A generalization of a logarithmic Sobolev inequality to the Hölder class

H. Ibrahim *, 1

December 9, 2010

Abstract
In a recent work of the author, a parabolic extension of the elliptic Ogawa type inequality has been established. This inequality is originated from the Brézis-Gallouët-Wainger logarithmic type inequalities revealing Sobolev embeddings in the critical case. In this paper, we improve the parabolic version of Ogawa inequality by allowing it to cover not only the class of functions from Sobolev spaces, but the wider class of Hölder continuous functions.

AMS subject classifications: 42B35, 54C35, 42B25, 39B05.
Key words: Littlewood-Paley decomposition, logarithmic Sobolev inequalities, parabolic BMO spaces, parabolic Lizorkin-Triebel spaces, parabolic Besov spaces.

1 Introduction and main results

In [9], a generalization of the Ogawa type inequality [16] to the parabolic framework has been shown. Ogawa inequality can be considered as a generalized version in the Lizorkin-Triebel spaces of the remarkable estimate of Brézis-Gallouët-Wainger [4, 5] that holds in a limiting case of the Sobolev embedding theorem. The inequality showed in [9, Theorem 1.1] provides an estimate of the L^∞ norm of a function in terms of its parabolic BMO norm, with the aid of the square root of the logarithmic dependency of a higher order Sobolev norm. More precisely, for any vector-valued function $f = \nabla g \in W_2^{2m,m}(\mathbb{R}^{n+1})$, $g \in L^2(\mathbb{R}^{n+1})$ with $m, n \in \mathbb{N}^*$, $2m > \frac{n+2}{2}$, there exists a constant $C = C(m,n) > 0$ such that:

$$\|f\|_{L^\infty(\mathbb{R}^{n+1})} \leq C\left(1 + \|f\|_{BMO(\mathbb{R}^{n+1})}\left(\log^+(\|f\|_{W_2^{2m,m}(\mathbb{R}^{n+1})} + \|g\|_{L^\infty(\mathbb{R}^{n+1})})\right)^{1/2}\right), \tag{1.1}$$

where $W_2^{2m,m}$ is the parabolic Sobolev space (we refer to [15] for the definition and further properties), and BMO is the parabolic bounded mean oscillation space (defined via parabolic balls instead of Euclidean ones [9, Definition 2.1]). The above inequality reflects a limiting case of Sobolev embeddings in the parabolic framework (see [10, 11] for similar type inequalities, and [4, 5, 6, 12, 13, 14, 16] for various elliptic versions). By considering functions $f \in W_2^{2m,m}(\Omega_T)$ defined on the bounded domain

$$\Omega_T = (0,1)^n \times (0,T), \quad T > 0,$$

*Lebanese University, Faculty of Sciences, Mathematics Department, Hadeth, Beirut, Lebanon.
1LaMA-Liban, Lebanese University, P. O. Box 37, Tripoli, Lebanon.
E-mail: hassan.ibrahim@liu.edu.lb

we have the following estimate (see [9, Theorem 1.2]):

$$\|f\|_{L^\infty(\Omega_T)} \le C\left(1 + \left(\|f\|_{BMO(\Omega_T)} + \|f\|_{L^1(\Omega_T)}\right)\left(\log^+ \|f\|_{W_2^{2m,m}(\Omega_T)}\right)^{1/2}\right). \tag{1.2}$$

The different norms of f appearing in inequalities (1.1) and (1.2) are finite since

$$W_2^{2m,m} \hookrightarrow C^{\gamma,\gamma/2} \hookrightarrow L^\infty \hookrightarrow BMO \quad \text{for some} \quad 0<\gamma<1, \tag{1.3}$$

where $C^{\gamma,\gamma/2}$ is the parabolic Hölder space that will be defined later. Moreover, it is easy to check that g is bounded and continuous.

The purpose of this paper to show that the condition $f = \nabla g \in W_2^{2m,m}$ (vector-valued case), or $f \in W_2^{2m,m}$ (scalar-valued case) can be relaxed. Indeed, inequalities (1.1) and (1.2) can be applied to a wider class of Hölder continuous functions $f = \nabla g \in C^{\gamma,\gamma/2}$, $0<\gamma<1$ (vector-valued case), or $f \in C^{\gamma,\gamma/2}$ (scalar-valued case). To be more precise, we now state the main results of this paper. Our first theorem is the following:

Theorem 1.1 *(Logarithmic Hölder inequality on $\mathbb{R}^{n+1}$). Let $0<\gamma<1$. For any $f = \nabla g \in C^{\gamma,\gamma/2}(\mathbb{R}^{n+1}) \cap L^2(\mathbb{R}^{n+1})$ with $g \in L^2(\mathbb{R}^{n+1})$, there exists a constant $C = C(\gamma,n) > 0$ such that*

$$\|f\|_{L^\infty(\mathbb{R}^{n+1})} \le C\left(1 + \|f\|_{BMO(\mathbb{R}^{n+1})}\left(\log^+(\|f\|_{C^{\gamma,\gamma/2}(\mathbb{R}^{n+1})} + \|g\|_{L^\infty(\mathbb{R}^{n+1})})\right)^{1/2}\right). \tag{1.4}$$

The second theorem deals with functions defined on the bounded domain Ω_T.

Theorem 1.2 *(Logarithmic Hölder inequality on a bounded domain). Let $0<\gamma<1$. For any $f \in C^{\gamma,\gamma/2}(\Omega_T)$, there exists a constant $C = C(\gamma,n,T) > 0$ such that*

$$\|f\|_{L^\infty(\Omega_T)} \le C\left(1 + \left(\|f\|_{BMO(\Omega_T)} + \|f\|_{L^1(\Omega_T)}\right)\left(\log^+(\|f\|_{C^{\gamma,\gamma/2}(\Omega_T)})\right)^{1/2}\right). \tag{1.5}$$

We notice that inequalities (1.4) and (1.5) directly imply (with the aid of the embeddings (1.3)) (1.1) and (1.2).

Remark 1.3 *The same inequality (1.4) still holds for scalar-valued functions $f = \frac{\partial g}{\partial x_i} \in C^{\gamma,\gamma/2}(\mathbb{R}^{n+1}) \cap L^2(\mathbb{R}^{n+1})$, $i \in 1,\dots,n+1$, with $g \in L^\infty(\mathbb{R}^{n+1})$.*

This paper is organized as follows: in Section 2, we give the definitions of some basic functional spaces used throughout this paper. Section 3 is devoted to the proofs of the main results.

2 Definitions

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^{n+1}$. A generic element $z \in \mathbb{R}^{n+1}$ has the form $z = (x,t)$ with $x = (x_1,\dots,x_n) \in \mathbb{R}^n$. We begin by defining parabolic Hölder spaces $C^{\gamma,\gamma/2}$.

Definition 2.1 *(Parabolic Hölder spaces). For $0<\gamma<1$, we define the parabolic space of Hölder continuous functions of order γ in the following way:*

$$C^{\gamma,\gamma/2}(\mathcal{O}) = \{f \in C(\overline{\mathcal{O}}),\ \|f\|_{C^{\gamma,\gamma/2}(\mathcal{O})} < \infty\},$$

where

$$\|f\|_{C^{\gamma,\gamma/2}(\mathcal{O})} = \|f\|_{L^\infty(\mathcal{O})} + \langle f\rangle^{(\gamma)}_{x,\mathcal{O}} + \langle f\rangle^{(\gamma/2)}_{t,\mathcal{O}}, \tag{2.1}$$

with

$$\langle f\rangle^{(\gamma)}_{x,\mathcal{O}} = \sup_{(x,t),(x',t)\in\mathcal{O},\, x\neq x'} \frac{|f(x,t)-f(x',t)|}{|x-x'|^\gamma}$$

and

$$\langle f\rangle^{(\gamma/2)}_{t,\mathcal{O}} = \sup_{(x,t),(x,t')\in\mathcal{O},\, t\neq t'} \frac{|f(x,t)-f(x,t')|}{|t-t'|^{\gamma/2}}.$$

For a detailed study of parabolic Hölder spaces, we refer the reader to [15]. We now briefly recall some basic facts about Littlewood-Paley decomposition which are crucial in obtaining our logarithmic inequalities. Given the expansive $(n+1)\times(n+1)$ matrix $A = \mathrm{diag}\{2,\ldots,2,2^2\}$ (parabolic anisotropy), the corresponding Littlewood-Paley decomposition asserts that any tempered distribution $f\in\mathcal{S}'(\mathbb{R}^{n+1})$ can be decomposed as

$$f = \sum_{j\in\mathbb{Z}} \varphi_j * f, \quad \text{where} \quad \varphi_j(z) = |\mathrm{det}A|^j \varphi(A^j z), \tag{2.2}$$

with the convergence in $\mathcal{S}'/\mathcal{P}$ (modulo polynomials). Here $\varphi\in\mathcal{S}(\mathbb{R}^{n+1})$ is a test function such that $\mathrm{supp}\,\hat{\varphi}$ is compact and bounded away from the origin, and $\sum_{j\in\mathbb{Z}}\hat{\varphi}(A^j z)=1$ for all $z\in\mathbb{R}^{n+1}\setminus\{0\}$, where $\hat{\varphi}$ is the Fourier transform of φ. The sequence $(\varphi_j)_{j\in\mathbb{Z}}$ is mainly used to define parabolic homogeneous Lizorkin-Triebel, Hardy and Besov spaces (see for instance [17, 18]). We only present here the spaces that are used throughout the analysis. For $1\le p\le\infty$, we define the parabolic homogeneous Lizorkin-Triebel space $\dot{F}^0_{p,2}$ as the space of functions $f\in\mathcal{S}'(\mathbb{R}^{n+1})$ with finite quasi-norms:

$$\|f\|_{\dot{F}^0_{p,2}(\mathbb{R}^{n+1})} = \Big\|\big(\sum_{|j|<\infty}|\varphi_j * f|^2\big)^{1/2}\Big\|_{L^p(\mathbb{R}^{n+1})} < \infty. \tag{2.3}$$

The space $\dot{F}^0_{p,2}$ can be identified with the parabolic Hardy space H^p, $1\le p<\infty$ having the following square function characterization stated informally as:

$$H^p(\mathbb{R}^{n+1}) = \{f\in\mathcal{S}'(\mathbb{R}^{n+1});\ \big(\sum_{|j|<\infty}|\varphi_j * f|^2\big)^{1/2}\in L^p\}.$$

This identification (see Bownik [2]) can be stated as follows: for all $1\le p<\infty$, we have:

$$\dot{F}^0_{p,2}(\mathbb{R}^{n+1}) \simeq H^p(\mathbb{R}^{n+1}). \tag{2.4}$$

Now, for defining the inhomogeneous parabolic Besov space $B^\gamma_{\infty,\infty}$ used later in obtaining our results, we use a slightly different sequence. Indeed, let $\theta\in C_0^\infty(\mathbb{R}^{n+1})$ be any cut-off function satisfying:

$$\theta(z) = \begin{cases} 1 & \text{if} \quad |z|_p\le 1 \\ 0 & \text{if} \quad |z|_p\ge 2, \end{cases} \tag{2.5}$$

where $|\cdot|_p$ is the parabolic quasi-norm associated to the matrix A (see [9]). Taking the new function (but keeping the same notation) φ_0 defined via the relation

$$\hat{\varphi}_0 = \theta, \tag{2.6}$$

we can give the definition of the Besov space $B^\gamma_{\infty,\infty}$.

Definition 2.2 *(Parabolic inhomogeneous Besov spaces). Take the smoothness parameter* $0 < \gamma < 1$. *Let* $(\varphi_j)_{j\in\mathbb{Z}}$ *be the sequence such that* φ_0 *is given by (2.6), while* φ_j *is given by (2.2) for all* $j \geq 1$. *We define the parabolic inhomogeneous Besov space* $B^{\gamma}_{\infty,\infty}$ *as the space of all functions* $f \in \mathcal{S}'(\mathbb{R}^{n+1})$ *with finite quasi-norms*

$$\|f\|_{B^{\gamma}_{\infty,\infty}} = \sup_{j\geq 0} 2^{\gamma j}\|\varphi_j * f\|_{L^\infty(\mathbb{R}^{n+1})}.$$

3 Proofs of theorems

The proof of Theorem 1.1 rely on the following two lemmas of different interest:

Lemma 3.1 *Let* $0 < \gamma < 1$ *and let* $N > 0$ *be a positive integer. Then for any* $f = \nabla g \in C^{\gamma,\gamma/2}(\mathbb{R}^{n+1}) \cap L^2(\mathbb{R}^{n+1})$ *with* $g \in L^2(\mathbb{R}^{n+1})$, *there exists a constant* $C = C(\gamma, n) > 0$ *such that*

$$\Big\|\Big(\sum_{j<-N} 2^{-2\gamma j}|\varphi_j * f|^2\Big)^{1/2}\Big\|_{L^\infty} \leq C\|g\|_{L^\infty}. \tag{3.1}$$

Proof. We provide a proof of (3.1) in the general case $N = 1$. We use the fact that $\partial_i g = f_i$ (for which we keep denoting it by f, i.e. $f = f_i$) for some $i = 1, \dots, n+1$, with $g \in L^\infty(\mathbb{R}^{n+1})$. For $z \in \mathbb{R}^{n+1}$, define

$$\Phi(z) = (\partial_i\varphi)(z), \tag{3.2}$$

and

$$\Phi_j(z) = |\det A|^j\,\Phi(A^j z) \quad \text{for all} \quad j \leq -1. \tag{3.3}$$

Using (2.2) we obtain:

$$(\partial_i\varphi_j)(z) = \begin{cases} 2^j\,\Phi_j(z) & \text{if} \quad i = 1, \dots, n \\ 2^{2j}\,\Phi_j(z) & \text{if} \quad i = n+1. \end{cases} \tag{3.4}$$

We now compute (see (3.3) and (3.4)):

$$\Big\|(\sum_{j\leq -1} 2^{-2\gamma j}|\varphi_j * f|^2)^{1/2}\Big\|_{L^\infty} \leq C \sup_{j\leq -1}\|\Phi_j * g\|_{L^\infty}, \tag{3.5}$$

where the constant C is given by:

$$C^2 = \begin{cases} \displaystyle\sum_{j\leq -1} 2^{2j(1-\gamma)} & \text{if} \quad i = 1, \dots, n \\ \displaystyle\sum_{j\leq -1} 2^{2j(2-\gamma)} & \text{if} \quad i = n+1, \end{cases}$$

which is finite $0 < C < +\infty$ under the choice

$$0 < \gamma < 1.$$

In order to terminate the proof, it suffices to show that

$$\|\Phi_j * g\|_{L^\infty} \leq C\|g\|_{L^\infty},$$

which can be deduced, by translation and dilation invariance, from the following estimate:

$$|(\Phi * g)(0)| \leq C\|g\|_{L^\infty}. \tag{3.6}$$

Indeed, define the positive radial decreasing function $h(r) = h(\|z\|)$ as follows:

$$h(r) = \sup_{\|z\| \geq r} |\Phi(z)|.$$

From (3.2), we remark that the function Φ is the inverse Fourier transform of a compactly supported function. Hence, we have:

$$h(0) = \|\Phi\|_{L^\infty} < +\infty, \tag{3.7}$$

and the asymptotic behavior

$$h(r) \leq \frac{C}{r^{n+2}} \quad \text{for all} \quad r \geq 1. \tag{3.8}$$

We compute (taking S_r^n as the n-dimensional sphere of radius r):

$$\begin{aligned} |(\Phi * g)(0)| &\leq \int_{\mathbb{R}^{n+1}} |\Phi(-z)||g(z)|dz \\ &\leq \int_0^\infty \left(\int_{S_r^n} |\Phi(-z)||g(z)|d\sigma(z) \right) dr \\ &\leq C \left(\int_0^\infty r^n h(r) dr \right) \|g\|_{L^\infty}. \end{aligned} \tag{3.9}$$

Using (3.7) and (3.8) we deduce that:

$$\begin{aligned} \int_0^\infty r^n h(r) dr &= \int_0^1 r^n h(r) dr + \int_1^\infty r^n h(r) dr \\ &\leq C \left(\int_0^1 h(0) dr + \int_1^\infty \frac{r^n}{r^{n+2}} dr \right) \\ &\leq C(\|\Phi\|_{L^\infty} + 1) \end{aligned}$$

which, together with (3.9), directly implies (3.6). As a conclusion, we obtain (see (3.5)):

$$\Big\| (\sum_{j \leq -1} 2^{-2\gamma j} |\varphi_j * f|^2)^{1/2} \Big\|_{L^\infty} \leq C \|g\|_{L^\infty},$$

and hence inequality (3.1) holds. □

Lemma 3.2 *Let $N > 0$ be a positive integer. Then for any $f \in BMO(\mathbb{R}^{n+1})$ there exists a constant $C = C(n) > 0$ such that:*

$$\Big\| (\sum_{|j| < N} |\varphi_j * f|^2)^{1/2} \Big\|_{L^\infty} \leq C \|f\|_{BMO}. \tag{3.10}$$

Proof. The proof provides inequality (3.10) for all $|j| < \infty$ by showing that $\dot{F}^0_{\infty,2} \simeq BMO$ and then using (2.3). Before starting the proof, we remind the reader that:

$$\|f\|_{BMO(\mathbb{R}^{n+1})} = \sup_{\mathcal{Q} \subseteq \mathbb{R}^{n+1}} \inf_{c \in \mathbb{R}} \left(\frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} |f - c| \right), \tag{3.11}$$

where $\mathcal{Q}$ denotes any arbitrary parabolic cube. Using the result of Bownik [3, Theorem 1.2], we have the following duality argument (that can be viewed as the parabolic extension of the well-known isotropic result of Triebel [17], and Frazier and Jawerth [8]):

$$\left(\dot{F}^0_{1,2}\right)' \simeq \dot{F}^0_{\infty,2}, \tag{3.12}$$

where $(\dot{F}^0_{1,2})'$ stands for the dual space of $\dot{F}^0_{1,2}$. Applying (2.4) with $p=1$ we obtain:

$$\dot{F}^0_{1,2} \simeq H^1. \tag{3.13}$$

Using the description of the dual of parabolic Hardy spaces H^p for $0<p\le 1$ (see Bownik [1, Theorem 8.3]), we get:

$$(H^p)' = \mathcal{C}^l_{q,s} \tag{3.14}$$

with the terms p, l, q, s chosen such that:

$$\begin{cases} l = \dfrac{1}{p} - 1, \\ 1 \le \dfrac{q}{q-1} \le \infty \quad \text{and} \quad p < \dfrac{q}{q-1}, \\ s \in \mathbb{N} \quad \text{and} \quad s \ge \lfloor l \rfloor, \quad \lfloor l \rfloor = \max\{n \in \mathbb{Z};\, n \le l\}. \end{cases} \tag{3.15}$$

The function space $\mathcal{C}^l_{q,s}$, $l \ge 0$, $1 \le q < \infty$ and $s \in \mathbb{N}$ (called the *Campanato space*), is the space of all $f \in L^q_{loc}(\mathbb{R}^{n+1})$ (defined up to addition by $P \in \mathcal{P}_s$; the set of all polynomials in $(n+1)$ variables of degree at most s) so that:

$$\|f\|_{\mathcal{C}^l_{s,q}(\mathbb{R}^{n+1})} = \sup_{\mathcal{Q}\subseteq\mathbb{R}^{n+1}} \inf_{P\in\mathcal{P}_s} |\mathcal{Q}|^l \left(\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}} |f-P|^q\right)^{1/q} < \infty. \tag{3.16}$$

Choosing $p=1$, $l=0$, $q=1$ and $s=0$, we can easily see that conditions (3.15) are all satisfied, and that (see (3.16) and (3.11)):

$$\mathcal{C}^0_{1,0} \simeq BMO.$$

This identification, together with (3.14), finally give:

$$\left(H^1\right)' \simeq BMO. \tag{3.17}$$

The proof then directly follows from (3.12), (3.13) and (3.17). □

Proof of Theorem 1.1. Let $N \in \mathbb{N}$ be any arbitrary integer. Using (2.2), we estimate $\|f\|_{L^\infty}$ in the following way:

$$\begin{aligned} \|f\|_{L^\infty} &\le \Big\|\sum_{j<-N} 2^{\gamma j}2^{-\gamma j}|\varphi_j * f|\Big\|_{L^\infty} + \Big\|\sum_{|j|\le N} |\varphi_j * f|\Big\|_{L^\infty} + \Big\|\sum_{j>N} 2^{-\gamma j}2^{\gamma j}|\varphi_j * f|\Big\|_{L^\infty} \\ &\le C_\gamma 2^{-\gamma N}\overbrace{\Big\|\Big(\sum_{j<-N} 2^{-2\gamma j}|\varphi_j * f|^2\Big)^{1/2}\Big\|_{L^\infty}}^{A_1} + (2N+1)^{1/2}\overbrace{\Big\|\Big(\sum_{|j|\le N} |\varphi_j * f|^2\Big)^{1/2}\Big\|_{L^\infty}}^{A_2} \\ &\quad + C'_\gamma 2^{-\gamma N}\overbrace{\big(\sup_{j>N} 2^{\gamma j}\|\varphi_j * f\|_{L^\infty}\big)}^{A_3}, \end{aligned} \tag{3.18}$$

where

$$C_\gamma = \left(\frac{1}{2^{2\gamma}-1}\right)^{1/2} \quad \text{and} \quad C'_\gamma = \frac{2^{-\gamma}}{1-2^{-\gamma}}.$$

Using (3.1), we assert that:

$$A_1 \le C\|g\|_{L^\infty}, \tag{3.19}$$

while (3.10) gives:

$$A_2 \le C\|f\|_{BMO}. \tag{3.20}$$

In order to estimate A_3, we proceed in the following way:

$$A_3 \le \sup_{j\ge 1} 2^{\gamma j}\|\varphi_j * f\|_{L^\infty} \le \sup_{j\ge 1} 2^{\gamma j}\|\varphi_j * f\|_{L^\infty} + \|\varphi_0 * f\|_{L^\infty}, \quad \varphi_0 \text{ is given by (2.6)},$$

hence (see Definition 2.2)

$$A_3 \le \|f\|_{B^\gamma_{\infty,\infty}}.$$

Using the well known result (see for instance [7])

$$B^\gamma_{\infty,\infty} = C^{\gamma,\gamma/2},$$

we finally obtain

$$A_3 \le \|f\|_{C^{\gamma,\gamma/2}}. \tag{3.21}$$

Inequalities (3.18), (3.19), (3.20) and (3.21) imply:

$$\|f\|_{L^\infty} \le C\left((2N+1)^{1/2}\|f\|_{BMO} + 2^{-\gamma N}(\|f\|_{C^{\gamma,\gamma/2}} + \|g\|_{L^\infty})\right). \tag{3.22}$$

We optimize (3.22) in N by setting:

$$N = 1 \quad \text{if} \quad \|f\|_{C^{\gamma,\gamma/2}} + \|g\|_{L^\infty} \le 2^\gamma\|f\|_{BMO}.$$

Then it is easy to check (using (3.22)) that

$$\|f\|_\infty \le C\|f\|_{BMO}\left(1 + \left(\log^+\frac{\|f\|_{C^{\gamma,\gamma/2}} + \|g\|_{L^\infty}}{\|f\|_{BMO}}\right)^{1/2}\right). \tag{3.23}$$

In the case where $\|f\|_{C^{\gamma,\gamma/2}} + \|g\|_{L^\infty} > 2^\gamma\|f\|_{BMO}$, we take $1 \le \beta < 2^\gamma$ such that

$$N = N(\beta) = \log^+_{2^\gamma}\left(\beta\frac{\|f\|_{C^{\gamma,\gamma/2}} + \|g\|_{L^\infty}}{\|f\|_{BMO}}\right) - \frac{1}{2} \in \mathbb{N}.$$

In fact this is valid since the function $N(\beta)$ varies continuously from $N(1)$ to $N(2^\gamma) = 1 + N(1)$ on the interval $[1, 2^\gamma]$. Using (3.22) with the above choice of N, we obtain:

$$\begin{aligned}
\|f\|_\infty &\le C\left[2^{1/2}\left(\log^+_{2^\gamma}\left(\beta\frac{\|f\|_{C^{\gamma,\gamma/2}} + \|g\|_{L^\infty}}{\|f\|_{BMO}}\right)\right)^{1/2}\|f\|_{BMO} + \frac{2^{\gamma/2}}{\beta}\|f\|_{BMO}\right] \\
&\le C\left[\frac{2}{(\gamma\log 2)^{1/2}}\left(\log^+\left(\frac{\|f\|_{C^{\gamma,\gamma/2}} + \|g\|_{L^\infty}}{\|f\|_{BMO}}\right)\right)^{1/2}\|f\|_{BMO} + \frac{2^{\gamma/2}}{\beta}\|f\|_{BMO}\right],
\end{aligned}$$

where for the second line we have used the fact that

$$\log^+ \beta < \log^+ \frac{\|f\|_{C^{\gamma,\gamma/2}} + \|g\|_{L^\infty}}{\|f\|_{BMO}}.$$

The above computations again imply (3.23). By using the inequality:

$$x\left(\log\left(e+\frac{y}{x}\right)\right)^{1/2} \leq \begin{cases} C(1+x(\log(e+y))^{1/2}) & \text{for} \quad 0 < x \leq 1, \\ Cx(\log(e+y))^{1/2} & \text{for} \quad x > 1, \end{cases}$$

in (3.23), we directly arrive to our result. □

We now present the proof of Theorem 1.2 that involve finer estimates on the Hölder norm.

Proof of Theorem 1.2. For the sake of simplifying the ideas of the proof, we only consider 1-spatial dimensions $x = x_1$. The general n-dimensional case can be easily deduced. Following the same notations of [9], we let $\widetilde{\Omega}_T = (-1,2) \times (-T, 2T)$, $\mathcal{Z}_1 \subseteq \mathcal{Z}_2 \subseteq \widetilde{\Omega}_T$ such that

$$\mathcal{Z}_1 = \{(x,t);\ -1/4 < x < 5/4 \text{ and } -T/4 < t < 5T/4\}$$

and

$$\mathcal{Z}_2 = \{(x,t);\ -3/4 < x < 7/4 \text{ and } -3T/4 < t < 7T/4\}.$$

We also take the cut-off function $\Psi \in C_0^\infty(\mathbb{R}^2)$, $0 \leq \Psi \leq 1$ satisfying:

$$\Psi(x,t) = \begin{cases} 1 & \text{for} \quad (x,t) \in \mathcal{Z}_1 \\ 0 & \text{for} \quad (x,t) \in \mathbb{R}^2 \setminus \mathcal{Z}_2. \end{cases} \tag{3.24}$$

The main idea of the proof consists in extending the function f to a suitable function of the form $\Psi\tilde{f}$ where $\tilde{f}$ is defined on $\widetilde{\Omega}_T$. We then apply inequality (1.4) (the scalar-valued version with $n = 1$) to $\Psi\tilde{f}$ and we estimate the different norms in order to get the result. However, away from the complicated extension (Sobolev extension) of the function $\tilde{f}$ that was done in [9], we here consider a simpler symmetric extension. Indeed, we first take the spatial symmetry of the function f:

$$\tilde{f}(x,t) = \begin{cases} f(-x,t) & \text{for} \quad -1 < x < 0, \quad 0 \leq t \leq T \\ f(2-x,t) & \text{for} \quad 1 < x < 2, \quad 0 \leq t \leq T, \end{cases} \tag{3.25}$$

and then the symmetry with respect to t:

$$\tilde{f}(x,t) = \begin{cases} f(x,-t) & \text{for} \quad -1 < x < 2, \quad -T < t \leq 0 \\ f(x,2T-t) & \text{for} \quad -1 < x < 2, \quad T \leq t < 2T. \end{cases} \tag{3.26}$$

We claim that $\Psi\tilde{f} \in C^{\gamma,\gamma/2}(\mathbb{R}^2)$ with

$$\|\Psi\tilde{f}\|_{C^{\gamma,\gamma/2}(\mathbb{R}^2)} \leq \|f\|_{C^{\gamma,\gamma/2}(\Omega_T)}. \tag{3.27}$$

In this case, we apply the scalar-valued version of inequality (1.4) (see Remark 1.3) to the function $\Psi\tilde{f}$ with $i = 1$ and $g(x,t) = \int_0^x \Psi(y,t)\tilde{f}(y,t)dy$. This, together with the fact that $\Psi = 1$ on Ω_T, lead to the following estimate:

$$\|f\|_{L^\infty(\Omega_T)} \leq \|\Psi\tilde{f}\|_{L^\infty(\mathbb{R}^2)} \leq C\left(1 + \|\Psi\tilde{f}\|_{BMO(\mathbb{R}^2)}\left(\log^+(\|\Psi\tilde{f}\|_{C^{\gamma,\gamma/2}(\mathbb{R}^2)} + \|g\|_{L^\infty(\mathbb{R}^2)})\right)^{1/2}\right). \tag{3.28}$$

It is worth noticing that choosing $i = 1$ above is somehow restrictive. In fact, we could also have used the inequality with $i = 2$ and $g(x,t) = \int_0^t \Psi(x,s)\tilde{f}(x,s)ds$.

In [11] it was shown that $\|\Psi\tilde{f}\|_{BMO(\mathbb{R}^2)} \le C(\|f\|_{BMO(\Omega_T)} + \|f\|_{L^1(\Omega_T)})$, while it is clear that $\|g\|_{L^\infty(\mathbb{R}^2)} \le C\|\tilde{f}\|_{L^\infty(\tilde{\Omega}_T)} \le C\|f\|_{C^{\gamma,\gamma/2}(\Omega_T)}$. These arguments, along with (3.27) and (3.28), directly terminate the proof. The only point left is to show the claim (3.27). Recall the norm

$$\|\Psi\tilde{f}\|_{C^{\gamma,\gamma/2}(\mathbb{R}^2)} = \|\Psi\tilde{f}\|_{L^\infty(\mathbb{R}^2)} + \langle\Psi\tilde{f}\rangle^{(\gamma)}_{x,\mathbb{R}^2} + \langle\Psi\tilde{f}\rangle^{(\gamma/2)}_{t,\mathbb{R}^2}.$$

It is evident that

$$\|\Psi\tilde{f}\|_{L^\infty(\mathbb{R}^2)} \le C\|f\|_{L^\infty(\Omega_T)},$$

hence we only need to estimate the two terms $\langle\Psi\tilde{f}\rangle^{(\gamma)}_{x,\mathbb{R}^2}$ and $\langle\Psi\tilde{f}\rangle^{(\gamma/2)}_{t,\mathbb{R}^2}$. We only deal with $\langle\Psi\tilde{f}\rangle^{(\gamma)}_{x,\mathbb{R}^2}$ since the second term can be treated similarly. We examine the different positions of $(x,t),(x',t) \in \mathbb{R}^2$. If $(x,t),(x',t) \in \mathbb{R}^2 \setminus \mathcal{Z}_2$, $x \neq x'$, then (since $\Psi = 0$ over $\mathbb{R}^2 \setminus \mathcal{Z}_2$):

$$\frac{|(\Psi\tilde{f})(x,t) - (\Psi\tilde{f})(x',t)|}{|x-x'|^\gamma} = 0. \tag{3.29}$$

If both $(x,t),(x',t) \in \widetilde{\Omega}_T$, $x \neq x'$, then the special extension (3.25) and (3.26) of the function f guarantees the existence of

$$(\bar{x},\bar{t}),(\bar{x}',\bar{t}) \in \Omega_T$$

such that:

$$\tilde{f}(x,t) = f(\bar{x},\bar{t}), \quad \tilde{f}(x',t) = f(\bar{x}',\bar{t}). \tag{3.30}$$

Two cases can be considered. Either $\bar{x} = \bar{x}'$ (see Figure 1), then we forcedly have

$$\tilde{f}(x,t) = \tilde{f}(x',t),$$

and therefore

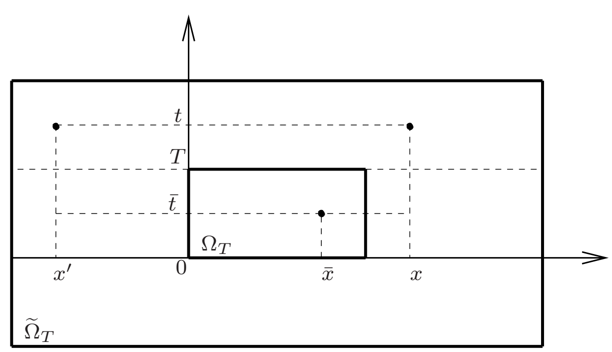

Figure 1: Case $(x,t),(x',t) \in \widetilde{\Omega}_T$ with $\bar{x} = \bar{x}'$.

$$\begin{aligned}\frac{|(\Psi\tilde{f})(x,t) - (\Psi\tilde{f})(x',t)|}{|x-x'|^\gamma} &\le \langle\Psi\rangle^{(\gamma)}_{x,\tilde{\Omega}_T}\|\tilde{f}\|_{L^\infty(\tilde{\Omega}_T)} \\ &\le C\|f\|_{L^\infty(\Omega_T)} \le C\|f\|_{C^{\gamma,\gamma/2}(\Omega_T)},\end{aligned} \tag{3.31}$$

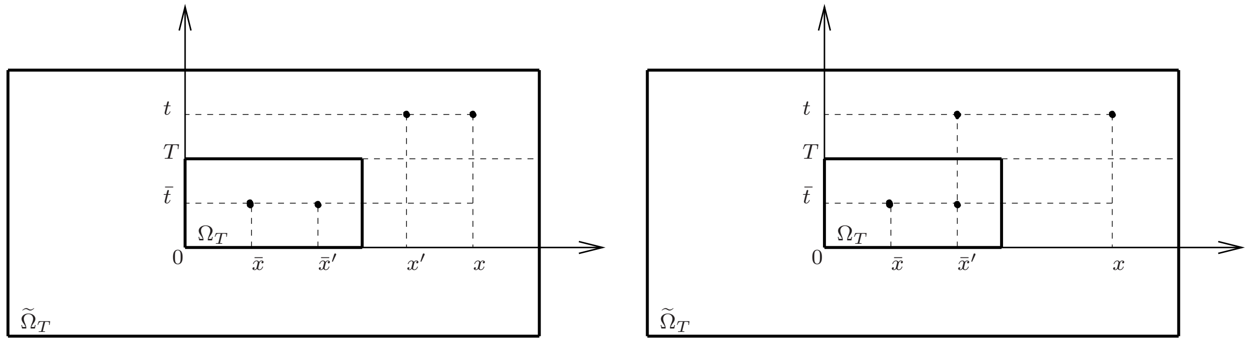

Figure 2: Case $(x,t),(x',t) \in \widetilde{\Omega}_T$ with $\bar{x} \neq \bar{x}'$. On the right: $x' = \bar{x}'$. On the left: $x' \neq \bar{x}'$.

or $\bar{x} \neq \bar{x}'$, then we forcedly have (see Figure 2)

$$|x - x'|^\gamma \geq |\bar{x} - \bar{x}'|^\gamma. \tag{3.32}$$

In this case, we compute:

$$\begin{aligned} \frac{|(\Psi\tilde{f})(x,t) - (\Psi\tilde{f})(x',t)|}{|x-x'|^\gamma} &\leq \frac{|\tilde{f}(x,t)||\Psi(x,t) - \Psi(x',t)|}{|x-x'|^\gamma} + \frac{|\Psi(x',t)||\tilde{f}(x,t) - \tilde{f}(x',t)|}{|x-x'|^\gamma} \\ &\leq \|\tilde{f}\|_{L^\infty(\widetilde{\Omega}_T)} \langle \Psi \rangle^{(\gamma)}_{x,\widetilde{\Omega}_T} + \frac{|\tilde{f}(x,t) - \tilde{f}(x',t)|}{|x-x'|^\gamma}. \end{aligned} \tag{3.33}$$

Using (3.30) and (3.32), we deduce that:

$$\frac{|\tilde{f}(x,t) - \tilde{f}(x',t)|}{|x-x'|^\gamma} = \frac{|f(\bar{x},\bar{t}) - f(\bar{x}',\bar{t})|}{|x-x'|^\gamma} \leq \frac{|f(\bar{x},\bar{t}) - f(\bar{x}',\bar{t})|}{|\bar{x}-\bar{x}'|^\gamma} \leq \langle f \rangle^{(\gamma)}_{x,\Omega_T},$$

therefore, by (3.33), we obtain:

$$\frac{|(\Psi\tilde{f})(x,t) - (\Psi\tilde{f})(x',t)|}{|x-x'|^\gamma} \leq \|\tilde{f}\|_{L^\infty(\widetilde{\Omega}_T)} \langle \Psi \rangle^{(\gamma)}_{x,\widetilde{\Omega}_T} + \langle f \rangle^{(\gamma)}_{x,\Omega_T} \leq C\|f\|_{C^{\gamma,\gamma/2}(\Omega_T)}. \tag{3.34}$$

The remaining case is when $(x,t) \in \mathcal{Z}_2$ and $(x',t) \in \mathbb{R}^2 \setminus \widetilde{\Omega}_T$ (see Figure 3). In this case, we have $(\Psi\tilde{f})(x',t) = 0$ and

$$|x - x'|^\gamma \geq \left(\frac{1}{4}\right)^\gamma, \tag{3.35}$$

hence

$$\frac{|(\Psi\tilde{f})(x,t) - (\Psi\tilde{f})(x',t)|}{|x-x'|^\gamma} \leq 4^\gamma \|\tilde{f}\|_{L^\infty(\mathcal{Z}_2)} \leq C\|f\|_{C^{\gamma,\gamma/2}(\Omega_T)}. \tag{3.36}$$

From (3.29), (3.31), (3.34) and (3.36), we finally deduce that

$$\langle \Psi\tilde{f} \rangle^{(\gamma)}_{x,\mathbb{R}^2} \leq C\|f\|_{C^{\gamma,\gamma/2}(\Omega_T)}.$$

Arguing in exactly the same way as above, we also find that:

$$\langle \Psi\tilde{f} \rangle^{(\gamma/2)}_{t,\mathbb{R}^2} \leq C\|f\|_{C^{\gamma,\gamma/2}(\Omega_T)},$$

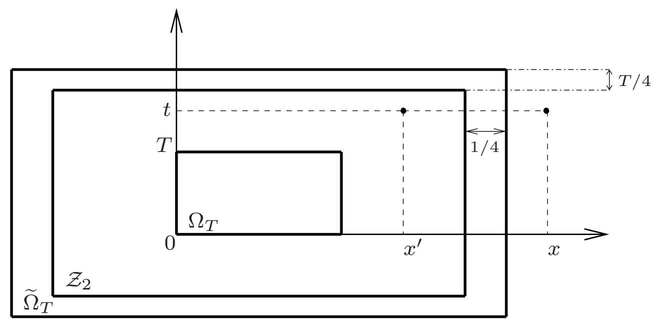

Figure 3: case $(x,t) \in \mathcal{Z}_2$ and $(x',t) \in \mathbb{R}^2 \setminus \widetilde{\Omega}_T$.

with a possibly different constant C that depend on T. Indeed, the term T enters in estimating $\langle \Psi \tilde{f} \rangle^{(\gamma/2)}_{t,\mathbb{R}^2}$ since (3.35) is now replaced (see again Figure 3) by

$$|t - t'|^\gamma \geq \left(\frac{T}{4}\right)^\gamma.$$

This shows the claim. □

Remark 3.3 *In the case of multi-spatial coordinates x_i, $i = 1, \ldots, n$, we simultaneously apply the extension (3.25) to each spatial coordinate while fixing all other coordinates including t. Finally, fixing the spatial variables, we make the extension with respect to t as in (3.26).*

Acknowledgements

The author is highly indebted to professor Lars-Erik Persson and to an anonymous referee for their useful comments on earlier versions of the manuscript.

References

[1] M. Bownik, Anisotropic Hardy spaces and wavelets, Mem. Amer. Math. Soc., 164 (2003), pp. vi+122.

[2] M. Bownik, Anisotropic Triebel-Lizorkin spaces with doubling measures, J. Geom. Anal., 17 (2007), pp. 387-424.

[3] M. Bownik, Duality and interpolation of anisotropic Triebel-Lizorkin spaces, Math. Z., 259 (2008), pp. 131-169.

[4] H. Brézis and T. Gallouët, Nonlinear Schrödinger evolution equations, Nonlinear Anal., 4 (1980), pp. 677-681.

[5] H. Brézis and S. Wainger, A note on limiting cases of Sobolev embeddings and convolution inequalities, Comm. Partial Differential Equations, 5 (1980), pp. 773-789.

[6] H. Engler, An alternative proof of the Brezis-Wainger inequality, Comm. Partial Differential Equations, 14 (1989), pp. 541-544.

[7] W. Farkas, J. Johnsen and W. Sickel, Traces of anisotropic Besov-Lizorkin-Triebel spaces—a complete treatment of the borderline cases, Math. Bohem., 125 (2000), pp. 1-37.

[8] M. Frazier and B. Jawerth, A discrete transform and decompositions of distribution spaces, J. Funct. Anal., 93 (1990), pp. 34-170.

[9] H. Ibrahim, A critical parabolic Sobolev embedding via Littlewood-Paley decomposition, preprint (2010).

[10] H. Ibrahim, M. Jazar and R. Monneau, Global existence of solutions to a singular parabolic/Hamilton-Jacobi coupled system with Dirichlet conditions, C. R. Math. Acad. Sci. Paris, 346 (2008), pp. 945-950.

[11] H. Ibrahim and R. Monneau, On a parabolic logarithmic sobolev inequality, J. Funct. Anal., 257 (2009), pp. 903-930.

[12] H. Kozono, T. Ogawa and Y. Taniuchi, The critical Sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations, Math. Z., 242 (2002), pp. 251-278.

[13] H. Kozono, T. Ogawa and Y. Taniuchi, Navier-Stokes equations in the Besov space near L^∞ and BMO, Kyushu J. Math., 57 (2003). pp. 303-324.

[14] H. Kozono and Y. Taniuchi, Limiting case of the Sobolev inequality in BMO, with application to the Euler equations, Comm. Math. Phys., 214 (2000), pp. 191-200.

[15] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Uralceva, Linear and quasilinear equations of parabolic type, Translated from the Russian by S. Smith. Transactions of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I., 1967.

[16] T. Ogawa, Sharp Sobolev inequality of logarithmic type and the limiting regularity condition to the harmonic heat flow, SIAM J. Math. Anal., 34 (2003), pp. 1318-1330 (electronic).

[17] H. Triebel, Theory of function spaces, vol. 78 of Monographs in Mathematics, Birkhäuser Verlag, Basel, 1983.

[18] H. Triebel, Theory of function spaces. III, vol. 100 of Monographs in Mathematics, Birkhäuser Verlag, Basel, 2006.