PACIFIC JOURNAL OF MATHEMATICS
Vol. 164, No. 2, 1994

A COUNTER-EXAMPLE CONCERNING THE PRESSURE IN THE NAVIER-STOKES EQUATIONS, AS $t \to 0^+$

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We show the existence of solutions of the Navier-Stokes equations for which the Dirichlet norm, $\|\nabla \mathbf{u}(t)\|_{L^2(\Omega)}$, of the velocity is continuous as $t = 0$, while the normalized L^2-norm, $\|p(t)\|_{L^2(\Omega)/R}$, of the pressure is not. This runs counter to the naive expectation that the relative orders of the spatial derivatives of $\mathbf{u}$, p and $\mathbf{u}_t$ should be the same in a priori estimates for the solutions as in the equations themselves.

1. Introduction. We consider the initial boundary value problem for the Navier-Stokes equations in an open bounded domain $\Omega \subset R^n$ $(n = 2$ or $3)$, with $\partial\Omega \in C^2$:

$$(1)\quad \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \Delta\mathbf{u} - \nabla p, \quad \nabla\cdot\mathbf{u} = 0, \quad \text{for } x \in \Omega \text{ and } t > 0,$$
$$\mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0.$$

For reference, let

$$D(\Omega) = \{\varphi \in C_0^\infty(\Omega) : \nabla\cdot\varphi = 0\},$$
$$J(\Omega) = \text{ completion of } D(\Omega) \text{ in the } L^2(\Omega)\text{-norm } \|\cdot\|,$$
$$J_1(\Omega) = \text{ completion of } D(\Omega) \text{ in the Dirichlet-norm } \|\nabla\cdot\|.$$

It is well known that if $\mathbf{u}_0 \in J_1(\Omega)$, then $\mathbf{u} \in C([0, T); J_1(\Omega))$ and

$$(2)\qquad \|\nabla\mathbf{u}(t)\|^2 + \int_0^t [\|\mathbf{u}(s)\|^2_{W_2^2(\Omega)} + \|\nabla p(s)\|^2 + \|\mathbf{u}_t(s)\|^2]\,ds \le C\|\nabla\mathbf{u}_0\|^2, \qquad 0 < t < T,$$

where T and C can be expressed as constants depending only on $\|\nabla\mathbf{u}_0\|$ and Ω (we are not concerned here with their optimal values).

It seems natural to expect that the relative orders of spatial differentiation of $\mathbf{u}$, p and $\mathbf{u}_t$ should be the same in a priori estimates for the Navier-Stokes equations as in the equations themselves. That is, p should appear with one less spatial derivative than $\mathbf{u}$, and $\mathbf{u}_t$ with two less, as they do under the integral sign in (2) and in many other known a priori estimates for the Navier-Stokes equations (see Heywood and Rannacher [3]).

In the analogue of (2) for the heat equation, the term $\|\nabla\mathbf{u}(t)\|^2$ on the left side is accompanied by the term $\|\mathbf{u}_t(t)\|_{-1}^2$, where $\|\cdot\|_{-1}$ is the negative Sobolev norm

$$\|\psi\|_{-1} \equiv \sup\{(\psi, \varphi) : \varphi \in \overset{\circ}{W}{}_2^1(\Omega),\ \|\nabla\varphi\| = 1\}.$$

Thus, with application in mind, we hoped that it might be possible to include the terms $\|p(t)\|^2_{L^2(\Omega)/R}$ and $\|\mathbf{u}_t(t)\|_{-1}^2$ along with $\|\nabla\mathbf{u}(t)\|^2$ on the left side of (2). We have found that this is not possible. The main result of this paper is the following:

Theorem. *There exists* $\mathbf{u}_0 \in J_1(\Omega)$ *such that the solution* $\mathbf{u}$, p *of* (1) *satisfies*

(i) $\limsup_{t\to 0} \|p(t)\|_{L^2(\Omega)/R} = \infty$,
(ii) $\limsup_{t\to 0} \|\mathbf{u}_t(t)\|_{-1} = \infty$.

Here, of course,

$$\|p\|_{L^2(\Omega)/R} \equiv \inf\{\|p - c\|_{L^2(\Omega)} : c \in \mathbf{R}\}.$$

For convenience, we will assume throughout this paper that the pressure $p(t)$ is normalized at every value of t by the condition $\int_\Omega p\,dx = 0$. This ensures that $\|p(t)\|_{L^2(\Omega)} = \|p(t)\|_{L^2(\Omega)/R}$. We make this same normalizing assumption of all pressure-like functions.

It seems worth offering a partial explanation of Theorem 1 from the point of view of function space decompositions. It is well known (see Solonnikov and Scadilov [7] for the second decomposition and related results) that

$$\mathbf{L}^2(\Omega) = \mathbf{J}(\Omega) \oplus \mathbf{G}(\Omega),$$
$$\overset{\circ}{\mathbf{W}}{}_2^1(\Omega) = \mathbf{J}_1(\Omega) \oplus \mathbf{R}(A),$$

where the second direct sum is relative to the Dirichlet inner product $(\nabla\varphi, \nabla\psi)$, and where

$$G(\Omega) \equiv \{\nabla p : p \in W_2^1(\Omega)\},$$
$$R(A) \equiv \{\mathbf{v} \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) : (\nabla\mathbf{v}, \nabla\varphi) = (p, \nabla\cdot\varphi)$$
$$\text{for some } p \in L^2(\Omega) \text{ and all } \varphi \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)\}.$$

Writing $Ap \equiv \mathbf{v}$ when p and $\mathbf{v}$ are related as above defines a homeomorphism between $L^2(\Omega)/R$ and $\mathbf{R}(A)$, i.e., there exist constants c_1, c_2, such that $c_1\|p\| \le \|\nabla Ap\| \le c_2\|p\|$.

Now, writing the Navier-Stokes equations, $\Delta \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{u}_t + \nabla p$, with the two terms on the right belonging to $\mathbf{J}(\Omega)$ and $\mathbf{G}(\Omega)$ respectively, it follows from the first decomposition that

$$\|\Delta \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u}\|^2 = \|\mathbf{u}_t\|^2 + \|\nabla p\|^2 .$$

Thus the estimates for $\mathbf{u}_t$ and p under the integral sign in (2) follow from that for $\mathbf{u}$. It is the estimate for $\mathbf{u}$ which is established first in proving (2).

On the other hand, if we write the Navier-Stokes equations in the generalized form

$$(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}, \varphi) = -(\nabla \mathbf{u}, \nabla \varphi) + (p, \nabla \cdot \varphi), \qquad \text{for all } \varphi \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega),$$

it is evident that the second decomposition implies

$$\|\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}\|_{-1}^2 = \|\nabla \mathbf{u}\|^2 + \|\nabla A p\|^2 .$$

Remembering that $\|\nabla A p\| \sim \|p\|$, we effectively have $\|\mathbf{u}_t\|_{-1}$ and $\|p\|$ on opposite sides of this equation. So it appears that both could be large, even when $\|\nabla \mathbf{u}\|$ is small. According to Theorem 1, that actually happens.

The behaviour of $\mathbf{u}_t$ and p that is demonstrated in Theorem 1 is not due to the nonlinearity in the Navier-Stokes equations. The same result is proved in the same way, and somewhat more simply, for the Stokes equations.

Proposition 2 below, which is proved at the end of the paper, provides a continuous dependence theorem that may be of independent interest.

We remark that our interest was drawn to the present problem in trying to determine whether the singular factor $t^{-1/2}$ in the pressure error estimate (1.3) of **[3]** is appropriate.

2. Preliminaries. We state here, as propositions, two results from general theory which will be needed in proving our main theorem. The first concerns the assumption of an initial value for the pressure, when the initial velocity belongs to $J_1(\Omega) \cap \mathbf{W}_2^2(\Omega)$. Its proof was given in **[3]**, and will be briefly described at the end of §4. The second proposition ensures that the pressure depends continuously on the initial value for the velocity in $\mathbf{J}_1(\Omega)$. Its proof is given in §4. In what follows, we make frequent use of the L^2-projection

$$P_G \colon \mathbf{L}^2(\Omega) \to \mathbf{G}(\Omega) .$$

PROPOSITION 1. *If* $\mathbf{u}_0 \in \mathbf{J}_1(\Omega) \cap \mathbf{W}_2^2(\Omega)$, *then the solution of* (1) *satisfies*

(i) $\mathbf{u} \in C([0, T); \mathbf{J}_1(\Omega) \cap \mathbf{W}_2^2(\Omega))$,

(ii) $p \in C([0, T); \mathbf{W}_2^1(\Omega))$,

where the initial pressure p_0 *is determined by the relation* $\nabla p_0 = P_G(\Delta \mathbf{u}_0 - \mathbf{u}_0 \cdot \nabla \mathbf{u}_0)$, *and the normalizing condition* $\int_\Omega p_0\, dx = 0$.

In the following proposition, $\mathbf{u}$, p *and* $\mathbf{v}$, q *are solutions of* (1) *taking initial values* $\mathbf{u}_0$ *and* $\mathbf{v}_0$, *respectively.*

PROPOSITION 2. *Given any* $\mathbf{u}_0 \in \mathbf{J}_1(\Omega)$, *there exist constants* T *and* B *depending only on* $\|\nabla \mathbf{u}_0\|$ *and* Ω, *such that for every* $\mathbf{v}_0 \in J_1(\Omega)$ *satisfying* $\|\nabla(\mathbf{v}_0 - \mathbf{u}_0)\| < 1$, *the difference of the ensuing solutions,* $\mathbf{w} \equiv \mathbf{v} - \mathbf{u}$, $r = q - p$, *satisfies*

$$\begin{aligned} \|\mathbf{w}(t)\|^2_{W_2^2} + \|r(t)\|^2_{W_2^1} + \|\mathbf{w}_t(t)\|^2 \\ \leq B t^{-1} \|\nabla \mathbf{w}_0\|^2, \qquad \text{for } 0 < t < T, \end{aligned} \tag{3}$$

where $\mathbf{w}_0 = \mathbf{v}_0 - \mathbf{u}_0$.

3. Construction of a counter-example. We begin with two lemmas. The first already shows the impossibility of including the term $\|p(t)\|^2_{L^2(\Omega)/R}$ on the left side of (2). Remember, in what follows, that the pressure and all pressure-like functions are normalized to have mean value zero.

LEMMA 1. *Given any positive numbers* ε *and* N, *there exists* $\mathbf{u}_0 \in J_1(\Omega) \cap W_2^2(\Omega)$ *such that* $\|\nabla \mathbf{u}_0\| < \varepsilon$ *and* $\|p_0\| > N$, *where* p_0 *is determined by the relation* $\nabla p_0 = P_G(\Delta \mathbf{u}_0 - \mathbf{u}_0 \cdot \nabla \mathbf{u}_0)$, *and the normalizing condition.*

Proof. First, choose $\mathbf{a} \in J_1(\Omega) \cap W_2^2(\Omega)$ such that $P_G \Delta \mathbf{a} \not\equiv 0$. It is possible to do this. For instance, $\mathbf{a}$ can be chosen as any eigenfunction of the stationary Stokes equations that has a non-zero corresponding eigenpressure. In fact, it is an amusing problem to show that there are such eigenfunctions, i.e., that not all solutions of $\Delta \mathbf{a} - \nabla q = \lambda \mathbf{a}$, $\nabla \cdot \mathbf{a} = 0$, $\mathbf{a}|_{\partial\Omega} = 0$ are solutions of $\Delta \mathbf{a} = \lambda \mathbf{a}$, $\mathbf{a}|_{\partial\Omega} = 0$, or to put it another way, that the study of the Stokes equations does not reduce trivially to that of the vector heat equation. We have a simple proof which is valid for a special class of domains, but will defer on this point to the reader's own devices and to forthcoming general results of Xie and of Grubb (private communications).

Next, choose $\mathbf{u}_1$ to be a multiple of $\mathbf{a}$ such that $\|p_1\| = 2N$, where p_1 is determined by the condition $\nabla p_1 = P_G \Delta \mathbf{u}_1$. This is possible since $\nabla q \equiv P_G \Delta \mathbf{a} \not\equiv 0$ implies $\|q\| \neq 0$.

We tentatively choose $\mathbf{u}_0 = \mathbf{u}_1 - \varphi$, where $\varphi \in D(\Omega)$ approximates $\mathbf{u}_1$ sufficiently well that $\|\nabla \mathbf{u}_0\| < \varepsilon/3$. Since $\varphi \in D(\Omega)$ implies that $\Delta\varphi \in D(\Omega)$, and hence that $P_G \Delta \varphi = 0$, we have $P_G \Delta \mathbf{u}_0 = P_G \Delta \mathbf{u}_1$. Thus, the relation $\nabla p_0 = P_G(\Delta \mathbf{u}_0 - \mathbf{u}_0 \cdot \nabla \mathbf{u}_0)$ implies that $p_0 = p_1 + p_2$, where $\nabla p_1 = P_G \Delta \mathbf{u}_1$ and $\nabla p_2 = P_G(\mathbf{u}_0 \cdot \nabla \mathbf{u}_0)$. Now, $\|p_0\| \geq |\, \|p_1\| - \|p_2\|\,|$. If this quantity exceeds N, we are done. If not, the nonlinear contribution satisfies $\|p_2\| \geq N$, in which case we obtain $\|p_0\| > N$ by simply tripling our original choice of $\mathbf{u}_0$.

LEMMA 2. *Given any positive numbers* ε *and* N, *and any* $\mathbf{a}_1 \in J_1(\Omega) \cap W_2^2(\Omega)$, *there exists* $\mathbf{a}_2 \in J_1(\Omega) \cap W_2^2(\Omega)$ *satisfying* $\|\nabla \mathbf{a}_2\| < \varepsilon$, *such that the pressure associated with the function* $\mathbf{u}_0 = \mathbf{a}_1 + \mathbf{a}_2$ *by the relation* $\nabla p_0 = P_G(\Delta \mathbf{u}_0 - \mathbf{u}_0 \cdot \nabla \mathbf{u}_0)$ *satisfies* $\|p_0\| > N$.

Proof. For any $\mathbf{a}_1$, $\mathbf{a}_2 \in J_1(\Omega) \cap W_2^2(\Omega)$, and $\mathbf{u}_0$, p_0 related to them as above, we have

$$\nabla p_0 = P_G[\Delta(\mathbf{a}_1 + \mathbf{a}_2) - (\mathbf{a}_1 + \mathbf{a}_2) \cdot \nabla(\mathbf{a}_1 + \mathbf{a}_2)] = \nabla p_1 + \nabla p_2 + \nabla r,$$

where

$$\nabla p_1 = P_G(\Delta \mathbf{a}_1 - \mathbf{a}_1 \cdot \nabla \mathbf{a}_1), \quad \nabla p_2 = P_G(\Delta \mathbf{a}_2 - \mathbf{a}_2 \cdot \nabla \mathbf{a}_2),$$
$$\nabla r = P_G(\mathbf{a}_1 \cdot \nabla \mathbf{a}_2 + \mathbf{a}_2 \cdot \nabla \mathbf{a}_1).$$

Using a Poincaré inequality for functions with mean value zero, Hölder's inequality, and several Sobolev inequalities which are valid in both two and three dimensions, we have

$$\|r\| \leq c_1 \|\nabla r\| = c_1 \|P_G(\mathbf{a}_1 \cdot \nabla \mathbf{a}_2 + \mathbf{a}_2 \cdot \nabla \mathbf{a}_1)\| \leq c_2 \|\mathbf{a}_1\|_{W_2^2} \|\nabla \mathbf{a}_2\|.$$

Now, given $\mathbf{a}_1$, we can use Lemma 1 to choose $\mathbf{a}_2$ such that $\|\nabla \mathbf{a}_2\| < \varepsilon$, and $\|p_2\| > \|p_1\| + \varepsilon c_2 \|\mathbf{a}_1\|_{W_2^2} + N$. Then, clearly, $\|p_0\| > \|p_2\| - \|p_1\| - \|r\| > N$.

Proof of Theorem. Let $\mathbf{u}_n(t)$, $p_n(t)$ be the solution of the initial value problem (1) corresponding to the initial data $\mathbf{u}_n(0) = \sum_{k=1}^{n} \mathbf{a}_k$, where the $\mathbf{a}_k$ are chosen as follows.

First, use Lemma 1 to choose $\mathbf{a}_1 \in J_1 \cap W_2^2$ such that $\|\nabla \mathbf{a}_1\| < 1/2$ and $\|p_1\| > 2$, where $\nabla p_1 = P_G(\Delta \mathbf{a}_1 - \mathbf{a}_1 \cdot \nabla \mathbf{a}_1)$. By Proposition 1, $\|p_1(t)\| > 2$ on some interval $[0, t_1]$.

Then, recursively, choose $\mathbf{a}_k \in J_1(\Omega) \cap W_2^2(\Omega)$ with $\|\nabla \mathbf{a}_k\| < (1/2)^k \sqrt{t_{k-1}/B}$, such that $\|p_k(t)\| > k+1$ on an interval $[0, t_k]$. The constant B here is from Proposition 2; for later convenience we assume that $B \geq 1$. We can also assume that the numbers t_k are chosen such that $t_{k+1} < t_k < 1$, and such that $t_k \to 0$ as $k \to \infty$.

Finally, let $\mathbf{u}(t)$, $p(t)$ be the solution of the initial value problem (1) corresponding to $\mathbf{u}_0 = \sum_{k=1}^{\infty} \mathbf{a}_k$. Since $\mathbf{u}_0, \mathbf{u}_n(0) \in J_1(\Omega)$, and $\|\nabla(\mathbf{u}_0 - \mathbf{u}_n(0))\| < 1$, Proposition 2 implies that

$$\|p(t_n) - p_n(t_n)\| \leq \sqrt{B/t_n}\left\|\nabla \sum_{k=n+1}^{\infty} \mathbf{a}_k\right\| \leq \sum_{k=n+1}^{\infty} (1/2)^k < 1.$$

Hence, $\|p(t_n)\| \geq \|p_n(t_n)\| - \|p(t_n) - p_n(t_n)\| > (n+1) - 1 = n$.

4. Proofs of the preliminary propositions. The proof of Proposition 1 will be briefly described at the end of this section. First, we give the proof of Proposition 2, beginning with two lemmas. Below, we frequently use inequalities of Sobolev's type without mention, in particular the inequalities $\|\mathbf{u}\|_{L^6} \leq c\|\nabla \mathbf{u}\|$, $\|\mathbf{u}\|_{L^3} \leq c\|\mathbf{u}\|^{1/2}\|\nabla \mathbf{u}\|^{1/2}$, $\|\nabla \mathbf{u}\|_{L^3} \leq c\|\nabla \mathbf{u}\|^{1/2}\|\mathbf{u}\|_{W_2^2}^{1/2}$, $\sup|\mathbf{u}| \leq c\|\nabla \mathbf{u}\|^{1/2}\|\mathbf{u}\|_{W_2^2}^{1/2}$. See **[1]** for the last of these. They are all dimensionally sharp in three-dimensions, but also valid in bounded two-dimensional domains. Everywhere, we use c as a generic constant that depends only on Ω.

Let $\widetilde{\Delta} = P_J \Delta$, where P_J is the L^2-projection $P_J \colon L^2(\Omega) \to J(\Omega)$. We will also frequently use without mention the well-known Cattabriga **[2]**/Solonnikov **[6]**, **[7]** estimate $\|\mathbf{u}\|_{W_2^2} \leq c\|\widetilde{\Delta}\mathbf{u}\|$, valid for solutions and generalized solutions $\mathbf{u} \in J_1(\Omega)$ of the Stokes equations $\Delta \mathbf{u} - \nabla p = \mathbf{f}$, $\nabla \cdot \mathbf{u} = 0$, $\mathbf{u}|_{\partial\Omega} = 0$, with $\mathbf{f}$ square-summable.

LEMMA 3. *Given any $\mathbf{u}_0 \in J_1(\Omega)$, there exist constants C and T depending only on $\|\nabla \mathbf{u}_0\|$ and Ω such that*

$$\|\nabla \mathbf{u}(t)\|^2 + t\|\mathbf{u}_t(t)\|^2 \leq C\|\nabla \mathbf{u}_0\|^2, \quad \textit{for } 0 < t < T. \tag{4}$$

Proof. The estimate (4) for $\|\nabla \mathbf{u}(t)\|$ is contained in (2), and is well known. To prove the estimate for $t\|\mathbf{u}_t(t)\|^2$, differentiate (1) with respect to t, multiply by $\mathbf{u}_t$, integrate over Ω and then by parts, and use the inequalities of Hölder, Sobolev and Young to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_t\|^2 + \|\nabla \mathbf{u}_t\|^2 &= (\mathbf{u}_t \cdot \nabla \mathbf{u}_t, \mathbf{u}) \leq c\|\nabla \mathbf{u}\|\, \|\mathbf{u}_t\|^{1/2}\|\nabla \mathbf{u}_t\|^{3/2} \\ &\leq c\|\nabla \mathbf{u}\|^4\|\mathbf{u}_t\|^2 + \|\nabla \mathbf{u}_t\|^2.\end{aligned}$$

Multiplying by t, we get

$$\frac{d}{dt}[t\|\mathbf{u}_t\|^2] \le \|\mathbf{u}_t\|^2 + c\|\nabla\mathbf{u}\|^4[t\|\mathbf{u}_t\|^2]. \tag{5}$$

The boundedness of the integral in (2) implies that $\liminf_{t\to 0} t\|\mathbf{u}_t(t)\|^2 = 0$. Hence, applying Gronwall's inequality to (5), using (2), we obtain the desired estimate (4). This completes the proof.

In what follows, let $\mathbf{u}$, p and $\mathbf{v}$, q be solutions of (1) taking initial values $\mathbf{u}_0$ and $\mathbf{v}_0$, respectively, and let $\mathbf{w} = \mathbf{v} - \mathbf{u}$, $r = q - p$, and $\mathbf{w}_0 = \mathbf{v}_0 - \mathbf{u}_0$. Then

$$\mathbf{w}_t - \Delta\mathbf{w} + \mathbf{w}\cdot\nabla\mathbf{w} + \mathbf{u}\cdot\nabla\mathbf{w} + \mathbf{w}\cdot\nabla\mathbf{u} = -\nabla r. \tag{6}$$

Multiplying by $-\widetilde{\Delta}\mathbf{w}$, integrating over Ω, and proceeding as in Lemma 2 of [4], we obtain the following continuous dependence theorem:

LEMMA 4. *Corresponding to any* $\mathbf{u}_0 \in J_1(\Omega)$, *there exist constants* C *and* T, *depending only on* $\|\nabla\mathbf{u}_0\|$ *and* Ω, *such that*

$$\|\nabla\mathbf{w}(t)\|^2 + \int_0^t (\|\widetilde{\Delta}\mathbf{w}\|^2 + \|\mathbf{w}_t\|^2)\,ds \le C\|\nabla\mathbf{w}_0\|^2, \quad \text{for } 0 < t < T, \tag{7}$$

provided $\mathbf{v}_0 \in J_1(\Omega)$, *and* $\|\nabla\mathbf{w}_0\| < 1$.

We proceed now with the proof of Proposition 2. Differentiating (6) with respect to t, multiplying by $\mathbf{w}_t$, integrating over Ω and then by parts, and applying Hölder's and Sobolev's inequalities, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{w}_t\|^2 + \|\nabla\mathbf{w}_t\|^2 &= (\mathbf{w}_t\cdot\nabla\mathbf{w}_t, \mathbf{w}) + (\mathbf{u}_t\cdot\nabla\mathbf{w}_t, \mathbf{w})\\ &\quad + (\mathbf{w}_t\cdot\nabla\mathbf{w}_t, \mathbf{u}) + (\mathbf{w}\cdot\nabla\mathbf{w}_t, \mathbf{u}_t)\\ &\le c\|\nabla\mathbf{w}\|\,\|\mathbf{w}_t\|^{1/2}\|\nabla\mathbf{w}_t\|^{3/2} + c\|\mathbf{u}_t\|\,\|\nabla\mathbf{w}\|^{1/2}\|\widetilde{\Delta}\mathbf{w}\|^{1/2}\|\nabla\mathbf{w}_t\|\\ &\quad + c\|\nabla\mathbf{u}\|\,\|\mathbf{w}_t\|^{1/2}\|\nabla\mathbf{w}_t\|^{3/2}.\end{aligned}$$

Using Young's inequality yields

$$\frac{d}{dt}\|\mathbf{w}_t\|^2 \le c\|\nabla\mathbf{w}\|^4\|\mathbf{w}_t\|^2 + c\|\mathbf{u}_t\|^2\|\nabla\mathbf{w}\|\,\|\widetilde{\Delta}\mathbf{w}\| + c\|\nabla\mathbf{u}\|^4\|\mathbf{w}_t\|^2.$$

Multiplying by t and using Young's inequality again, we get

$$\begin{aligned}\frac{d}{dt}[t\|\mathbf{w}_t\|^2] \le \|\mathbf{w}_t\|^2 &+ ct^2\|\mathbf{u}_t\|^4\|\nabla\mathbf{w}\|^2 + \|\widetilde{\Delta}\mathbf{w}\|^2\\ &+ c[\|\nabla\mathbf{w}\|^4 + \|\nabla\mathbf{u}\|^4][t\|\mathbf{w}_t\|^2].\end{aligned}$$

Let $\mu(t) = c\int_0^t [\|\nabla \mathbf{w}\|^4 + \|\nabla \mathbf{u}\|^4]\,ds$. Together, Lemmas 3 and 4 provide an interval $[0, T]$ on which $\mu(t)$ remains bounded, with T depending only on $\|\nabla \mathbf{u}_0\|$ and Ω. This choice of T is fixed in what follows. Since (7) implies that $\liminf_{t\to 0} t\|\mathbf{w}_t(t)\|^2 = 0$, we can apply Gronwall's inequality, using (4) and (7), to obtain

$$
\begin{aligned}
(8)\qquad t\|\mathbf{w}_t\|^2 &\le e^{\mu(t)} \int_0^t e^{-\mu(s)}[\|\mathbf{w}_t\|^2 + cs^2\|\mathbf{u}_t\|^4\|\nabla \mathbf{w}\|^2 + \|\tilde{\Delta} \mathbf{w}\|^2]\,ds \\
&\le C\|\nabla \mathbf{w}_0\|^2,
\end{aligned}
$$

on $[0, T]$, proving part of (3). Multiplying (6) by $-\tilde{\Delta}\mathbf{w}$, we get

$$
\begin{aligned}
\|\tilde{\Delta}\mathbf{w}\|^2 &= (\mathbf{w}_t, \tilde{\Delta}\mathbf{w}) + (\mathbf{w}\cdot\nabla\mathbf{w}, \tilde{\Delta}\mathbf{w}) + (\mathbf{u}\cdot\nabla\mathbf{w}, \tilde{\Delta}\mathbf{w}) + (\mathbf{w}\cdot\nabla\mathbf{u}, \tilde{\Delta}\mathbf{w}) \\
&\le \|\mathbf{w}_t\|\,\|\tilde{\Delta}\mathbf{w}\| + c\|\nabla\mathbf{w}\|^{3/2}\|\tilde{\Delta}\mathbf{w}\|^{3/2} + c\|\nabla\mathbf{u}\|\,\|\nabla\mathbf{w}\|^{1/2}\|\tilde{\Delta}\mathbf{w}\|^{3/2} \\
&\le c\|\mathbf{w}_t\|^2 + c\|\nabla\mathbf{w}\|^6 + c\|\nabla\mathbf{u}\|^4\|\nabla\mathbf{w}\|^2 + \tfrac{1}{2}\|\tilde{\Delta}\mathbf{w}\|^2.
\end{aligned}
$$

Thus

$$
(9)\qquad \|\tilde{\Delta}\mathbf{w}\|^2 \le c\|\mathbf{w}_t\|^2 + c[\|\nabla\mathbf{w}\|^4 + \|\nabla\mathbf{u}\|^4]\|\nabla\mathbf{w}\|^2.
$$

Multiplying (6) by ∇r yields

$$
\begin{aligned}
\|\nabla r\|^2 &= (\Delta\mathbf{w}, \nabla r) - (\mathbf{w}\cdot\nabla\mathbf{w}, \nabla r) - (\mathbf{u}\cdot\nabla\mathbf{w}, \nabla r) - (\mathbf{w}\cdot\nabla\mathbf{u}, \nabla r) \\
&\le \|\tilde{\Delta}\mathbf{w}\|^2 + c\|\nabla\mathbf{w}\|^3\|\tilde{\Delta}\mathbf{w}\| + c\|\nabla\mathbf{u}\|^2\|\nabla\mathbf{w}\|\,\|\tilde{\Delta}\mathbf{w}\| + \tfrac{1}{2}\|\nabla r\|^2,
\end{aligned}
$$

which combined with (9) gives

$$
\begin{aligned}
(10)\qquad \|\nabla r\|^2 &\le c\|\tilde{\Delta}\mathbf{w}\|^2 + c[\|\nabla\mathbf{w}\|^4 + \|\nabla\mathbf{u}\|^4]\|\nabla\mathbf{w}\|^2 \\
&\le c\|\mathbf{w}_t\|^2 + c[\|\nabla\mathbf{w}\|^4 + \|\nabla\mathbf{u}\|^4]\|\nabla\mathbf{w}\|^2.
\end{aligned}
$$

Finally, combining (8), (9), and (10), and assuming that $\|\nabla\mathbf{w}_0\| < 1$, we get

$$
t\|\tilde{\Delta}\mathbf{w}\|^2 + t\|\nabla r\|^2 \le ct\|\mathbf{w}_t\|^2 + ct[\|\nabla\mathbf{w}\|^4 + \|\nabla\mathbf{u}\|^4]\|\nabla\mathbf{w}\|^2 \le C\|\nabla\mathbf{w}_0\|^2
$$

on the interval $[0, T]$ chosen above. This completes the proof of Proposition 2.

In outlining the proof of Proposition 1 (from Theorem 2.3 and Proposition 2.1 of **[3]**) we take it as well known that $\mathbf{u} \in C([0, T); J_1(\Omega))$ if $\mathbf{u}_0 \in J_1(\Omega)$, and that $\|\tilde{\Delta}\mathbf{u}(t)\|$ is bounded on $[0, T)$ if $\mathbf{u}_0 \in J_1(\Omega) \cap W_2^2(\Omega)$. To show, further, that the initial velocity is assumed strongly in $W_2^2(\Omega)$, it suffices to show that $\limsup_{t\to 0}\|\tilde{\Delta}\mathbf{u}(t)\| \le \|\tilde{\Delta}\mathbf{u}_0\|$. The following is a formal argument which can be carried out rigorously in the construction of the solution by Galerkin approximation. Multiplying (1) by $\tilde{\Delta}\mathbf{u}_t$, one obtains

$$\begin{aligned}(11)\qquad \|\nabla\mathbf{u}_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\tilde{\Delta}\mathbf{u}\|^2 &= (\mathbf{u}\cdot\nabla\mathbf{u}\,,\,\tilde{\Delta}\mathbf{u}_t)\\ &= \frac{d}{dt}(\mathbf{u}\cdot\nabla\mathbf{u}\,,\,\tilde{\Delta}\mathbf{u}) - (\mathbf{u}_t\cdot\nabla\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}_t\,,\,\tilde{\Delta}\mathbf{u}).\end{aligned}$$

Since $|(\mathbf{u}_t\cdot\nabla\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}_t\,,\,\tilde{\Delta}\mathbf{u})| \le c\|\nabla\mathbf{u}_t\|\,\|\nabla\mathbf{u}\|^{1/2}\|\tilde{\Delta}\mathbf{u}\|^{3/2} \le c\|\nabla\mathbf{u}\|\,\|\tilde{\Delta}\mathbf{u}\|^3 + \|\nabla\mathbf{u}_t\|^2$, and since $\|\nabla\mathbf{u}(t)\|$ and $\|\tilde{\Delta}\mathbf{u}(t)\|$ are bounded on $[0\,,\,T)$, we can integrate (11) to get

$$\|\tilde{\Delta}\mathbf{u}(t)\|^2 \le \|\tilde{\Delta}\mathbf{u}_0\|^2 + 2(\mathbf{u}(t)\cdot\nabla\mathbf{u}(t)\,,\,\tilde{\Delta}\mathbf{u}(t)) - 2(\mathbf{u}_0\cdot\nabla\mathbf{u}_0\,,\,\tilde{\Delta}\mathbf{u}_0) + Ct.$$

This implies the desired result, i.e., that $\tilde{\Delta}\mathbf{u}\to\tilde{\Delta}\mathbf{u}_0$ strongly in $L^2(\Omega)$, since $\tilde{\Delta}\mathbf{u}\to\tilde{\Delta}\mathbf{u}_0$ weakly in $L^2(\Omega)$, and $\mathbf{u}\cdot\nabla\mathbf{u}\to\mathbf{u}_0\cdot\nabla\mathbf{u}_0$ strongly in $L^2(\Omega)$. Finally, from this, it follows that

$$\nabla p - \nabla p_0 = P_G[(\Delta\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u}) - (\Delta\mathbf{u}_0 - \mathbf{u}_0\cdot\nabla\mathbf{u}_0)] \to 0\,, \quad \text{as } t\to 0.$$

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Received November 5, 1991.

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