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WEAK TYPE ESTIMATES FOR COMMUTATORS GENERATED BY THE RIESZ TRANSFORM ASSOCIATED WITH SCHRÖDINGER OPERATORS

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1. Introduction

In recent years, many authors have been interested in the problems of harmonic analysis associated with Schrödinger operators. The Schrödinger operator is the operator defined by

$$A = -\Delta + V(x), \ \ x \in \mathbb{R}^n,$$

where Δ denotes the Laplacian operator. The function V is called a potential function. One topic of interest is the boundedness of Riesz transforms associated with Schrödinger operators on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ with nonnegative potential functions, which are defined as

$$R_j = \frac{\partial}{\partial x_j} A^{-\frac{1}{2}}, \ j = 1, 2, \cdots, n. \tag{1}$$

In [**14**], J. Zhong proved that if V is a nonnegative polynomial then Riesz transforms, defined at (1), are Calderon-Zygmund operators. Later, potential functions which are local integrable and in some inverse Hölder class were investigated by several authors. For $q > 1$, a local integrable function $V(\geq 0)$ belongs to B_q, the inverse Hölder class, if

$$\left(\frac{1}{|B|}\int_B V^q(x)dx\right)^{1/q} \leq C\frac{1}{|B|}\int_B V(x)dx,$$

holds for every ball $B \subset \mathbb{R}^n$. Z. Shen, [**13**], showed the relation between the inverse Hölder index q, n and L^p spaces on which the Riesz transforms are bounded. In [**13**], when the index $q \geq \frac{n}{2}$ we know that the Riesz transforms are bounded on some L^p spaces. In [**1**], Auscher et al improve Shen's results and show that $q \geq \frac{n}{2}$ is not a necessary restriction for the boundedness of Riesz transforms. For $p = 1$, the associated Hardy space and its dual BMO_A were introduced and studied by Dziubanski et al in [**4, 5, 6, 7**].

Another topic in harmonic analysis associated with Schrödinger operators is that of the commutators of Riesz transforms and BMO_A functions. In classical harmonic analysis, commutators of singular integral operators have been extensively studied

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and it well known that BMO functions are bounded on L^p for $1 < p < \infty$ (see [**3, 10, 12**]). Let T be a linear operator. The commutator of T and b is defined as

$$[b,\ T]f(x) = b(x)Tf(x) - T(bf)(x).$$

In [**3**], Coifman, Rochberg and Weiss proved that $[b,\ T]$ is a bounded operator on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ if and only if b is a BMO function. Furthermore, for $p = 1$, C. Perez obtained the weak type estimate and the boundedness from $H^1_b(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ in [**11**]. The boundedness properties of commutators of Riesz transforms associated with Schrödinger operators and BMO_A were investigated by Guo etc (in [**9**]) and Chu and Ma([**2**]). By giving the estimates of the kernels of Riesz transform associated with Schrödinger operators, Guo etc. obtained the $L^p(\mathbb{R}^n)(1 < p < p_0)$-boundedness of commutator generated by Riesz transform associated with the Schrödinger operator and BMO_A type functions. By using the sharp maximal function estimates and establishing the "good-λ inequality", Chu and Ma proved the same result. Motivated by their work, we study the endpoint estimates when $p = 1$. In this paper we will give a weak type estimate and discuss the boundedness of the commutators on Hardy spaces.

2. Notation and the Main Results.

Henceforth, Q will always denote a cube with sides parallel to the axes. λQ, where $\lambda > 0$, denotes the cube with the same center as Q and dilated by λ. For a locally integrable function f, we denote the average of f on Q by

$$f_Q = \frac{1}{|Q|}\int_Q f(y)dy.$$

Also $B = B(x,\ r)$ will denote a ball centered at x with radius r and corresponding notation applies for λB and f_B.

In this paper, we assume that $V \in B_q$ is a nonnegative function.

We now define function space BMO_A associated with A. To this end, define an auxiliary function

$$\rho(x,V) = \rho(x) = \frac{1}{m(x,V)} = \sup\left\{r > 0 : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy \le 1\right\}.$$

Definition 1. *([**7**]) Let $f \in L_{loc}(\mathbb{R}^n)$. We say that f belongs to BMO_A if there is a constant $C \ge 0$, so that $\frac{1}{|B_s|}\int_{B_s} |f - f_{B_s}| \le C$ and $\frac{1}{|B_r|}\int_{B_r} |f| \le C$ for all balls $B_s = B(x,s)$, $B_r = B(x,r)$ such that $s \le \rho(x) \le r$.*

The infimum of all such C is denoted by $\|f\|_{BMO_A}$.

Definition 2. *([**11**]) A function a is a b-*atom *if there is a cube Q for which:*

(1) $\operatorname{supp} a \subset Q$

(2) $\|a\|_{L^\infty} \le \frac{1}{|Q|}$

(3) $\int_Q a(y)dy = 0,$

(4) $\int_Q a(y)b(y)dy = 0.$

The space $H_b^1(\mathbb{R}^n)$ consists of the subspace of $L^1(\mathbb{R}^n)$ functions, f, which can be written as $f = \sum_j \lambda_j a_j$ where a_j are b-atoms and λ_j are complex numbers with $\sum_j |\lambda_j| < \infty$. We define its space norm as

$$\|f\|_{H_b^1} = \inf \left(\sum_j |\lambda_j| \right).$$

Definition 3. *([**2**]) For $\delta > 0$ we define the δ-sharp maximal operator $M_\delta^\sharp(f)$ associated with Schrödinger operator as $M_\delta^\sharp(f) = M_A^\sharp(|f|^\delta)^{1/\delta}$, where*

$$M_A^\sharp f(x) = \begin{cases} \sup_{x \in B} \dfrac{1}{|B|} \displaystyle\int_B |f(t) - f_B| dt, \ s \le \rho(x) \\ \sup_{x \in B} \dfrac{1}{|B|} \displaystyle\int_B |f(t)| dt, \ s > \rho(x) \end{cases}$$

for any $B = B(x, s)$.

Furthermore, a function $A : [0, \infty) \to [0, \infty)$ is a Young function if it is continuous, convex, and increasing with $A(0) = 0$, and $A(t) \to \infty$ as $t \to \infty$. Define the A-average of a function f over a cube Q by the following Luxemburg norm

$$\|f\|_{A,Q} = \inf \{\lambda > 0 : \frac{1}{|Q|} \int_Q A\Big(\frac{|f(y)|}{\lambda}\Big) dy \le 1\}.$$

Let T be a Riesz transform associated with Schrödinger operator A as at (1). We can now state our main results.

Theorem 4. *Let $V \in B_q$, $\frac{n}{2} < q < n$ and suppose $V(x)$ satisfies the following inequality*

$$\int_{B(x,R)} \frac{V(y)}{|x-y|^{n-1}} dy \le \frac{C}{R^{n-1}} \int_{B(x,R)} V(y) dy \tag{2}$$

and $b \in BMO_A$. Then there exists a positive constant C such that for every smooth function f with compact support and for all $\lambda > 0$,

$$|\{y \in \mathbb{R}^n : \ |[b, T] f(y)| > \lambda\}| \le C \|b\|_{BMO_A} \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} \Big(1 + \log^+ \Big(\frac{|f(y)|}{\lambda}\Big)\Big) dy.$$

Theorem 5. *Let V and b satisfy the same conditions as in Theorem 4. Then $[b,\ T]$ is a bounded operator from $H_b^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.*

3. Preliminary Results

In this section, we state the preliminary results we shall need.

Lemma 6. *([**7**]) Let $p \in [1, \infty)$. Then there exists a constant $C = C(n, p)$ such that for any $f \in BMO_A$ we have*

$$\Big(\frac{1}{|B|} \int_B |f(x) - f_B|^p dx\Big)^{1/p} \le C \|f\|_{BMO_A}, \quad \text{for any ball } B$$

and

$$\Big(\frac{1}{|B|} \int_B |f(x)|^p dx\Big)^{1/p} \le C \|f\|_{BMO_A}, \quad \text{for } B = (x, r),\ r \ge \rho(x).$$

Lemma 7. *([**2**]) Let $V \in B_q$ and $q \geq \frac{n}{2}$. Then for any $\gamma \in \mathbb{R}$, $(-\Delta + V)^{i\gamma}$ is of weak type $(1,1)$. Furthermore, $\nabla(-\Delta+V)^{-1/2}$ is of weak type $(1,1)$ when $V \in B_n$.*

The proof of Lemma 7 is similar to the proof of the standard Calderón-Zygmund operator given in [**8**].

The proof of the next lemma can be found in [**2**]. However, for completeness, we give a sketch of the proof.

Lemma 8. *([**2**]) Let $V \in B_q$ and $\frac{n}{2} \leq q < n$ and suppose $V(x)$ satisfies the inequality (2). Then the kernel of the operator $\nabla(-\Delta+V)^{-1/2}$ satisfies the required estimates of $C-Z$ operator kernel.*

Proof. Write

$$\nabla(-\Delta + V)^{-1/2} f(x) = \int_{\mathbb{R}^n} K(y,x) f(y) dy,$$

where

$$K(x,y) = -\frac{1}{2\pi} \int_R (-i\tau)^{-1/2} \nabla_x \Gamma(x,y,\tau) d\tau,$$

and $\Gamma(x,y,\tau)$ is the fundamental solution for $-\Delta + V(x) + i\tau$, $\tau \in \mathbb{R}$. From the proof of Theorem 0.8 in [**13**], we know that

$$|K(x,y)| \leq \frac{C_k}{\{1 + m(x,V)|x-y|\}^k} \cdot \frac{1}{|x-y|^n}.$$

Next, we fix $x_0, y_0 \in \mathbb{R}^n$, $h \in \mathbb{R}^n$ and $|h| < |x_0 - y_0|/4$. Let $R = |x_0 - y_0|/4$ and $u(x) = \nabla_y \Gamma(x, y_0, \tau)$. Since $\nabla_y \Gamma(x,y,\tau) = \nabla_x \Gamma(y,x,-\tau)$ (this can be seen by using a similar to the proof of Theorem 0.4 in [**13**]) for $\frac{1}{t} = \frac{1}{q} - \frac{1}{n}$, it follows from the imbedding theorem of Morrey that

$$\begin{aligned} |u(x_0 + h) - u(x_0)| &\leq C|h|^{1-\frac{n}{t}} \left(\int_{B(x_0,R)} |\nabla u|^t dx \right)^{1/t} \\ &\leq C \left(\frac{|h|}{R} \right)^{2-\frac{n}{q}} \{1 + Rm(x_0,V)\}^{k_0} \sup_{B(x_0,2R)} |u| \\ &\leq C \left(\frac{|h|}{R} \right)^{2-\frac{n}{q}} \frac{1}{(1 + |\tau|^{1/2}|x_0 - y_0|)^3} \cdot \frac{1}{|x_0 - y_0|^{n-1}}. \end{aligned}$$

Thus we have proved that for $x, y \in \mathbb{R}^n$, $h \in \mathbb{R}^n$ and $|h| < |x-y|/4$.

$$|\nabla_y \Gamma(x+h,y,\tau) - \nabla_y \Gamma(x,y,\tau)| \leq \frac{C}{(1 + |\tau|^{1/2}|x-y|)^3} \cdot \frac{|h|^\delta}{|x-y|^{n-1+\delta}},$$

where $\delta = 2 - \frac{n}{q} > 0$. This estimate also holds for $|x-y|/4 \leq |h| < |x-y|/2$. Therefore,

$$|K(x+h,y) - K(x,y)| \leq \frac{C|h|^\delta}{|x-y|^{n+\delta}}.$$

This concludes the proof. □

Lemma 9. *([**11**]) Let $Mf(x)$ be the Hardy-Littlewood maximal function of f. For any function f and for all $\lambda > 0$, there exists a positive constant C such that*

$$|\{y \in \mathbb{R}^n : M^2 f(y) > \lambda\}| \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda}\Big(1 + log^+\Big(\frac{|f(y)|}{\lambda}\Big)\Big)dy, \tag{3}$$

here $M^2 = M \circ M$.

Lemma 10. *([**2**]) Let $\gamma > 0$, $\lambda > 0$. For any $f \in L_{loc}(\mathbb{R}^n)$, the "good-$\lambda$ inequality" holds. That is*

$$|\{x \in \mathbb{R}^n : Mf(x) > 2\lambda, M_A^\sharp f(x) \leq \gamma\lambda\}| \leq 2^{n+1}\gamma|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}|. \tag{4}$$

Proof. Let $\Omega_\lambda = \{x \in \mathbb{R}^n : Mf(x) > \lambda\}$, suppose the measure of this set is finite, otherwise the inequality is obviously true. Then, for any $x \in \Omega_\lambda$, there exist a maximal cube Q^x contained x such that

$$\frac{1}{|Q^x|}\int_{Q^x} |f(y)|dy > \lambda. \tag{5}$$

Write $Q_j = \{Q^x : x \in \Omega_\lambda\}$. Since the Q^x's are mutually disjoint, so are the Q_j's. Therefore,

$$\Omega_\lambda = \cup_j Q_j.$$

Now, to prove Lemma 10, we only need to prove that for every Q_j,

$$|\{x \in Q_j : Mf(x) > 2\lambda, M_A^\#(f)(x) \leq \gamma\lambda\}| \leq 2^n\gamma|Q_j|. \tag{6}$$

Fix j, $x \in Q_j$. Suppose $Mf(x) > 2\lambda$. By the definition of maximal function of f, it should be that $Q \supset Q_j$ or $Q \subset Q_j$. If $Q \supset Q_j$, then the average of $|f|$ on Q is less than or equal to λ, but this will be contradiction to $Mf(x) > 2\lambda$. It implies $Q \subset Q_j$ for some j. For $x \in Q_j$, we have

$$\begin{aligned} M\left((f - \frac{1}{|Q_j^*|}\int_{Q_j^*}|f(t)|dt)\chi_{Q_j}\right)(x) \quad &\geq M(f\chi_{Q_j})(x) - \frac{1}{|Q_j^*|}\int_{Q_j^*}|f(t)|dt \\ &> 2\lambda - \lambda = \lambda. \end{aligned}$$

Here Q_j^* denotes the cube with the same center as Q_j and with twice the length of its side.

Thus,

$$|\{x \in Q_j : Mf(x) > 2\lambda\}| \leq \left|\{x \in Q_j : M\left((f - \frac{1}{|Q_j^*|}\int_{Q_j^*}|f(t)|dt)\chi_{Q_j}\right)(x) > \lambda\}\right|$$

Since M is of weak type (1,1), we obtain

$$\begin{aligned} \frac{1}{\lambda}\int_{Q_j}\left|f(y) - \frac{1}{|Q_j^*|}\int_{Q_j^*} f(t)dt\right| dy \quad &\leq \quad \frac{2^n|Q_j|}{\lambda}\frac{1}{|Q_j^*|}\int_{Q_j^*}\left|f(y) - \frac{1}{|Q_j^*|}\int_{Q_j^*} f(t)dt\right| dy \\ &\leq \quad \frac{2^n|Q_j|}{\lambda}M^\#(f)(\xi_j) \\ &\leq \quad \frac{2^n|Q_j|}{\lambda}M_A^\#(f)(\xi_j), \end{aligned}$$

where $\xi_j \in Q_j$.

To prove the inequality (6), we suppose $M_A^\#(f)(\xi_j) \leq \gamma\lambda$ for some $\xi_j \in Q_j$, otherwise the set on the left side in inequality (6) is empty, and the inequality (6) holds. By using the last inequality, we can deduce that inequality (6) holds for above $\xi_j \in Q_j$. This finish the proof of Lemma 10. $\square$

4. Proof of the Main Theorems

In this section we first give the δ-sharp type estimate which is the key estimate for Lemma 12.

Lemma 11. *Let T be a Riesz transform associated with the Schrödinger operator, $b \in BMO_A$. Then, for $0 < \delta < \varepsilon$, there exists a positive constant $C = C_{\delta,\varepsilon}$ such that*

$$M^{\sharp}_{\delta}([b, T]f)(x) \leq C\|b\|_{BMO_A}(M_\varepsilon(Tf)(x) + M^2 f(x)),$$

for all smooth function f .

Proof. We prove the Lemma 11 by separating it into two cases.

Let $B = B(x, r_0)$ be an arbitrary ball. We recall that $0 < \delta < 1$ implies $||\alpha|^\delta - |\beta|^\delta| \leq |\alpha - \beta|^\delta$ for $\alpha, \beta \in \mathbb{R}$.

Case 1, $r_0 \leq \rho(x)$. Then for all $c \in \mathbb{R}$, we have

$$\Big(\frac{1}{|B|}\int_B ||[b, T]f(y)|^\delta - |c|^\delta|dy\Big)^{1/\delta} \leq \Big(\frac{1}{|B|}\int_B |[b, T]f(y) - c|^\delta dy\Big)^{1/\delta}. \quad (7)$$

Let $f = f_1 + f_2 = f\chi_{2B} + f\chi_{(2B)^c}$, for an arbitrary constant a we can write

$$[b, T]f = (b - a)Tf - T((b - a)f_1) - T((b - a)f_2),$$

Choose $c = (T((b - a)f_2))_B,\ a = b_{2B}$, then we can estimate the left hand side of (7) by a multiple of

$$\begin{aligned}
&\Big(\frac{1}{|B|}\int_B ||[b, T]f(y)|^\delta - |c|^\delta|\, dy\Big)^{1/\delta}\\
\leq\ & C(\frac{1}{|B|}\int_B |(b(y) - b_{2B})Tf(y)|^\delta dy)^{1/\delta}\\
&+C(\frac{1}{|B|}\int_B |T((b - b_{2B})f_1)(y)|^\delta dy)^{1/\delta}\\
&+C(\frac{1}{|B|}\int_B |T((b - b_{2B})f_2)(y) - (T((b - b_{2B})f_2))_B|^\delta dy)^{1/\delta}\\
=\ & \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

To estimate I we use Hölder's inequality with $1 < r < \frac{\varepsilon}{\delta}$, $\frac{1}{r} + \frac{1}{r'} = 1$ and Lemma 6. We obtain

$$\begin{aligned}
\mathrm{I}\ &\leq\ C\Big(\frac{1}{|B|}\int_B |b(y) - b_{2B}|^{\delta r'} dy\Big)^{1/\delta r'}(\frac{1}{|B|}\int_B |Tf(y)|^{\delta r} dy)^{1/\delta r}\\
&\leq\ C\|b\|_{BMO_A} M_{\delta r}(Tf)(x) \leq C\|b\|_{BMO_A} M_\varepsilon(Tf)(x).
\end{aligned}$$

For part II we apply Lemma 7 and Kolmogorov's inequality. Then

$$\begin{aligned}
\mathrm{II}\ &\leq\ \frac{C}{|B|}\int_{2B} |T((b(y) - b_{2B})f)(y)|dy \leq \frac{C}{|2B|}\int_{2B} |(b(y) - b_{2B})f(y)|dy\\
&\leq\ C\|b - b_{2B}\|_{expL,2B}\|f\|_{LlogL,2B} \leq C\|b\|_{BMO_A} M_{LlogL} f(x),
\end{aligned}$$

here we used the fact $\|b-b_B\|_{expL,B}\le C\|b\|_{BMO_A}$ (see details in the proof of 3.1 in [**11**]).

For part III, we use the properties of kernel function $K(x,y)$ and Fubini's theorem. For any $y,z\in B$, $w\in(2B)^c$, we have following estimate

$$\begin{aligned}
\mathrm{III} &\le \frac{1}{|B|}\int_B |T((b-b_{2B})f_2)(y)-(T((b-b_{2B})f_2))_B|dy\\
&= \frac{1}{|B|}\int_B \Big|\int_{\mathbb{R}^n} K(y,w)(b(w)-b_{2B})f_2(w)dw\\
&\quad -\frac{1}{|B|}\int_B\int_{\mathbb{R}^n} K(z,w)(b(w)-b_{2B})f_2(w)dwdz\Big|dy\\
&\le \frac{1}{|B|^2}\int_B\int_B\int_{\mathbb{R}^n\setminus 2B}|K(y,w)-K(z,w)||(b(w)-b_{2B})f(w)|dwdzdy\\
&\le \frac{1}{|B|^2}\int_B\int_B\sum_{j=1}^{\infty}\int_{2^{j+1}B\setminus 2^jB}\frac{|y-z|^\delta}{|z-w|^{n+\delta}}|b(w)-b_{2B}||f(x)|dwdzdy\\
&\le C\sum_{j=1}^{\infty}\frac{r_0^\delta}{(2^jr_0)^{n+\delta}}\int_{2^{j+1}B}|b(w)-b_{2B}||f(w)|dw\\
&\le C\sum_{j=1}^{\infty}\frac{2^{-j\delta}}{(2^jr_0)^{n}}\int_{2^{j+1}B}|b(w)-b_{2^{j+1}B}||f(w)|dw\\
&\quad +C\sum_{j=1}^{\infty}2^{-j\delta}|b_{2^{j+1}B}-b_{2B}|\frac{1}{(2^jr_0)^n}\int_{2^{j+1}B}|f(w)|dw\\
&\le C\sum_{j=1}^{\infty}2^{-j\delta}\|b(w)-b_{2^{j+1}B}\|_{\exp L,2^{j+1}B}\|f\|_{L\log L,2^{j+1}B}+C\|b\|_{BMO_A}\sum_{j=1}^{\infty}\frac{j}{2^{j\delta}}Mf(x)\\
&\le C\|b\|_{BMO_A}M_{LlogL}f(x)+C\|b\|_{BMO_A}Mf(x)\le C\|b\|_{BMO_A}M_{LlogL}f(x).
\end{aligned}$$

Here we have used the fact that

$$\begin{aligned}
|b_{2^{j+1}B}-b_{2B}| &\le |b_{2^{j+1}B}-b_{2^jB}+b_{2^jB}-\ldots+b_{2^2B}-b_{2B}|\\
&\le |b_{2^{j+1}B}-b_{2^jB}|+\ldots+|b_{2^2B}-b_{2B}|\le Cj\|b\|_{BMO_A}.
\end{aligned}$$

Case 2, $r_0>\rho(x)$. Write $[b,\ T]f=bT(f)-T(bf_1)-T(bf_2)$. Then

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B|[b,\ T]f(y)|^\delta dy\right)^{1/\delta}\\
&\le \left(\frac{1}{|B|}\int_B|bTf(y)|^\delta dy\right)^{1/\delta}+\left(\frac{1}{|B|}\int_B|T(bf_1)(y)|^\delta dy\right)^{1/\delta}\\
&\quad +\left(\frac{1}{|B|}\int_B|T(bf_2)(y)|^\delta dy\right)^{1/\delta}\\
&= \mathrm{I}'+\mathrm{II}'+\mathrm{III}'.
\end{aligned}$$

To estimate I', we again use Hölder's inequality to deduce that

$$\begin{aligned} \mathrm{I}' &\leq \left(\frac{1}{|B|}\int_B |b(y)|^{\delta r'}dy\right)^{1/\delta r'}\left(\frac{1}{|B|}\int_B |T(f)(y)|^{\delta r}dy\right)^{1/\delta r} \\ &\leq \|b\|_{BMO_A} M_{\delta r}(T(f))(x) \leq \|b\|_{BMO_A} M_\varepsilon(Tf)(x). \end{aligned}$$

Following the argument giving the estimate for II we obtain the desired estimate for II'.

Now we turn to estimate III'. Making use of the estimation of the kernel, we have

$$\begin{aligned} &\mathrm{III}' \\ &\leq \frac{1}{|B|}\int_B \Big|\int_{\mathbb{R}^n\setminus 2B} K(y,w)b(w)f(w)dw\Big|dy \\ &\leq C\frac{1}{|B|}\int_B\int_{\mathbb{R}^n\setminus 2B}\Big|\frac{C_k}{\{1+|y-w|m(y,V)\}^k}\frac{1}{|y-w|^n}b(w)f(w)\Big|dwdy \\ &\leq C\frac{1}{|B|}\int_B\sum_{j=0}^{\infty}\int_{2^{j+2}B\setminus 2^{j+1}B}\Big|\frac{C_k}{\{1+|y-w|m(y,V)\}^k}\Big|\cdot\frac{1}{|y-w|^n}|b(w)f(w)| \\ &\leq C\frac{1}{|B|}\int_B\sum_{j=1}^{\infty}\int_{2^{j+1}B\setminus 2^jB}\Big|\frac{C_k}{\{|y-w|m(y,V)\}^k}\frac{1}{|y-w|^n}b(w)f(w)\Big|dwdy \\ &\leq C\frac{1}{|B|}\int_B\sum_{j=1}^{\infty}\frac{1}{2^{kj}(2^jr_0)^n}\int_{2^{j+1}B\setminus 2^jB}|b(w)f(w)|dwdy \\ &\leq C\sum_{j=1}^{\infty}\frac{1}{2^{kj}|2^jB|}\int_{2^{j+1}B}|b(w)f(w)|dw \\ &\leq C\sum_{j=1}^{\infty}\frac{1}{2^{kj}}\|b\|_{expL,2^{j+1}B}\|f\|_{LlogL,2^{j+1}B} \\ &\leq C\|b\|_{BMO_A}M_{LlogL}f(x)\leq C\|b\|_{BMO_A}M^2f(x), \end{aligned}$$

where we have used the fact $Mf(x) \leq M_{LlogL}f(x) \approx M^2f(x)$ (see [**11**] p.170). This completes the proof of the lemma 11. □

Lemma 12. *Let $\Phi(t) = t(1+\log^+ t)$ where T is a Riesz transform associated with the Schrödinger operator and $b \in BMO_A$. Suppose $V \in B_q$, $\frac{n}{2} < q < n$ and V still satisfies the inequality (2). Then there exists a constant C such that*

$$\begin{aligned} &\sup_{t>0}\frac{1}{\Phi(1/t)}|\{y\in\mathbb{R}^n:\ |[b,\ T]f(y)|>t\}| \\ &\quad\leq\ C\|b\|_{BMO_A}\sup_{t>0}\frac{1}{\Phi(1/t)}|\{y\in\mathbb{R}^n: M^2f(y)>t\}|, \end{aligned}$$

for all smooth functions with compact support.

Proof. Let f be a smooth function with compact support. We have to prove the inequality above with a constant C is independent of f. Instead of working with the functional on the left hand side of the inequality, we consider the following functional. For some $\delta > 0$, $L_{\Phi,\delta}(f) = L_\delta(f)$ is defined by

$$L_\delta(f) = \sup_{t>0} \frac{1}{\Phi(1/t)} |\{y \in \mathbb{R}^n : M_\delta([b,\ T]f)(y) > t\}|.$$

We claim that the operator $[b,\ T]$ satisfies the following inequality. For any $0 < \delta < 1,\ \varepsilon > 0$,

$$L_\delta(f) \le \varepsilon C L_\delta(f) + C(\varepsilon)\|b\|_{BMO_A} \sup_{t>0} \frac{1}{\Phi(1/t)} |\{y \in \mathbb{R}^n : M^2 f(y) > t\}| \quad (8)$$

To prove (8), for any $t > 0,\ \delta > 0$, by Lemma 10, we have

$$\begin{aligned}
&|\{y \in \mathbb{R}^n : M_\delta([b,\ T]f)(y) > t\}| \\
&= |\{y \in \mathbb{R}^n : M(|[b,\ T]f|^\delta)(y) > t^\delta\}| \\
&\le \varepsilon C_n |\{y \in \mathbb{R}^n : M(|[b,T]f|^\delta) > \frac{t^\delta}{2}\}| + |\{y \in \mathbb{R}^n : M_A^\sharp(|[b,\ T]f|^\delta)(y) > \varepsilon t^\delta\}| \\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

To estimate II, let $\varepsilon = r\delta,\ 1 < r < \frac{1}{\delta}$, applying Lemma 11 we can write

$$\begin{aligned}
\mathrm{II} &= |\{y \in \mathbb{R}^n : M_\delta^\sharp([b,\ T]f)(y) > \varepsilon^{1/\delta} t\}| \\
&\le |\{y \in \mathbb{R}^n : M_{\delta r}(Tf)(y) + M^2 f(y) > \frac{\varepsilon^{1/\delta} t}{C\|b\|_{BMO_A}}\}| \\
&\le |\{y \in \mathbb{R}^n : M_{\delta r}(Tf)(y) > \frac{\varepsilon^{1/\delta} t}{2C\|b\|_{BMO_A}}\}| \\
&\quad + |\{y \in \mathbb{R}^n : M^2 f(y) > \frac{\varepsilon^{1/} t}{2C\|b\|_{BMO_A}}\}|.
\end{aligned}$$

Choose $a = \dfrac{\varepsilon^{1/\delta}}{2C\|b\|_{BMO_A}}$. Then, dividing the above inequality by $\Phi(1/t)$ and using that $\Phi(t)$ is doubling, we have

$$\begin{aligned}
&\frac{1}{\Phi(1/t)}|\{y \in \mathbb{R}^n : M_\delta([b,\ T]f)(y) > t\}| \\
\le\ & \frac{\varepsilon C}{\Phi(1/t)}|\{y \in \mathbb{R}^n : M_\delta([b,\ T]f)(y) > \frac{t}{2^{1/\delta}}\}| \\
&+\frac{1}{\Phi(1/t)}|\{y \in \mathbb{R}^n : M_{\delta r}(Tf)(y) > at\}| + \frac{1}{\Phi(1/t)}|\{y \in \mathbb{R}^n : M^2 f(y) > at\}| \\
\le\ & \frac{\varepsilon C}{\Phi(2^{1/\delta}/t)}|\{y \in \mathbb{R}^n : M_\delta([b,\ T]f)(y) > \frac{t}{2^{1/\delta}}\}| \\
&+\frac{C\|b\|_{BMO_A}}{\Phi(1/at)}|\{y \in \mathbb{R}^n : M_{\delta r}(Tf)(y) > at\}| \\
&+\frac{C\|b\|_{BMO_A}}{\Phi(1/at)}|\{y \in \mathbb{R}^n : M^2 f(y) > at\}| \\
\le\ & \varepsilon C L_\delta(f) + C\|b\|_{BMO_A} \sup_{t>0} \frac{1}{\Phi(1/t)}|\{y \in \mathbb{R}^n : M_{\delta r}(Tf)(y) > t\}| \\
&+C\|b\|_{BMO_A} \sup_{t>0} \frac{1}{\Phi(1/t)}|\{y \in \mathbb{R}^n : M^2 f(y) > t\}|.
\end{aligned}$$

Now, since $0 < r\delta < 1$, we can use the estimate $M^\sharp_\alpha(Tf)(y) \le CMf(y)$, which holds for all $0 < \alpha < 1$. Then,

$$\begin{aligned}
L_\delta(f) &\le \varepsilon C L_\delta(f) + C\|b\|_{BMO_A} \sup_{t>0} \frac{1}{\Phi(1/t)}|\{y \in \mathbb{R}^n : M^\sharp_{\delta r}(Tf)(y) > t\}| \\
&\quad +C\|b\|_{BMO_A} \sup_{t>0} \frac{1}{\Phi(1/t)}|\{y \in \mathbb{R}^n : M^2 f(y) > t\}| \\
&\le \varepsilon C L_\delta(f) + C\|b\|_{BMO_A} \sup_{t>0} \frac{1}{\Phi(1/t)}|\{y \in \mathbb{R}^n : Mf(y) > t\}| \\
&\quad +C\|b\|_{BMO_A} \sup_{t>0} \frac{1}{\Phi(1/t)}|\{y \in \mathbb{R}^n : M^2 f(y) > t\}| \\
&\le \varepsilon C L_\delta(f) + C\|b\|_{BMO_A} \sup_{t>0} \frac{1}{\Phi(1/t)}|\{y \in \mathbb{R}^n : M^2 f(y) > t\}|.
\end{aligned}$$

To finish the proof of Lemma 12 we need to show that $L_\delta(f)$ is finite so that we can choose $\varepsilon < \dfrac{1}{C}$ to conclude that

$$L_\delta(f) \le C\|b\|_{BMO_A} \sup_{t>0} \frac{1}{\Phi(1/t)}|\{y \in \mathbb{R}^n : M^2 f(y) > t\}|. \tag{9}$$

For each $m = 1, 2, 3\cdots$ we let $b_m = \inf\{b,\ m\}$. Since $\|b_m\|_{BMO_A} \le C\|b\|_{BMO_A}$ with C independent of m, we shall prove that $L_{\Phi,\delta,b}$ is finite with b replaced by b_m. For $b_m \to b$ as $m \to \infty$, we shall let $m \to \infty$ to conclude the proof of inequality (9).

Now, since f is bounded and has compact support, we may assume that $\operatorname{supp} f \subset B(0, s)$ for some $s > 0$. Recalling that $b = b_m$, $\|b\|_{L^r} \le m$, for $|x| > 2s$, by Lemma

8 we have,

$$\begin{aligned} |[b,\ T]f(x)| &\leq C\int_{B(0,s)} \frac{|b(x)-b(y)|}{|x-y|^n}|f(y)|dy \\ &\leq \frac{2Cm}{|x|^n}\int_{B(0,C|x|)} |f(y)|dy \leq CmMf(x). \end{aligned}$$

Using the above inequality and $0<\delta<1$, $t>0$, we obtain

$$\begin{aligned} &\frac{1}{\Phi(1/t)}|\{x\in \mathbb{R}^n : M_\delta([b,\ T]f(x)>t\}| \\ \leq\ & \frac{1}{\Phi(1/t)}|\{x\in \mathbb{R}^n : M(\chi_{B(0,2s)}[b,\ T]f)(x)>\frac{t}{2}\}| \\ &+\frac{1}{\Phi(1/t)}|\{x\in \mathbb{R}^n : M(\chi_{\mathbb{R}^n\setminus B(0,2s)}[b,\ T]f)(x)>\frac{t}{2}\}| \\ \leq\ & \frac{1}{\Phi(1/t)}\ \frac{1}{t}\int_{B(0,2s)} |[b,\ T]f(x)|dx + \frac{1}{\Phi(1/t)}|\{x\in \mathbb{R}^n :\ M^2f(x)>Cmt\}| \\ \leq\ & C|B(0,2s)|\left(\frac{1}{|B(0,2s)|}\int_{B(0,2s)} |[b,\ T]f(y)|^2dy\right)^{1/2} \\ &+\frac{C}{\Phi(1/t)}\int_{\mathbb{R}^n}\Phi\left(\frac{|Cmf(y)|}{t}\right)dy \\ \leq\ & C|B(0,2s)|\|b\|_{BMO_A}\left(\frac{1}{|B(0,s)|}\int_{B(0,s)} |f(y)|^2dy\right)^{1/2} \\ &+C\int_{B(0,s)}\Phi(f(y))dy, \end{aligned}$$

here we used the fact that M is of weak type (1,1) and the analog for M^2 given in Lemma 9. Since f is smooth with compact support the last expression is finite, and the Lemma 12 is proved. □

Proof of Theorem 4 We prove that for each f and $\lambda>0$, there exists a positive constant C such that

$$|\{y\in \mathbb{R}^n :\ |[b,\ T]f(y)|>\lambda\}| \leq C\|b\|_{BMO_A}\int_{\mathbb{R}^n}\frac{|f(y)|}{\lambda}(1+log^+(\frac{|f(y)|}{\lambda}))dy.$$

By homogeneity it is sufficient to consider the case of $\lambda=1$. Namely,

$$|\{y\in \mathbb{R}^n :\ |[b,\ T]f(y)|>1\}| \leq C\|b\|_{BMO_A}\int_{\mathbb{R}^n}|f(y)|(1+log^+|f(y)|)dy.$$

Assume that f is smooth with compact support, by Lemma 12 we have

$$\begin{aligned} &\sup_{t>0}\frac{1}{\Phi(1/t)}|\{y\in \mathbb{R}^n :\ \ |[b,\ T]f(y)>t\}| \\ \leq\ & C\|b\|_{BMO_A}\sup_{t>0}\frac{1}{\Phi(1/t)}|\{y\in \mathbb{R}^n\ :\ M^2f(y)>t\}|. \end{aligned}$$

For all $t>0$, making use of Lemma 9, we deduce that

$$|\{y\in \mathbb{R}^n:\ M^2f(y)>t\}|\le C\int_{\mathbb{R}^n}\Phi(\frac{|f(y)|}{t})dy\le C\int_{\mathbb{R}^n}\Phi\left(|f(y)|\right)\Phi(1/t)dy,$$

Since Φ is submultiplicative. Hence,

$$\begin{aligned}|\{y\in \mathbb{R}^n:\ |[b,\ T]f(y)|>1\}| &\le \sup_{t>0}\frac{1}{\Phi(1/t)}|\{y\in \mathbb{R}^n:\ |[b,\ T]f(y)|>t\}|\\ &\le C\|b\|_{BMO_A}\sup_{t>0}\frac{1}{\Phi(1/t)}|\{y\in \mathbb{R}^n: M^2f(y)>t\}|\\ &\le C\|b\|_{BMO_A}\int_{\mathbb{R}^n}\Phi(|f(y)|)dy\\ &= C\|b\|_{BMO_A}\int_{\mathbb{R}^n}|f(y)|(1+log^+|f(y)|)dy,\end{aligned}$$

and this yields the desired estimate. □

Recall that BMO may be defined by

$$BMO(\mathbb{R}^n)=\{f\in L^1_{loc}(\mathbb{R}^n):\|f\|_{BMO}=\|M^\sharp f\|_\infty<\infty\}.$$

Two basic facts about BMO will be used in the proof of Theorem 5. First

$$|f_{2^kB}-f_B|\le C(k+1)\|f\|_{BMO},\ k>0.$$

and then the John-Nirenberg inequality

$$\|f\|_{BMO}\sim\sup_B(\frac{1}{|B|}\int_B|f(y)-f_B|^pdy)^{1/p},\ \forall p\ge 1.$$

Proof of Theorem 5 Let $b\in BMO$. By the atomic decomposition of Hardy space, we only need to prove that there exists a constant C such that

$$\int_{\mathbb{R}^n}|[b,\ T]a(y)|dy\le C\|b\|_{BMO}\|f\|_{H^1_b(\mathbb{R}^n)},$$

for each b-atom a.

To prove this, suppose $\operatorname{supp} a\subset B(x,r)$ for some ball B. Then

$$\int_{\mathbb{R}^n}|[b,\ T]a(y)|dy=\int_{2B}|[b,\ T]a(y)|dy+\int_{\mathbb{R}^n\backslash 2B}|[b,\ T]a(y)|dy=\mathrm{I}+\mathrm{II}.$$

The estimate of I follows the boundedness of $[b,\ T]$ on $L^2(\mathbb{R}^n)$ (see [6]) and the size condition of atom a. That is,

$$\begin{aligned}\mathrm{I}\ &\le\ C|B|\left(\frac{1}{|2B|}\int_{2B}|[b,\ T]a(y)|^2dy\right)^{1/2}\le C\|b\|_{BMO}|B|(\frac{1}{|B|}\int_B|a(y)|^2dy)^{1/2}\\ &\le\ C\|b\|_{BMO}|B|\ \|a\|_{L^2(\mathbb{R}^n)}\le C\|b\|_{BMO}.\end{aligned}$$

Now, to estimate II, we split $[b,\ T]$ as $[b,\ T]a=(b-b_B)Ta-T((b-b_B)a)$, then

$$\mathrm{II}\le\int_{\mathbb{R}^n\backslash 2B}|(b(x)-b_B)Ta(x)|dx+\int_{\mathbb{R}^n\backslash 2B}|T((b-_B)a)(x)|dx=\mathrm{III}+\mathrm{IV}.$$

By Lemma 8, we know that the kernel of T satisfies the kernel's estimate of Calderón-Zygmund operator. Let $B = B(x_B, r)$ using the cancellation condition $\int_B a(y)dy = 0$, then

$$\begin{aligned}
\mathrm{III} &\leq \int_B |a(y)| \int_{\mathbb{R}^n\setminus 2B} |K(x,y) - K(x,x_B)||b(x) - b_B|dxdy \\
&= \int_B |a(y)| \sum_{j=1}^{\infty} \int_{2^j r\leq |x-x_B|<2^{j+1}r} \frac{|y-x_B|^\delta}{|x-x_B|^{n+\delta}} |b(x) - b_B|dxdy \\
&\leq \int_B |a(y)|dy \sum_{j=1}^{\infty} \frac{2^{-j\delta}}{|2^{j+1}B|} \int_{2^{j+1}B} |b(x) - b_B|dx \\
&\leq C\sum_{j=1}^{\infty} 2^{-j\delta}[\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B} |b(x) - b_{2^{j+1}B}|dx + |b_{2^{j+1}B} - b_B|] \\
&\leq C\sum_{j=1}^{\infty} 2^{-j\delta}[\|b\|_{BMO} + (j+1)\|b\|_{BMO}] \\
&\leq C\|b\|_{BMO}.
\end{aligned}$$

By the definition of a, we have

$$\int_B (b(y) - b_B)a(y)dy = \int_B a(y)b(y)dy - b_B \int_B a(y) = 0,$$

Moreover, similar to Lemma 3.3 from [9, p.413], we obtain the estimate of the final part. That is,

$$\begin{aligned}
\mathrm{IV} &= \int_{\mathbb{R}^n\setminus 2B} |T((b-b_B)a)(x)|dx = \int_{\mathbb{R}^n\setminus 2B} \left| \int_B k(x,y)(b(y) - b_B)a(y)dy \right| dx \\
&\leq \int_{\mathbb{R}^n\setminus 2B} \left| \int_B (K(x,y) - K(x,x_B))(b(y) - b_B)a(y)dy \right| dx \\
&\leq \int_B |b(y) - b_B)|\,|a(y)| \int_{\mathbb{R}^n\setminus 2B} |K(x,y) - K(x,x_B)|dxdy \\
&\leq \int_B |b(y) - b_B||a(y)| \sum_{j=1}^{\infty} \int_{2^j r\leq |x-x_B|\leq 2^{j+1}r} \frac{|y-x_B|^\delta}{|x-x_B|^{n+\delta}} dxdy \\
&\leq \int |b(y) - b_B||a(y)| \sum_{j=1}^{\infty} \frac{2^{-j\delta}}{|2^{j+1}B|} \int_{2^{j+1}B} \leq \sum_{j=1}^{\infty} 2^{-j\delta} \int_B |b(y) - b_B||a(y)|dy \\
&\leq C\int_B |b(y) - b_B|dy \leq C\|b\|_{BMO},
\end{aligned}$$

where $\delta = 2 - \dfrac{n}{q} > 0$. This concludes the proof of Theorem 5. □

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