CLASSICAL SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

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1. Introduction

The simplest, most elementary proofs of the existence of solutions of the Navier-Stokes equations are given via Galerkin approximation. The core of such proofs lies in obtaining estimates for the approximations from which one can infer their convergence (or at least the convergence of a subsequence of the approximations) as well as some degree of regularity of the resulting solution. The first to use this approach was Hopf [5], who based an existence theorem for the initial boundary value problem on an energy estimate for Galerkin approximations. However, based on this single estimate, Hopf's theorem provides very little regularity of the solution, in fact, insufficient regularity to prove the solution's uniqueness if the domain is three-dimensional. To remedy this situation, Kiselev and Ladyzhenskaya [7] introduced a second estimate for the approximations which yields enough further regularity for a uniqueness theorem. As is well known, this second estimate holds only locally in time unless the data are small or the domain is two-dimensional, a circumstance which has stimulated much speculation over the question of "unique solvability in the large". On the other hand, even during the time interval for which it holds, the estimate of Kiselev and Ladyzhenskaya provides far less than the full classical regularity of the solution.

An interesting varient of the Galerkin method, yielding a somewhat more regular solution, under weaker assumptions on the data, has been given by Prodi [11]. Prodi's existence theorem is based on an estimate, entirely different from those of Hopf and of Kiselev and Ladyzhenskaya, which is available when eigenfunctions of the Stokes operator are used as a basis for the approximations. This estimate is somewhat less elementary than those of Hopf and of Kiselev and Ladyzhenskaya, as it requires an L^2-theory of regularity for the steady Stokes equations. Still, like the estimate of Kiselev and Ladyzhenskaya, Prodi's holds only locally in time and yields only a generalized solution. Until now, the classical regularity of such generalized solutions has been proved only by resort to entirely different and more complicated methods, methods which have invariably depended on potential theoretic results for the Stokes equations. In this regard, we cite particularly the important

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contributions of Ito [6], Fujita and Kato [3], Ladyzhenskaya [8,9], and Solonnikov [15,16].

The main point of the present paper is to show how the Galerkin approach to existence theorems can be pushed further, through additional estimates, to give the classical regularity of the solution directly and easily, with minimal reliance on the regularity theory for the Stokes equations. In fact, the only result which will be needed concerning the regularity of solutions of the Stokes equations is the L^2-estimate of the second derivatives of stationary solutions, which is already needed for Prodi's estimate of the Galerkin approximations, and which has been recently proved in a relatively simple way, independently of potential theory, by Solonnikov and Ščadilov [17].

Our procedure begins with the introduction of two infinite sequences of differential inequalities for the Galerkin approximations. Integration of the first inequality of one sequence, over a time interval [0,T), gives Prodi's estimate. Integration of the first two inequalities of the other sequence, over [0,T), gives, respectively, the estimates of Hopf and of Kiselev and Ladyzhenskaya. To proceed further, with the integration of the succeeding inequalities, it is necessary to work with both sequences simultaneously, using recursively the estimates obtained by integrating the inequalities of one sequence to linearize and integrate those of the other. To avoid the necessity of compatibility conditions for the data, which for the Navier-Stokes equations are of a very complicated non-local nature, these subsequent differential inequalities are integrated over time intervals $[\varepsilon,T)$, with $\varepsilon > 0$. For this, it is necessary to obtain "initial estimates" at $t = \varepsilon$, which we do utilizing yet another sequence of identities and inequalities for the Galerkin approximations.

Combining these estimates for the Galerkin approximations, one can infer the existence of a solution $u \in C^\infty(0,T;W_2^2(\Omega)) \cap L^\infty(0,T';W_2^2(\Omega))$, where Ω is the spatial domain and T' is any number less than T. With this degree of regularity in hand, the solution's classical regularity follows by a standard argument, which is again based on only an L^2-estimate for the steady Stokes equations.

Although the procedure just described is simple, we will not attempt to give all the details here. The details are given in [4], along with a number of extensions and related results. One extension is the existence theorem, for classical solutions, in the case of unbounded three-dimensional domains with possibly non-compact boundaries. For such domains the result is new. Also in [4], the local existence theorem is proved for initial velocities merely required to possess a finite Dirichlet integral. This result is new in the case of unbounded domains, where, unless Poincaré's inequality holds, the initial velocity need not belong to L^2. One of the related topics studied in [4] is the decay of solutions, in unbounded domains, as $t \to \infty$. If the initial velocity is square-summable and the forces and boundary values are homogeneous, the decay is shown to be of order $t^{-1/2}$. The proof of this is outlined in the final part of the present paper.

2. Galerkin Approximations

Let $\Omega \subset R^3$ be a bounded domain with boundary $\partial\Omega$ of class C^3. We consider the initial boundary value problem

$$u_t + u\cdot\nabla u = -\nabla p + \Delta u \tag{1a}$$

$$\nabla\cdot u = 0 \tag{1b}$$

$$u(x,0) = u_o(x) \tag{1c}$$

$$u|_{\partial\Omega} = 0 \tag{1d}$$

for the vector velocity $u(x,t)$ and scalar pressure $p(x,t)$ of a viscous incompressible fluid. The problem has been normalized so that the density and viscosity are equal to one. It is required that the equations (1a), (1b) should be satisfied in a space-time cylinder $\Omega \times (0,T)$. The initial velocity is u_o. For simplicity, we have taken the external force and boundary values to be homogeneous; inhomogeneous boundary values and forces are considered in [4]. We call u,p a classical solution of problem (1) if $u \in C(\bar{\Omega} \times [0,T))$, if $u_t, \nabla u, \Delta u, \nabla p \in C(\Omega \times (0,T))$ and if the conditions of the problem are satisfied continuously.

Employing the Galerkin method, we consider approximate solutions

$$u^n(x,t) = \sum_{k=1}^{n} c_{kn}(t)\, a^k(x)$$

developed in terms of a system of functions $\{a^k\}$ which is complete in the space $J_1^*(\Omega)$ of divergence-free vector-valued functions from $\overset{\circ}{W}{}_2^1(\Omega)$. A special choice of the functions $\{a^k\}$ will be made shortly, but for now they are merely taken to be smooth and orthonormal in $L^2(\Omega)$. The coefficients $c_{kn}(t)$ are determined by the system of ordinary differential equations

$$(u_t^n, a^\ell) - (\Delta u^n, a^\ell) = -(u^n\cdot\nabla u^n, a^\ell)\ , \tag{2}$$

$\ell = 1,\dots,n$, with initial conditions $c_{kn}(0) = (a_k, u_o)$. Here, (ϕ,ψ) denotes the L^2 inner product $\int_\Omega \phi\cdot\psi\, dx$.

Hopf's energy identity for the Galerkin approximations is obtained by multiplying (2) by $c_{\ell n}(t)$, summing $\Sigma_{\ell=1}^n$, and integrating several terms by parts, noting in particular that $(u^n\cdot\nabla u^n, u^n) = 0$. The result is

$$\frac{1}{2}\frac{d}{dt}\|u^n\|^2 + \|\nabla u^n\|^2 = 0\ , \tag{3}$$

where $\|\cdot\|$ denotes the L^2-norm.

Clearly, if $u_o \in L^2(\Omega)$, one has a bound for the initial values

$$\|u^n(0)\| \leq \|u_o\| ,$$

which is uniform n. Hence (3) can be integrated from 0 to t, yielding the energy estimate

$$\frac{1}{2}\|u^n(t)\|^2 + \int_0^t \|\nabla u^n\|^2 d\tau \leq \frac{1}{2}\|u_o\|^2 \tag{4}$$

on which Hopf's existence theorem is based.

Kiselev and Ladyzhenskaya's estimate is based on an identity found by differentiating (2) with respect to t, multiplying by $d/dt\, c_{\ell n}$, summing, and integrating several terms by parts. The result is

$$\frac{1}{2}\frac{d}{dt}\|u_t^n\|^2 + \|\nabla u_t^n\|^2 = -(u_t^n \cdot \nabla u^n, u_t^n) . \tag{5}$$

The right side of (5) can be estimated by using successively Hölder's inequality, Sobolev's inequality, Young's inequality, and the inequality $\|\nabla u^n\|^2 \leq \|u^n\| \cdot \|u_t^n\|$, which follows from (3):

$$\begin{aligned}
|(u_t^n \cdot \nabla u^n, u_t^n)| &\leq \|\nabla u^n\| \cdot \|u_t^n\|_4^2 \\
&\leq c\|\nabla u^n\| \cdot \|u_t^n\|^{1/2} \cdot \|\nabla u_t^n\|^{3/2} \\
&\leq c\|\nabla u^n\|^4 \cdot \|u_t^n\|^2 + \frac{1}{2}\|\nabla u_t^n\|^2 \\
&\leq c\|u_o\|^2 \|u_t^n\|^4 + \frac{1}{2}\|\nabla u_t^n\|^2 .
\end{aligned}$$

Here, $\|\cdot\|_p$ denotes the L^p-norm. Using this estimate for its right side, (5) becomes

$$\frac{d}{dt}\|u_t^n\|^2 + \|\nabla u_t^n\|^2 \leq c\|u_o\|^2 \|u_t^n\|^4 . \tag{6}$$

To obtain estimates for the Galerkin approximations by integrating (6), one needs a bound for the initial values $\|u_t^n(0)\|$, which is uniform in n. Using (2) one obtains

$$\|u_t^n(0)\| \leq \|\tilde{\Delta} u^n(0)\| + \|u^n(0) \cdot \nabla u^n(0)\| . \tag{7}$$

Here, $\tilde{\Delta} = P\Delta$, where P is the orthogonal projection of $L^2(\Omega)$ onto its subspace $J(\Omega)$, formed by completing the set of solenoidal test functions. A bound for the right side of (7) is found almost trivially if $u_o \in J_1^*(\Omega) \cap W_2^2(\Omega)$, provided the functions $\{a^k\}$ are chosen so that $a^1 = u_o/\|u_o\|$. However, we shall need to choose the $\{a^k\}$ differently, as eigenfunctions of the Stokes operator $\tilde{\Delta}$. In this case, a bound of the form $2\|\tilde{\Delta}u_o\| + c\|\nabla u_o\|^2 + c\|\nabla u_o\|^3$ is obtained for the right side of (7) using the inequality (18), below, and the orthogonality of the eigenfunctions $\{a^k\}$ in the inner-products $(\nabla\phi,\nabla\psi)$ and $(\tilde{\Delta}\phi,\tilde{\Delta}\psi)$. It follows, if $u_o \in J_1^*(\Omega) \cap W_2^2(\Omega)$, that by integrating (6) one obtains estimates of the form

$$\|u_t^n(t)\|, \quad \int_0^t \|\nabla u_t^n\|^2 \, d\tau, \quad \|\nabla u^n(t)\| \le G(t), \tag{8}$$

for t in some interval $[0,T)$. Here, the inequality $\|\nabla u^n\|^2 \le \|u_o\|\cdot\|u_t^n\|$ has been used again, in a final step, to get the estimate for $\|\nabla u^n(t)\|$. The estimates (8) are the ones on which the existence theorem of Kiselev and Ladyzhenskaya is based.

Prodi's estimate for the Galerkin approximations is based on the identity

$$\frac{1}{2}\frac{d}{dt}\|\nabla u^n\|^2 + \|\tilde{\Delta}u^n\|^2 = (u^n\cdot\nabla u^n, \tilde{\Delta}u^n), \tag{9}$$

which holds, simultaneously with (3) and (5), if the basis functions $\{a^k\}$ are taken to be the eigenfunctions of the eigenvalue problem

$$-\Delta a = \lambda a + \nabla p, \quad x \in \Omega \tag{10a}$$

$$\nabla\cdot a = 0, \quad x \in \Omega \tag{10b}$$

$$a|_{\partial\Omega} = 0. \tag{10c}$$

It follows from the regularity theory for the Stokes equations, discussed below, that the eigenfunctions a^k belong to $W_2^2(\Omega)$, so that one can write $\tilde{\Delta}a^k = -\lambda_k a^k$, where λ_k is the k^{th} eigenvalue. Thus, multiplying (2) by λ_ℓ and summing $\Sigma_{\ell=1}^n$, one obtains

$$(u_t^n, -\tilde{\Delta}u^n) + (\Delta u^n, \tilde{\Delta}u^n) = (u^n\cdot\nabla u^n, \tilde{\Delta}u^n)$$

and hence (9).

The regularity theory needed above, and again below, consists of L^2-estimates of the general form

$$\|D^2u\|_{\Omega\cap G''} \le c\|f\|_{\Omega\cap G'} + c\|\nabla u\|_{\Omega\cap G'} + c\|u\|_{\Omega\cap G'} \tag{11}$$

for solutions of Stokes' problem:

$$\Delta u = \nabla p - f, \quad x \in \Omega \tag{12a}$$

$$\nabla\cdot u = 0, \quad x \in \Omega \tag{12b}$$

$$u|_{\partial\Omega} = 0. \tag{12c}$$

Here, G'' and G' are bounded open subsets of R^3, with $\overline{G''} \subset G'$, and $\|D^2u\|^2 \equiv \Sigma_{i,j=1}^{3} \|\partial^2 u/\partial x_i \partial x_j\|^2$. For a simple proof of the "interior estimate", i.e., the case $G' \subset \Omega$, see Ladyzhenskaya [9 ,p.38]. For a relatively simple proof of the "estimate up to the boundary", i.e., the case that $G'' \cap \partial\Omega$ is nonempty, see Solonnikov and Ščadilov [17]. For a potential theoretic proof, giving general L^p-estimates, see Cattabriga [1].

To estimate the right side of (9) we shall need several consequences of (11). If Ω is a bounded domain, the global estimate

$$\|D^2u\| \leq c(\|\tilde{\Delta}u\| + \|\nabla u\|) , \tag{13}$$

for solutions of (12), follows almost immediately from (11), setting $-f = \tilde{\Delta}u$. Using a slightly refined version of (11), we have also proved (13) for unbounded domains, even those with noncompact boundaries; see [4]. Of course, if Ω is unbounded, we require $u(x) \to 0$ as $|x| \to \infty$, in a generalized sense. Using, in addition to (13), the Sobolev inequality $\|\phi\|_3 \leq c\|\nabla\phi\|^{\frac{1}{2}} \cdot \|\phi\|^{\frac{1}{2}}$ for $\phi \in C_o^\infty(R^3)$, and estimates for the Sobolev norms of a function continued beyond its original domain of definition, one can show solutions of (12) satisfy

$$\|\nabla u\|_3 \leq c(\|\tilde{\Delta}u\|^{\frac{1}{2}} \cdot \|\nabla u\|^{\frac{1}{2}} + \|\nabla u\|). \tag{14}$$

This inequality, like (13), is valid for general unbounded domains; see [4].

The right side of (9) can now be estimated using successively Holder's inequality, Sobolev's inequality $\|\phi\|_6 \leq \|\nabla\phi\|$, the inequality (14), and Young's inequality:

$$\begin{aligned} |(u\cdot\nabla u , \tilde{\Delta}u)| &\leq \|u\|_6 \cdot \|\nabla u\|_3 \cdot \|\tilde{\Delta}u\| \\ &\leq c\|\nabla u\|(\|\tilde{\Delta}u\|^{\frac{1}{2}} \cdot \|\nabla u\|^{\frac{1}{2}} + \|\nabla u\|)\|\tilde{\Delta}u\| \\ &\leq c\|\nabla u\|^4 + c'\|\nabla u\|^6 + \frac{1}{2}\|\tilde{\Delta}u\|^2 . \end{aligned}$$

Thus we have

$$\frac{d}{dt}\|\nabla u^n\|^2 + \|\tilde{\Delta}u^n\|^2 \leq c\|\nabla u^n\|^4 + c'\|\nabla u^n\|^6 . \tag{15}$$

Also, if $u_o \in J_1^*(\Omega)$, we have a bound for the initial values,

$$\|\nabla u^n(0)\| \leq \|\nabla u_o\| ,$$

because of the orthogonality of the eigenfunctions $\{a^k\}$ in the inner product $(\nabla\phi,\nabla\psi)$. Hence, (15) can be integrated, yielding estimates of the form

$$\|\nabla u^n(t)\|, \int_0^t \|\tilde{\Delta}u^n\|^2 d\tau \leq F(t) , \tag{16}$$

for t in some interval $[0,T)$.

An estimate for u_t^n is obtained by noting (2) implies

$$\|u_t^n\|^2 = (\tilde{\Delta}u^n, u_t^n) - (u^n\cdot\nabla u^n, u_t^n), \tag{17}$$

so that, using (14),

$$\begin{aligned}\|u_t^n\| &\leq \|\tilde{\Delta}u^n\| + \|u^n\|_6\cdot\|\nabla u^n\|_3 \\ &\leq \|\tilde{\Delta}u^n\| + c\|\nabla u^n\|\cdot(\|\tilde{\Delta}u^n\|^{\frac{1}{2}}\cdot\|\nabla u_n\|^{\frac{1}{2}} + \|\nabla u^n\|) \\ &\leq 2\|\tilde{\Delta}u^n\| + c\|\nabla u^n\|^3 + c\|\nabla u^n\|^2,\end{aligned} \tag{18}$$

and hence, using (16),

$$\int_0^t \|u_t^n\|^2\, d\tau \leq \tilde{F}(t), \tag{19}$$

for $t \in [0,T)$. The estimates (16) and (19) are the ones on which Prodi's existence theorem in [11] is based. Here, we have derived them in a manner independent of the "size" of either Ω or $\partial\Omega$. These estimates, that is, the functions F and $\tilde{F}$, depend only on $\|\nabla u_o\|$ and the regularity of $\partial\Omega$. In contrast, the estimates (8) depend on $\|u_o\|$, $\|\nabla u_o\|$ and $\|D^2 u_o\|$, but are independent of the regularity of $\partial\Omega$.

To obtain a classical solution, we need estimates of the solution's higher order derivatives. We work first to establish the regularity of u with respect to t, more precisely, to show $u \in C^\infty(0,T; W_2^2(\Omega))$ by obtaining estimates

$$\|\nabla D_t^k u^n(t)\|, \int_\varepsilon^t \|\tilde{\Delta}D_t^k u^n\|^2\, d\tau \leq F_k(t,\varepsilon), \tag{20.k}$$

for $k = 0, 1, 2, \ldots$, and $0 < \varepsilon < t < T$. To this end, we write down three infinite sequences of identities for the Galerkin approximations (for brevity, the superscripts n are omitted):

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \|\tilde{\Delta}u\|^2 = (u\cdot\nabla u, \tilde{\Delta}u) \tag{21.0}$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_t\|^2 + \|\tilde{\Delta}u_t\|^2 = (u_t\cdot\nabla u, \tilde{\Delta}u_t) + (u\cdot\nabla u_t, \tilde{\Delta}u_t) \tag{21.1}$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_{tt}\|^2 + \|\tilde{\Delta}u_{tt}\|^2 = (u_{tt}\cdot\nabla u, \tilde{\Delta}u_{tt}) + 2(u_t\cdot\nabla u_t\cdot\tilde{\Delta}u_{tt}) + (u\cdot\nabla u_{tt}, \tilde{\Delta}u_{tt}) \tag{21.2}$$

etc.

$$\|u_t\|^2 = (\tilde{\Delta}u, u_t) - (u\cdot\nabla u, u_t) \tag{22.1}$$

$$\|u_{tt}\|^2 = (\tilde{\Delta}u_t , u_{tt}) - (u_t\cdot\nabla u , u_{tt}) - (u\cdot\nabla u_t , u_{tt}) \tag{22.2}$$

$$\|u_{ttt}\|^2 = (\tilde{\Delta}u_{tt} , u_{ttt}) - (u_{tt}\cdot\nabla u , u_{ttt}) - 2(u_t\cdot\nabla u_t , u_{ttt}) - (u\cdot\nabla u_{tt} , u_{ttt}) \tag{22.3}$$

etc.

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2 + \|\nabla u_t\|^2 = -(u_t\cdot\nabla u , u_t) \tag{23.1}$$

$$\frac{1}{2}\frac{d}{dt}\|u_{tt}\|^2 + \|\nabla u_{tt}\|^2 = -(u_{tt}\cdot\nabla u , u_{tt}) - 2(u_t\cdot\nabla u_t , u_{tt}) \tag{23.2}$$

$$\frac{1}{2}\frac{d}{dt}\|u_{ttt}\|^2 + \|\nabla u_{ttt}\|^2 = -(u_{ttt}\cdot\nabla u , u_{ttt}) - 3(u_{tt}\cdot\nabla u_t , u_{ttt}) - 3(u_t\cdot\nabla u_{tt} , u_{ttt}) \tag{23.3}$$

etc.

Notice, we have already derived the leading identity of each sequence; (21.0) is just (9), (22.1) is (17), and (23.1) is (5). The succeeding identities are derived similarly, first differentiating (2) an appropriate number of times with respect to t.

Our basic plan now is to estimate the right sides of the identities (21.k), $k = 0, 1, 2, \ldots$, and integrate them with respect to t, obtaining the estimates (20.k). We have already done this for $k = 0$. To proceed, one thing which must be done, in order to be able to integrate each identity up to the same right limit T as (21.0), is to estimate the right side of (21.k), $k = 1, 2, 3, \ldots$, in such a way that it becomes a linear differential inequality when the estimates (20.ℓ), $\ell = 0, 1, \ldots, k-1$, are taken into account. This is easily done using the inequalities $\sup_\Omega |u| \le c(\|\tilde{\Delta}u\| + \|\nabla u\|)$, $\|\nabla u\|_3 \le c(\|\tilde{\Delta}u\| + \|\nabla u\|)$ and $\|u\|_6 \le c\|\nabla u\|$, which are derived from (13) and various Sobolev inequalities. For instance, for the right side of (21.1) we have

$$|(u_t\cdot\nabla u , \tilde{\Delta}u_t) + (u\cdot\nabla u_t , \tilde{\Delta}u_t)| \le \|u_t\|_6\cdot\|\nabla u\|_3\cdot\|\tilde{\Delta}u_t\| + \sup_\Omega|u|\cdot\|\nabla u_t\|\cdot\|\tilde{\Delta}u_t\|$$
$$\le c(\|\tilde{\Delta}u\|^2 + \|\nabla u\|^2)\|\nabla u_t\|^2 + \tfrac{1}{2}\|\tilde{\Delta}u_t\|^2,$$

so that (21.1) becomes

$$\frac{d}{dt}\|\nabla u_t\|^2 + \|\tilde{\Delta}u_t\|^2 \le c(\|\tilde{\Delta}u\|^2 + \|\nabla u\|^2)\|\nabla u_t\|^2 . \tag{24}$$

The right sides of (21.k), $k \ge 2$, are estimated similarly; we omit the details.

Finally, we must find estimates, independent of n, for the "initial values" of

$\|\nabla D_t^k u^n\|$. This presents a rather severe difficulty; it is to deal with it that we have introduced the identities (22.k) and (23.k), $k = 1, 2, \ldots$. The difficulty is that for initial values $u_o \in \overset{*}{J}_1(\Omega)$, or even for $u_o \in D(\Omega)$, one must generally expect, for the solution u, that $\|\nabla u_t(t)\| \to \infty$ as $t \to 0^+$, and hence, for the Galerkin approximations u^n, that $\|\nabla u_t^n(0)\| \to \infty$ as $n \to \infty$. To see this, suppose $u_o \in D(\Omega)$, i.e., $u_o \in C_o^\infty(\Omega)$ and $\nabla \cdot u_o = 0$. Then conditions (1) imply $\Delta p = -\nabla\cdot(u_o \cdot \nabla u_o)$ and $\nabla p|_{\partial\Omega} = 0$, at $t = 0$. This is an overdetermined Neumann problem for the initial pressure. In general, the tangential components of ∇p will not vanish on $\partial\Omega$ at $t = 0$, and yet, initially, $u_t = \nabla p$ in a neighborhood of $\partial\Omega$, if $u_o \in D(\Omega)$. Thus, one can not expect $\lim_{t\to 0^+} u_t(t) \in \overset{\circ}{W}{}_2^1(\Omega)$. The compatibility condition which would be needed is a non-local one, expressible as an integral identity involving the Neumann function, and virtually uncheckable in practice.

Instead of imposing such conditions on the initial velocity, we use (22.k) and (23.k) to obtain estimates for $\|\nabla D_t^k u^n(\varepsilon)\|$, at arbitrarily small values of ε. For this, we only need to assume $u_o \in J_1^*(\Omega)$. The procedure is as follows. First, (21.0) is integrated giving (20.0), that is, (15) is integrated giving (16), which we have already done. Then, (22.1), i.e., (18), is integrated giving (19), which we have already done. Now, from (19), we see that for every $\varepsilon > 0$ and every positive integer n, there exists a number τ_n, $0 < \tau_n < \varepsilon$, such that

$$\|u_t^n(\tau_n)\|^2 \le \tilde{F}(\varepsilon)/\varepsilon. \tag{25}$$

Also, the right side of (23.1) can be estimated, using part of the derivation of (6), giving

$$\frac{d}{dt}\|u_t^n\|^2 + \|\nabla u_t^n\|^2 \le c\|\nabla u^n\|^4\|u_t^n\|^2 , \tag{26}$$

which is a linear differential inequality when (20.0) is taken into account. Thus, using (25), we can integrate (26) over the interval $[\tau_n, t]$, obtaining

$$\int_\varepsilon^t \|\nabla u_t^n\|^2 d\tau \le \int_{\tau_n}^t \|\nabla u_t^n\|^2 d\tau \le G(t;\varepsilon) , \tag{27}$$

for $\varepsilon < t < T$. Now, from (27), we see that for every $\varepsilon > 0$ and every positive integer n, there exists a number σ_n, $\varepsilon < \sigma_n < 2\varepsilon$, such that

$$\|\nabla u_t^n(\sigma_n)\|^2 \le G(2\varepsilon;\varepsilon)/\varepsilon . \tag{28}$$

This provides the estimate of the "initial values" needed for integrating (21.1), or rather the corresponding differential inequality (24). Integrating (24) over

$[\sigma_n, t]$, one obtains estimates of the form

$$\|\nabla u_t^n(t)\|^2, \quad \int_{2\varepsilon}^{t} \|\tilde{\Delta} u_t^n\|^2 d\tau \quad \leq \quad F_1(t;2\varepsilon) , \tag{29}$$

for $2\varepsilon < t < T$. This is just (20.1).

We have come full cycle. One can continue by integrating (22.2), using (20.1), to obtain estimates of $\|u_{tt}^n(\rho_n)\|$, with $2\varepsilon < \rho_n < 3\varepsilon$. Then, one can integrate (23.2) to obtain estimates of $\|\nabla u_{tt}^n(\delta_n)\|$, with $3\varepsilon < \delta_n < 4\varepsilon$. And then, one can integrate (21.2) obtaining (20.2), etc., etc.. The full argument is given by induction in [4].

To prove the continuous assumption of the initial values, one last estimate for the Galerkin approximations will be needed. We derive it here assuming $u_o \in W_2^2(\Omega)$, though the condition $u_o \in L^2(\Omega)$ is not necessary; see [4]. Multiplying (2) by λ_ℓ and summing, we obtain

$$(\Delta u^n , \tilde{\Delta} u^n) \;=\; (u_t^n , \tilde{\Delta} u^n) + (u^n \cdot \nabla u^n , \tilde{\Delta} u^n) ,$$

and hence, using (14) as in the derivation of (18),

$$\|\tilde{\Delta} u^n\| \quad \leq \quad 2\|u_t^n\| + c\|\nabla u^n\|^3 + c\|\nabla u^n\|^2 . \tag{30}$$

Thus, (8) and (16) imply

$$\|\tilde{\Delta} u^n(t)\| \quad \leq \quad \hat{G}(t) , \tag{31}$$

for $t \in [0,T']$, for some $T' > 0$.

3. Passage from the Approximations to a Classical Solution

Using only the estimates (16) and (19), one can show the Galerkin approximations converge to a generalized solution $u \in L^\infty(0,T' ; J_1^*(\Omega))$ with u_t, $D_x^2 u$, $\nabla p \in L^2(0,T';L^2(\Omega))$, for $0 < T' < T$. Here, we only need the convergence of a subsequence of the approximations, which is proved by a compactness argument in [4]; in fact, the whole sequence of approximations is known to converge; see Rautmann [12,13] and Foias [2]. Using (13) and Sobolev's inequality, the estimates (20.k) imply $u \in C^\infty(0,T;W_2^2(\Omega))$. It follows, of course, that $u \in C(\bar{\Omega} \times (0,T))$. In passing to the limit on the basis of estimates (16) and (19), the solution is only shown to satisfy the initial condition in a generalized sense: $u(t) \to u_o$ in $\overset{\circ}{W}{}_2^1(\Omega)$, as $t \to 0^+$. However, the estimate (31) implies $u \in L^\infty(0,T';W_2^2(\Omega))$, and hence that $u(t) \to u_o$

weakly in $W_2^2(\Omega)$. So, from the compactness of the imbedding $W_2^2(\Omega) \subset C(\Omega)$, it follows that $u(\bar{x},t) \to u_0(x)$ continuously as $(\bar{x},t) \to (x,0)$. Thus $u \in C(\bar{\Omega} \times [0,T))$, if $u_0 \in W_2^2(\Omega)$.

We noted above, the estimates (20.k) imply $u \in C^\infty(0,T;W_2^2(\Omega))$. To establish further interior regularity with respect to the spatial variables, one observes, for any fixed t, and for $k = 0, 1, 2, \ldots$, that $D_t^k u$ is a solution of (12), with force

$$f_k = -D_t^{k+1}u - \sum_{\beta=0}^{k} c(D_t^{k-\beta}u)\cdot\nabla(D_t^{\beta}u) .$$

From the known regularity of u, it is clear $f_k \in C^\infty(0,T;W_2^1(\Omega))$. Thus, viewing $D_x^1 D_t^k u$ as a solution of (12a), (12b) with force $D_x^1 f_k \in C^\infty(0,T;L^2(\Omega))$, the interior estimate (11) implies $u \in C^\infty(0,T;W_2^3(G))$, for every $G \subset\subset \Omega$, i.e., for every bounded set G with closure $\bar{G} \subset \Omega$. This, in turn, implies $f_k \in C^\infty(0,T;W_2^2(G))$, for every $G \subset\subset \Omega$. And thus, viewing $D_x^2 D_t^k u$ as a solution of (12a), (12b) with force $D_x^2 f_k \in C^\infty(0,T;L^2(\Omega))$, (11) implies $u \in C^\infty(0,T;W_2^4(G))$, for every $G \subset\subset \Omega$. By induction, one sees $u \in C^\infty(0,T;W_2^\ell(G))$, for every $G \subset\subset \Omega$, and for $\ell = 3, 4, \ldots$. It follows that $u \in C^\infty(\Omega \times (0,T))$.

4. Decay, as $t \to \infty$, in Unbounded Domains

Using Poincaré's inequality, i.e., the inequality $\|\phi\| \le C_\Omega \|\nabla\phi\|$ for $\phi \in \overset{\circ}{W}{}_2^1(\Omega)$, once can show the Galerkin approximations of section 2 decay exponentially as $t \to \infty$. More precisely, there exists a number T^* dependent on C_Ω and $\|u_0\|$, such that for every $\gamma < C_\Omega^{-2}$, one obtains an estimate of the form

$$\sup_{x\in\Omega} |u^n(x,t)| \le c(\|\tilde{\Delta} u^n(t)\| + \|\nabla u^n(t)\|) \le Ce^{-\gamma t} , \qquad (32)$$

for $t \ge T^*$. From this follows the global existence and exponential decay of the classical solution constructed in section 3, provided $T^* < T$, with T as in either (8) or (16). One can show $T^* < T$ under various hypotheses, for instance, if $\|u_0\|_{W_2^1(\Omega)}$ and C_Ω are sufficiently small. In general, if $T^* > T$, one still has the global existence of Hopf's generalized solution, and its classical regularity can be proved for $t \in (0,T) \cup (T^*,\infty)$. During the interval $[T,T^*]$, it is classical, except perhaps, for values of t belonging to a set of t-measure zero, whose complement consists of intervals. These results for bounded domains are rather standard, see [4,14]. Instead of giving the details, we will describe some analogous results for unbounded domains.

All the estimates given in section 2 are independent of the size of Ω and $\partial\Omega$, though some of them depend on the C^3-regularity of $\partial\Omega$. This makes possible the construction of a solution of problem (1) in any three-dimensional domain with uniformly C^3 boundary, by considering an expanding sequence of subdomains. All the estimates of section 2 remain valid for the eventual solution. Also, assuming $u_o \in J_1(\Omega)$, the solution which is constructed belongs to $J_1(\Omega)$, for almost every t.

Of course, the estimate (32) generally fails in unbounded domains. Still, an explicit estimate for the solution's rate of decay can be obtained from the energy estimate (4) and the differential inequality (15), i.e., from

$$\int_o^\infty \|\nabla u^n\|^2 dt \le \frac{1}{2}\|u_o\|^2 \equiv E_o \tag{33}$$

and

$$\frac{d}{dt}\|\nabla u^n\|^2 \le c\|\nabla u^n\|^4 + c'\|\nabla u^n\|^6 . \tag{34}$$

If $\|\nabla u^n(t)\|$ were known to be monotonically decreasing, (33) would clearly imply $\|\nabla u^n(t)\|^2 \le E_o t^{-1}$. In fact, (34) implies such a slow rate of growth of $\|\nabla u^n\|$, when $\|\nabla u^n\|$ is small, that one gets a similar result for large t, namely

$$\|\nabla u^n(t)\|^2 \le H(t) \le \left(\frac{\exp(cE+1) - 1}{c + E^{-1}}\right) t^{-1} \tag{35}$$

for $t \ge c'E^2\exp(cE+1) \ge T^*$. Here c and c' are the same as in (34). This estimate is proved by comparison with solutions of the differential equation $\phi' = \alpha\phi^2$, which are of the form $\phi = \alpha^{-1}(t_o - t)^{-1}$. It is easily checked that if such a function ϕ is defined for $t \in [0,\tau]$ and satisfies $\int_o^\tau \phi\, dt < E$, then $\phi(\tau) < (\exp \alpha E - 1)/\alpha\tau$. The more complicated form of the estimate (35) is due to the presence of the term $c'\|\nabla u^n\|^6$ in (34); the details are given in [4]. Once (35) is proven, (15) can be integrated to give

$$\int_t^\infty \|\tilde{\Delta} u^n\|^2 d\tau \le C t^{-1},$$

for $t \ge T^*$. Then, integration of (18) gives

$$\int_t^\infty \|u_t^n\|^2 d\tau \le C t^{-1},$$

for $t \ge T^*$. Since (6) is a differential inequality of the form $\phi' \le \alpha\phi^2$, this implies

$$\|u_t^n(t)\|^2 \le C_\delta t^{-1}, \tag{36}$$

for $t > T^* + \delta$, for any $\delta > 0$. The estimates (35) and (36), together with (30), imply

$$\|\tilde{\Delta}u^n(t)\|^2 \leq C_\delta t^{-1} , \tag{37}$$

for $t > T^* + \delta$. Finally, (35) and (37) imply

$$\sup_{x\in\Omega}|u(x,t)| \leq c(\|\tilde{\Delta}u^n\| + \|\nabla u^n\|) \leq c_\delta t^{-\frac{1}{2}} , \tag{38}$$

for $t \geq T^* + \delta$. This estimate is a variant of one proved by Masuda [10] for exterior domains. Masuda's estimate is based essentially on (6) rather than (15); while it gives a slower rate of decay, it remains valid in the case of nonhomogeneous boundary values.

The relation between the estimates (16) and (35) is shown in Figure 1. For every n, $\|\nabla u^n(t)\|^2$ would be represented by a smooth curve defined for all $t \geq 0$ and bounded by the graphs of $F(t)$ and $H(t)$. T depends on $\|\nabla u_o\|$, and T^* on $\|u_o\|$.

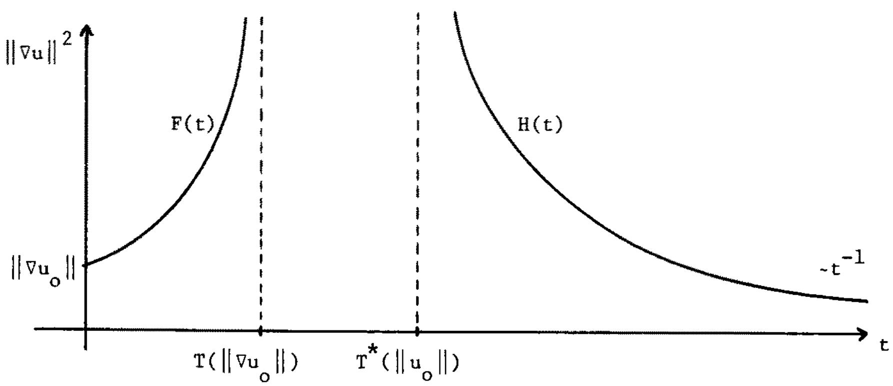

Figure 1. Estimates (16) and (35) for the Galerkin approximations.

If $T \leq T^*$, we lack a bound for $\|\nabla u^n(t)\|$, uniform in n, on the interval $[T,T^*]$; so, during this time interval, the regularity of solutions constructed from the Galerkin approximations may break down. If $T^* < T$, there is only one solution and it is regular in the classical sense for all $t \geq 0$. It is shown in [4] that $T^* < T$, if, for some number β,

$$\frac{1}{2}\|u_o\|^2 \leq \frac{\log(\beta/\|\nabla u_o\|^2)}{c + c'\beta} .$$

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