Global Weak Solutions of the Navier-Stokes System with Nonzero Boundary Conditions

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Consider the Navier-Stokes equations in a smooth bounded domain $\Omega \subset \mathbb{R}^3$ and a time interval $[0,T)$, $0 < T \leq \infty$. It is well-known that there exists at least one global weak solution u with vanishing boundary values $u\big|_{\partial\Omega} = 0$ for any given initial value $u_0 \in L^2_\sigma(\Omega)$, external force $f = \operatorname{div} F$, $F \in L^2\big(0,T; L^2(\Omega)\big)$, and satisfying the strong energy inequality. In this paper we extend this existence result to the case of inhomogeneous boundary values $u\big|_{\partial\Omega} = g \neq 0$. Given f as above and $u_0 \in L^2(\Omega)$ satisfying the necessary compatibility conditions $\operatorname{div} u_0 = 0$ and $N \cdot u_0\big|_{\partial\Omega} = N \cdot g$, where N denotes the exterior normal vector on $\partial\Omega$, we prove as a main result the existence of a weak solution u satisfying $u\big|_{\partial\Omega} = g$, the strong energy inequality and an energy estimate.

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1 Introduction and main results

Throughout this paper $\Omega \subset \mathbb{R}^3$ denotes a bounded domain with boundary $\partial\Omega$ of class $C^{1,1}$ and $[0,T)$, $0 < T \leq \infty$, is a given time interval. We are interested in the Navier-Stokes system

$$\begin{aligned} u_t - \Delta u + u \cdot \nabla u + \nabla p = f, \quad & \operatorname{div} u = 0 \\ u\big|_{\partial\Omega} = g, \quad & u(0) = u_0 \end{aligned} \tag{1.1}$$

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in $[0,T)\times\Omega$ where the initial value u_0, the boundary values g and the external force f satisfy the properties

$$u_0 \in L^2(\Omega), \quad g \in W^{\frac{1}{2},2}(\partial\Omega), \quad f = \operatorname{div} F, \quad F \in L^2\big(0,T;L^2(\Omega)\big) \tag{1.2}$$

and the compatibility conditions

$$\operatorname{div} u_0 = 0 \text{ in } \Omega, \quad N\cdot u_0\big|_{\partial\Omega} = N\cdot g. \tag{1.3}$$

Here $N = N(x)$ denotes the exterior normal vector at $x \in \partial\Omega$, so that (1.3) yields the flux condition $\int_{\partial\Omega} N\cdot g\, d\sigma = 0$.

Before discussing (1.1) with $g \neq 0$ let us recall some classical results in the case $g = 0$.

Definition 1.1 Let $u_0 \in L^2_\sigma(\Omega)$ and $f = \operatorname{div} F$, $F \in L^2\big(0,T;L^2(\Omega)\big)$. Then a vector field

$$u \in L^\infty\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;W_0^{1,2}(\Omega)\big) \tag{1.4}$$

is called a *weak (Leray-Hopf) solution* of (1.1) in $[0,T)\times\Omega$ with data u_0, $g = 0$, f if

$$-\langle u, w_t\rangle_{\Omega,T} + \langle \nabla u, \nabla w\rangle_{\Omega,T} - \langle uu, \nabla w\rangle_{\Omega,T} = \langle u_0, w(0)\rangle_\Omega - \langle F, \nabla w\rangle_{\Omega,T}$$

is satisfied for each test function $w \in C_0^\infty\big([0,T);\, C_{0,\sigma}^\infty(\Omega)\big)$, and u satisfies the *energy inequality*

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2\, d\tau \leq \frac{1}{2}\|u_0\|_2^2 - \int_0^t \langle F, \nabla u\rangle_\Omega\, d\tau \tag{1.5}$$

for $0 \leq t < T$.

In this definition $\langle\cdot,\cdot\rangle_\Omega$ denotes the pairing of vector fields in Ω, and $\langle\cdot,\cdot\rangle_{\Omega,T}$ means the corresponding pairing in $[0,T)\times\Omega$. Given a vector field u on Ω, let $u\cdot\nabla u = (u\cdot\nabla)u = u_1D_1u + u_2D_2u + u_3D_3u$ where $D_j = \partial/\partial x_j$, $j = 1,2,3$. For a matrix field $F = (F_{ij})_{i,j=1}^3$ let $f = \operatorname{div} F = (D_1F_{1j} + D_2F_{2j} + D_3F_{3j})_{j=1}^3$ so that when $\operatorname{div} u = 0$ and $uu = (u_iu_j)_{i,j=1}^3$, we have $u\cdot\nabla u = \operatorname{div}(uu)$. With $C_{0,\sigma}^\infty(\Omega) = \{v \in C_0^\infty(\Omega) : \operatorname{div} v = 0\}$ we define $L^2_\sigma(\Omega) = \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_2}$ where $\|\cdot\|_q$ denotes the norm of the Lebesgue space $L^q(\Omega)$, $1 \leq q \leq \infty$. For vector fields u, it will be convenient to define the L^q-norm by the L^q-norm of the scalar function $|u|$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^3$. Further, $W^{k,q}(\Omega)$, $k \in \mathbb{N}_0$, $1 \leq q < \infty$, and $W_0^{k,q}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{k,q}(\Omega)}}$ denote the usual Sobolev spaces. Finally we need the Bochner spaces $L^s\big(0,T;L^q(\Omega)\big)$, $1 < s,q < \infty$, with norm

$$\|\cdot\|_{L^s(0,T;L^q(\Omega))} = \|\cdot\|_{q,s;T} = \Big(\int_0^T \|\cdot\|_q^s\, d\tau\Big)^{1/s},$$

and correspondingly the spaces $L^\infty\big(0,T;L^2(\Omega)\big)$, $L^\infty_{\rm loc}\big([0,T);L^2(\Omega)\big)$, and $L^2\big(0,T;W_0^{1,2}(\Omega)\big)$. The trace space of functions in $W^{1,q}(\Omega)$, $1<q<\infty$, is denoted by $W^{1-\frac{1}{q},q}(\partial\Omega)$, its dual space by $W^{-\frac{1}{q'},q'}(\partial\Omega)$, $q'=\frac{q}{q-1}$. The surface measure on $\partial\Omega$ is called $d\sigma$.

As is well-known there always exists at least one weak solution u in the sense of Definition 1.1 of the Navier-Stokes system (1.1) with $g=0$. We may assume without loss of generality (after modifying u on a null set in $(0,T)$) that $u:[0,T)\to L^2_\sigma(\Omega)$ is weakly continuous. Thus $u\big|_{t=0}=u_0$ is well-defined. Further, there exists a distribution p in $(0,T)\times\Omega$ such that

$$u_t-\Delta u+u\cdot\nabla u+\nabla p=f \quad \text{in } (0,T)\times\Omega$$

in the sense of distributions. Finally, from (1.5) and the Cauchy-Schwarz inequality we obtain the *energy estimate*

$$\|u(t)\|_2^2+\int_0^t\|\nabla u\|_2^2\,d\tau\le\|u_0\|_2^2+\int_0^t\|F\|_2^2\,d\tau,\ 0\le t<T. \tag{1.6}$$

The extension from homogeneous boundary values $g=0$ to the case $g\neq 0$ in (1.1) requires some obvious modifications caused by the compatibility conditions (1.3). Further, the form of the energy inequalities (1.5), (1.6) will be different, see (1.13), (1.15) below, but as before, they are formally obtained by testing the Navier-Stokes system with the weak solution u itself.

For this purpose we have to find a suitable extension $E\in W^{1,2}(\Omega)$ of the given boundary data

$$g\in W^{\frac{1}{2},2}(\partial\Omega) \quad \text{with} \int_{\partial\Omega}N\cdot g\,d\sigma=0, \tag{1.7}$$

e.g., as the uniquely determined weak solution of the stationary Stokes system

$$-\Delta E+\nabla\tilde p=f_0=\operatorname{div}F_0,\quad \operatorname{div}E=0,\quad E\big|_{\partial\Omega}=g \tag{1.8}$$

with data g, $f_0=\operatorname{div}F_0$ where $F_0\in L^2(\Omega)$, and pressure $\tilde p\in L^2(\Omega)$ satisfying the estimate

$$\|E\|_{W^{1,2}(\Omega)}\le c\big(\|F_0\|_2+\|g\|_{W^{1/2,2}(\partial\Omega)}\big) \tag{1.9}$$

with a constant $c=c(\Omega)>0$. In particular, if $F_0=0$, then the map $g\mapsto E$ is a well-defined linear bounded extension operator from $W^{\frac{1}{2},2}(\partial\Omega)$ to $W^{1,2}(\Omega)$. However, we will see that it is reasonable to consider also the inhomogeneous Stokes system (1.8) with $f_0\neq 0$.

Now, setting $v=u-E$ and $h=p-\tilde p$ we can write (1.1) in the form

$$\begin{aligned} v_t-\Delta v+(v+E)\cdot\nabla(v+E)+\nabla h=f-f_0,\quad &\operatorname{div}v=0\\ v\big|_{\partial\Omega}=0,\qquad\quad &v(0)=v_0\end{aligned} \tag{1.10}$$

where $v_0 = u_0 - E$. Since

$$(v+E)\cdot\nabla(v+E) = v\cdot\nabla v + (v\cdot\nabla E + E\cdot\nabla v) + E\cdot\nabla E,$$

the system (1.10) may be considered as a perturbation of the usual Navier-Stokes system with zero boundary conditions, using the perturbation terms $E\cdot\nabla E$ and $v\cdot\nabla E + E\cdot\nabla v$. Moreover, formally testing (1.10) with a weak solution v itself and noting the identity $\langle (v+E)\cdot\nabla v, v\rangle_\Omega = \frac{1}{2}\int_\Omega (v+E)\cdot\nabla |v|^2\,dx = 0$, we get the energy inequality (1.13) below. Since $v : (0,T] \to L^2_\sigma(\Omega)$ is weakly continuous (as in the case above), we conclude that $v\big|_{t=0} = v_0$ is well-defined, $v_0 \in L^2_\sigma(\Omega)$ and $\operatorname{div} u_0 = 0$, $N\cdot u_0\big|_{\partial\Omega} = N\cdot g$ as in (1.3).

These considerations lead to the following definition:

Definition 1.2 (Navier-Stokes system with $u\big|_{\partial\Omega} = g$) *Let $u_0 \in L^2(\Omega)$, $g \in W^{\frac{1}{2},2}(\partial\Omega)$ satisfy the compatibility conditions $\operatorname{div} u_0 = 0$ in Ω, $N\cdot u_0\big|_{\partial\Omega} = N\cdot g$, cf. (1.3), let $f = \operatorname{div} F$, $F \in L^2\big(0,T;L^2(\Omega)\big)$, be given, and let $E \in W^{1,2}(\Omega)$ be the weak solution of the Stokes system (1.8) with data g, $f_0 = \operatorname{div} F_0$ where $F_0 \in L^2(\Omega)$, satisfying the a priori estimate (1.9). Then a vector field*

$$u \in L^\infty_{\mathrm{loc}}\big([0,T);L^2(\Omega)\big) \cap L^2_{\mathrm{loc}}\big([0,T);W^{1,2}(\Omega)\big) \tag{1.11}$$

is called a weak (Leray-Hopf) solution *of the system (1.1) in $[0,T)\times\Omega$ with data u_0, g and f if the relation*

$$-\langle u, w_t\rangle_{\Omega,T} + \langle \nabla u, \nabla w\rangle_{\Omega,T} - \langle uu, \nabla w\rangle_{\Omega,T} = \langle u_0, w(0)\rangle_\Omega - \langle F, \nabla w\rangle_{\Omega,T} \tag{1.12}$$

is satisfied for all $w \in C_0^\infty\big([0,T);C^\infty_{0,\sigma}(\Omega)\big)$, and if the energy inequality

$$\frac{1}{2}\|u(t)-E\|_2^2 + \int_0^t \|\nabla(u-E)\|_2^2\,d\tau \le \frac{1}{2}\|u_0-E\|_2^2 - \int_0^t \langle F-F_0-uE, \nabla(u-E)\rangle_\Omega\,d\tau \tag{1.13}$$

holds for all $0 \le t < T$.

Now our main result reads as follows:

Theorem 1.3 *Let $u_0 \in L^2(\Omega)$ and $g \in W^{\frac{1}{2},2}(\partial\Omega)$ satisfy the compatibility conditions (1.3), let $f = \operatorname{div} F$ with $F \in L^2\big(0,T;L^2(\Omega)\big)$ and $f_0 = \operatorname{div} F_0$ where $F_0 \in L^2(\Omega)$ be given, and let $E \in W^{1,2}(\Omega)$ satisfy (1.8), (1.9). Then there exists at least one weak solution u of the Navier-Stokes system (1.1) with data u_0, g, f in $[0,T)\times\Omega$ in the sense of Definition 1.2. This solution u satisfies the* strong energy inequality

$$\begin{aligned} n\frac{1}{2}\|u(t)-E\|_2^2 &+ \int_s^t \|\nabla(u-E)\|_2^2\,d\tau \\ &\le \frac{1}{2}\|u(s)-E\|_2^2 - \int_s^t \langle F-F_0-uE, \nabla(u-E)\rangle_\Omega\,d\tau \end{aligned} \tag{1.14}$$

for almost all $s \in (0,T)$ *and all* $t \in (s,T)$*, and the energy estimate*

$$\begin{aligned} &\tfrac{1}{2}\|u-E\|^2_{2,\infty;T'} + \|\nabla(u-E)\|^2_{2,2;T'} \\ &\quad \le 2e^{\alpha T'\|E\|_4^8}\big(\|u_0-E\|_2^2 + 2\|F-F_0\|^2_{2,2;T'} + 4T'\|E\|_4^4\big) \end{aligned} \tag{1.15}$$

for all finite $0 < T' \le T$*, where* $\alpha \ge 0$ *is an absolute constant.*

Theorem 1.3 considers the worst and most general case in which the energies $\frac{1}{2}\|u-E\|^2_{2,\infty,T'}$ and $\|\nabla(u-E)\|^2_{2,2;T'}$ may grow exponentially in time with an exponent proportional to $\|E\|_4^8$, i.e., when $\alpha > 0$. Here, we may simply take E as the solution of the Stokes system (1.8) with data g and $f_0 = 0$. However, a modification of its proof will show that under certain assumptions on the boundary data and a careful choice of E, better to say, of F_0 in (1.8), the exponent α in (1.15) may vanish. Since generally the vector field E will not be the limit of the weak solution $u(t)$ as $t \to \infty$, the energy term $\|\nabla(u-E)\|^2_{2,2;T'}$ necessarily implies a linear growth in time in the energy estimate.

Corollary 1.4 *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded and possibly multiply connected domain with boundary components* $\Gamma_0, \ldots, \Gamma_m$ *of class* $C^{1,1}$*. Further let* $u_0 \in L^2(\Omega)$*,* $f = \operatorname{div} F$*,* $F \in L^2\big(0,T; L^2(\Omega)\big)$ *and* $g \in W^{\frac{1}{2},2}(\partial\Omega)$ *satisfy*

$$\operatorname{div} u_0 = 0, \quad u_0\big|_{\partial\Omega} = g$$

and the flux condition

$$\int_{\Gamma_j} N \cdot g \, d\sigma = 0 \quad \text{for } j = 0, \ldots, m. \tag{1.16}$$

Then there exists a vector field $E \in W^{1,2}(\Omega)$ *and a weak (Leray-Hopf) solution* u *of the Navier-Stokes system* (1.1) *satisfying* (1.11) – (1.12) *and the energy estimate*

$$\begin{aligned} &\|u-E\|^2_{2,\infty;T'} + \|\nabla(u-E)\|^2_{2,2;T'} \\ &\quad \le \|u_0-E\|_2^2 + 8\|F\|^2_{2,2;T'} + cT'\big(\|g\|^2_{W^{1/2,2}(\partial\Omega)} + \|g\|^4_{W^{1/2,2}(\partial\Omega)}\big) \end{aligned} \tag{1.17}$$

for all finite $0 < T' \le T$ *where* $c = c(\Omega) > 0$*.*

Before proving Theorem 1.3 and Corollary 1.4 in Section 2 we summarize some well-known results and introduce further notations.

For a bounded smooth domain $\Omega \subset \mathbb{R}^3$ as in Section 1 let $P : L^2(\Omega) \to L^2_\sigma(\Omega)$ denote the Helmholtz projection, and let $A : \mathcal{D}(A) \to L^2_\sigma(\Omega)$, $A = -P\Delta$, denote the Stokes operator with domain $D(A) = W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \cap L^2_\sigma(\Omega)$ and range $\mathcal{R}(A) = L^2_\sigma(\Omega)$. Then $A^\alpha : \mathcal{D}(A^\alpha) \to L^2_\sigma(\Omega)$, $-1 \le \alpha \le 1$, denote the fractional powers of A; it holds

$$D(A) \subseteq D(A^\alpha) \subseteq L^2_\sigma(\Omega), \quad \mathcal{R}(A^\alpha) = L^2_\sigma(\Omega) \quad \text{for } 0 \le \alpha \le 1,$$

and $(A^\alpha)^{-1} = A^{-\alpha}$ for $-1 \le \alpha \le 1$. In particular, the square root $A^{1/2}$ of A satisfies

$$\langle A^{\frac12} v, A^{\frac12} w\rangle = \langle \nabla v, \nabla w\rangle \quad \text{for } v, w \in \mathcal{D}(A^{\frac12}) = W^{1,2}_{0,\sigma}(\Omega) := W^{1,2}_0(\Omega) \cap L^2_\sigma(\Omega).$$

Moreover, A^α is a selfadjoint operator in $L^2_\sigma(\Omega)$ for every $\alpha \in [-1, 1]$.

Let $v \in \mathcal{D}(A^\alpha)$, $0 \le \alpha \le \frac12$, and let $2 \le q < \infty$ satisfy $2\alpha + \frac3q = \frac32$. Then we obtain the embedding estimate

$$\|v\|_q \le c\|A^\alpha v\|_2 \tag{1.18}$$

with $c = c(\alpha) > 0$. Moreover, if $v \in \mathcal{D}(A)$ and $0 \le \alpha \le 1$, then

$$\|A^\alpha v\|_2 \le \|Av\|_2^\alpha \, \|v\|_2^{1-\alpha} \tag{1.19}$$

and, if $v \in \mathcal{D}(A^{\frac12})$, $0 \le \alpha \le 1$, then

$$\|A^{\frac{\alpha}{2}} v\|_2 \le \|A^{\frac12} v\|_2^\alpha \, \|v\|_2^{1-\alpha}, \tag{1.20}$$

where $\|A^{\frac12} v\|_2 = \|\nabla v\|_2$. As consequence of (1.18) and (1.20) we see that for $2 \le q < \infty$ and $\beta = \frac32 - \frac3q$ there exists a constant $c = c(\beta) > 0$ such that

$$\|v\|_q \le c\|\nabla v\|_2^\beta \, \|v\|_2^{1-\beta} \quad \text{for } v \in W^{1,2}_{0,\sigma}(\Omega). \tag{1.21}$$

Let $F = (F_{ij})_{i,j=1}^3 \in L^2(\Omega)$ be given. Then the term $A^{-\frac12} P \operatorname{div} F$ is well-defined (in a generalized sense) in $L^2_\sigma(\Omega)$ by the relation

$$\langle A^{-\frac12} P \operatorname{div} F, v\rangle_\Omega = \langle F, \nabla A^{-\frac12} v\rangle, \quad v \in L^2_\sigma(\Omega),$$

and it holds

$$\|A^{-\frac12} P \operatorname{div} F\|_2 \le \|F\|_2\,; \tag{1.22}$$

indeed, $w := A^{-\frac12}(A^{-\frac12} P \operatorname{div} F)$ is the weak solution of the Stokes system $-\Delta w + \nabla h = \operatorname{div} F$ in $W^{1,2}_{0,\sigma}(\Omega)$ with pressure h. For the latter results we refer to [9, Chapter III].

The Yosida approximation, based on the operator $A^{\frac12}$, is defined by the sequence of operators

$$J_k = \big(I + \frac1k A^{\frac12}\big)^{-1}, \quad k \in \mathbb{N},$$

where I denotes the identity. As is well-known,

$$\|J_k v\|_2 \le \|v\|_2, \quad \|\frac1k A^{\frac12} J_k v\|_2 \le \|v\|_2 \quad \text{for } k \in \mathbb{N},\ v \in L^2_\sigma(\Omega), \tag{1.23}$$

and $J_k v \to v$ in $L^2(\Omega)$ as $k \to \infty$ for every $v \in L^2_\sigma(\Omega)$. Moreover, for $k \in \mathbb{N}$,

$$\|\nabla J_k v\|_2 = \|A^{\frac12} J_k v\|_2 = \|J_k A^{\frac12} v\|_2 \le \|A^{\frac12} v\|_2 = \|\nabla v\|_2,\ v \in D(A^{\frac12}). \tag{1.24}$$

The operator $-A$ generates a bounded analytic semigroup $e^{-tA} : L^2_\sigma(\Omega) \to L^2_\sigma(\Omega)$, $0 \le t < \infty$, such that for $0 \le \alpha \le 1$

$$\|A^\alpha e^{-tA} v\|_2 \le t^{-\alpha} \|v\|_2 \quad \text{for } t > 0,\, v \in L^2_\sigma(\Omega), \tag{1.25}$$

see [9, Chapter IV.1]

Finally we need some properties of the linear instationary Stokes system

$$\begin{aligned} v_t - \Delta v + \nabla h = \hat{f}, \quad & \operatorname{div} v = 0 \quad \text{in } \Omega \times (0,T) \\ v\big|_{\partial\Omega} = 0, \quad & v(0) = v_0 \quad \text{at } t = 0 \end{aligned} \tag{1.26}$$

with data $v_0 \in L^2_\sigma(\Omega)$, $\hat{f} = \operatorname{div} \hat{F}$, where $\hat{F} \in L^2\big(0,T; L^2(\Omega)\big)$. There exists a unique weak solution $v \in L^\infty\big(0,T; L^2_\sigma(\Omega)\big) \cap L^2\big(0,T; W^{1,2}_0(\Omega)\big)$ defined by the relation

$$-\langle v, w_t \rangle_{\Omega,T} + \langle \nabla v, \nabla w \rangle_{\Omega,T} = \langle v_0, w(0) \rangle - \langle \hat{F}, \nabla w \rangle_{\Omega,T},$$

$w \in C^\infty_0\big([0,T);\, C^\infty_{0,\sigma}(\Omega)\big)$, which has the well-defined integral representation

$$v(t) = e^{-tA} v_0 + \int_0^t A^{\frac{1}{2}} e^{-A(t-\tau)}\, A^{-\frac{1}{2}}\, P \operatorname{div} \hat{F}\, d\tau, \quad 0 \le t < T. \tag{1.27}$$

Moreover, the solution v satisfies the energy inequality

$$\frac{1}{2}\, \|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2\, d\tau \le \frac{1}{2}\, \|v_0\|_2^2 - \int_0^t \langle \hat{F}, \nabla v \rangle_\Omega\, d\tau, \quad 0 \le t < T, \tag{1.28}$$

and consequently the energy estimate

$$\|v\|^2_{2,\infty;T} + \|\nabla v\|^2_{2,2;T} \le \|v_0\|_2^2 + \|\hat{F}\|^2_{2,2;T}\,. \tag{1.29}$$

Further, $v : [0,T) \to L^2_\sigma(\Omega)$ is strongly continuous, see [9, Chapter IV].

2 Proofs

There are several proofs of Theorem 1.3 when $u\big|_{\partial\Omega} = g = 0$, see e.g. [6] - [9], [11]. Usually, in a first step, a sequence of approximate equations yields approximate solutions u_k, $k \in \mathbb{N}$. In a second step energy estimates for u_k with a bound independent of $k \in \mathbb{N}$ are derived. Hence a subsequence of (u_k) will converge in a weak sense to an element u which is shown to be a solution of the original problem. One possibility is to use the Yosida approximation yielding the approximate system

$$\begin{aligned} u_t - \Delta u + (J_k u) \cdot \nabla u + \nabla p = f, \;\; & \operatorname{div} u = 0 \\ u\big|_{\partial\Omega} = 0, \;\; & u(0) = u_0. \end{aligned}$$

In the following we will use a modification of this procedure and consider the mollified perturbed system

$$\begin{aligned} v_t - \Delta v + (J_k v + E)\cdot\nabla(v+E) + \nabla p = f, \;\; \operatorname{div} v = 0 \\ v\big|_{\partial\Omega} = 0, \;\; v(0) = v_0 \end{aligned} \tag{2.1}$$

where u_0, E are given as in Definition 1.2, and

$$\begin{aligned} v_0 = u_0 - E \in L^2_\sigma(\Omega),\; f = \operatorname{div}(F - F_0), \\ F \in L^2\big(0,T;L^2(\Omega)\big),\; F_0 \in L^2(\Omega). \end{aligned} \tag{2.2}$$

We may assume in the following that $E \neq 0$. Then we are looking for a weak solution

$$v = v_k \in L^\infty_{\rm loc}\big([0,T);L^2_\sigma(\Omega)\big) \cap L^2_{\rm loc}\big([0,T);W^{1,2}_0(\Omega)\big) \tag{2.3}$$

of (2.1) with data (2.2) in the sense that

$$\begin{aligned} -\langle v, w_t\rangle_{\Omega,T} + \langle \nabla v, \nabla w\rangle_{\Omega,T} - \langle (J_k v + E)\cdot(v+E), \nabla w\rangle_{\Omega,T} \\ = \langle v_0, w(0)\rangle_\Omega - \langle F - F_0, \nabla w\rangle_{\Omega,T} \end{aligned} \tag{2.4}$$

for all $w \in C^\infty_0\big([0,T;C^\infty_{0,\sigma}(\Omega)\big)$, and satisfying the energy inequality

$$\frac{1}{2}\|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2\,d\tau \le \frac{1}{2}\|v_0\|_2^2 - \int_0^t \langle F - F_0 - (J_k v + E)E, \nabla v\rangle_\Omega\,d\tau, \tag{2.5}$$

$0 \le t < T$.

Lemma 2.1 *For every $k \in \mathbb{N}$ there exists some $0 < T_k = T(k, \|v_0\|_2, \|F - F_0\|_{2,2;T}, \|E\|_4) \le \min(1,T)$ such that the perturbed mollified Navier-Stokes system* (2.1) *has a unique weak solution*

$$v = v_k \in X_{T_k} := L^\infty\big(0,T_k;L^2_\sigma(\Omega)\big) \cap L^2\big(0,T_k;W^{1,2}_0(\Omega)\big) \tag{2.6}$$

in the sense (2.4), (2.5).

Proof First assume that $v = v_k \in X_k := X_{T_k}$ is a weak solution of (2.1) satisfying (2.3), (2.4), (2.5). To estimate its norm in the space X_k, i.e., the norm

$$\|v\|_{X_k} = \|v\|_{2,\infty;T_k} + \|\nabla v\|_{2,2;T_k}\,,$$

we have to analyze the nonlinear term $(J_k v + E)(v+E)$. By Hölder's inequality and the properties (1.18), (1.20), (1.23) with $\sigma = 8$, $s = \frac{8}{3}$, $\beta = \frac{3}{8}$ we get when $0 < T_k \le 1$, that

$$\begin{aligned} \|(J_k v + E)(v+E)\|_{2,2;T_k} &\le \|J_k v + E\|_{4,\sigma;T_k}\|v + E\|_{4,s;T_k} \\ &\le c(\|A^\beta J_k v\|_{2,\sigma;T_k} + \|E\|_{4,\sigma;T_k})(\|A^\beta v\|_{2,s;T_k} + \|E\|_{4,s;T_k}) \\ &\le c\big((\|A^{\frac{1}{2}} J_k v\|_2^{\frac{3}{4}}\|J_k v\|_2^{\frac{1}{4}})_{\sigma;T_k} + \|E\|_{4,\sigma;T_k}\big) \\ &\quad \times \big((\|A^{\frac{1}{2}} v\|_2^{\frac{3}{4}}\|v\|_2^{\frac{1}{4}})_{s;T_k} + \|E\|_{4,s;T_k}\big) \\ &\le c(k^{\frac{3}{4}}\|v\|_{2;8;T_k} + T_k^{\frac{1}{8}}\|E\|_4)(\|v\|_{X_k} + T_k^{\frac{1}{8}}\|E\|_4), \end{aligned}$$

where $c > 0$ is an absolute constant. Hence we obtain the estimate

$$\|(J_k v + E)(v + E)\|_{2,2;T_k} \le c\, k^{\frac{3}{4}}\, T_k^{\frac{1}{8}} \left(\|v\|_{X_k} + \|E\|_4\right)^2. \tag{2.7}$$

Defining the nonlinear operators

$$F_k(v) = F - F_0 - (J_k v + E)(v + E), \quad f_k(v) = \operatorname{div} F_k(v), \tag{2.8}$$

we can write (2.1) in the form

$$\begin{aligned} v_t - \Delta v + \nabla p = f_k(v), \quad & \operatorname{div} v = 0, \\ v\big|_{\partial\Omega} = 0, \quad & v(0) = v_0. \end{aligned} \tag{2.9}$$

By the above estimate (2.7), $F_k(v) \in L^2\big(0, T_k; L^2(\Omega)\big)$ so that v may be considered as the weak solution of the Stokes system (2.9). Hence (1.26), (1.27) yield the representation and fixed point problem

$$v = \mathcal{F}_k(v) \quad \text{in } X_k \tag{2.10}$$

where

$$\mathcal{F}_k(v)(t) = e^{-tA} v_0 + \int_0^t A^{\frac{1}{2}} e^{-(t-\tau)} A^{-\frac{1}{2}} P \operatorname{div} F_k(v)(\tau)\, d\tau, \tag{2.11}$$

$0 \le t < T_k$. Moreover, by the energy estimate (1.29)

$$\|\mathcal{F}_k(v)\|_{X_k} \le c\left(\|v_0\|_2 + \|F - F_0\|_{2,2;T_k}\right) + c\, k^{\frac{3}{4}} T_k^{\frac{1}{8}} \left(\|v\|_{X_k} + \|E\|_4\right)^2,$$

or, for short,

$$\|\mathcal{F}_k(v)\|_{X_k} + d \le a \left(\|v\|_{X_k} + d\right)^2 + b \tag{2.12}$$

where

$$a = c\, k^{\frac{3}{4}} T_k^{\frac{1}{8}}, \quad b = c\big(\|v_0\|_2 + \|F - F_0\|_{2,2;T}\big) + d, \quad d = \|E\|_4,$$

with an absolute constant $c > 0$.

Up to now v was a given solution of (2.1) in the sense of (2.4), (2.5). In the next step we solve the fixed point problem (2.10) in X_k by Banach's fixed point theorem provided that $T_k > 0$ is sufficiently small. For any $0 < T_k \le \min(1, T)$ and $v \in X_{T_k}$ we know that $F_k(v) \in L^2\big(0, T_k; L^2(\Omega)\big)$, that $\mathcal{F}_k(v)$ is well-defined and satisfies the estimate (2.12). For fixed $k \in \mathbb{N}$ choose $T_k = T\big(k, \|v_0\|_2, \|F - F_0\|_{2,2;T_k}, \|E\|_4\big)$ in $\big(0, \min(1, T)\big)$ such that

$$4ab < 1.$$

Then the quadratic equation $y = ay^2 + b$ has a minimal positive root y_1, namely $y_1 = 2b\big(1 + \sqrt{1 - 4ab}\big)^{-1}$, satisfying $d \le b < y_1 < 2b$. Hence the closed ball

$B_k = B = \{v \in X_k : \|v\|_{X_k} \le y_1 - d\}$ is not empty, and from (2.12) we conclude that $\mathcal{F}_k$ maps B into B. Moreover, if $v, v' \in B$, then

$$\begin{aligned}\big(\mathcal{F}_k(v) - \mathcal{F}_k(v')\big)(t) = -\int_0^t A^{\frac12} e^{-(t-\tau)A} A^{-\frac12} P\mathrm{div}\,\big((J_k v + E)(v - v')\\ + (J_k(v - v'))(v' + E)\big)\, d\tau,\end{aligned}$$

and the same arguments as used for (2.12) lead to the estimate

$$\begin{aligned}\|\mathcal{F}_k(v) - \mathcal{F}_k(v')\|_{X_k} &\le a\,\big(\|v\|_{X_k} + \|v'\|_{X_k} + 2d\big)\|v - v'\|_{X_k}\\ &\le 2ay_1\,\|v - v'\|_{X_k} < 4ab\,\|v - v'\|_{X_k}.\end{aligned} \tag{2.13}$$

Hence $\mathcal{F}_k$ is a strict contraction on B. Now Banach's fixed point theorem yields the existence of a unique $v = v_k \in B$ satisfying $v = \mathcal{F}_k(v)$, i.e.,

$$v(t) = e^{-tA}v_0 + \int_0^t A^{\frac12} e^{-(t-\tau)A} A^{\frac12} P\mathrm{div}\, F_k(v)(\tau)\, d\tau, \quad 0 \le t < T_k.$$

Obviously, v is the unique weak solution of the Stokes system (2.9) with data $f_k(v), v_0$; in particular, $v : [0, T_k) \to L^2_\sigma(\Omega)$ is strongly continuous, and the energy equality

$$\frac12\|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2\, d\tau = \frac12\|v_0\|_2^2 - \int_0^t \langle F_k(v), \nabla v\rangle d\tau \tag{2.14}$$

holds for every $0 \le t < T_k$.

Let us consider the term $\langle F_k(v), \nabla v\rangle_\Omega$ in (2.14) with $F_k(v)$ as in (2.8) more closely. Since by (2.7) $(J_k v + E)(v + E)(\tau) \in L^2(\Omega)$ and $\nabla v(\tau) \in L^2(\Omega)$ for almost all $\tau \in [0, T_k)$, we get for these τ

$$\begin{aligned}\langle (J_k v + E)(v + E), \nabla v\rangle_\Omega &= \langle (J_k v + E)E, \nabla v\rangle_\Omega + \frac12\int_\Omega (J_k v + E)\nabla(|v|^2)\, dx\\ &= \langle (J_k v + E)E, \nabla v\rangle_\Omega.\end{aligned}$$

This identity and (2.14) yield the energy identity

$$\frac12\|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2\, d\tau = \frac12\|v_0\|_2^2 - \int_0^t \langle F - F_0 - (J_k v + E)E, \nabla v\rangle_\Omega\, d\tau, \tag{2.15}$$

i.e., (2.5) with "=" instead of "$\le$" for $t \in [0, T)$. Moreover, v satisfies (2.4) in $\Omega \times (0, T_k)$. Thus v is a weak solution of (2.1) with data (2.2) in $(0, T_k) \times \Omega$.

To prove the uniqueness of this solution v not only in the ball $B = B_k \subset X_k$, but in the whole of X_k, let $w \in X_k$ be another weak solution of (2.1), (2.2) in $\Omega \times (0, T_k)$. Then $w = \mathcal{F}_k(w)$ with $\mathcal{F}_k$ as in (2.11), and the estimate (2.13) with $\|\cdot\|_{X_{T'}}$ replacing $\|\cdot\|_{X_{T_k}}$ for any $0 < T' \le T_k$ implies that

$$\|v - w\|_{X_{T'}} = \|\mathcal{F}_k(v) - \mathcal{F}_k(w)\|_{X_{T'}} \le a'\big(\|v\|_{X_{T'}} + \|w\|_{X_{T'}} + 2d\big)\|v - w\|_{X_{T'}} \tag{2.16}$$

where $a' = ck^{\frac{3}{4}}(T')^{\frac{1}{8}}$ and $d = \|E\|_4$. Now we choose

$$T' = T'\big(k, \|v\|_{X_k}, \|w\|_{X_k}, \|E\|_4\big) \in (0, T_k)$$

in such a way that $a'\big(\|v\|_{X_k} + \|w\|_{X_k} + 2d\big) \le \frac{1}{2}$. Then we obtain from (2.15) that $v \equiv w$ in $[0, T')$. Repeating this procedure finitely many times with the same T' we finally get that $v = w$ in $[0, T_k)$. This completes the proof of Lemma 2.1. ■

For the final passage to the limit $k \to \infty$ in the proof of Theorem 1.3 we need an energy estimate which holds uniformly in $k \in \mathbb{N}$.

Lemma 2.2 *For $k \in \mathbb{N}$ let $v = v_k$ be a weak solution of the perturbed mollified Navier-Stokes system* (2.1) *with data* (2.2) *in $[0,T) \times \Omega$, $0 < T < \infty$. Then there exists an absolute constant $\alpha \ge 0$ not depending on $k \in \mathbb{N}$ such that the energy estimate*

$$\frac{1}{2}\|v\|^2_{2,\infty;T} + \|\nabla v\|^2_{2,2;T} \le 2e^{\alpha T\|E\|_4^8}\big(\|v_0\|_2^2 + 2\|F - F_0\|^2_{2,2;T} + 4T\|E\|_4^4\big) \quad (2.17)$$

holds.

Proof By a slight modification of the proof of (2.7) where $0 < T_k \le 1$ was assumed we get that $(J_k v + E)(v + E) \in L^2\big(0, T; L^2(\Omega)\big)$ and consequently that

$$F_k(v) \in L^2\big(0, T; L^2(\Omega)\big),$$

see (2.8). Moreover, v is the weak solution of the Stokes system (2.9) so that $v : [0, T) \to L^2_\sigma(\Omega)$ is strongly continuous.

We will use the following notation for $0 \le t_0 < t_1 \le T$:

$$\|v\|_{2,\infty;t_0,t_1} = \max_{t_0 \le t \le t_1} \|v(t)\|_2, \quad \|\nabla v\|_{2,2;t_0,t_1} = \Big(\int_{t_0}^{t_1} \|\nabla v\|_2^2\, d\tau\Big)^{\frac{1}{2}},$$

and correspondingly $\|F\|_{2,2;t_0,t_1}$ etc.

From the energy inequality (2.5) we obtain the estimate

$$\frac{1}{2}\|v\|^2_{2,\infty;0,T} + \|\nabla v\|^2_{2,2;0,T} \le \frac{1}{2}\|v_0\|_2^2 + \int_0^T \big(\|F - F_0\|_2 + \|(J_k v + E)E\|_2\big)\|\nabla v\|_2\, d\tau. \quad (2.18)$$

Next, by Young's inequality

$$\int_0^T \|F - F_0\|_2\, \|\nabla v\|_2\, d\tau \le \frac{1}{4}\|\nabla v\|^2_{2,2;0,T} + \|F - F_0\|^2_{2,2;0,T},$$

and by Hölder's inequality and (1.21)

$$\begin{aligned}\int_0^T \|(J_k v + E)E\|_2\, \|\nabla v\|_2\, d\tau \;&\le\; \int_0^T \big(\|J_k v\|_4\, \|E\|_4 + \|E\|_4^2\big)\|\nabla v\|_2\, d\tau \\ &\le c\int_0^T \|\nabla (J_k v\|_2^{\frac{3}{4}}\, \|J_k v\|_2^{\frac{1}{4}}\, \|E\|_4\, \|\nabla v\|_2\, d\tau + \int_0^T \|E\|_4^2\, \|\nabla v\|_2\, d\tau.\end{aligned}$$

Hence, by (1.23), (1.24) and Young's inequality,

$$\begin{aligned}
&\int_0^T \|(J_k v + E)E\|_2 \, \|\nabla v\|_2 \, d\tau \\
&\quad \le c \int_0^T \|\nabla v\|_2^{\frac{7}{4}} \, \|v\|_2^{\frac{1}{4}} \, \|E\|_4 \, d\tau + \int_0^T \|E\|_4^2 \, \|\nabla v\|_2 \, d\tau \\
&\quad \le \frac{1}{4}\|\nabla v\|_{2,2;0,T}^2 + c' \int_0^T \|v\|_2^2 \, \|E\|_4^8 \, d\tau + 2\int_0^T \|E\|_4^4 \, d\tau \\
&\quad \le \frac{1}{4}\|\nabla v\|_{2,2;0,T}^2 + T\big(c'\|v\|_{2,\infty;0,T}\|E\|_4^8 + 2\|E\|_4^4\big)
\end{aligned}$$

with an absolute constant $c' > 0$. Summarizing the previous estimates we deduce from (2.18) the inequality

$$\big(1-2c'T\|E\|_4^8\big)\|v\|_{2,\infty;0,T}^2+\|\nabla v\|_{2,2;0,T}^2 \le \|v_0\|_2^2+2\|F-F_0\|_{2,2;0,T}^2+4T\|E\|_4^4. \quad (2.19)$$

The estimate (2.19) holds for T replaced by any $0 < T' \le T$. Now choose $T' = \min\big(T, \frac{1}{4c'\|E\|_4^8}\big)$ so that $1 - 2c'T'\|E\|_4^8 \ge \frac{1}{2}$. If $T' = T$, Lemma 2.2 is proved (use (2.19) to get (2.17) with $\alpha = 0$). Otherwise, (2.19) yields the estimate

$$\frac{1}{2}\|v\|_{2,\infty;0,T'}^2 + \|\nabla v\|_{2,2;0,T'}^2 \le \|v_0\|_2^2 + \|h\|_{0,T'} \quad (2.20)$$

where

$$h(\tau) = 2\|(F - F_0)(\tau)\|_2^2 + 4\|E\|_4^4,$$

$$\|h\|_{t_1,t_2} = \int_{t_1}^{t_2} h \, d\tau \text{ for any } 0 \le t_1 < t_2 < T.$$

Then choose $m \in \mathbb{N}$ such that $mT' < T \le (m+1)T'$ and use (2.20) on the consecutive intervals $(0,T')$, $(T',2T')$, ..., $\big((m-1)T', mT'\big)$ and (mT',T) together with the initial values $v(0), v(T'), \ldots, v\big((m-1)T'\big)$ and $v(mT')$. Since e.g. $\|h\|_{0,T'} + \|h\|_{T',2T'} = \|h\|_{0,2T'}$ etc., we easily get that

$$\frac{1}{2}\|v\|_{2,\infty;(j-1)T',jT'}^2 + \|\nabla v\|_{2,2;(j-1)T',jT'}^2 \le 2^{j-1}\big(\|v(0)\|_2^2 + \|h\|_{0,jT'}\big), \quad j = 1, \ldots, m.$$

Then a final estimate on (mT',T) and the previous estimates for $j = 1, \ldots, m$ imply that

$$\frac{1}{2}\|v\|_{2,\infty;0,T}^2 + \|\nabla v\|_{2,2;0,T}^2 \le 2^{m+1}\big(\|v(0)\|_2^2 + \|h\|_{0,T}\big).$$

Since $m \le \frac{T}{T'}$, we see that $2^m = e^{m\log 2} \le e^{\alpha T\|E\|_4^8}$ with $\alpha = 4c'\log 2$. Now Lemma 2.2 is proved. ■

Lemma 2.3 *For every $k \in \mathbb{N}$ the perturbed mollified Navier-Stokes equation system* (2.1) *with data* (2.2) *has a unique global weak solution v_k in $[0,T) \times \Omega$, $0 < T \le \infty$, satisfying* (2.3), (2.4)*, the energy inequality* (2.5) *and the energy estimate* (2.17) *with T replaced by each finite $0 < T' \le T$.*

Proof By Lemmata 2.1 and 2.2 there exists a weak solution v_k in some interval $[0,T_k)$ with $0 < T_k \le T$. Let $[0,T^*)$, $T_k \le T^* \le T$, be the largest interval within $[0,T)$ such that a weak solution with the above properties exists. If $T^* = T$, then the proof is complete; the energy estimate (1.15) follows from (2.17).

Thus suppose that $0 < T^* < T$. Then $T^* < \infty$ and we apply Lemma 2.2 and the energy estimate (2.17) with T replaced by T^*. Since v_k is strongly $L^2_\sigma(\Omega)$-continuous in $[0,T^*)$, we can choose some $T_0 \in (0,T^*)$ close to T^* such that Lemma 2.1 yields the existence of a unique weak solution of (2.1) with initial value $v_k(T_0) \in L^2_\sigma(\Omega)$ in some interval $[T_0,T_1)$ with $T_1 > T^*$. Indeed, the length of the interval of existence, $T_1 - T_0$, is determined by $T(k, \|v_k(T_0)\|_2, \|F\|_{2,2,T}, \|F_0\|_2, \|E\|_4)$, see Lemma 2.1, where by (2.17)

$$\|v_k(T_0)\|_2^2 \le 4e^{\alpha T^* \|E\|_4^8}\big(\|v_0\|_2^2 + 2\|F - F_0\|^2_{2,2;0,T^*} + 4T^*\|E\|_4^4\big).$$

Thus there exists a $\delta > 0$ independent of T_0 chosen sufficiently close to T^* such that the existence of the weak solution starting at T_0 is guaranteed on $[T_0, T_0+\delta)$ with $T_1 = T_0 + \delta > T^*$.

This procedure allows to extend the given weak solution from $[0,T^*)$ to $[0,T_1) \supset [0,T^*]$ in contradiction to choice of T^*. The uniqueness of the extended solution v_k follows in the same way as in the proof of Lemma 2.1. ■

Now we are in a position to pass to the limit $k \to \infty$ for the functions v_k and to prove Theorem 1.3.

Proof of Theorem 1.3. First consider the case $0 < T < \infty$, so that the sequence of weak solutions, (v_k), constructed in Lemma 2.3, satisfies the energy estimate

$$\frac{1}{2}\|v_k\|^2_{2,\infty;T} + \|\nabla v_k\|^2_{2,2;T} \le c\left(\|v_0\|_2^2 + \|F - F_0\|^2_{2,2;0,T} + \|E\|_4^4\right) \tag{2.21}$$

with a constant $c = c(T, \|E\|_4) > 0$, see (2.17). Then there exists

$$v \in L^\infty\big(0,T; L^2_\sigma(\Omega)\big) \cap L^2\big(0,T; W_0^{1,2}(\Omega)\big) \tag{2.22}$$

and a subsequence of (v_k), which for simplicity is again denoted by (v_k), with the following properties:

$$\begin{array}{llll} v_k & \text{converges weakly to} & v & \text{in } L^2\big(0,T; W_0^{1,2}(\Omega)\big) \\ v_k & \text{converges strongly to} & v & \text{in } L^2\big(0,T; L^2(\Omega)\big) \\ v_k(t) & \text{converges strongly to} & v(t) & \text{in } L^2_\sigma(\Omega) \text{ for a.a. } t \in [0,T) \end{array} \tag{2.23}$$

as $k \to \infty$. For the proof of (2.23) we refer to [9, V. 3.2, 3.3].

To prove that v is a weak solution of the perturbed Navier-Stokes system, we still have to show that v solves the variational problem

$$-\langle v, w_t\rangle_{\Omega,T} + \langle \nabla v, \nabla w\rangle_{\Omega,T} + \langle (v+E)(v+E), \nabla w\rangle_{\Omega,T} \qquad (2.24)$$
$$= \langle u_0 - E, w(0)\rangle_\Omega - \langle F - F_0, \nabla w\rangle_{\Omega,T}$$

for any test function $w \in C_0^\infty\big([0,T); C_{0,\sigma}^\infty(\Omega)\big)$. Due to (2.23), (1.23)

$$\begin{aligned} \langle v_k, w_t\rangle_{\Omega,T} &\to \langle v, w_t\rangle_{\Omega,T}\,, \\ \langle \nabla v_k, \nabla w\rangle_{\Omega,T} &\to \langle \nabla v, \nabla w\rangle_{\Omega,T}\,, \\ \langle (J_k v_k + E)(v_k+E), \nabla w\rangle_{\Omega,T} &\to \langle (v+E)(v+E), \nabla w\rangle_{\Omega,T}\,, \\ \langle (J_k v_k + E)E, \nabla v_k\rangle_{\Omega,T} &\to \langle (v+E)E, \nabla v\rangle_{\Omega,T}, \quad 0 \le t < T, \end{aligned}$$

as $k \to \infty$; for the fourth property we also need (1.21) and Lebesgue's theorem on dominated convergence to get that $\int_0^T \|J_k v_k - v\|_4^2\, d\tau \to 0$. Hence $v = u - E$ satisfies (2.24) and $v(0) = v_0$.

Further we obtain that

$$\|\nabla v\|_{2,2;T} \le \liminf_{k\to\infty} \|\nabla v_k\|_{2,2;T}\,.$$

Now the energy identity (2.15) for v_k yields with the help of $(2.23)_4$ for almost all $s \in (0,T)$ including $s = 0$ the strong energy inequality

$$\frac{1}{2}\|v(t)\|_2^2 + \int_s^t \|\nabla v\|_2^2\, d\tau \le \frac{1}{2}\|v(s)\|_2^2 - \int_s^t \langle F - F_0 - (v+E)E, \nabla v\rangle_\Omega\, d\tau\,.$$

First, this inequality is proved only for a.a. $t \in (0,T)$, cf. $(2.23)_3$; however, since v is weakly L^2-continuous, the estimate holds for all $t \in (0,T)$. Since $v = u - E$ satisfies (2.22) and $E \in W^{1,2}(\Omega)$, we also get that u satisfies (1.4). This proves Theorem 1.3 when $0 < T < \infty$.

Now let $T = \infty$. This case can be reduced to the situation above by using the method of diagonal sequences, see e.g. [9, p. 133]. Choose an strictly increasing sequence $(T_j)_{j=1}^\infty$ with $\lim_{j\to\infty} T_j = \infty$ and let (v_k) denote a sequence of approximate solutions in $[0,\infty) \times \Omega$ as in Lemma 2.3. For $T = T_1$ we find a subsequence $\big(v_k^{(1)}\big)$ of (v_k) with the properties (2.15), (2.17), (2.21) - (2.23) and obtain a solution $v^{(1)}$ in $[0,T_1) \times \Omega$. Then we choose a subsequence $\big(v_k^{(2)}\big)$ of $\big(v_k^{(1)}\big)$ to obtain a solution $v^{(2)}$ in $[0,T_2) \times \Omega$ such that $v^{(2)}\big|_{[0,T_1]} = v^{(1)}$. Proceeding in this way and finally choosing the diagonal sequence of the sequences $\big(v_k^{(j)}\big)$, $j \in \mathbb{N}$, we get by passing to the limit a global in time weak solution v of the perturbed Navier-Stokes system (2.1), such that $v\big|_{[0,T_j)} = v^{(j)}$, $j \in \mathbb{N}$. Obviously $v = u - E$ satisfies the energy inequality (1.13) and the energy estimate (1.15) in $[0,\infty)$.

Now the proof of Theorem 1.3 is complete. ■

Proof of Corollary 1.4 Let the boundary data $g \in W^{\frac{1}{2},2}(\partial\Omega)$ satisfy the flux condition (1.16). Then a classical result [11] yields for any $\varepsilon > 0$ a solenoidal vector field $E_\varepsilon \in W^{1,2}(\Omega)$ such that $E_\varepsilon\big|_{\partial\Omega} = g$, $\|E_\varepsilon\|_2 \le c\|g\|_{W^{\frac{1}{2},2}(\partial\Omega)}$, $c = c(\varepsilon,\Omega) > 0$, and such that the trilinear form

$$b(u,v,w) = \int_\Omega u \cdot \nabla v \cdot w \, dx \,, \quad u,v,w \in W^{1,2}(\Omega),$$

satisfies the estimate

$$|b(v,E_\varepsilon,v)| \le \varepsilon \|\nabla v\|_2^2, \quad v \in W_0^{1,2}(\Omega).$$

To apply Theorem 1.3 let $F_0 := -\nabla E_\varepsilon \in L^2(\Omega)$ such that

$$\|F_0\|_2 \le c\,\|g\|_{W^{\frac{1}{2},2}(\partial\Omega)}.$$

We use the strong energy inequality (1.14) with $s = 0$ for $v = u - E$, $E = E_\varepsilon$, in the form

$$\frac{1}{2}\|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2 \, d\tau \le \frac{1}{2}\|v_0\|_2^2 - \int_0^t \langle F - F_0 - EE, \nabla v\rangle_\Omega \, d\tau - \int_0^t b(v,E,v)\, d\tau$$

and get by the estimate of b with $\varepsilon = \frac{1}{4}$ for any finite $T' \in (0,T)$ the inequality

$$\begin{aligned} \frac{1}{2}\|v\|_{2,\infty;0,T'}^2 + \|\nabla v\|_{2,2;0,T'}^2 &\le \frac{1}{2}\|v_0\|_2^2 + \frac{1}{4}\|\nabla v\|_{2,2;0,T'}^2 \\ &\quad + \int_0^{T'} \left(\|F - F_0\|_2 + \|E\|_4^2\right) \|\nabla v\|_2 \, d\tau. \end{aligned}$$

A further application of Young's inequality and the above estimate of F_0 imply (1.17). ■

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