ERROR ESTIMATES FOR A SINGLE-PHASE NONLINEAR STEFAN PROBLEM IN ONE SPACE DIMENSION

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ABSTRACT. In this paper we introduce the semidiscrete solution of a single-phase nonlinear Stefan problem. We analyze the optimal convergence of the semidiscrete solution in H^1 and H^2 normed spaces and also we derive the error estimates in L^2 normed space.

1. Introduction

The mathematical formulation of many problems arising in practice leads to a free boundary problem—a Stefan problem. In one space dimension a single-phase nonlinear Stefan problem with zero forcing term can be described as follows:

Find a pair of $\{(U,S); U = U(y,\tau)$ and $S = S(\tau)\}$ such that U satisfies

$$(1.1) \qquad U_\tau - (a(U)U_y)_y = 0 \quad \text{in} \quad \Omega(\tau) \times (0, T_0],$$

$$(1.2) \qquad U(y,0) = g(y) \quad \text{for} \quad y \in I,$$

$$(1.3) \qquad U_y(0,\tau) = U(S(\tau),\tau) = 0 \quad \text{for} \quad 0 < \tau \le T_0,$$

$$(1.4) \qquad S_\tau + (a(U)U_y)_{|y=S(\tau)} = 0 \quad \text{for} \quad 0 < \tau \le T_0$$

$$S(0) = 1,$$

where $\Omega(\tau) = \{y; 0 < y < S(\tau)\}$ for each $\tau \in (0, T_0]$ and $I = (0, 1)$. For simplicity, we suppress τ in $\Omega(\tau)$ and write $\Omega(\tau)$ as Ω only. For a single-phase linear Stefan problem, the study of semidiscrete finite element error analysis was initiated with the fixed domain method by Nitsche [4, 5]. Das & Pani [1] extended the error analyis to the

problem (1.1)-(1.4) and derived error estimates in H^1 and H^2 norms for semidiscrete Galerkin approximations to the problem (1.1)-(1.4) using the fixed domain method. And when the temperature was given at the fixed boundary instead of the flux condition of (1.3), Pani & Das [6, 7] obtained error estimates for semidiscrete Galerkin approximations to the problem (1.1)-(1.4). Also error estimates for fully discrete Galerkin approximations to the problem (1.1)-(1.4), depending on the backward Euler method in time, were derived in [7]. Lee & Lee [3] adopted the modifed Crank-Nicolson method to improve the rate of convergence in the temporal direction.

In this paper, we not only improve the previous error estimates in H^1 and H^2 norms for the semi-discrete case in [1], but we also derive the error estimates in L^2. In section 2, the weak formulation and Galerkin approximations are considered. In section 3, the auxiliary projection and related estimates are given. In section 4, error estimates for the semidiscrete case are established. In section 5, the global existence of the Galerkin approximation is considered.

Throughout this paper, we assume the followings:
  (i) The pair $\{U, S\}$ is the unique smooth solution of (1.1)-(1.4) with $S(\tau) \geq \nu > 0$ for all $\tau \in [0, T_0]$.
  (ii) The function $a(\cdot)$, only depending on U, is $C^4(R)$ and has bounded derivatives up to order 4, bounded by a common constant K_1, say. Further, the function $a(\cdot)$ is bounded below on R by a positive constant α.
  (iii) The initial function g is sufficiently smooth and non-negative and satisfies the compatibility conditions $g(0) = g(1) = 0$.

For an integer $m \geq 0$ and $1 \leq p \leq \infty$, $W^{m,p}(\Omega)$ denotes the usual Sobolev space of measurable functions which, together with their distributional derivatives of order up to m, are in L^p. For $\Omega = I$ and $p = 2$, we shall use the symbol H^m in place of $W^{m,2}(I)$ with norm $\|\cdot\|_m$. Let X be a Banach space with norm $\|\cdot\|_X$ and let $v : [0, T] \to X$ be a function. The following notations are used:

$$\|v\|_{L^p(0,T;X)} = \left(\int_0^T \|v(\tau)\|_X^p d\tau \right)^{\frac{1}{p}}, \quad \text{for} \quad 1 \leq p < \infty$$

$$\|v\|_{L^\infty(0,T;X)} = \sup_{0 \leq \tau \leq T} \|v(\tau)\|_X.$$

We assume that $\{U, S\}$ satisfies the following regularity condition: For $r \geq 1$,
Condition $\check{R}$:

(1.5) $\qquad U \in W^{1,\infty}(0, T_0; H^{r+1}(\Omega))$ and $S \in W^{1,\infty}(0, T_0)$.

Let $\check{K}_2$ be a bound for $\{U, S\}$ in the space appeared in the condition $\check{R}$.

2. Weak formulation and Galerkin approximations

With the help of the Landau transformations

(2.1) $\qquad x = yS^{-1}$ and $t = t(\tau) = \int_0^\tau S^{-2}(\tau')d\tau'$,

the problem (1.1)-(1.4) can be transformed into the following problem with the fixed domain:

Find a pair of $\{(u, s); u(x, t) = U(y, \tau)$ and $s(t) = S(\tau)\}$ such that u satisfies

(2.2) $\qquad u_t - (a(u)u_x)_x = -xa(u(1, t))u_x(1, t)u_x$

$\qquad$ in $I \times (0, T]$,

(2.3) $\qquad u(x, 0) = g(x)$ for $x \in I$,

(2.4) $\qquad u_x(0, t) = u(1, t) = 0$ for $0 < t \leq T$,

(2.5) $\qquad \dfrac{ds}{dt} = -a(u(1, t))u_x(1, t)s$ for $0 < t \leq T$,

$\qquad s(0) = 1$.

Here, $t = T$ corresponds to $\tau = T_0$. Note that all the regularity properties for $\{u, s\}$, denoted by the condition R, are inherited from (1.5) for $\{U, S\}$ and that K_2 is a bound for $\{u, s\}$. Note that the integral in (2.1) can be rewritten as

(2.6) $\qquad \dfrac{d\tau}{dt} = s^2(t)$ for $0 < t \leq T$

$\qquad \tau(0) = 0$.

To obtain the weak formulation of the problem (2.2)-(2.5), we consider the space

(2.7) $\dot{H}^2(I) = \{v \in H^2(I) : v_x(0) = v(1) = 0\}.$

Multiplying both sides of (2.2) by w_{xx} and integrating the first term of (2.2) with respect to x, we obtain

(2.8) $(u_{tx}, w_x) + ((a(u)u_x)_x, w_{xx}) = a(u(1,t))u_x(1,t)(xu_x, w_{xx}),$

for $t > 0$, $w \in \dot{H}^2(I)$ with $u(x,0) = g(x)$.

To get Galerkin approximation of u, let $\dot{S}_h$ be a finite-dimensional subspace of $\dot{H}^2(I)$ with the following properties:

(i) The approximation property: for $v \in H^k(I) \bigcap \dot{H}^2(I)$, there exists a constant K_0, independent of h and v, such that

(2.9) $\inf_{\chi \in \dot{S}_h} \|v - \chi\|_j \leq K_0 h^{k-j} \|v\|_k$, for $0 \leq j \leq 2$, $2 \leq k \leq r+1$,

where r is a positive integer.

(ii) The inverse property:

$$\|\chi\|_2 \leq K_0 h^{-1} \|\chi\|_1, \chi \in \dot{S}_h.$$

Then a Galerkin approximation of u can be defined as follows:
Find $u^h = u^h(\cdot, t) \in \dot{S}_h$ such that for $t > 0$, $\chi \in \dot{S}_h$

(2.10) $(u_{tx}^h, \chi_x) + ((a(u^h)u_x^h)_x, \chi_{xx}) = a(u^h(1,t))u_x^h(1,t)(xu_x^h, \chi_{xx}),$

with

$$u^h(x,0) = Q_h g(x),$$

where Q_h is an appropriate projection of u onto $\dot{S}_h$ at $t = 0$ to be defined later in section 4. Moreover, Galerkin approximations s_h and τ_h of s and τ, respectively are given by

(2.11)
$$\frac{ds_h}{dt} = -a(0)u_x^h(1,t)\, s_h \text{ for } 0 < t \leq T,$$
$$s_h(0) = 1$$

and

(2.12)
$$\frac{d\tau_h}{dt} = s_h^2(t) \text{ for } 0 < t \leq T,$$
$$\tau_h(0) = 0.$$

3. Auxiliary projection and related estimates

For u, v, $w \in \dot{H}^2(I)$, we define a trilinear form

$$A(u; v, w) = ((a(u)v_x + a_u(u)u_x v)_x, w_{xx}) - a(0)u_x(1)(xv_x, w_{xx}),$$

as in [7]. Then it is easy to show that
the boundedness of A :

(3.1) $$|A(u; v, w)| \leq K_3 \|v_{xx}\| \|w_{xx}\|$$

a Garding-type inequality for A :

(3.2) $$A(u; v, v) \geq \alpha \|v_{xx}\|^2 - \Lambda \|v_x\|^2$$

for u, v, and $w \in \dot{H}^2(I)$ where K_3, α, and Λ are constants and K_3 and Λ may depend on $\|u\|_2$.

Let

(3.3) $$A_\Lambda(u; v, w) = A(u; v, w) + (v_x, w_x).$$

Let $\tilde{u}(x, t) \in \dot{S}_h$ be the auxiliary projection of u with respect to A_Λ :

(3.4) $$A_\Lambda(u; u - \tilde{u}, \chi) = 0, \quad \chi \in \dot{S}_h.$$

Due to [2], we obtain the following result.

THEOREM 3.1. *For a given $u \in \dot{H}^2(I)$, there exists a unique solution $\tilde{u} \in \dot{S}_h$ to (3.4).*

Define $\eta = u - \tilde{u}$ and $\zeta = u^h - \tilde{u}$. Then we obtain the following estimates for η and η_t whose proofs are similar to those of Lemma 4.2 - Lemma 4.7 in [1].

THEOREM 3.2. *For $t \in [0, T]$, there exists a constant*

$$K_4 = K_4(\alpha, \Lambda, K_0, K_1, K_2, K_3)$$

such that

$$\|\eta\|_j \leq K_4 h^{m-j} \|u\|_m,$$

$$\|\eta_t\|_j \leq K_4 h^{m-j} \{\|u\|_m + \|u_t\|_m\},$$

and

$$|\eta_x(1, t)| \leq K_4 h^{2(m-2)} \|u\|_m$$

for $j = 0, 1, 2$ and $2 \leq m \leq r + 1$.

Due to the conditions on u and Theorem 3.2, there exists K_5 such that

$$\|\tilde{u}\|_{L^\infty(H^2)} + \|\tilde{u}_t\|_{L^\infty(H^2)} \leq K_5.$$

4. Error estimates for the semidiscrete case

Throughout this section, it is assumed that there exist constants K^* and h_0 such that a Galerkin approximation $u^h \in \dot{S}_h$ of (2.10) exists and satisfies

(4.1) $$\|u^h\|_{L^\infty(H^2)} \leq K^* \quad \text{for} \quad 0 \leq h \leq h_0,$$

where $u^h(x,0)$ is defined as $Q_h g(x)$, satisfying

$$A_\Lambda(g; g - Q_h g, \chi) = 0, \quad \chi \in \dot{S}_h.$$

Clearly, $u^h(x,\,0) \equiv \tilde{u}(x,0)$.

Following the standard notations for nonlinear problems, we define

$$e = u - u^h, \ \eta = u - \tilde{u}, \quad \text{and} \quad \zeta = u^h - \tilde{u}.$$

Then we obtain

$$e = \eta - \zeta$$

and

(4.2) $$a(u^h)u^h_x - a(\tilde{u})\tilde{u}_x$$
$$= a(u^h)\zeta_x + (a(u^h) - a(\tilde{u}))\tilde{u}_x,$$

(4.3) $$a(u)u_x - a(\tilde{u})\tilde{u}_x$$
$$= a(u)\eta_x + a_u(u)\eta u_x - \tilde{a}_u\eta\eta_x - \tilde{a}_{uu}\eta^2 u_x,$$

where

$$\tilde{a}_u = \int_0^1 \frac{\partial a}{\partial u}(u - \xi\eta)d\xi,$$

$$\tilde{a}_{uu} = \int_0^1 (1 - \xi)\frac{\partial^2 a}{\partial u^2}(u - \xi\eta)d\xi.$$

THEOREM 4.1. *There exists a constant* $K_6 = K_6(\alpha, \Lambda, K_1, K_2, K_3, K_4, K_5, K^*)$ *such that*

(4.4) $$\|\zeta\|_{L^\infty(H^1)} + \beta\|\zeta\|_{L^2(H^2)} \leq K_6 h^m$$

for $4 \leq m \leq r+1$.

Proof. Substracting

$$(\tilde{u}_{tx}, \chi_x) + ((a(\tilde{u})\tilde{u}_x)_x, \chi_{xx}) = a(0)u_x(1)(x\tilde{u}_x, \chi_{xx})$$

from (2.8) and (2.10) and using (4.2) and (4.3), we have

$$(\zeta_{tx}, \chi_x) + \left(\frac{\partial}{\partial x}[a(u^h)\zeta_x + (a(u^h) - a(u))\tilde{u}_x], \chi_{xx}\right)$$
$$= a(0)u_x^h(1)(xu_x^h, \chi_{xx}) + (\eta_{tx}, \chi_x)$$
$$+ \left(\frac{\partial}{\partial x}[a(u)\eta_x + a_u(u)\eta u_x - \tilde{a}_u \eta \eta_x - \tilde{a}_{uu}\eta^2 u_x], \chi_{xx}\right)$$
$$- a(0)u_x(1)(xu_x, \chi_{xx}),$$

which implies that

$$
\begin{aligned}
(4.5) \quad & (\zeta_{tx}, \chi_x) + ((a(u^h)\zeta_x)_x, \chi_{xx}) \\
& = (\eta_{tx}, \chi_x) - \Lambda(\eta_x, \chi_x) + a(0)u_x(1)(x\zeta_x, \chi_{xx}) \\
& + a(0)\zeta_x(1)(xu_x^h, \chi_{xx}) - (((a(u^h) - a(\tilde{u}))\tilde{u}_x)_x, \chi_{xx}) \\
& - ((\tilde{a}_u \eta \eta_x + \tilde{a}_{uu}\eta^2 u_x)_x, \chi_{xx}).
\end{aligned}
$$

Taking $\chi = \zeta$ in (4.3), integrating by parts the first term on the right-hand side, using Schwartz inequality and Sobolev imbedding theorem, we obtain

$$
\begin{aligned}
(4.6) \quad & \frac{1}{2}\frac{d}{dt}\|\zeta_x\|^2 + \alpha\|\zeta_{xx}\|^2 \\
& \leq K_1 K^*\|\zeta_x\|\|\zeta_{xx}\| + \|\eta_t\|\|\zeta_{xx}\| + \Lambda\|\eta\|\|\zeta_{xx}\| + K_1 K_2\|\zeta_x\|\|\zeta_{xx}\| \\
& + K_1 K^*|\eta_x(1)|\|\zeta_{xx}\| + K(\epsilon)\|\zeta_{xx}\|^2 + K(K_1, K^*, \epsilon)\|\zeta\|_1^2 \\
& + 2K_1 K_5\|\zeta\|_1\|\zeta\|_2 + K_1 K_5\|\zeta\|_1\|\zeta_{xx}\| + [K_1(K_2 + \|\eta\|_2)\|\eta\|_1^2 \\
& + 2K_1\|\eta\|_1\|\eta\|_2 + K_1(K_2 + \|\eta\|_2)\|\eta\|_1^3 + 2K_1\|\eta\|_1\|\eta\|_2 \\
& + K_1(K_2 + \|\eta\|_2)\|\eta\|_1^3 + 3K_1\|\eta\|_1^2]\|\zeta_{xx}\|.
\end{aligned}
$$

From (4.6), we obtain

$$
\begin{aligned}
(4.7) \quad & \frac{d}{dt}\|\zeta\|_1^2 + 2\alpha\|\zeta\|_2^2 \\
& \leq K(\varepsilon)\|\zeta\|_2^2 + K_7(K_1, K_2, \Lambda, K^*, K_5, \varepsilon)\|\zeta\|_1^2 \\
& + K_7(K_1, K_2, \Lambda, K^*, K_5, \varepsilon)\big[\|\eta_t\|^2 + \|\eta\|^2 + |\eta_x(1)|^2 \\
& + \|\eta\|_1^4 + \|\eta\|_2^2\|\eta\|_1^4 + \|\eta\|_2^2\|\eta\|_1^2\big].
\end{aligned}
$$

Choosing ε sufficiently small so that $2\alpha - K(\varepsilon) > 0$ and applying Gronwall's inequality to (4.7), we obtain

$$(4.8) \quad \|\zeta\|_1^2 + \beta \int_0^t \|\zeta\|_2^2 dt' \leq K_7 \exp(K_7 t) \int_0^t \left[\|\eta_t\|^2 + \|\eta\|^2 + |\eta_x(1)|^2 \right. $$
$$\left. + \|\eta\|_1^4 + \|\eta\|_2^2 \|\eta\|_1^4 + \|\eta\|_2^2 \|\eta\|_1^2 \right] dt'.$$

The desired result can be obtained for $m \geq 4$ if we take the supremum over all t in $[0, T]$ in (4.8) and if we use the results of Theorem 3.2. $\square$

COROLLARY 4.2. *For $m \geq 4$, the following estimate holds:*

$$(4.9) \quad \|\zeta\|_{L^\infty(L^2)} \leq K_8 h^m.$$

From Theorem 3.2, Theorem 4.1, and Corollary 4.2, the following theorem is obtained. The estimates in H^2 and H^1 norms for e are the same as those in [1]. However, the order of h in the previous estimate in L^2 norm in [1] is improved by 1.

THEOREM 4.3. *Let the solution $u \in \dot{H}^2(I)$ of (2.8) with (2.3) be sufficiently smooth so that the regularity condition R is satisfied. Further, let there exist constants h_0 and $K^*(K^* \geq 2K_2)$ such that a Galekin approximation $u^h \in \dot{S}_h$ of (2.10) satisfying (4.1) exists in $I \times (0, t]$ for $0 \leq h \leq h_0$. Then we obtain the following estimate:*

$$(4.10) \quad \|e\|_{L^\infty(H^j)} \leq K_9 h^{r+1-j} \quad \text{for} \quad r \geq 3 \quad \text{and} \quad j = 0, 1, 2,$$

where $K_9 = K_9(K_4, K_6)$. Besides, for sufficiently small h and $r \geq 3$,

$$\|u^h\|_{L^\infty(H^2)} \leq 2K_2 \leq K^*$$

and consequently K_9 can be chosen independent of K^*.

Finally, the Galerkin approximation of the solution $\{U(y, \tau), S(\tau)\}$ of the problem (1.1)-(1.4) can be defined as

$$U^h(y, \tau) = u^h(x, t),$$
$$S_h(\tau) = s_h(t),$$

where

$$y = s_h(t)x,$$
$$\tau = \tau_h(t).$$

Here s_h and τ_h are given by (2.5) and (2.6), respectively. Then we obtain the following error estimates for the Galerkin approximation in which the order of h in the previous estimate for S and τ in [1] is improved by 1.

THEOREM 4.4. *Under the assumptions of Theorem 4.3 and the regularity condition $\check{R}$, we obtain the following estimates:*

$$\|S - S_h\|_{L^\infty(0,T_0)} = O(h^{r+1})$$
$$\|\tau - \tau_h\|_{L^\infty(0,T_0)} = O(h^{r+1})$$
$$\|U - U^h\|_{L^\infty(0,T_0;H^j(\overline{\Omega}(\tau)))} = O(h^{r+1-j}), \text{ for } r \geq 3 \text{ and } j = 0,1,2,$$

where $\overline{\Omega}(\tau) = (0, \min(S(\tau), S_h(\tau))$ for $\tau \in (0, T_0)$.

Proof. The proof is similar to that of Theorem 5.5 in [7]. $\qquad\square$

5. Global existence of the Galerkin approximation

To obtain the unique existence of the Galerkin approximation $u^h \in \dot{S}_h$ of (2.10) in the domain of existence of the solution u and a priori estimates of $u - u^h$, we consider the following linear ordinary differential equation of ζ in time t with $\zeta(x,0) = 0$

$$
\begin{aligned}
&(\zeta_{tx}, \chi_x) + ((a(u - E)\zeta_x)_x, \chi_{xx}) \\
=&(\eta_{tx}, \chi_x) - \Lambda(\eta_x, \chi_x) + a(0)u_x(1)(x\zeta_x, \chi_{xx}) \\
&-a(0)\eta_x(1)(x(u_x - E_x), \chi_{xx}) + a(0)\zeta_x(1)(x(u_x - E_x), \chi_{xx}) \\
&-(((a(u - E) - a(u - \eta))(u_x - \eta_x))_x, \chi_{xx}) \\
&-((\tilde{a}_u \eta \eta_x + \tilde{a}_{uu}\eta^2 u_x)_x, \chi_{xx}), \quad \text{for } \chi \in \dot{S}_h,
\end{aligned}
$$

(5.1)

where $E \in L^\infty(\dot{H}^2(I))$,

$$\tilde{a}_u = \int_0^1 \frac{\partial a}{\partial u}(u - \xi\eta)d\xi, \text{ and } \tilde{a}_{uu} = \int_0^1 (1 - \xi)\frac{\partial^2 a}{\partial u^2}(u - \xi\eta)d\xi.$$

Then, for any $E = E(x, t)$, the existence of a unique solution of (5.1) with $\zeta(x, 0) = 0$ in $(0, T]$ can be established, see [4]. Therefore we can define an operator $\Im$ such that $\zeta = \Im E$ for each $E \in L^\infty(\dot{H}^2(I))$. Since $e = \eta - \zeta$, we obatin

$$(5.2) \qquad e = \eta - \Im E \quad \text{for each} \quad E \in L^\infty(\dot{H}^2(I)).$$

To show the existence of a solution u^h of (2.10), it is sufficient to show that the operator equation (5.2) has a fixed point, i.e., $e(E) = E$.

THEOREM 5.1. *If the solution u satisfies the regularity condition R and K is any positive constant, then, for sufficiently small h and $r \geq 3$, there exists a unique solution $u^h \in \dot{S}_h$ of (2.10) in the ball $\{\|u - u^h\|_{L^\infty(H^2(I))} \leq K\}$.*

Proof. Letting $\chi = \zeta$ in (5.1), we obtain

$$
\begin{aligned}
&(\zeta_{tx}, \zeta_x) + ((a(u - E)\zeta_x)_x, \zeta_{xx}) \\
&= (\eta_{tx}, \zeta_x) - \Lambda(\eta_x, \zeta_x) + a(0)u_x(1)(x\zeta_x, \zeta_{xx}) \\
&\quad -a(0)\eta_x(1)(x(u_x - E_x), \zeta_{xx}) + a(0)\zeta_x(1)(x(u_x - E_x), \zeta_{xx}) \\
&\quad -(((a(u - E) - a(u - \eta))(u_x - \eta_x))_x, \zeta_{xx}) \\
&\quad -((\tilde{a}_u \eta \eta_x + \tilde{a}_{uu} \eta^2 u_x)_x, \zeta_{xx}),
\end{aligned}
$$

(5.3)

which implies that

$$
\begin{aligned}
&\frac{d}{dt}\|\zeta_x\|^2 + 2\alpha\|\zeta_{xx}\|^2 \\
&\leq K(\varepsilon)\|\zeta_{xx}\|^2 + K(\varepsilon, K_1, \Lambda, K_2)\big[\|\eta_t\|^2 + \|\eta\|^2 \\
&\quad +(1 + \|E\|_2^2)|\eta_x(1)|^2 + (1 + \|E\|_2^2)(1 + \|\eta\|_1^2) \\
&\quad +(1 + \|\eta\|_2^2)(1 + \|\eta\|_1^2) + (\|\eta\|_1^2 + \|E\|_2^2)(1 + \|\eta\|_2^2) \\
&\quad +(1 + \|\eta\|_2^2)\|\eta\|_1^2 + \|\eta\|_2^2\|\eta\|_1^2 + \|\eta\|_1^2\|\eta\|_2^2 \\
&\quad +(1 + \|\eta\|_2^2)\|\eta\|_1^2 + \|\eta\|_2^2\|\eta\|_1^2 + \|\eta\|_1^4\big] \\
&\quad +K(K_1, K_2, \epsilon)(1 + \|E\|_2^2)\|\zeta_x\|^2.
\end{aligned}
$$

(5.4)

Choosing $\varepsilon > 0$ so that $2\alpha - K(\varepsilon) > 0$, we get

(5.5)
$$
\frac{d}{dt}\|\zeta_x\|^2
$$
$$
\leq K(\varepsilon, K_1, \Lambda, K_2)\big[\|\eta_t\|^2 + \|\eta\|^2 + (1 + \|E\|_2^2)(|\eta_x(1)|^2 + \|\eta\|_1^2)
$$
$$
+\|\eta\|_2^2\|\eta\|_1^2 + \|\eta\|_2^2 + \|\eta\|_1^2 + \|E\|_2^2\|\eta\|_2^2 + \|\eta\|_1^4\big]
$$
$$
+K(K_1, K_2, \varepsilon)(1 + \|E\|_2^2)\|\zeta_x\|^2,
$$

which implies that

(5.6)
$$
\|\zeta\|_1^2(t)
$$
$$
\leq K(\varepsilon, K_1, \Lambda, K_2) \exp\big[K(K_1, K_2, \varepsilon)(1 + \|E\|_2^2)t\big]
$$
$$
\cdot \int_0^t \big[\|\eta_t\|^2 + \|\eta\|^2 + (1 + \|E\|_2^2)(|\eta_x(1)|^2 + \|\eta\|_1^2)
$$
$$
+ \|\eta\|_2^2\|\eta\|_1^2 + \|\eta\|_2^2 + \|\eta\|_1^2 + \|E\|_2^2\|\eta\|_2^2 + \|\eta\|_1^4\big]\,dt'
$$
$$
\leq K \exp\big[K(1 + \|E\|_2^2)T\big] \cdot \big[h^{2m} + (1 + \|E\|_2^2)(h^{4(m-2)}
$$
$$
+ h^{2(m-1)}) + h^{4m-6} + h^{2(m-2)} + \|E\|_2^2 h^{2(m-2)} + h^{4(m-1)}\big].
$$

If $\|E\|_{L^\infty(H^2)} \leq \delta$, then from (5.6) we obtain

$$
\|\zeta\|_{L^\infty(H^1)} \leq K(K_1, K_2, \varepsilon, \Lambda, \delta)h^{m-2} \leq K(K_1, K_2, \varepsilon, \Lambda, \delta)h^{r-1}
$$

for $m \geq 4$ and so

$$
\|e\|_{L^\infty(H^2)} \leq \|\eta\|_{L^\infty(H^2)} + \|\zeta\|_{L^\infty(H^2)}
$$
$$
\leq \|\eta\|_{L^\infty(H^2)} + K_0 h^{-1}\|\zeta\|_{L^\infty(H^1)}
$$
$$
\leq Kh^{m-3} \leq Kh^{r-2}.
$$

Therefore, for sufficiently small h,

$$
\|e\|_{L^\infty(H^2)} \leq \delta.
$$

Thus the operator $\mathfrak{S}$ defined by (5.2) maps a ball

$$
B_\delta = \{v \in L^\infty(H^2) : \|v\|_{L^\infty(H^2)} \leq \delta\}
$$

into itself for sufficiently small h and so by Schauder's fixed point theorem the operator equation (5.2) has a fixed point, i.e., $e(E) = E$. This completes the proof. $\qquad\square$

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