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On the Eigenvalues of the Electrostatic Integral Operator

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A fundamental result in scattering and potential theory in R^3 states that the eigenvalues of the electrostatic integral operator lie in the interval $[-1, 1)$, provided the surface of integration is sufficiently smooth. In the case of a sphere it is known that the eigenvalues lie in the interval $[-1, 0)$. In this paper the case when the surface is a prolate spheroid is considered. The eigenvalues of the electrostatic integral operator are calculated explicitly and it is proven that these eigenvalues also lie in the interval $[-1, 0)$. © 1986 Academic Press, Inc.

1. INTRODUCTION

Plemelj [8] derived a fundamental result in the area of scattering and potential theory in R^3 which states that the eigenvalues λ_i of the electrostatic integral operator, which we denote by K', satisfy the following inequality:

$$-1 \leqslant \lambda_i < 1. \tag{1.1}$$

(Both our meaning of an eigenvalue and the exact definition of K' will be given in Section 2.)

Using an integral equations approach, Kleinman [5] has given a simple method for "optimally" solving exterior Neumann potential and low frequency scattering problems in R^3 for the case when K' has only nonpositive eigenvalues. (For clarification of what is meant here by optimally see Kleinman [5] or the discussion of Kleinman's method in [3, pp. 152–153].) If on the other hand, there exist some positive eigenvalues of K', then Kleinman has shown that the situation becomes far more difficult. In this latter case it can be shown that Kleinman's method for optimally solving the exterior Neumann problem reduces to evaluating the supremum of the set of all positive eigenvalues of K'. Unfortunately, this supremum becomes a formidable task to compute.

Apart from Plemelj's result (1.1) and with the exception when the surface of integration for K' is a sphere, explicit representations and information

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about the location of the eigenvalues of K' are not known. For the case of a sphere, however, it is known (e.g., see [3, p. 153]) that the eigenvalues of K' are negative, and by a straightforward calculation it can be shown that the eigenvalues are given by

$$\lambda_i = -1/(2i+1), \qquad i = 0, 1, 2, \ldots. \tag{1.2}$$

In view of Kleinman's results, there has been considerable interest in whether or not the sphere is the only geometry for which K' has no positive eigenvalues.

In the present note we make what we believe is an important contribution to that area of scattering and potential theory dealing with the location of the eigenvalues of the electrostatic integral operator. We calculate explicitly the eigenvalues of K' for the case when the surface of integration for K' is a prolate spheroid and demonstrate that they lie in the interval $[-1, 0)$.

In the next section we give our notation and pertinent definitions and state some basic results which we will need. In Section 3 we consider the specific example of a prolate spheroid and calculate explicitly the eigenvalues of K'. In the last section we show that these eigenvalues indeed lie in the interval $[-1, 0)$.

2. Notation and Preliminary Results

In this section we give our notation and state some results which we shall require. Let D_i be a bounded domain in R^3 containing the origin, with a closed, simply connected C^2 boundary ∂D and let D_e denote the region exterior to $\bar{D}_i$. Let $\hat{n}$ denote a unit normal on ∂D directed out of D_i. Let x and y denote typical points in R^3. Let $\Phi(x, y)$ be defined by

$$\Phi(x, y) := \frac{1}{4\pi} \frac{1}{|x-y|}, \qquad x, y \in R^3,\ x \neq y. \tag{2.1}$$

We now define the following standard integral operators of potential theory:

$$(Ku)(x) := 2 \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial n(y)} u(y)\, dS_y, \qquad x \in \partial D, \tag{2.2}$$

$$(K'u)(x) := 2 \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial n(x)} u(y)\, dS_y, \qquad x \in \partial D, \tag{2.3}$$

$$(Du)(x) := 2 \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial n(y)} u(y)\, dS_y, \qquad x \in R^3 \backslash \partial D. \tag{2.4}$$

Let $C(\partial D)$ denote the Banach space of complex-valued continuous functions defined on ∂D equipped with the maximum norm and let $\langle C(\partial D), C(\partial D)\rangle$ denote the dual system with the bilinear form

$$\langle u, v\rangle = \int_{\partial D} u(y)\, v(y)\, dS_y, \qquad u, v \in C(\partial D). \tag{2.5}$$

By interchanging the order of integration it can be shown that K and K' are adjoint, that is, $\langle Ku, v\rangle = \langle u, K'v\rangle$. Furthermore, it can be shown (e.g., see [3, Theorem 2.30]) that K and K' are compact on $C(\partial D)$.

Let $+$ and $-$ denote the limits obtained for the double layer potential $(Du)(x)$ by approaching the boundary ∂D from D_e and D_i, respectively, that is,

$$(D_+ u)(x) = \lim_{\substack{x_e \to x \\ x_e \in D_e}} (Du)(x_e),\ (D_- u)(x) = \lim_{\substack{x_i \to x \\ x_i \in D_i}} (Du)(x_i), \qquad x \in \partial D. \tag{2.6}$$

It can be shown (e.g., see [3, Theorem 2.13]) that for $u \in C(\partial D)$

$$(D_\pm u)(x) = (Ku)(x) \pm u(x), \qquad x \in \partial D, \tag{2.7}$$

where the integral $(Ku)(x)$ exists as an improper integral. Consequently, we have

$$(Ku)(x) = \frac{1}{2}\left[(D_+ u)(x) + (D_- u)(x)\right], \qquad x \in \partial D. \tag{2.8}$$

Let A denote any bounded linear operator mapping a Banach space X into itself. By an eigenvalue of A we mean a complex number λ such that the nullspace $N(\lambda I - A) \neq \{0\}$, where I denotes the identity operator. Let $\sigma(A)$ denote the spectrum of A. It is known (e.g., see [2, Chap. 18] or [3, Theorem 1.34]) that if X is an infinite dimensional Banach space and if A is a compact linear operator then $\lambda = 0$ lies in $\sigma(A)$ and $\sigma(A)\backslash\{0\}$ consists of at most a countable set of eigenvalues, with $\lambda = 0$ the only possible limit point.

We now state the following important results of Plemelj alluded to in Section 1 (see [3, Theorem 5.1], [4, pp. 309–310], or [8]):

$$\sigma(K) = \sigma(K') \subset [-1, 1); \tag{2.9}$$

$$\dim N(I + K) = \dim N(I + K') = 1. \tag{2.10}$$

In the subsequent sections we shall compute the eigenvalues of K for the case when ∂D is a prolate spheroid and obtain an inequality that these eigenvalues satisfy. In view of (2.9), the same results will be true for the eigenvalues of the electrostatic integral operator K'. We choose to work with K rather than K', because, for the case when ∂D is a prolate spheroid the direct computation of these eigenvalues is easier for K than it is for K'.

3. Calculating the Eigenvalues for a Prolate Spheroidal Surface

In this section we take the surface ∂D to be a prolate spheroid. We first give some basic results for this geometry and then determine the eigenvalues of the integral operator K in this case.

With respect to rectangular coordinates, the prolate spheroid is oriented with its axis of revolution along the z-axis and the origin at its geometric center. The relationship between the rectangular coordinates (y_1, y_2, y_3) of the point y and prolate spheroidal coordinates (ξ, η, ϕ) is

$$y_1 = \frac{d}{2}[(\xi^2-1)(1-\eta^2)]^{1/2} \cos\phi, \tag{3.1}$$

$$y_2 = \frac{d}{2}[(\xi^2-1)(1-\eta^2)]^{1/2} \sin\phi, \tag{3.2}$$

$$y_3 = \frac{d}{2}\xi\eta, \tag{3.3}$$

where d is the interfocal distance of the spheroid and $1 \leqslant \xi < \infty$, $-1 \leqslant \eta \leqslant 1$, $0 \leqslant \phi \leqslant 2\pi$. The surfaces $\xi = \text{constant}$ represent confocal prolate spheroids. In terms of rectangular coordinates, the foci are at $(0, 0, \pm d/2)$ and the limiting case $\xi = 1$ is a degenerate case corresponding to the line segment between the foci. Let (ξ_x, η_x, ϕ_x) and (ξ, η, ϕ) denote the prolate spheroidal coordinates of the points x and y, respectively. Finally, let ξ_∂ denote the surface coordinate of our prolate spheroid ∂D.

From (2.1) and [1] we have that

$$\Phi(x, y) = \frac{1}{2\pi d} \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^m \varepsilon_m (2n+1) \left[\frac{(n-m)!}{(n+m)!}\right]^2 \cos m(\phi - \phi_x)$$
$$\times P_n^m(\eta)\, P_n^m(\eta_x)\, P_n^m(\xi_<)\, Q_n^m(\xi_>), \tag{3.4}$$

where

$$\varepsilon_m = 1, \qquad m = 0 \tag{3.5}$$
$$= 2, \qquad m = 1, 2, \ldots,$$

$$\xi_< = \min(\xi, \xi_x) \qquad \text{and} \qquad \xi_> = \max(\xi, \xi_x) \tag{3.6}$$

and $P_n^m(\eta)$, $P_n^m(\xi)$ and $Q_n^m(\xi)$ are the associated Legendre functions (see [6] or [7] for the definitions of these functions). Furthermore, it is known (see [1])

$$\left.\frac{\partial}{\partial n}\right|_{\xi=\xi_\partial} = \frac{2}{d}(\xi_\partial^2 - 1)^{1/2}(\xi_\partial^2 - \eta^2)^{-1/2}\frac{\partial}{\partial \xi_\partial}, \tag{3.7}$$

$$dS = \left(\frac{d}{2}\right)^2 [(\xi_\partial^2 - 1)(\xi_\partial^2 - \eta^2)]^{1/2} d\eta\, d\phi. \tag{3.8}$$

For $y \in \partial D$ we have from (3.4) and (3.7)

$$\frac{\partial \Phi(x, y)}{\partial n(y)} = \frac{1}{\pi d^2}\left[\frac{\xi_\partial^2 - 1}{\xi_\partial^2 - \eta^2}\right]^{1/2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^m \varepsilon_m (2n+1)\left[\frac{(n-m)!}{(n+m)!}\right]^2$$
$$\times \cos m(\phi - \phi_x)\, P_n^m(\eta)\, P_n^m(\eta_x) \times \begin{cases} P_n^m(\xi_\partial)'\, Q_n^m(\xi_x), & x \in D_e, \\ P_n^m(\xi_x)\, Q_n^m(\xi_\partial)', & x \in D_i, \end{cases} \tag{3.9}$$

where the prime denotes differentiation with respect to the argument.

Define

$$C_n^m(\eta, \phi) := P_n^m(\eta) \cos m\phi, \qquad m, n \in \mathbb{Z}, 0 \leqslant m \leqslant n, \tag{3.10}$$

$$S_n^m(\eta, \phi) := P_n^m(\eta) \sin m\phi, \qquad m, n \in \mathbb{Z}, 0 \leqslant m \leqslant n, n \geqslant 1. \tag{3.11}$$

For the convenience of the reader we list the following well-known orthogonality relations:

$$\int_0^{2\pi} \cos m(\phi - \phi_x) \cos r\phi\, d\phi = \frac{2\pi \cos m\phi_x}{\varepsilon_m}, \qquad m = r \tag{3.12}$$
$$= 0, \qquad m \neq r,$$

$$\int_0^{2\pi} \cos m(\phi - \phi_x) \sin r\phi\, d\phi = \pi \sin m\phi_x, \qquad m = r = 1, 2... \tag{3.13}$$
$$= 0, \qquad m \neq r,$$

$$\int_{-1}^{1} P_n^m(\eta)\, P_s^m(\eta)\, d\eta = \frac{(n+m)!}{(n-m)!}\frac{2}{2n+1}, \qquad n = s \tag{3.14}$$
$$= 0, \qquad n \neq s.$$

From Eqs. (2.2), (2.8) and (3.8) through (3.14) it follows after some calculations that

$$KC_n^m(\eta_x, \phi_x) = \lambda_{mn}(\xi_\partial)\, C_n^m(\eta_x, \phi_x), \tag{3.15}$$

$$KS_n^m(\eta_x, \phi_x) = \lambda_{mn}(\xi_\partial)\, S_n^m(\eta_x, \phi_x), \tag{3.16}$$

where, dropping the subscript ∂,

$$\lambda_{mn}(\xi) := (-1)^m \frac{(n-m)!}{(n+m)!} (\xi^2 - 1) [P_n^m(\xi) Q_n^m(\xi)]'. \tag{3.17}$$

From (3.15) and (3.16) it follows that $\lambda_{mn}(\xi)$ is an eigenvalue of the integral operator K with corresponding eigenfunctions C_n^m and S_n^m. Both C_n^m and S_n^m lie in our underlying Banach space $C(\partial D)$. Moreover, the set of functions $\{C_n^m, S_n^m : m, n \in \mathbb{Z},\ 0 \leqslant m \leqslant n\}$ forms a complete orthogonal system in the Hilbert space $L_2(\partial D)$. Consequently, we have the following theorem:

THEOREM 3.1. *Let ∂D represent the surface of a prolate spheroid with surface coordinate ξ. Let $K: C(\partial D) \to C(\partial D)$ be the compact linear operator defined in* (2.2). *Then the $\lambda_{mn}(\xi)$'s defined in* (3.17) *are the only eigenvalues of K.*

Proof. Suppose λ^* is an eigenvalue of K and let $f^* \in C(\partial D) \subset L_2(\partial D)$ be a corresponding eigenfunction. Furthermore, suppose $\lambda^* \neq \lambda_{mn}$ for all $m, n \in \mathbb{Z}$, $0 \leqslant m \leqslant n$. From the completeness and the orthogonality of the set $\{C_n^m, S_n^m\}$ it follows that

$$f^* = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (\alpha_{mn} C_n^m + \beta_{mn} S_n^m) \qquad \text{in } L_2(\partial D) \tag{3.18}$$

for suitably chosen constants α_{mn} and β_{mn}.

From (3.15), (3.16) and (3.18) it follows that

$$\lambda^* f^* = Kf^* = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \lambda_{mn}(\alpha_{mn} C_n^m + \beta_{mn} S_n^m), \tag{3.19}$$

where the termwise action of K is justified by (2.6), (2.8) and the fact that the kernel of Du in (2.4) as a function of y is in $L_2(\partial D)$.

Consequently, from (3.18) and (3.19) we have

$$\lambda^* \alpha_{mn} = \lambda_{mn} \alpha_{mn}, \qquad \lambda^* \beta_{mn} = \lambda_{mn} \beta_{mn}, \qquad m, n \in \mathbb{Z}, 0 \leqslant m \leqslant n \tag{3.20}$$

and this leads to a contradiction to the assumptions we have made about λ^*. ∎

4. AN INEQUALITY FOR THE EIGENVALUES $\lambda_{mn}(\xi)$

In this section we show that the eigenvalues $\lambda_{mn}(\xi)$ defined in (3.17) satisfy

$$\lambda_{00}(\xi) = -1, \qquad -1 < \lambda_{mn}(\xi) < 0 \qquad \text{for } n > 0, \tag{4.1}$$

where here and throughout this section it is understood that $1 < \xi$ and that m and n are integers such that $0 \leqslant m \leqslant n$. To this end we now establish some preliminary results which we shall require.

From [7, p. 165] we have the following Wronskian identity for the associated Legendre functions $P_n^m(\xi)$ and $Q_n^m(\xi)$:

$$W\{P_n^m, Q_n^m\} = \frac{w_{mn}}{1-\xi^2}, \tag{4.2}$$

where

$$w_{mn} := (-1)^m \frac{(n+m)!}{(n-m)!}. \tag{4.3}$$

From the definition of a Wronskian and (4.2) it can be seen that

$$\left[\frac{Q_n^m(\xi)}{P_n^m(\xi)}\right]' = \frac{W}{[P_n^m(\xi)]^2} = \frac{w_{mn}}{[P_n^m(\xi)]^2(1-\xi^2)}. \tag{4.4}$$

From the result in (4.4) and the asymptotic behavior of the associated Legendre functions at $+\infty$ (e.g., see [7, p. 197]), it follows that

$$Q_n^m(\xi) = w_{mn} P_n^m(\xi) \int_\xi^\infty \frac{dx}{[P_n^m(x)]^2(x^2-1)}, \qquad 1 < \xi < \infty. \tag{4.5}$$

After some calculations we obtain from (3.17), (4.2) and (4.5)

$$\lambda_{mn}(\xi) = -1 + 2(\xi^2-1)\, P_n^m(\xi)\, P_n^m(\xi)' \int_\xi^\infty \frac{dx}{[P_n^m(x)]^2(x^2-1)}. \tag{4.6}$$

We now prove the following important theorem which will establish (4.1):

THEOREM 4.1. *Let $P(\xi) := P_n^m(\xi)$. Then*

$$0 \leqslant 2(\xi^2-1)\, P(\xi)\, P'(\xi) \int_\xi^\infty \frac{dx}{(x^2-1)\, P^2(x)} < 1, \qquad \xi > 1, \tag{4.7}$$

where m and n are integers and $0 \leqslant m \leqslant n$, and where equality holds only for $n = 0$.

Proof. From [7, p. 174] we have

$$P(\xi) = (\xi^2-1)^{m/2} P_n^{(m)}(\xi), \tag{4.8}$$

where $P_n^{(m)}(\xi)$ is the mth derivative of the nth order Legendre polynomial $P_n(\xi)$.

For $n=0$, $P_0(\xi)\equiv 1$, $P_0'(\xi)\equiv 0$ and equality in (4.7) holds trivially. From now on suppose $n>0$. Define the function $F(\xi)$ by

$$F(\xi):=[2(\xi^2-1)\,P(\xi)\,P'(\xi)]^{-1}. \tag{4.9}$$

From the asymptotic behavior of $P(\xi)$ at $+\infty$ (see [7, p. 197]) we have $\lim_{\xi\to+\infty} F(\xi)=0$ and consequently

$$F(\xi)=-\int_\xi^\infty F'(x)\,dx. \tag{4.10}$$

Define the function $D(\xi)$ by

$$D(\xi):=F(\xi)-\int_\xi^\infty \frac{dx}{(x^2-1)\,P^2(x)}. \tag{4.11}$$

Then from (4.9), (4.10) and (4.11) we have

$$\begin{aligned} D(\xi)&=\int_\xi^\infty \left[-F'(x)-\frac{1}{(x^2-1)\,P^2(x)}\right]dx \\ &=\int_\xi^\infty \frac{[(x^2-1)\,PP']'-2(x^2-1)(P')^2}{2[(x^2-1)\,PP']^2}\,dx. \end{aligned} \tag{4.12}$$

After some calculations, it can be seen that

$$D(\xi)=\int_\xi^\infty \frac{Q'(x)}{N(x)}\,dx, \qquad \xi>1, \tag{4.13}$$

where

$$Q(x):=(x^2-1)\,P'/P, \qquad x>1 \tag{4.14}$$

and

$$N(x):=2[(x^2-1)\,P']^2, \qquad x>1. \tag{4.15}$$

From the definitions (4.9) and (4.11) of $F(\xi)$ and $D(\xi)$, resepctively, it is seen that the statement of the theorem is equivalent to proving that $D(\xi)>0$ for $\xi>1$. Noting that $N(x)>0$ for $x>1$, it suffices to show that $Q'(x)>0$ for $x>1$.

From (4.8) and (4.14) we have

$$Q(x)=mx+(x^2-1)\,\frac{P_n^{(m+1)}(x)}{P_n^{(m)}(x)} \tag{4.16}$$

$$=mx+(x^2-1)\,\frac{[(x^2-1)^n]^{(n+m+1)}}{[(x^2-1)^n]^{(n+m)}}, \tag{4.17}$$

where this last step follows from Rodrigue's formula (see [7, p. 232])

$$P_n(x) = \frac{2^{-n}}{n!}\,[(x^2-1)^n]^{(n)}. \tag{4.18}$$

From Leibnitz's formula we have

$$\begin{aligned}[(x^2-1)^n]^{(n+m)} &= \sum_{\substack{r,s\geqslant 0\\ r+s=n+m}} \frac{(n+m)!}{r!s!}\,[(x-1)^n]^{(r)}[(x+1)^n]^{(s)} \qquad (4.19)\\ &= (n+m)! \sum_{r+s=m+m} \binom{n}{r}\binom{n}{s}(x-1)^{n-r}(x+1)^{n-s}.\end{aligned}$$

After some manipulations we have

$$[(x^2-1)^n]^{(n+m)} = (n+m)!\,A(x), \tag{4.20}$$

where

$$A(x) := \sum_{\substack{r,s\geqslant 0\\ r+s=n-m}} c_{rs}(x-1)^r(x+1)^s > 0, \qquad x>1 \tag{4.21}$$

and

$$c_{rs} := \binom{n}{r}\binom{n}{s} > 0. \tag{4.22}$$

Furthermore, from (4.20) we have

$$(x^2-1)\,[(x^2-1)^n]^{(n+m+1)} = (n+m)!\,(x^2-1)\,A'(x) \tag{4.23}$$

and after some calculations we find

$$(x^2-1)\,A'(x) = (n-m)\,xA(x) + B(x), \tag{4.24}$$

where

$$B(x) := \sum_{r+s=n-m} c_{rs}(r-s)(x-1)^r(x+1)^s. \tag{4.25}$$

A formula similar to (4.24) holds for $B(x)$. It is

$$(x^2-1)\,B'(x) = (n-m)\,xB(x) + C(x), \tag{4.26}$$

where

$$C(x) := \sum_{r+s=n-m} c_{rs}(r-s)^2(x-1)^r(x+1)^s. \tag{4.27}$$

From Eqs. (4.17), (4.20), (4.23) and (4.24) it follows that

$$Q(x) = nx + \frac{B(x)}{A(x)}. \tag{4.28}$$

Consequently,

$$Q'(x) = n + \frac{AB' - BA'}{A^2} \tag{4.29}$$

and from (4.24), (4.26) and (4.29) we obtain

$$Q'(x) = n + \frac{A(x)\, C(x) - B^2(x)}{(x^2-1)\, A^2(x)}, \qquad x > 1. \tag{4.30}$$

From Eqs. (4.25) and (4.27) it is seen that $B(x) = C(x) \equiv 0$, if $m = n$. Also, for $u \in \mathbb{R}$, we have from (4.21), (4.25) and (4.27)

$$u^2 A(x) + 2uB(x) + C(x) = \sum_{r+s=n-m} c_{rs}(u + r - s)^2 (x-1)^r (x+1)^s > 0, \tag{4.31}$$

if $m < n$ and $x > 1$. Hence $A(x)\, C(x) - B^2(x) > 0$, since the roots $u = (-B \pm \sqrt{B^2 - AC})/A$ of $u^2 A + 2uB + C = 0$ here cannot be real for any $x > 1$. Thus from (4.30) we have that $Q'(x) \geqslant n > 0$ for $0 \leqslant m \leqslant n$, $x > 1$. ∎

From (2.9), (2.10), (4.6) and Theorem 4.1 we have the following theorem:

THEOREM 4.2. *Let ∂D be the surface of a prolate spheroid with surface coordinate $\xi > 1$. Then the eigenvalues $\lambda_{mn}(\xi)$ of the integral operator K satisfy*

$$\lambda_{00}(\xi) = -1 \qquad \text{and} \qquad -1 < \lambda_{mn}(\xi) < 0$$

for $n > 0$, $0 \leqslant m \leqslant n$.

From this theorem and the fact that K is a linear compact operator on the infinite dimensional Banach space $C(\partial D)$ and (2.9) we have the following result:

THEOREM 4.3. *Let ∂D be the surface of a prolate spheroid with surface coordinate $\xi > 1$. Then the spectrum of K and of K' is given by*

$$\sigma(K) = \sigma(K') = \{\lambda_{mn}(\xi) \colon m, n \in \mathbb{Z}, 0 \leqslant m \leqslant n\} \cup \{0\},$$

where the $\lambda_{mn}(\xi)$'s denote the eigenvalues of K and are given in (3.17).

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