

NUMERICAL SOLUTION OF  
HYPERSENSINGULAR INTEGRAL EQUATIONS

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**Abstract:** In this article, an accurate numerical solution for solving hypersingular integral equation is presented. Chebyshev orthogonal polynomials of the second kind are used to approximate the unknown function. The regular kernel is interpolated using Chebyshev interpolation formula of the first kind. Numerical examples are solved using the proposed numerical technique. Numerical results show the accuracy of the present numerical solution.

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1. Introduction

Consider the hypersingular integral equation (HSIE) of the form

$$\int_{-1}^1 \frac{\varphi(t)}{(t-x)^2} dt + \int_{-1}^1 K(x,t)\varphi(t) dt = f(x), \quad -1 < x < 1, \quad (1)$$

with the conditions

$$\varphi(\pm 1) = 0, \quad (2)$$

where  $K$  and  $f'$  are assumed to be Hölder-continues functions and  $\varphi$  is unknown function to be determined. The first integral in the left-hand side of equation

(1) is understood as a Hadamard finite-part integral, which is defined by (see Lifanov et al [1] and Martin [2])

$$\int_{-1}^1 \frac{\varphi(t)}{(t-x)^2} dt = \lim_{\epsilon \rightarrow 0} \left\{ \int_{-1}^{x-\epsilon} \frac{\varphi(t)}{(t-x)^2} dt + \int_{x+\epsilon}^1 \frac{\varphi(t)}{(t-x)^2} dt - \frac{2\varphi(x)}{\epsilon} \right\}. \quad (3)$$

The Hadamard finite-part integral (3) is also defined by the Cauchy principal-value integral as

$$\int_{-1}^1 \frac{\varphi(t)}{(t-x)^2} dt = -\frac{d}{dx} \int_{-1}^1 \frac{\varphi(t)}{t-x} dt. \quad (4)$$

where the Cauchy principal-value integral is defined as

$$\int_{-1}^1 \frac{\varphi(t)}{t-x} dt = \lim_{\epsilon \rightarrow 0} \left\{ \int_{-1}^{x-\epsilon} \frac{\varphi(t)}{t-x} dt + \int_{x+\epsilon}^1 \frac{\varphi(t)}{t-x} dt \right\}. \quad (5)$$

Equation (1) arises in a variety of mixed boundary value problems in mathematical physics such as water wave scattering and radiation problems involving thin submerged, fluid mechanics, fracture mechanics, acoustics and elasticity (see Chan et al [3], Mandal et al [4] and Parsons et al [5]).

In this paper, we present a numerical solution for solving the HSIE(1). In the approximation, we use the Chebyshev polynomials of the second kind,  $U_i$ , to approximate the unknown function  $\varphi(t)$  and Chebyshev polynomials of the first kind,  $T_i$ , to interpolate the regular kernel  $K(x, t)$  with respect to  $t$  in the zeros of  $T_{M+1}$ .

### 2. Numerical Technique

Using (4), equation (1) can be converted into

$$-\frac{d}{dx} \int_{-1}^1 \frac{\varphi(t)}{t-x} dt + \int_{-1}^1 K(x, t)\varphi(t) dt = f(x), \quad -1 < x < 1. \quad (6)$$

Rewriting the unknown function  $\varphi(t)$  in equation (6) which satisfies the conditions in (2) as

$$\varphi(t) = \sqrt{1-t^2} \psi(t), \quad (7)$$

where the function  $\psi(t)$  is well behavior on the interval  $[-1, 1]$ .

Now, approximating  $\psi(t)$  by using Chebyshev polynomials of the second kind as follows

$$\psi(t) \approx \sum_{i=0}^N a_i U_i(t), \quad (8)$$

where  $a_i, i = 0, 1, \dots, N$  are the unknown coefficients to be determined. Due to (7-8), the numerical solution of equation (1) is of the form

$$\varphi(t) \approx \sqrt{1-t^2} \sum_{i=0}^N a_i U_i(t). \tag{9}$$

Substituting (9) into equation (6) yields

$$\sum_{i=0}^N a_i [P_i(x) + Q_i(x)] = f(x), \tag{10}$$

where

$$P_i(x) = -\frac{d}{dx} \int_{-1}^1 \frac{\sqrt{1-t^2} U_i(t)}{t-x} dt \tag{11}$$

and

$$Q_i(x) = \int_{-1}^1 \sqrt{1-t^2} K(x,t) U_i(t) dt. \tag{12}$$

It is known that (see Mason et al [6])

$$\left. \begin{aligned} \int_{-1}^1 \frac{\sqrt{1-t^2} U_i(t)}{t-x} dt &= -\pi T_{i+1}(x), \quad -1 < x < 1, \\ \frac{d}{dx}(T_{i+1}(x)) &= (i+1) U_i(x), \quad -1 < x < 1. \end{aligned} \right\} \tag{13}$$

Taking into account (13) we obtain

$$P_i(x) = -\pi (i+1) U_i(x), \quad i = 0, 1, \dots, N. \tag{14}$$

In order to evaluate  $Q_i(x)$ , the integral in (12) may be calculated analytically. If we can not evaluate the integral in (12) analytically, we may do so approximately, for instant, by interpolating the regular kernel  $K(x, t)$ , with respect to  $t$ , in the zeros of  $T_{M+1}$  as a sum of Chebyshev polynomials in the form (see Mason et al [6])

$$K(x, t) \approx \sum_{k=0}^M c_k(x) T_k(t), \tag{15}$$

where

$$c_k(x) = \frac{2}{M+1} \sum_{\ell=1}^{M+1} K(x, t_\ell) T_k(t_\ell), \tag{16}$$

and

$$t_\ell = \frac{(2\ell - 1)\pi}{2(M + 1)}, \quad \ell = 1, 2, \dots, M + 1. \tag{17}$$

Then

$$Q_i(x) = \sum'_{k=0}^M c_k(x) \int_{-1}^1 \sqrt{1 - t^2} T_k(t) U_i(t) dt. \tag{18}$$

It is easy to verify that (see Mason et al [6] and Abdulkawi et al [7])

$$T_k(t) U_i(t) = \frac{1}{2} [U_{k+i}(t) + U_{i-k}(t)], \quad i, k = 0, 1, 2, \dots \tag{19}$$

Using (19) into (18) yields

$$Q_i(x) = \frac{1}{2} \sum'_{k=0}^M c_k(x) [b_{i,k} + d_{i,k}], \tag{20}$$

where

$$b_{i,k} = \int_{-1}^1 \sqrt{1 - t^2} U_{k+i}(t) dt = \begin{cases} \frac{\pi}{2}, & k + i = 0, \\ 0, & k + i \neq 0. \end{cases} \tag{21}$$

and

$$d_{i,k} = \int_{-1}^1 \sqrt{1 - t^2} U_{i-k}(t) dt = \begin{cases} \frac{\pi}{2}, & i = k, \\ -\frac{\pi}{2}, & k - i = 2, \\ 0, & \text{Otherwise.} \end{cases} \tag{22}$$

Then, choosing the suitable collocation points  $x_j, j = 0, 1, \dots, N$  for equation (10) such as the zeros of  $U_{N+1}$ . These lead to a system of linear equations

$$\sum_{i=0}^N a_i [P_i(x_j) + Q_i(x_j)] = f(x_j), \quad j = 0, 1, \dots, N. \tag{23}$$

Solving the system (23) for the unknown coefficients  $a_i, i = 0, 1, \dots, N$  and substituting the values of  $a_i$  into (9) we obtain the numerical solution of equation (1). Note that the dash in  $\sum'$  denotes that the first term in the sum is to be halved.

### 3. Numerical Results

**Example 1.** Consider the following HSIE

$$\int_{-1}^1 \frac{\varphi(t)}{(t-x)^2} dt + 16 \int_{-1}^1 x^3 t^3 \varphi(t) dt = -\pi[31x^3 - 16x], \tag{24}$$

with the conditions

$$\varphi(\pm 1) = 0. \tag{25}$$

It is not difficult to verify that the exact solution of equation (24) is

$$\varphi(t) = \sqrt{1-t^2} [8t^3 - 4t]. \tag{26}$$

Due to (12) we have

$$Q_i(x) = 16 \int_{-1}^1 \sqrt{1-t^2} x^3 t^3 U_i(t) dt. \tag{27}$$

With the help of the recurrence relation of Chebyshev polynomials of the second kind

$$\left. \begin{aligned} U_0(x) &= 1, & U_1(x) &= 2x, \\ U_n(x) &= 2x U_{n-1}(x) - U_{n-2}(x), & n &\geq 2. \end{aligned} \right\} \tag{28}$$

we have

$$t^3 = \frac{1}{8} [U_3(t) + 2U_1(t)]. \tag{29}$$

Using (29) into (27) yields

$$Q_i(x) = 2x^3 \int_{-1}^1 \sqrt{1-t^2} [U_3(t) + 2U_1(t)] U_i(t) dt. \tag{30}$$

It is known that

$$\int_{-1}^1 \sqrt{1-t^2} U_i(t) U_j(t) dt = \begin{cases} 0, & i \neq j, \\ \frac{\pi}{2}, & i = j. \end{cases} \tag{31}$$

From (30-31) we obtain

$$\left. \begin{aligned} Q_0(x) &= 0, & Q_1(x) &= 2\pi x^3, & Q_2(x) &= 0, \\ Q_3(x) &= \pi x^3, & Q_i(x) &= 0, & i &\geq 4. \end{aligned} \right\} \tag{32}$$

Thus, the system of linear equations (23) for  $N = 3$  becomes

$$\left. \begin{aligned} \sum_{i=0}^3 a_i [-\pi(i+1)U_i(x_j)] + 2\pi a_1 x_j^3 + \pi a_3 x_j^3 \\ = -\pi[31x_j^3 - 16x_j], \quad j = 0, 1, 2, 3. \end{aligned} \right\} \quad (33)$$

which is equivalent to

$$\left. \begin{aligned} [2\pi a_1 - 31\pi a_3]x_j^3 - 12\pi a_2 x_j^2 + [-4\pi a_1 + 16\pi a_3]x_j \\ + 3\pi a_2 - \pi a_0 = -31\pi x_j^3 + 16\pi x_j, \quad j = 0, 1, 2, 3. \end{aligned} \right\} \quad (34)$$

Comparing the coefficients of various powers of  $x_j$  from the both sides of equation (34) yields the following system

$$\left. \begin{aligned} 3\pi a_2 - \pi a_0 &= 0, \\ -4\pi a_1 + 16\pi a_3 &= 16\pi, \\ -12\pi a_2 &= 0, \\ 2\pi a_1 - 31\pi a_3 &= -31\pi. \end{aligned} \right\} \quad (35)$$

It is easy to see that the solution of the above system (35) is

$$a_0 = a_1 = a_2 = 0, \quad a_3 = 1. \quad (36)$$

Substituting the values of the coefficients  $a_i, i = 0, 1, 2, 3$ , into (9) we obtain the numerical solution of equation (24) which is identical to the exact solution given by (26).

**Example 2.** Consider the following HSIE

$$\int_{-1}^1 \frac{\varphi(t)}{(t-x)^2} dt + \int_{-1}^1 \sin(x) t^4 \varphi(t) dt = f(x) \quad (37)$$

with the conditions

$$\varphi(\pm 1) = 0, \quad (38)$$

where

$$f(x) = -\pi \left[ 5(16x^4 - 12x^2 + 1) - \frac{\sin(x)}{32} \right].$$

The exact solution of equation (37) is

$$\varphi(x) = \sqrt{1-x^2} (16x^4 - 12x^2 + 1). \quad (39)$$

By using (20-22) with  $M = 5$ , we obtain

$$\left. \begin{aligned} Q_0(x) &= 0.06249999990 \pi \sin(x), & Q_1(x) &= 0, \\ Q_2(x) &= 0.09374999990 \pi \sin(x), & Q_3(x) &= 0, \\ Q_4(x) &= 0.03125000012 \pi \sin(x), & Q_5(x) &= 0. \end{aligned} \right\} \quad (40)$$

Choosing the zeros of  $U_{N+1}$  as the collocation points  $x_j$  for equation (23) where  $N = 5$ , i.e.

$$x_j = \cos\left(\frac{(j+1)\pi}{N+2}\right), \quad j = 0, 1, \dots, 5, \quad (41)$$

and solving the obtained system for the unknown coefficients  $a_i, i = 0, 1, \dots, 5$ . Substituting the values of  $a_i$  into (9) we obtain the numerical solution of equation (37). The errors of numerical solution of equation (37) are given by Table 1.

**Example 3.** Consider the following HSIE

$$\int_{-1}^1 \frac{\varphi(t)}{(t-x)^2} dt + \int_{-1}^1 e^x t^2 \varphi(t) dt = -\pi \left[ 12x^2 - \frac{1}{8}e^x - 3 \right]. \quad (42)$$

with the conditions

$$\varphi(\pm 1) = 0. \quad (43)$$

It is easy to see that the exact solution of equation (37) is

$$\varphi(x) = \sqrt{1-x^2} (4x^2 - 1). \quad (44)$$

Using (20-22) with  $M = 5$ , we obtain

$$\left. \begin{aligned} Q_0(x) &= 0.125 \pi e^x, & Q_1(x) &= 0, \\ Q_2(x) &= 0.1249999999 \pi e^x, & Q_3(x) &= 0, \\ Q_4(x) &= 1.75 \times 10^{-10} \pi e^x, & Q_5(x) &= 0. \end{aligned} \right\} \quad (45)$$

Using  $x_j$  in (41) as the collocation points for equation (23) where  $N = 5$  and solving the obtained system for the unknown coefficients  $a_i, i = 0, 1, \dots, 5$ . Substituting the values of  $a_i$  into (9) we obtain the numerical solution of equation (42). The errors of numerical solution of equation (42) are given by Table 2.

$x$	$N = M = 5$	Error
-1.0		0
-0.9		$5.00 \times 10^{-10}$
-0.8		$2.10 \times 10^{-09}$
-0.7		$3.60 \times 10^{-09}$
-0.6		$4.00 \times 10^{-09}$
-0.5		$2.90 \times 10^{-09}$
-0.4		$3.40 \times 10^{-09}$
-0.3		$2.81 \times 10^{-09}$
-0.2		$2.30 \times 10^{-09}$
-0.1		$2.10 \times 10^{-09}$
0.0		$2.00 \times 10^{-09}$
0.1		$2.10 \times 10^{-09}$
0.2		$2.30 \times 10^{-09}$
0.3		$2.72 \times 10^{-09}$
0.4		$3.30 \times 10^{-09}$
0.5		$2.90 \times 10^{-09}$
0.6		$4.00 \times 10^{-09}$
0.7		$3.60 \times 10^{-09}$
0.8		$2.70 \times 10^{-09}$
0.9		$4.00 \times 10^{-10}$
1.0		0

Table 1: Errors of numerical solution of equation (37)

#### 4. Conclusion

We have developed collocation method based on the Chebyshev orthogonal polynomials for solving the hypersingular integral equations. Chebyshev interpolation formula helped us to approximate the regular kernel. The collocation points in examples 2 – 3 are chosen to be the zeros of Chebyshev polynomials of the second kind  $U_N + 1$ . The errors of the collocation method shown in the Tables 2-3 are computed as the absolute value of the difference between the exact and numerical solutions. We used Maple 13 to carry the computations. The developed collocation method gives a very accurate numerical results for any singular point  $x \in [-1, 1]$  with only small number of collocation points  $N = 5$  (Tables 1-2). Moreover, the developed collocation method gives the exact solution for some hypersingular integral equations (Example 1).

$x$	$N = M = 5$	Error
-1.0		0
-0.9		$5.0 \times 10^{-10}$
-0.8		$6.0 \times 10^{-10}$
-0.7		$5.0 \times 10^{-10}$
-0.6		$6.0 \times 10^{-10}$
-0.5		$9.6 \times 10^{-10}$
-0.4		$1.1 \times 10^{-09}$
-0.3		$1.0 \times 10^{-09}$
-0.2		$9.0 \times 10^{-10}$
-0.1		$1.0 \times 10^{-09}$
0.0		$1.0 \times 10^{-09}$
0.1		$1.0 \times 10^{-9}$
0.2		$9.0 \times 10^{-10}$
0.3		$1.0 \times 10^{-09}$
0.4		$1.1 \times 10^{-9}$
0.5		$9.3 \times 10^{-10}$
0.6		$6.0 \times 10^{-10}$
0.7		$4.0 \times 10^{-10}$
0.8		$6.0 \times 10^{-10}$
0.9		$5.0 \times 10^{-10}$
1.0		0

Table 2: Errors of numerical solution of equation (42)

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