$ for $\boldsymbol{x} \in \Omega_1 - \Omega_2$ and $\mu(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \notin \Omega_1$. Let $\boldsymbol{a}$ be a unit vector and $\boldsymbol{x} \in \Omega_2$. Then

$$\begin{aligned}
\boldsymbol{a}\cdot\nabla p(\boldsymbol{x},t) \,&=\, \boldsymbol{a}\cdot\mu(\boldsymbol{x})\,\nabla p(\boldsymbol{x},t) \,=\, -\frac{1}{4\pi}\int_{\mathbb{R}^3}\boldsymbol{a}\cdot\frac{\Delta_y[\mu(\boldsymbol{y})\,\nabla_y\,p(\boldsymbol{y},t)]}{|\boldsymbol{y}-\boldsymbol{x}|}\,d\boldsymbol{y}\\
&=\, \frac{1}{4\pi}\int_\Omega \Delta_y\biggl(\boldsymbol{a}\,\frac{\mu(\boldsymbol{y})}{|\boldsymbol{y}-\boldsymbol{x}|}\biggr)\cdot\nabla_y\,p(\boldsymbol{y},t)\,d\boldsymbol{y} \,+\, \frac{\boldsymbol{a}}{4\pi}\cdot\int_\Omega\frac{\mu(\boldsymbol{y})}{|\boldsymbol{y}-\boldsymbol{x}|}\,\nabla_y\,[u_{i,j}(\boldsymbol{y},t)\,u_{j,i}(\boldsymbol{y},t)]\,d\boldsymbol{y}\\
&=\, \frac{1}{4\pi}\int_\Omega \nabla_y\,\varphi^{x,a}(\boldsymbol{y})\cdot\nabla_y\,p(\boldsymbol{y},t)\,d\boldsymbol{y} \,+\, \frac{\boldsymbol{a}}{4\pi}\cdot\boldsymbol{I}(x,t)
\end{aligned}$$

where $\boldsymbol{I}$ belongs to $L^\infty(t_1+\zeta, t_2-\zeta;\, L^\infty(\Omega_2)^3)$ (due to Serrin's results) and

$$\nabla_y\,\varphi^{x,a}(\boldsymbol{y}) \,=\, (I-P_\sigma)\,\Delta_y\biggl(\boldsymbol{a}\,\frac{\mu(\boldsymbol{y})}{|\boldsymbol{y}-\boldsymbol{x}|}\biggr) \,=\, (I-P_\sigma)\,\Delta_y\biggl(\boldsymbol{a}\,\frac{\mu(\boldsymbol{y})-\mu(\boldsymbol{x})}{|\boldsymbol{y}-\boldsymbol{x}|}\biggr).$$

The already mentioned commutativity of projection P_σ with operators A and A^2 implies that if we apply $(I-P_\sigma)$ to the Navier–Stokes equation (1), written in a more usual way, i.e. with the convective term in the form $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ and ∇p instead of $\nabla(p+\frac{1}{2}\,|\boldsymbol{u}|^2)$, we obtain

$$\nabla p \,=\, -(I-P_\sigma)\,(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}. \tag{43}$$

$(I-P_\sigma)$ is the orthogonal projection of $L^2(\Omega)^3$ onto the subspace $\{\boldsymbol{v} = \nabla q$ for some $q \in W^{1,2}(\Omega)\}$. Thus,

$$\begin{aligned}
\boldsymbol{a}\cdot\nabla p(\boldsymbol{x},t) \,&=\, -\frac{1}{4\pi}\int_\Omega \nabla_y\,\varphi^{x,a}(\boldsymbol{y})\,(I-P_\sigma)\,(\boldsymbol{u}\cdot\nabla_y\,)\boldsymbol{u}\,d\boldsymbol{y} \,+\, \boldsymbol{a}\cdot\boldsymbol{I}(\boldsymbol{x},t)\\
&=\, -\frac{1}{4\pi}\int_\Omega \nabla_y\,\varphi^{x,a}(\boldsymbol{y})\,(\boldsymbol{u}\cdot\nabla_y\,)\boldsymbol{u}\,d\boldsymbol{y} \,+\, \boldsymbol{a}\cdot\boldsymbol{I}(\boldsymbol{x},t)\\
&=\, \frac{1}{4\pi}\int_\Omega \varphi^{x,a}_{,ij}(\boldsymbol{y})\,u_j(\boldsymbol{y},t)\,u_i(\boldsymbol{y},t)\,d\boldsymbol{y} \,+\, \boldsymbol{a}\cdot\boldsymbol{I}(\boldsymbol{x},t).
\end{aligned}$$

This shows that ∇p belongs to $L^\infty(t_1+\zeta, t_2-\zeta;\, L^\infty(\Omega_2)^3)$. The same statement about the space derivatives of ∇p can be obtained analogously, if we deal with $D_x^{|k|}\nabla p$ (where $D_x^{|k|} = \partial^{|k|}/\partial x_1^{k_1}\,\partial x_2^{k_2}\,\partial x_3^{k_3}$, $k = (k_1, k_2, k_3)$ is a multi–index) instead of ∇p. □

C. Foias, C. Guillope and R. Temam [6] (in a space–periodic case in $\mathbb{R}^3$) and G. F. D. Duff [4] (in a bounded domain Ω and with the Dirichlet boundary condition) proved that if the initial velocity, in addition, belongs to $W^{1,2}(\Omega)^3$ then the global in time weak solution to the problem (1)–(3) can be constructed so that

$$\int_0^T \|\boldsymbol{u}\|_{2,2}^{2/3}\,dt \,<\, +\infty. \tag{44}$$

This was generalized by H. Bellout, F. Bloom and J. Nečas [1], [2] to non–Newtonian fluids (in a space periodic case) and space dimension $n = 3$. Using the approach of [1], J. Málek, J. Nečas and M. Růžička established the results of [1] to a general space dimension $n \ge 3$ in [22]. We will outline below a proof based on the approach of [2]. The crucial point is the usage of $\boldsymbol{w} = P_\sigma \Delta \boldsymbol{u}/(1+\|\boldsymbol{u}\|_{1,2}^2)^2$ as a test function which multiplies the Navier–Stokes equation (in this form on the apriori level or with $\boldsymbol{u}^n$ instead of $\boldsymbol{u}$ on the level of approximations). Since $P_\sigma \Delta \boldsymbol{u} = \Delta \boldsymbol{u} = -\mathbf{curl}^2 \boldsymbol{u}$ in the case of the boundary conditions (4), we can use $\boldsymbol{w} = \mathbf{curl}^2 \boldsymbol{u}/(1+\|\mathbf{curl}\,\boldsymbol{u}\|_2^2)^2$ as a test function in our case. This possibility substantially simplifies the proof of (44) in comparison with [4], nevertheless there still remain several important steps. Let us explain them, for simplicity, on the formal level. Assume that $\boldsymbol{u}_0 \in D^1$. After integrating by parts and using the boundary conditions (4), we obtain

$$-\frac{1}{2}\frac{d}{dt}(1+\|\mathbf{curl}\,\boldsymbol{u}\|_2^2)^{-1} + \int_\Omega (\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u})\cdot\frac{\mathbf{curl}^2\boldsymbol{u}}{(1+\|\mathbf{curl}\,\boldsymbol{u}\|_2^2)^2}\,d\boldsymbol{x} + \nu\int_\Omega \frac{|\mathbf{curl}^2\boldsymbol{u}|^2}{(1+\|\mathbf{curl}\,\boldsymbol{u}\|_2^2)^2}\,d\boldsymbol{x} \;\leq\; 0. \tag{45}$$

The convective term can be treated as follows:

$$\begin{aligned}\int_\Omega (\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u})\cdot\mathbf{curl}^2\boldsymbol{u}\,d\boldsymbol{x} &= \int_{\partial\Omega}(\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u})\cdot(\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n})\,dS \\ &\quad + \int_\Omega \mathbf{curl}(\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n})\cdot\mathbf{curl}\,\boldsymbol{u}\,d\boldsymbol{x} = \int_\Omega [(\boldsymbol{u}\cdot\nabla)\mathbf{curl}\,\boldsymbol{u} - (\mathbf{curl}\,\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\mathbf{curl}\,\boldsymbol{u}\,d\boldsymbol{x} \\ &= -\int_\Omega (\mathbf{curl}\,\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{u}\,d\boldsymbol{x}.\end{aligned}$$

(The integral on $\partial\Omega$ is zero because $\mathbf{curl}\,\boldsymbol{u}$ and $\boldsymbol{u}$ are both tangent vectors to $\partial\Omega$ and so their cross product is normal, while the cross product $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}$ is tangent.) Hence

$$\begin{aligned}\left|\int_\Omega (\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u})\cdot\frac{\mathbf{curl}^2\boldsymbol{u}}{(1+\|\mathbf{curl}\,\boldsymbol{u}\|_2^2)^2}\,d\boldsymbol{x}\right| &\leq c\,(1+\|\mathbf{curl}\,\boldsymbol{u}\|_2^2)^{-2}\,\|\boldsymbol{u}\|_{1,3}^3 \\ &\leq c\,(1+\|\mathbf{curl}\,\boldsymbol{u}\|_2^2)^{-2}\,\|\boldsymbol{u}\|_{1,2}^{3/2}\,\|\boldsymbol{u}\|_{2,2}^{3/2} \leq c\,(1+\|\mathbf{curl}\,\boldsymbol{u}\|_2^2)^{-2}\,\|\boldsymbol{u}\|_{1,2}^{3/2}\,\|\Delta\boldsymbol{u}\|_2^{3/2} \\ &\leq \frac{\nu}{4}\,(1+\|\mathbf{curl}\,\boldsymbol{u}\|_2^2)^{-2}\,\|\Delta\boldsymbol{u}\|_2^2 + c\,\|\mathbf{curl}\,\boldsymbol{u}\|_2^2.\end{aligned} \tag{46}$$

We have used the interpolation–embedding inequality $\|\boldsymbol{u}\|_{1,3} \leq c\,\|\boldsymbol{u}\|_{1,2}^{1/2}\,\|\boldsymbol{u}\|_{2,2}^{1/2}$ (see H. Triebel [29], p. 186) and inequalities (19), (20). Integrating (45) with respect to t between 0 and T and using (19), (46), we are led to the result:

$$\int_0^T (1+\|\mathbf{curl}\,\boldsymbol{u}\|_2^2)^{-2}\,\|\Delta\boldsymbol{u}\|_2^2\,dt \;\leq\; c_{12}. \tag{47}$$

This finally implies that

$$\begin{aligned}\int_0^T \|\Delta\boldsymbol{u}\|_2^{2/3}\,dt &= \int_0^T \frac{\|\Delta\boldsymbol{u}\|_2^{2/3}}{(1+\|\mathbf{curl}\,\boldsymbol{u}\|_2^2)^{2/3}}\,(1+\|\mathbf{curl}\,\boldsymbol{u}\|_2^2)^{2/3}\,dt \\ &\leq \left(\int_0^T (1+\|\mathbf{curl}\,\boldsymbol{u}\|_2^2)^{-2}\,\|\Delta\boldsymbol{u}\|_2^2\,dt\right)^{1/3}\left(\int_0^T (1+\|\mathbf{curl}\,\boldsymbol{u}\|_2^2)\,dt\right)^{2/3} \leq c_{13}.\end{aligned} \tag{48}$$

Deriving the same estimate for the Galerkin approximations and considering an appropriate limit transition, we can prove the theorem:

Theorem 6.2 *If $\boldsymbol{u}_0 \in D^1$ then the weak solution of the problem (1)–(4), mentioned in Theorem 5.1, can be constructed so that it satisfies estimate (44).*

Note that the same result can also be obtained for $\boldsymbol{f} \neq \boldsymbol{0}$, $\boldsymbol{f} \in L^2(Q_T)^3$.

The weak solution $\boldsymbol{u}$ that satisfies estimate (44) has $\mathbf{curl}^2\boldsymbol{u}$ in $L^2_\sigma(\Omega)^3$ for a.a. $t \in (0,T)$. Thus, it fulfills the third boundary condition $\mathbf{curl}^2\boldsymbol{u}\cdot\boldsymbol{n}\,|_{\partial\Omega} = 0$ in the sense of equality in the space $W^{-1/2,2}(\partial\Omega)^3$. (This can be derived analogously as in the proof of Lemma 4.1.)

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