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Integral Equations and Operator Theory

HYPERSINGULAR INTEGRAL OPERATORS ALONG SURFACES

HUNG VIET LE

In this note, we estimate the boundedness for singular integral operators along curves and surfaces with highly singular kernels.

Introduction:

In the past several decades, many mathematicians have studied the well-known Hilbert transform along curves. One may find this interesting subject in several literatures. A few of them are listed in the reference of this note. It is commonly known that the Hilbert transform along curves

$$H^{\gamma}f(x) = p.v. \int_{-\infty}^{\infty} f(x - \gamma(t))\frac{dt}{t} \ \ (x \in \mathbb{R}^n)$$

is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, for some appropriate curves (see [3], [5], [6], [11], [12], [16], etc). Recently, Sharad Chandarana studied the following hypersingular integral operators along curves

$$T_{\alpha,\beta}f(x,y) = p.v. \int_{-1}^{1} f(x - t, y - \gamma(t))\frac{e^{-2\pi i|t|^{-\beta}}}{t|t|^{\alpha}}\, dt;\ x,\ y \in \mathbb{R};\ \alpha, \beta > 0.$$

Observe that the singularity at the origin for the above operator is worse than that of the Hilbert transform. To compensate for this profound singularity, the author cleverly introduced the oscillation factor $e^{-2\pi i|t|^{-\beta}}$. As a consequence, the author proved that for $\gamma(t) = |t|^k$ or $|t|^k$ sgnt, $k \geq 2$ and $\beta > 3\alpha > 0$, the operator $T_{\alpha,\beta}f$ is bounded on $L^p(\mathbb{R}^2)$ for

$$1 + \frac{3\alpha(\beta+1)}{\beta(\beta+1) + (\beta - 3\alpha)} < p < \frac{\beta(\beta+1) + (\beta - 3\alpha)}{3\alpha(\beta+1)} + 1.$$

Chandarana's work has motivated us to investigate the natural minimal conditions of the curve γ, which will allow the boundedness of the above operator. Furthermore, we would like to generalize the results to higher dimensions. We now state the main results of this paper as follows:

Theorem 1:
Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous bounded, measurable, and even function, which is differentiable a.e. on $\mathbb{R}$. Assume that either h is monotone or $h' \in L^1(\mathbb{R})$. Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a measurable, even function such that $|\gamma'(r)|$ is increasing on *supp* $\gamma' \cap [0, \infty)$. Suppose that either $\gamma' \in L^1(\mathbb{R})$ or $\gamma \in L^{\infty}(\mathbb{R})$ and $\gamma(r)$ is monotone on $[0, \infty)$.

Let the singular integral operator Tf be defined by

$$Tf(x_1, x_2) = p.v. \int \frac{f(x_1 - y, x_2 - \gamma(y))e^{i|y|^{-\beta}}h(y)}{y|y|^{\alpha}}\, dy,$$

where $x_1, x_2, y \in \mathbb{R}$ and $0 < 2\alpha < \beta$. Then Tf is bounded on $L^2(\mathbb{R}^2)$. Moreover, Tf is bounded on $L^p(\mathbb{R}^2)$ for

$$\frac{\beta}{\beta - \alpha} < p < \frac{\beta}{\alpha} \text{ with } 0 < 2\alpha < \beta,$$

provided that the one-dimensional maximal function

$$M^{\gamma}g(x_n) = \sup_{r>0} \left\{ \frac{1}{r} \int_{|t| \le r} |g(x_n - \gamma(t))|\, dt \right\}$$

is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$.

Theorem 2:
Let the functions h and γ, defined on $\mathbb{R}^{n-1}(n \geq 3)$, be real-valued, measurable, radial and differentiable a.e. on $[0, \infty)$. Assume that h is continuous, bounded; and either h is monotone or $h' \in L^1(\mathbb{R})$. Suppose that $|\gamma'(r)|$ is increasing on $supp \gamma' \cap [0, \infty)$, and that either $\gamma' \in L^1(\mathbb{R})$ or $\gamma \in L^{\infty}(\mathbb{R})$ and $\gamma(r)$ is monotone on $[0, \infty)$. Define the singular integral operator Tf by

$$Tf(x, x_n) = p.v. \int \frac{f(x - y, x_n - \gamma(y))e^{i|y|^{-\beta}}\Omega(y)h(y)}{|y|^{n-1+\alpha}}\, dy,$$

where $x, y \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$, and $0 < 2\alpha < \beta$. Here Ω, defined on $\mathbb{R}^{n-1}$, satisfies the following conditions:
a) Ω is homogeneous of degree zero
b) Ω has mean value zero over the sphere S^{n-2}, and
c) $\Omega \in L^q(S^{n-2})$ for some q with $1 < q < \infty$.
Then Tf is bounded on $L^2(\mathbb{R}^n)$. Moreover, Tf is bounded on $L^p(\mathbb{R}^n)$ for

$$\frac{\beta}{\beta - \alpha} < p < \frac{\beta}{\alpha} \text{ with } 0 < 2\alpha < \beta,$$

provided that the maximal function $M^{\gamma}g(x_n)$ (defined in Theorem 1) is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$.

Corollary:
Let $\gamma : [0, \infty) \to \mathbb{R}$ be a measurable C^1 function, which has compact support and is strictly increasing on its compact support. If γ' is increasing on its support, then the singular integral operators in Theorems 1 and 2 are bounded on L^p for

$$\frac{\beta}{\beta - \alpha} < p < \frac{\beta}{\alpha}, \text{ with } 0 < 2\alpha < \beta.$$

The proof of our theorems depends on Theorems C and D' in [3]. For convenience, we state Theorems C and D' below

Theorem C:[3]
Let $\{\mu_k\}_{-\infty}^{\infty}$ be probability measures in $\mathbb{R}^n$ such that

$$\left|\widehat{\mu}_k\left(\zeta^0,\overline{\zeta}\right)-\widehat{\mu}_k\left(\zeta^0,0\right)\right|\leq C\left|a_{k+1}\overline{\zeta}\right|^{\alpha}$$
$$\left|\widehat{\mu}_k\left(\zeta^0,\overline{\zeta}\right)\right|\leq C\left|a_k\overline{\zeta}\right|^{-\alpha}$$

where α is a fixed postive constant.
Suppose that $\sup_k\left|\mu_k^{(0)} * g(x^0)\right|$ is a bounded operator in $L^p(\mathbb{R}^m)(1\leq m<n)$ for all $p>1$. Then $\sup_k|\mu_k * f(x)|$ is also bounded in $L^p(\mathbb{R}^n)$ for all $p>1$. Here, $\{a_k\}_{k\in\mathbb{Z}}$ stands for a lacunary sequence of positive numbers: $a_k>0$ and $\inf_{k\in\mathbb{Z}}\left\{\frac{a_{k+1}}{a_k}\right\}=a>1$.

Theorem D':[3]
Suppose that $||\sigma_k||\leq 1$ and the measures $\{\sigma_k\}_{k\in\mathbb{Z}}$ satisfy the estimates
$\widehat{\sigma}_k(\zeta^0,0)=0,\left|\widehat{\sigma}_k\left(\zeta^0,\overline{\zeta}\right)\right|\leq C\min\left\{\left|a_{k+1}\overline{\zeta}\right|^{\alpha},\left|a_k\overline{\zeta}\right|^{-\alpha}\right\}$, for all $k\in\mathbb{Z}$ and some fixed $\alpha>0$.
If $\sup_k||\sigma_k| * f|$ and $\sup_k\left||\sigma_k^{(0)}| * g\right|$ are bounded in $L^q(\mathbb{R}^n)$ and $L^q(\mathbb{R}^m)$ respectively, then Tf and $g(f)$ are bounded in $L^p(\mathbb{R}^n)$ for $\left|\frac{1}{p}-\frac{1}{2}\right|<\frac{1}{2q}$.
Here $Tf(x)=\sum_{k=-\infty}^{\infty}(\sigma_k * f)(x)$, $g(f)(x)=\left(\sum_k|\sigma_k * f(x)|^2\right)^{\frac{1}{2}}$, and $|\sigma_k|$ denotes the total variation of the measure σ_k, and similar definition for $|\sigma_k^{(0)}|$.

Throughout the rest of this paper, we will denote C as a constant, which is not necessarily the same at each occurence. However, C does not depend on any essential variable. Note that one may assume the radial function $h\geq 0$ in the proofs of both Theorems 1 and 2. This is true because h is bounded. Before we prove the theorems, we need the following lemmas:

Lemma 1:
Let $\gamma:[0,\infty)\to\mathbb{R}$ satisfy the following conditions:
a) $|\gamma'(y)|$ is increasing on $supp\gamma'$, and
Either
b) $\gamma(y)$ is monotone and $\gamma(y)\in L^{\infty}([0,\infty))$
Or
c) $\gamma'(y)\in L^1([0,\infty))$.
Then for $0\leq a\leq b\leq\infty$, $\zeta_n\in\mathbb{R}$

$$\int_a^b|\zeta_n|\,|\gamma'(y)|\,dy\leq C,$$

where the constant C is independent of ζ_n.

Lemma 2:
Let $z = \sigma + i\tau$ be a complex number, with $\sigma < \dfrac{\beta - 2\alpha}{2}$ and $\tau \in \mathbb{R}$. Let G be defined on $[0, b]$ by

$$G(y) = \int_0^y \frac{e^{i\zeta s}\, e^{is^{-\beta}}}{s^{1+\alpha+z}}\, ds; \quad \zeta \in \mathbb{R},\ 0 \leq y \leq b$$

Then $|G(y)| \leq C(1 + |z|)y^{\frac{\beta-2\alpha-2\sigma}{2}}$ for $0 \leq y \leq b$.

Lemma 3:
Let $\phi(t) = 2^k\zeta t + (2^k)^{-\beta}t^{-\beta}$, with $\zeta \in \mathbb{R},\ k \in \mathbb{Z}$ and $|2^k\zeta| > 1$. Then

$$\left| \int_1^R e^{i\phi(t)}\, dt \right| \leq C|2^k\zeta|^{-\frac{1}{2}} \text{ with } 1 \leq R \leq 2.$$

Proof of Lemma 1:
If $|\zeta_n| \leq 1$, then $\int_a^b |\zeta_n||\gamma'(y)|\, dy \leq \int_a^b |\gamma'(y)|\, dy \leq C$, where C is either $2||\gamma||_{L^\infty}$ or $||\gamma'||_{L^1}$. If $|\zeta_n| > 1$, then because $|\gamma'|$ is increasing, we have

$$\begin{aligned} \int_a^b |\zeta_n||\gamma'(y)|\, dy &= \int_{a|\zeta_n|}^{b|\zeta_n|} \left|\gamma'\left(\frac{y}{|\zeta_n|}\right)\right| dy \\ &\leq \int_{a|\zeta_n|}^{b|\zeta_n|} |\gamma'(y)|\, dy \leq C \end{aligned}$$

where C is again either $2||\gamma||_{L^\infty}$ or $||\gamma'||_{L^1}$. Lemma 1 is proved.

Proof of Lemma 2:
Let $\phi(t) = \zeta t + t^{-\beta}$, and define $g(s)$ by

$$g(s) = \int_0^s e^{i\phi(t)}\, dt \text{ so that } G(y) = \int_0^y \frac{g'(s)}{s^{1+\alpha+z}}\, ds.$$

Observe that $\phi'(t) = \zeta - \frac{\beta}{t^{\beta+1}}$, which is monotone. Also, $\phi''(t) = \dfrac{\beta(\beta+1)}{t^{\beta+2}}$. Thus $|\phi''(t)| \geq \dfrac{C}{t^{\beta+2}} \geq \dfrac{C}{s^{\beta+2}}$ for $0 \leq t \leq s$. By van der Corput's lemma, $|g(s)| \leq \left(\dfrac{C}{s^{\beta+2}}\right)^{-\frac{1}{2}} = C \cdot s^{\frac{\beta+2}{2}}$. Now

$$\begin{aligned} G(y) &= \int_0^y \frac{g'(s)\, ds}{s^{1+\alpha+z}} = \frac{g(s)}{s^{1+\alpha+z}}\Bigg]_0^y + (1+\alpha+z)\int_0^y \frac{g(s)\, ds}{s^{2+\alpha+z}} \\ \text{and thus}\quad |G(y)| &\leq C\left\{ \frac{s^{\frac{\beta+2}{2}}}{s^{1+\alpha+\sigma}}\Bigg|_0^y + (1+\alpha+|z|)\int_0^y s^{\frac{\beta-2\alpha-2\sigma}{2}-1}\, ds \right\}. \end{aligned}$$

Hence, $|G(y)| \leq C(1+|z|)y^{\frac{\beta-2\alpha-2\sigma}{2}}$, provided that $\sigma < \dfrac{\beta-2\alpha}{2}$. Lemma 2 is proved.

Proof of Lemma 3:
Notice that

$$\begin{aligned}\phi'(t) &= 2^k\zeta + \frac{-\beta(2^k)^{-\beta}}{t^{\beta+1}}\\ &\equiv 2^k\zeta + \psi(t)\end{aligned}$$

Observe that $\psi(t)$ is negative for all $t \in [1,2]$, and that $t\psi(t)$ is increasing on $[1,2]$. We divide the proof in two cases.

Case 1: $\zeta < 0$
Then $\phi'(t) \le 2^k\zeta$ for all $t \in [1,2]$, and so $|\phi'(t)| \ge |2^k\zeta|$. By van der Corput's lemma, $\left|\int_1^R e^{i\phi(t)}\,dt\right| \le C|2^k\zeta|^{-1}, \;\; 1 \le R \le 2.$

Case 2: $\zeta > 0$
Since $\phi'(t)$ is strictly increasing, there is a unique t_0 such that $\phi'(t_0) = 0$. That is, $\psi(t_0) = -2^k\zeta$. Let $\delta = |2^k\zeta|^{-\frac{1}{2}}$ if $|2^k\zeta| > 1$. Let $t_1 = \min(t_0, 2)$. Write

$$\int_1^R e^{i\phi(t)}\,dt = \int_{J_1} + \int_{J_2} + \int_{J_3} \equiv I_1 + I_2 + I_3$$

with $J_1 = [1,R] \cap [t_1-\delta, t_1+\delta]$, $J_2 = [1, t_1-\delta]$ and $J_3 = [t_1+\delta, R]$. Trivially, $|I_1| \le 2\delta = 2|2^k\zeta|^{-\frac{1}{2}}$. For I_2, we have for $t \in J_2$,

$$\begin{aligned}\phi'(t) &\le 2^k\zeta + \psi(t_1)\frac{t_1}{t}\\ &\le 2^k\zeta(1-\frac{t_1}{t}) \le 2^k\zeta(-\frac{\delta}{2})\end{aligned}$$

So $|\phi'(t)| \ge |2^k\zeta|\frac{\delta}{2} = \frac{1}{2}|2^k\zeta|^{\frac{1}{2}}$. Therefore, by van der Corput's lemma, $|I_2| \le 2|2^k\zeta|^{-\frac{1}{2}}$. Now for the estimate of I_3, observe that $J_3 = \phi$ unless $t_1 = t_0$. For $t \in J_3$,

$$\begin{aligned}\phi'(t) &\ge 2^k\zeta + \psi(t_0)\frac{t_0}{t}\\ &= 2^k\zeta(1-\frac{t_0}{t}) \ge 2^k\zeta\cdot\frac{\delta}{2} = \frac{1}{2}|2^k\zeta|^{\frac{1}{2}}\end{aligned}$$

Another appeal to van der Corput's lemma yields $|I_3| \le 2|2^k\zeta|^{-\frac{1}{2}}$.
Consequently, $\left|\int_1^R e^{i\phi(t)}\,dt\right| \le C|2^k\zeta|^{-\frac{1}{2}}$, $1 \le R \le 2$. Lemma 3 is proved.

Remark: The idea in the proof of this Lemma 3 is similar the one in [3] (see the Lemma on page 558 of [3]).

Proof of Theorem 1:
We first show that Tf is bounded on $L^2(\mathbb{R}^2)$. Observe that $\widehat{Tf}(\zeta,\zeta_n) = m(\zeta,\zeta_n)\widehat{f}(\zeta,\zeta_n)$, where $\zeta,\zeta_n \in \mathbb{R}$ and

$$\begin{aligned} m(\zeta,\zeta_n) &= \int \frac{e^{i\zeta y}\, e^{i\zeta_n\gamma(y)}\, e^{i|y|^{-\beta}}\, h(y)}{y|y|^{\alpha}}\, dy \\ &= \int_0^\infty \frac{e^{i\zeta y}\, e^{i\zeta_n\gamma(y)}\, e^{iy^{-\beta}}\, h(y)}{y^{1+\alpha}}\, dy - \int_0^\infty \frac{e^{-i\zeta y}\, e^{i\zeta_n\gamma(y)}\, e^{iy^{-\beta}}\, h(y)}{y^{1+\alpha}}\, dy \\ &\equiv m^+(\zeta,\zeta_n) - m^-(\zeta,\zeta_n) \end{aligned}$$

By Plancherel's Theorem, it suffices to show that m is uniformly bounded on $\mathbb{R}^2$. Write

$$m^+(\zeta,\zeta_n) = \int_0^1 \ldots dy + \int_1^\infty \ldots dy \equiv I_1 + I_2.$$

Trivially $|I_2| \le C$.
Let $G(y) = \displaystyle\int_0^y \frac{e^{i\zeta s}\, e^{is^{-\beta}}}{s^{1+\alpha}}\, ds$ for $0 \le y \le 1$. Then $I_1 = \displaystyle\int_0^1 G'(y)\, e^{i\zeta_n\gamma(y)}\, h(y)\, dy$. So $I_1 = G(y)h(y)e^{i\zeta_n\gamma(y)}\Big|_0^1 - \displaystyle\int_0^1 G(y)e^{i\zeta_n\gamma(y)}\, \{i\zeta_n\gamma'(y)h(y) + h'(y)\}\, dy$. By Lemma 2 with $z=0$, we have $|G(y)| \le y^{\frac{\beta-2\alpha}{2}}$. Thus

$$\begin{aligned} |I_1| &\le C\left\{|G(1)| + |G(1)|\int_0^1 |\zeta_n||\gamma'(y)||h(y)|\, dy + |G(1)|\int_0^1 |h'(y)|\, dy\right\} \\ &\le C, \end{aligned}$$

where Lemma 1 has been applied to get the last inequality. Thus $|m^+(\zeta,\zeta_n)| \le C$. Similarly, $|m^-(\zeta,\zeta_n)| \le C$. Therefore, $||Tf||_{L^2(\mathbb{R}^2)} \le C||f||_{L^2(\mathbb{R}^2)}$.
For the remaining part of Theorem 1, we need the following lemmas.

Lemma 4:
For $z = \sigma + i\tau$ a complex number with $-\alpha < \sigma < \dfrac{\beta - 2\alpha}{2}$ and $\tau \in \mathbb{R}$, define a family of operators $\{T_z f\}$ by $\widehat{T_z f}(\zeta,\zeta_n) = m_z(\zeta,\zeta_n)\widehat{f}(\zeta,\zeta_n)$, where $\zeta,\zeta_n \in \mathbb{R}$ and

$$m_z(\zeta,\zeta_n) = \int \frac{e^{i\zeta y}\, e^{i\zeta_n\gamma(y)}\, e^{i|y|^{-\beta}}\, h(y)}{y|y|^{\alpha+z}}\, dy.$$

Then $||T_z f||_{L^2(\mathbb{R}^2)} \le C(1+|z|)||f||_{L^2(\mathbb{R}^2)}$.

Lemma 5:
Consider the family of operators $\{T_z f\}$ defined in Lemma 4. If $z = \sigma + i\tau$ with $\sigma = -\alpha$ and $\tau \in \mathbb{R}$, then $||T_z f||_{L^p(\mathbb{R}^2)} \le C(1+|z|)||f||_{L^p(\mathbb{R}^2)}$ for $1 < p < \infty$.

Assume for the moment that $\{T_z f\}$ is an admissible family of operators and that Lemmas 4 and 5 are valid. Notice that $T_0 f = Tf$ (at least for f a Schwartz function).

Note also that the bounds for T_z in both lemmas grow at most as fast as the first degree polynomial in $|z|$ variable. Therefore, we may apply the Analytic Interpolation Theorem to obtain the desired results. To finish the proof of Theorem 1, we need to prove Lemmas 4 and 5 and show that $\{T_z f\}$ is an admissible family of operators. We omit the proof of Lemma 4, since it is just a repetition of the proof which shows the L^2-boundedness of Tf. For the proof of Lemma 5, we will apply Theorems C and D' in [3].

Proof of Lemma 5:
We write $T_z f(x, x_n) = \sum_k \sigma_k * f(x, x_n);\ \ x, x_n \in \mathbb{R}$ where

$$\widehat{\sigma_k}(\zeta, \zeta_n) = \int_{|y|\simeq 2^k} \frac{e^{i\zeta y}\, e^{i\zeta_n \gamma(y)}\, e^{i|y|^{-\beta}}\, h(y)}{y|y|^{\alpha+z}}\, dy = \int_{|y|\simeq 2^k} \frac{e^{i\zeta y}\, e^{i\zeta_n \gamma(y)}\, e^{i|y|^{-\beta}}\, h(y)}{y|y|^{i\tau}}\, dy.$$

Then $\{\sigma_k\}$ are finite Borel measures with $||\sigma_k|| \le C$, $\int d\sigma_k = 0$, and $\widehat{\sigma_k}(0, \zeta_n) = 0$ for all $k \in \mathbb{Z}$. Let $\mu_k = |\sigma_k|$ be the total variation of σ_k, i.e.,

$$\widehat{\mu_k}(\zeta, \zeta_n) = \int_{|y|\simeq 2^k} \frac{e^{i\zeta y}\, e^{i\zeta_n \gamma(y)}\, h(y)}{|y|}\, dy.$$

Then $\{\mu_k\}$ are positive finite Borel measures with $||\mu_k|| \le C$ for all $k \in \mathbb{Z}$.
We need to show the following:

$$|\widehat{\sigma_k}(\zeta, \zeta_n)| \le C(1 + |z|) \min\{|2^k\zeta|^{\frac{1}{2}},\ |2^k\zeta|^{-\frac{1}{2}}\} \tag{1}$$

$$|\widehat{\mu_k}(\zeta, \zeta_n) - \widehat{\mu_k}(0, \zeta_n)| \le C|2^k\zeta| \tag{2}$$

$$|\widehat{\mu_k}(\zeta, \zeta_n)| \le C|2^k\zeta|^{-1} \tag{3}$$

and $\sup_k |\mu_k^{(0)} * g(x_n)|$ is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$, where $\widehat{\mu}_k^{(0)}(\zeta_n) = \widehat{\mu_k}(0, \zeta_n)$. But observe that

$$\mu_k^{(0)} * g(x_n) = \int_{|y|\simeq 2^k} g(x_n - \gamma(y)) h(y) \frac{dy}{|y|}$$

It is clear that $\sup_k |\mu_k^{(0)} * g(x_n)| \le CM^\gamma g(x_n)$, which is bounded on $L^p(\mathbb{R})$, $1 < p < \infty$, by hypothesis. We now prove inequality (1). Because $\widehat{\sigma_k}(0, \zeta_n) = 0$, we have on the one hand

$$|\widehat{\sigma_k}(\zeta, \zeta_n)| = \left| \int_{|y|\simeq 2^k} (e^{i\zeta y} - 1) \frac{e^{i\zeta_n \gamma(y)}\, e^{i|y|^{-\beta}}\, h(y)}{y \cdot |y|^{i\tau}}\, dy \right| \le C \int_{|y|\simeq 2^k} \frac{|i\zeta y|}{|y|}\, dy \le C|2^k\zeta| \le C|2^k\zeta|^{\frac{1}{2}} \ \text{ if } |2^k\zeta| \le 1$$

On the other hand, we write

$$\widehat{\sigma}_k(\zeta,\zeta_n) = \int_{y\simeq 2^k} \frac{e^{i\zeta y}\, e^{i\zeta_n\gamma(y)}\, e^{iy^{-\beta}}\, h(y)}{y|y|^{i\tau}}\,dy - \int_{y\simeq 2^k} \frac{e^{-i\zeta y}\, e^{i\zeta_n\gamma(y)}\, e^{iy^{-\beta}}\, h(y)}{y|y|^{i\tau}}\,dy$$
$$\equiv \widehat{\sigma}_k^+(\zeta,\zeta_n) - \widehat{\sigma}_k^-(\zeta,\zeta_n)$$

After a change of variable, we obtain

$$\begin{aligned}\widehat{\sigma}_k^+(\zeta,\zeta_n) &= \frac{1}{(2^k)^{i\tau}} \int_1^2 \frac{e^{i(2^k\zeta)y}\, e^{i\zeta_n\gamma(2^k y)}\, e^{i(2^k)^{-\beta}y^{-\beta}}\, h(2^k y)}{y|y|^{i\tau}}\,dy \\ &= \frac{1}{(2^k)^{i\tau}} \int_1^2 \frac{G'(y)e^{i\zeta_n\gamma(2^k y)}\, h(2^k y)}{y^{1+i\tau}}\,dy,\end{aligned}$$
where $G(y) = \int_1^y e^{i(2^k\zeta t+(2^k)^{-\beta}t^{-\beta})}\,dt.$

By Lemma 3, $|G(y)| \le C|2^k\zeta|^{-\frac{1}{2}}$ for $1 \le y \le 2$. Integrating by parts yields

$$\widehat{\sigma}_k^+(\zeta,\zeta_n) = \frac{1}{(2^k)^{i\tau}} \left\{ \frac{G(y)e^{i\zeta_n\gamma(2^k y)}\, h(2^k y)}{y^{1+i\tau}} \Bigg|_1^2 \right\} -$$
$$\int_1^2 \frac{G(y)\, e^{i\zeta_n\gamma(2^k y)}}{(2k)^{i\tau} y^{2+2i\tau}} \left\{(2^k)[i\zeta_n\gamma'(2^k y)h(2^k y) + h'(2^k y)]\, y^{1+i\tau} - (1+i\tau)y^{i\tau}h(2^k y)\right\} dy$$
$$\equiv A - B$$

By Lemma 3, $|A| \le C|2^k\zeta|^{-\frac{1}{2}}$, and

$$\begin{aligned}|B| \le C|2^k\zeta|^{-\frac{1}{2}} \Bigg\{ &\int_1^2 2^k \left[|\zeta_n|\,|\gamma'(2^k y)h(2^k y)| + |h'(2^k y)|\right] \frac{dy}{y} \\ &+ \int_1^2 (1+\alpha+|z|)\,\frac{|h(2^k y)|}{y^2}\,dy \Bigg\}.\end{aligned}$$

It's clear that the second integral on the RHS of the above inequality is bounded by $C(1+|z|)$. If we make a change of variable $y \to 2^k y$, then the first integral on the RHS of the above inequality becomes

$$\begin{aligned}&2^k \int_{2^k}^{2^{k+1}} \left\{ \frac{|\zeta_n\,\gamma'(y)\,h(y)| + |h'(y)|}{y} \right\} dy \\ &\le C \left\{ \int_{2^k}^{2^{k+1}} |\zeta_n|\,|\gamma'(y)|\,dy + \int_{2^k}^{2^{k+1}} |h'(y)|\,dy \right\} \\ &\le C.\end{aligned}$$

Thus $|B| \le C(1+|z|)|2^k\zeta|^{-\frac{1}{2}}$. Hence, $|\widehat{\sigma}_k^+(\zeta,\zeta_n)| \le |A| + |B| \le C(1+|z|)|2^k\zeta|^{-\frac{1}{2}}$. A similar estimate holds for $\widehat{\sigma}_k^-(\zeta,\zeta_n)$. Therefore, $|\widehat{\sigma}_k(\zeta,\zeta_n)| \le C(1+|z|)|2^k\zeta|^{-\frac{1}{2}}$.

It remains to show inequalities (2) and (3). Inequality (2) is obvious. For inequality (3), one writes

$$\begin{aligned}\widehat{\mu}_k(\zeta,\zeta_n) &= \int_{2^k}^{2^{k+1}} e^{i\zeta y}\, e^{i\zeta_n\gamma(y)}\frac{h(y)}{y}\,dy \;+\; \int_{2^k}^{2^{k+1}} e^{-i\zeta y}\, e^{i\zeta_n\gamma(y)}\,\frac{h(y)}{y}\,dy \\ &\equiv \widehat{\mu}_k^{+}(\zeta,\zeta_n)+\widehat{\mu}_k^{-}(\zeta,\zeta_n).\end{aligned}$$

Integrating by parts yields

$$\begin{aligned}\widehat{\mu}_k^{+}(\zeta,\zeta_n) &= \frac{e^{i\zeta y}\, e^{i\zeta_n\gamma(y)}\, h(y)}{i\zeta y}\Bigg|_{2^k}^{2^{k+1}} - \\ &\int_{2^k}^{2^{k+1}} \frac{e^{i\zeta y}\, e^{i\zeta_n\gamma(y)}}{i\zeta}\left\{\frac{i\zeta_n\gamma'(y)h(y)+h'(y)}{y}-\frac{h(y)}{y^2}\right\}dy.\end{aligned}$$

$$\begin{aligned}\text{Thus } |\widehat{\mu}_k^{+}(\zeta,\zeta_n)| &\le \frac{C}{|2^k\zeta|}\left\{1+\int_{2^k}^{2^{k+1}}|\zeta_n\gamma'(y)h(y)|dy+\int_{2^k}^{2^{k+1}}\frac{h(y)}{y}\,dy\right\} \\ &\le C|2^k\zeta|^{-1}.\end{aligned}$$

Similarly, $|\widehat{\mu}_k^{-}(\zeta,\zeta_n)\le C|2^k\zeta|^{-1}$, whence $|\widehat{\mu}_k(\zeta,\zeta_n)\le C|2^k\zeta|^{-1}$. Lemma 5 is proved.

Finally, we need to show that the family of operators $\{T_z f\}$ is admissible. That is, whenever f and g are simple functions in $L^1(\mathbb{R}^2)$, the mapping $z \to \int_{\mathbb{R}^2}(T_z f)g\,dx$ is analytic in the interior of S and continuous on S. Here S is the strip in the complex plane bounded by the lines $\sigma = b < \dfrac{\beta-2\alpha}{2}$ and $\sigma=-\alpha$. Observe that $T_z f(x,x_n) = \int \dfrac{f(x-y,x_n-\gamma(y))\, e^{i|y|^{-\beta}}\, h(y)}{y|y|^{\alpha+z}}\,dy$. If f is a simple function, it is a finite linear combination of characteristic functions. In order for f to be in $L^1(\mathbb{R}^2)$, the domains of these characteristic functions must have finite Lebesgue measures. Thus, it suffices to show that the integral

$$F(z)=\int_0^d \frac{e^{iy^{-\beta}}h(y)}{y^{1+\alpha+z}}\,dy,\quad 0<d<\infty$$

is continuous on S and analytic on S^0, the interior of S.

Define $F_n(z)=\int_{d/n}^{d}\dfrac{e^{iy^{-\beta}}h(y)}{y^{1+\alpha+z}}\,dy,\quad n=1,2,3,\ldots$ It's clear that F_n is continuous on S and analytic on S^0 for all n. Moreover, $|F_n(z)-F(z)|\le C(1+|z|)n^{\alpha+\sigma-\beta}$, which tends to zero as n approaches infinity (independent of z in S). This shows that $F_n \to F$ uniformly on every compact subset $K_1 \subset S\backslash S^0$, and on every compact subset $K_2 \subset S^0$. Therefore F is analytic on S^0 and continuous on S. Theorem 1 is proved.

Proof of Theorem 2:
We begin to show that Tf is bounded on $L^2(\mathbb{R}^n)$. Write $\widehat{Tf}(\zeta,\zeta_n)=m(\zeta,\zeta_n)\widehat{f}(\zeta,\zeta_n)$, where $\zeta\in\mathbb{R}^{n-1},\zeta_n\in\mathbb{R}$, and

$$m(\zeta,\zeta_n)=\int_{\mathbb{R}^{n-1}}\frac{e^{i\zeta\cdot y}\, e^{i\zeta_n\gamma(y)}\, e^{i|y|^{-\beta}}\,\Omega(y)h(y)}{|y|^{n-1+\alpha}}\,dy$$

$$= \int_{S^{n-2}} \left(\int_0^{\infty} \frac{e^{i|\zeta|r(\zeta'\cdot y')}\, e^{i\zeta_n\gamma(r)}\, e^{ir^{-\beta}} h(r)}{r^{1+\alpha}}\, dr \right) \Omega(y')\, dv(y')$$
$$\equiv \int_{S^{n-2}} I_r(y')\Omega(y')dv(y');\ y' \in S^{n-2},\ \zeta' = \frac{\zeta}{|\zeta|}.$$

Note that $|I_r(y')| \le C$ for all $y' \in S^{n-2}$. The proof for this inequality is again a repetition of the proof of the L^2-boundedness for Tf in Theorem 1. Hence $|m(\zeta,\zeta_n)| \le C||\Omega||_{L^1(S^{n-2})}$, and so $||Tf||_{L^2(\mathbb{R}^n)} \le C||f||_{L^2(\mathbb{R}^n)}$. For the remaining part of this theorem, it suffices to prove the following lemmas:

Lemma 6:
For $z = \sigma + i\tau$ a complex number, with $-\alpha < \sigma < \dfrac{\beta - 2\alpha}{2}$ and $\tau \in \mathbb{R}$, define a family of operators $\{T_z f\}$ by $\widehat{T_z f}(\zeta,\zeta_n) = m_z(\zeta,\zeta_n)\widehat{f}(\zeta,\zeta_n)$, where

$$m_z(\zeta,\zeta_n) = \int \frac{e^{i\zeta\cdot y}\, e^{i\zeta_n\gamma(y)}\, e^{i|y|^{-\beta}}\Omega(y)h(y)}{|y|^{n-1+\alpha+z}}\, dy.$$

Then $||T_z f||_{L^2(\mathbb{R}^n)} \le C(1+|z|)||f||_{L^2(\mathbb{R}^n)}$.

Lemma 7:
Consider the family of operators $\{T_z f\}$ defined in Lemma 6. If $z = \sigma + i\tau$ with $\sigma = -\alpha$ and $\tau \in \mathbb{R}$, then $||T_z f||_{L^p(\mathbb{R}^n)} \le C(1+|z|)||f||_{L^p(\mathbb{R}^n)}$ for $1 < p < \infty$.

We omit the proof of Lemma 6 since it is similar to the proof of Lemma 4 in Theorem 1. For the proof of lemma 7, we will apply Theorems C and D' in [3].

Proof of Lemma 7:
Write $T_z f(x,x_n) = \sum_k \sigma_k * f(x,x_n);\ x \in \mathbb{R}^{n-1}, x_n \in \mathbb{R}$, where

$$\widehat{\sigma}_k(\zeta,\zeta_n) = \int_{|y|\simeq 2^k} \frac{e^{i\zeta\cdot y}\, e^{i\zeta_n\gamma(y)}\, e^{i|y|^{-\beta}}\,\Omega(y)h(y)}{|y|^{n-1+i\tau}}\, dy$$
$$= \int_{S^{n-2}} \Omega(y') \left(\int_{2^k}^{2^{k+1}} \frac{e^{i|\zeta|(\zeta'\cdot y')r}\, e^{i\zeta_n\gamma(r)}\, e^{ir^{-\beta}}\, h(r)}{r^{1+i\tau}}\, dr \right) dv(y')$$
$$\equiv \int_{S^{n-2}} \Omega(y')I_r(y')dv(y');\ y' \in S^{n-2}, \zeta' = \zeta/|\zeta|.$$

Let $\mu_k = |\sigma_k|$ be the total variation of σ_k, i.e.,

$$\widehat{\mu}_k(\zeta,\zeta_n) = \int_{S^{n-2}} |\Omega(y')| \left(\int_{2^k}^{2^{k+1}} e^{i|\zeta|(\zeta'\cdot y')r}\, e^{i\zeta_n\gamma(r)}\, \frac{|h(r)|}{r}\, dr \right) dv(y')$$

Again, we must show

$$|\widehat{\sigma}_k(\zeta,\zeta_n)| \le C \min\left\{|2^k\zeta|, |2^k\zeta|^{-b}\right\} \text{ for some } b > 0 \qquad (4)$$

$$|\widehat{\mu}_k(\zeta,\zeta_n) - \widehat{\mu}_k(0,\zeta_n) \leq C|2^k\zeta| \tag{5}$$

$$|\widehat{\mu}_k(\zeta,\zeta_n)| \leq C|2^k\zeta|^{-\frac{1}{2q'}} \text{ where } q' > 1 \tag{6}$$

and $\sup_k |\mu_k^{(0)} * g(x_n)|$ is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$. Here $\widehat{\mu}_k^{(0)}(\zeta_n) = \widehat{\mu}_k(0,\zeta_n)$. But notice that

$$\begin{aligned}\mu_k^{(0)} * g(x_n) &= \int_{|y|\simeq 2^k} g(x_n - \gamma(y)) \frac{|\Omega(y)h(y)|}{|y|^{n-1}}\, dy \\ &= \int_{S^{n-2}} |\Omega(y')| \left(\int_{2^k}^{2^{k+1}} \frac{g(x_n - \gamma(r))|h(r)|}{r}\, dr \right) dv(y')\end{aligned}$$

Thus $\sup_k |\mu_k^{(0)} * g(x_n)| \leq C||\Omega||_{L^1(S^{n-2})} M^\gamma g(x_n)$, which is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$, by hypothesis. We now proceed to prove inequality (4). By the cancellation property of Ω, we have

$$\begin{aligned}|\widehat{\sigma}_k(\zeta,\zeta_n)| &= \left| \int_{|y|\simeq 2^k} (e^{i\zeta\cdot y} - 1) \frac{e^{i\zeta_n\gamma(y)}\, e^{i|y|^{-\beta}} \Omega(y)h(y)}{|y|^{n-1+i\tau}}\, dy \right| \\ &\leq \int_{|y|\simeq 2^k} \frac{|\zeta\cdot y|\, |\Omega(y)h(y)|}{|y|^{n-1}}\, dy \\ &\leq C|\zeta| \int_{S^{n-2}} |\Omega(y')| \,(\int_{2^k}^{2^{k+1}} dr)\, dv(y') \\ &\leq C|2^k\zeta|.\end{aligned}$$

On the other hand,

$$\begin{aligned}\widehat{\sigma}_k(\zeta,\zeta_n) &= \int_{S^{n-2}} \Omega(y') \left(\int_{2^k}^{2^{k+1}} \frac{e^{i|\zeta|(\zeta'\cdot y')r}\, e^{i\zeta_n\gamma(r)}\, e^{ir^{-\beta}} h(r)}{r^{1+i\tau}}\, dr \right) dv(y') \\ &\equiv \int_{S^{n-2}} \Omega(y') I_r(y')\, dv(y').\end{aligned}$$

By Hölder's inequality, $|\widehat{\sigma}_k(\zeta,\zeta_n)| \leq ||\Omega(y')||_{L^q(S^{n-2})}\, ||I_r(y')||_{L^{q'}(S^{n-2})}$ where q and q' are conjugate. Observe that

$$\begin{aligned}||I_r(y')||^{q'}_{L^{q'}(S^{n-2})} &= \int_{S^{n-2}} |I_r(y')|^{q'}\, dv(y') \\ &= \int_{S^{n-2}} \left| \int_{2^k}^{2^{k+1}} \frac{e^{i|\zeta|(\zeta'\cdot y')r}\, e^{i\zeta_n\gamma(r)}\, e^{ir^{-\beta}} h(r)}{r^{1+i\tau}}\, dr \right|^{q'} dv(y') \\ &= C\int_0^\pi \left| \int_{2^k}^{2^{k+1}} \frac{e^{i|\zeta|r\cos\theta}\, e^{i\zeta_n\gamma(r)}\, e^{ir^{-\beta}}\, h(r)}{r^{1+i\tau}} \right|^{q'} (\sin\theta)^{n-3}\, d\theta\end{aligned}$$

$$
\begin{aligned}
&\equiv C\int_0^\pi |I_r(\cos\theta)|^{q'}(\sin\theta)^{n-3}\,d\theta\\
&\leq C\int_0^\pi |I_r(\cos\theta)|^{q'}\,d\theta\\
&= C\left\{\int_0^{\frac{\pi}{2}-\delta}\cdots+\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta}\cdots+\int_{\frac{\pi}{2}+\delta}^{\pi}\cdots\right\}\\
&\equiv C\{I_1+I_2+I_3\},\text{ where } 0<\delta<1.
\end{aligned}
$$

This δ will be chosen later. Trivially, $I_2 \leq C\delta$. Observe that $I_r(\cos\theta)$ looks like $\widehat{\sigma}_k^+(\zeta,\zeta_n)$ in the proof of Theorem 1, with ζ being replaced by $|\zeta|\cos\theta$. Thus we also get a similar estimate for $|I_r(\cos\theta)|$ as the estimate for $|\widehat{\sigma}_k^+(\zeta,\zeta_n)|$ in Theorem 1. That is,

$$
\begin{aligned}
|I_r(\cos\theta)| &\leq C(1+|z|)|2^k\zeta\cos\theta|^{-\frac{1}{2}}\\
&\equiv C_z|2^k\zeta\cos\theta|^{-\frac{1}{2}}\\
\text{Hence, }\ I_1 &\leq C_z^{q'}\int_0^{\frac{\pi}{2}-\delta}|2^k\zeta\cos\theta|^{-\frac{q'}{2}}\,d\theta\\
&\leq C_z^{q'}\int_0^{\frac{\pi}{2}-\delta}|2^k\zeta|^{-\frac{q'}{2}}\,|\cos(\frac{\pi}{2}-\delta)|^{-\frac{q'}{2}}\,d\theta\\
&\leq C_z^{q'}|2^k\zeta|^{-\frac{q'}{2}}|\sin\delta|^{-\frac{q'}{2}}\\
&\leq C_z^{q'}|2^k\zeta|^{-\frac{q'}{2}}\,\delta^{-\frac{q'}{2}}
\end{aligned}
$$

The last inequality follows because $\sin t \geq \dfrac{2t}{\pi}$ for $0\leq t\leq\dfrac{\pi}{2}$. Similarly, $I_3 \leq C_z^{q'}|2^k\zeta|^{\frac{-q'}{2}}\,\delta^{-\frac{q'}{2}}$. So, $||I_r(y')||^{q'}_{L^{q'}(S^{n-2})} \leq C_z^{q'}\left\{\delta+|2^k\zeta|^{-\frac{1}{2}}\delta^{-\frac{q'}{2}}\right\}$. If $|2^k\zeta|>1$, we choose $\delta=|2^k\zeta|^{-\frac{1}{2}}$ so that

$$
\begin{aligned}
||I_r(y')||_{L^{q'}(s^{n-2})} &\leq C_z^{q'}\left\{|2^k\zeta|^{-\frac{1}{2}}+|2^k\zeta|^{-\frac{q'}{4}}\right\}\\
&\leq C_z^{q'}\,|2^k\zeta|^{-bq'},\text{ where } b=\begin{cases}\dfrac{1}{2q'} & \text{if } q'\geq 2\\[2mm] \dfrac{1}{4} & \text{if } 1<q'<2\end{cases}\\
\text{Thus } |\widehat{\sigma}_k(\zeta,\zeta_n)| &\leq ||\Omega(y')||_{L^q(S^{n-2})}\,||I_r(y')||_{L^{q'}(S^{n-2})}\\
&\leq C_z|2^k\zeta|^{-b}\equiv C(1+|z|)|2^k\zeta|^{-b}
\end{aligned}
$$

Consequently, $|\widehat{\sigma}_k(\zeta,\zeta_n)\leq C(1+|z|)\min\{|2^k\zeta|,\ |2^k\zeta|^{-b}\}$, $b>0$. Inequality (4) is proved. It is easy to see that inequality (5) holds. Thus it remains to show that inequality (6) holds. To prove inequality (6), we write

$$
\begin{aligned}
\widehat{\mu}_k(\zeta,\zeta_n) &= \int_{S^{n-2}}|\Omega(y')|\left(\int_{2^k}^{2^{k+1}} e^{i|\zeta|(\zeta'\cdot y')r}\,e^{i\zeta_n\gamma(r)}\,\frac{h(r)}{r}\,dr\right)dv(y')\\
&\equiv \int_{S^{n-2}}|\Omega(y')|I_r(y')dv(y')
\end{aligned}
$$

$$\leq ||\Omega||_{L^q(S^{n-2})}\, ||I_r||_{L^{q'}(S^{n-2})}, \text{ where } q \text{ and } q' \text{ are conjugate.}$$

$$\begin{aligned}
||I_r||_{L^{q'}}^{q'} &= \int_{S^{n-2}} \left| \int_{2^k}^{2^{k+1}} e^{i\zeta(\zeta'\cdot y')r}\, e^{i\zeta_n\gamma(r)}\, \frac{h(r)}{r}\, dr \right|^{q'} dv(y') \\
&= C \int_0^{\pi} \left| \int_{2^k}^{2^{k+1}} e^{i\zeta r\cos\theta}\, e^{i\zeta_n\gamma(r)}\, \frac{h(r)}{r}\, dr \right|^{q'} (\sin\theta)^{n-3}\, d\theta \\
&\equiv C \int_0^{\pi} |I_r(\cos\theta)|^{q'} (\sin\theta)^{n-3}\, d\theta \\
&\leq C \int_0^{\pi} |I_r(\cos\theta)|^{q'}\, d\theta \\
&= C \left\{ \int_0^{\frac{\pi}{2}-\delta} \ldots\, d\theta + \int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} \ldots d\theta + \int_{\frac{\pi}{2}+\delta}^{\pi} \ldots d\theta \right\} \\
&\equiv C\{I_1 + I_2 + I_3\}
\end{aligned}$$

where $0 < \delta < 1$. This δ will be chosen later. It's clear that $I_2 \leq C\delta$. Observe that

$$\begin{aligned}
I_r(\cos\theta) &= \int_{2^k}^{2^{k+1}} \frac{h(r)\, e^{i\zeta r\cos\theta}\, e^{i\zeta_n\gamma(r)}}{r}\, dr \\
&= \frac{e^{i\zeta r cos\theta}}{i\zeta\cos\theta}\, \frac{e^{i\zeta_n\gamma(r)}h(r)}{r}\Bigg]_{2^k}^{2^{k+1}} - \\
&\int_{2^k}^{2^{k+1}} \frac{e^{i\zeta r\cos\theta}\, e^{i\zeta_n\gamma(r)}}{i\zeta\cos\theta}\, \frac{\{[i\zeta_n\gamma'(r)h(r) + h'(r)]\, r - h(r)\}}{r^2}\, dr.
\end{aligned}$$

$$\begin{aligned}
|I_r(\cos\theta)| &\leq \frac{C}{|2^k\zeta|\,|\cos\theta|} \left\{ 1 + \int_{2^k}^{2^{k+1}} |\zeta_n\gamma'(r)h(r)|dr + \int_{2^k}^{2^{k+1}} |h'(r)|dr \right. \\
&\qquad \left. + \int_{2^k}^{2^{k+1}} \frac{h(r)}{r}\, dr \right\} \\
&\leq \frac{C}{|2^k\zeta|\,|\cos\theta|}
\end{aligned}$$

$$\begin{aligned}
\text{Thus } I_1 &= \int_0^{\frac{\pi}{2}-\delta} |I_r(\cos\theta)|^{q'}\, d\theta \\
&\leq \frac{C}{|2^k\zeta|^{q'}} \int_0^{\frac{\pi}{2}-\delta} \frac{d\theta}{|\cos\theta|^{q'}} \\
&\leq \frac{C}{|2^k\zeta|^{q'}}\, \frac{1}{(\sin\delta)^{q'}} \leq \frac{C}{|2^k\zeta|^{q'}\delta^{q'}}
\end{aligned}$$

Similarly, $I_3 \leq \dfrac{C}{|2^k\zeta|^{q'}\delta^{q'}}$ We choose $\delta = |2^k\zeta|^{-\frac{1}{2}}$.

$$\begin{aligned}
\text{Then } \|I_r\|_{L^{q'}}^{q'} &\le C\{I_1 + I_2 + I_3\} \\
&\le C\left\{\delta + \frac{1}{(|2^k\zeta|\delta)^{q'}}\right\} \\
&= C\{\,|2^k\zeta|^{-\frac{1}{2}} + |2^k\zeta|^{-\frac{q'}{2}}\} \\
&\le C|2^k\zeta|^{-\frac{1}{2}} \text{ if } |2^k\zeta| > 1.
\end{aligned}$$

Therefore, $|\widehat{\mu}_k(\zeta, \zeta_n)| \le \|\Omega\|_{L^q(S^{n-2})}\, \|I_r\|_{L^{q'}(S^{n-2})} \le C|2^k\zeta|^{-\frac{1}{2q'}}$. Theorem 2 is proved.

Proof of the Corollary:
It suffices to show that the one-dimensional maximal function $M^\gamma g(x_n)$ is bounded on $L^p(\mathbb{R})$ for all $p > 1$. But this proof can be found in the corollary of [7] or [2].

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Department of Mathematics
Southwestern Oklahoma State University
100 Campus Drive
Weatherford, Oklahoma 73096
Email: leh@swosu.edu

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