REGULARITY OF LERAY-HOPF SOLUTIONS TO NAVIER-STOKES EQUATIONS (II)–BLOW UP RATE WITH SMALL $L^2(\mathbb{R}^3)$ DATA

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ABSTRACT. An upper bound of blow up rate for impressible Navier-Stokes equations with small data in $L^2(\mathbb{R}^3)$ is obtained.

1. Introduction

We consider the blow up rate of weak solutions to impressible Navier-Stokes equations

$$\begin{cases} \partial_t u - \Delta u + u\cdot\nabla u + \nabla p = 0, & \text{in} \quad \mathbb{R}^3\times(0,T) \\ \text{div} u = 0, & \text{in} \quad \mathbb{R}^3\times(0,T) \\ u(x,0) = u_0(x), & \text{in} \quad \mathbb{R}^3 \end{cases} \tag{1.1}$$

where u and p denote the unknown velocity and pressure of incompressible fluid respectively.

In this paper, we shall estimate the upper bound of blow up rate for the Navier-Stokes equations.

Theorem 1.1. *There is $\delta > 0$ such that if $\|u_0\|_{L^2(\mathbb{R}^3)} \le \delta$, and if u is a Leray-Hopf solution to the problem (1.1) and blows up at $t = T$, then for any small $\epsilon > 0$, there is $t_0 \in (0,T)$, such that*

$$\|u(t)\|_{L^\infty(\mathbb{R}^3)} \le \frac{\epsilon}{(T-t)^{1/2}}, \quad \textit{for all} \quad t\in(t_0,T). \tag{1.2}$$

Here $u:\ (x,t)\in\mathbb{R}^3\times(0,T)\to\mathbb{R}^3$ is called a weak solution of (1.1) if it is a Leray-Hopf solution. Precisely, it satisfies

(1) $u\in L^\infty(0,T;L^2(\mathbb{R}^3))\cap L^2(0,T;H^1(\mathbb{R}^3))$,

(2) $\text{div}u = 0$ in $\mathbb{R}^3\times(0,T)$,

(3) $\displaystyle\int_0^T\int_{\mathbb{R}^3}\{-u\cdot\partial_t\phi + \nabla u\cdot\nabla\phi + (u\cdot\nabla u)\cdot\phi\}dxdt = 0$

for all $\phi\in C_0^\infty(\mathbb{R}^3\times(0,T))$ with $\text{div}\phi = 0$ in $\mathbb{R}^3\times(0,T)$.

Combining Theorem 1.1 with my former result in [31], we have

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Corollary 1.2. *There is $\delta > 0$ such that if $\|u_0\|_{L^2(\mathbb{R}^3)} \le \delta$, and if u is a Leray-Hopf solution of the Navier-Stokes equations (1.1), then u is regular in $\mathbb{R}^3 \times (0,\infty)$.*

Because (1.1) is invariant under a group action, if $(u(x,t),\ p(x,t))$ is a solution, the same is true for

$$u_\lambda(x,t) := \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x,t) := \lambda^2 p(\lambda x, \lambda^2 t)$$

for all $\lambda \in (0,\infty)$. So for any $u_0 \in L^2(\mathbb{R}^3)$, there is $\lambda \in (0,\infty)$ such that the $L^2(\mathbb{R}^3)$ norm of $\lambda u_0(\lambda x)$ satisfies the condition of Corollary 1.2. Then we have

Corollary 1.3. *For all $u_0 \in L^2(\mathbb{R}^3)$, if u is a Leray-Hopf solution of the Navier-Stokes equations (1.1), then u is regular in $\mathbb{R}^3 \times (0,\infty)$. Moreover, u is the unique solution of the Navier-Stokes equations (1.1).*

Since Leray(1934)[19] and Hopf(1951)[15] obtained the global existence of weak solutions, it has been a fundamental open problem to prove the uniqueness and regularity of weak solutions to the Navier-Stokes equations.

2. Energy estimates

As in [7][8][9] where Giga and Kohn introduced similar transformations for the blow-up problem of semi-linear heat equations, we apply

$$y = \frac{1}{(T-t)^{1/2}}x, \quad \tau = -\ln(T-t), \quad w(y,\tau) = (T-t)^{1/2}u(x,t), \tag{2.1}$$

to (1.1) and consider the following new problem

$$\begin{cases} \partial_\tau w = \Delta_y w - \frac{y}{2}\cdot\nabla_y w - \frac{1}{2}w - w\cdot\nabla_y w - \nabla_y q, \quad \forall y \in \mathbb{R}^3, \quad \tau > -\ln T \\ \mathrm{div}_y\ w(y,\tau) = 0, \quad \text{in} \quad \mathbb{R}^3 \times (-\ln T, \infty) \\ w(y, -\ln T) = T^{1/2}u_0(T^{1/2}y), \quad \text{in} \quad \mathbb{R}^3 \end{cases} \tag{2.2}$$

where

$$q(y,\tau) = (T-t)p(x,t).$$

Without loss generality, in this section we take $T=1$. Multiplying the first one of (2.2) by w and integrating it over $\mathbb{R}^3$, by using the second equation of (2.2) we have

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{R}^3} \partial_\tau |w(y,\tau)|^2 dy = (-1)\int_{\mathbb{R}^3} |\nabla_y w(y,\tau)|^2 - \frac{1}{4}|w(y,\tau)|^2 dy \\ -\frac{1}{4}\int_{\mathbb{R}^3} \mathrm{div}\ (y|w(y,\tau)|^2)dy. \end{aligned} \tag{2.3}$$

Noting that

$$\int_{\mathbb{R}^3} \mathrm{div}\ (y|w(y,\tau)|^2)dy = \lim_{R\to\infty}\int_{\partial B_R} |y||w(y,\tau)|^2 d\sigma(y) \ge 0 \tag{2.4}$$

we obtain

Lemma 2.1. *For any $\tau > 0$, we have*

$$\frac{1}{2}\frac{d}{d\tau}\|w(\tau)\|^2_{L^2(\mathbb{R}^3)} \le (-1)\{\|\nabla_y w(\tau)\|^2_{L^2(\mathbb{R}^3)} - \frac{1}{4}\|w(\tau)\|^2_{L^2(\mathbb{R}^3)}\}. \tag{2.5}$$

Furthermore, we take differential in the equations of (2.2) and obtain

$$\begin{aligned}\partial_\tau \partial_j w = \Delta \partial_j w - \frac{1}{2} y \cdot \nabla \partial_j w - \partial_j w \\ -(\partial_j w \cdot \nabla) w - (w \cdot \nabla)\partial_j w - \nabla_y \partial_j q.\end{aligned} \tag{2.6}$$

By the same strategy as in the proof of Lemma 2.1, from (2.6) as well as the equation

$$\partial_\tau \Delta w = \Delta^2 w - \frac{1}{2}(y\cdot\nabla)\Delta w - \frac{3}{2}\Delta w - \Delta((w\cdot\nabla)w) - \nabla\Delta q$$

by taking twice differential in (2.2), we have

Lemma 2.2. *For all $\tau > 0$*

$$\begin{aligned}\frac{d}{d\tau}\int_{\mathbb{R}^3} |\nabla w(y,\tau)|^2 dy \le -2\int_{\mathbb{R}^3} |\nabla^2 w(y,\tau)|^2 dy - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla w(y,\tau)|^2 dy \\ -2\sum_{j,k,l=1}^{3}\int_{\mathbb{R}^3} \partial_j w_k(y,\tau)\partial_j w_l(y,\tau)\partial_l w_k(y,\tau) dy\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\frac{d}{d\tau}\int_{\mathbb{R}^3} |\Delta w(y,\tau)|^2 dy \le -2\int_{\mathbb{R}^3} |\nabla\Delta w(y,\tau)|^2 dy - \frac{3}{2}\int_{\mathbb{R}^3} |\Delta w(y,\tau)|^2 dy \\ -2\int_{\mathbb{R}^3} (\Delta w(y,\tau)) \cdot \Delta((w(y,\tau)\cdot\nabla) w(y,\tau)) dy.\end{aligned} \tag{2.8}$$

Remark 2.3. (1) For any $t_1 > 0$, there is $t_0 \in (0, t_1)$ such that $u(\cdot, t_0) \in H^1(\mathbb{R}^3)$. With the initial data $u(x, t_0)$, the Leray-Hopf solution $u(x,t)$ is regular at least in a short time interval after t_0 (see [19][24]). We are discussing the blow-up problem for these short time regular solutions.

(2) As a blow-up argument, we assume that $u(x,t)$ is bounded for $t < T$ and blows up at $t = T$. As a direct corollary, we can prove that $\|u(t)\|_{H^3(\mathbb{R}^3)}$ and $\partial_t\|u(t)\|_{H^m(\mathbb{R}^3)}$ ($m = 0,1,2$), as well as $\|\partial_t u(t)\|_{L^2(\mathbb{R}^3)}$, $\|\partial_t \nabla_x u(t)\|_{L^2(\mathbb{R}^3)}$ are bounded for $t < T$. So we have the same results for $\|w(\tau)\|_{H^3(\mathbb{R}^3)}$ and $\partial_\tau\|w(\tau)\|_{H^m(\mathbb{R}^3)}$ ($m = 0,1,2$) for $\tau < \infty$, as well as the similar results for q by the boundedness of Riesz transformation.

(3) Since $u(x,t)$, $\partial_t u(x,t) \in L^2(\mathbb{R}^3)$ for $t < T$,

$$\int_0^t \int_{\mathbb{R}^3} |\partial_h u(x,h)||u(x,h)| dx dh < \infty,$$

we can use Fubini theorem to obtain

$$
\begin{aligned}
2\int_{\mathbb{R}^3}\partial_t u(x,t)\cdot u(x,t)dx &= \frac{d}{dt}\int_0^t\int_{\mathbb{R}^3}\partial_h|u(x,h)|^2dxdh \\
&= \frac{d}{dt}\int_{\mathbb{R}^3}\int_0^t\partial_h|u(x,h)|^2dhdx = \frac{d}{dt}\int_{\mathbb{R}^3}|u(x,t)|^2dx.
\end{aligned} \tag{2.9}
$$

Noting that

$$
\partial_t u(x,t) = (T-t)^{-\frac{3}{2}}\{\partial_\tau w(\frac{x}{(T-t)^{1/2}},\tau)+\frac{x}{2(T-t)^{1/2}}\cdot\nabla_y w(\frac{x}{(T-t)^{1/2}},\tau)+\frac{1}{2}w(\frac{x}{(T-t)^{1/2}},\tau)\}
$$

where $\tau = (-)\ln(T-t)$, from

$$
\int_{\mathbb{R}^3}|\partial_t u(x,t)|^2dx = (T-t)^{-\frac{3}{2}}\int_{\mathbb{R}^3}|\partial_\tau w(y,\tau)+\frac{y}{2}\cdot\nabla_y w(y,\tau)+\frac{1}{2}w(y,\tau)|^2dy \tag{2.10}
$$

and

$$
\int_{\mathbb{R}^3}|u(x,t)|^2dx = (T-t)^{1/2}\int_{\mathbb{R}^3}|w(y,\tau)|^2dy \tag{2.11}
$$

we have for $t < T$

$$
\int_{\mathbb{R}^3}|\partial_\tau w(y,\tau)+\frac{y}{2}\cdot\nabla_y w(y,\tau)|^2dy < \infty. \tag{2.12}
$$

Moreover, from (2.9), we get

(2.13)

$$
\begin{aligned}
&(T-t)^{-\frac{1}{2}}\{\partial_\tau\int_{\mathbb{R}^3}|w(y,\tau)|^2dy - \frac{1}{2}\int_{\mathbb{R}^3}|w(y,\tau)|^2dy\} \\
&= \frac{d}{dt}\{(T-t)^{\frac{1}{2}}\int_{\mathbb{R}^3}|w(y,\tau)|^2dy\} = \frac{d}{dt}\int_{\mathbb{R}^3}|u(x,t)|^2dx = 2\int_{\mathbb{R}^3}\partial_t u(x,t)\cdot u(x,t)dx \\
&= 2\int_{\mathbb{R}^3}(T-t)^{-\frac{3}{2}}\{\partial_\tau w(\frac{x}{(T-t)^{1/2}},\tau)+\frac{x}{2(T-t)^{1/2}}\cdot\nabla_y w(\frac{x}{(T-t)^{1/2}},\tau) \\
&\qquad +\frac{1}{2}w(\frac{x}{(T-t)^{1/2}},\tau)\}\cdot(T-t)^{-\frac{1}{2}}w(\frac{x}{(T-t)^{1/2}},\tau)dx \\
&= (T-t)^{-\frac{1}{2}}\int_{\mathbb{R}^3}\{2\partial_\tau w(y,\tau)\cdot w(y,\tau)+(y\cdot\nabla_y w(y,\tau))\cdot w(y,\tau)+|w(y,\tau)|^2dy.
\end{aligned}
$$

By using (2.13), from (2.3) we get (2.5) again.

(4) From Leray's theorem (see V. Scheffer, Pacific J. Math. Vol. 66, No.2, pp 535-552(1976)), there is a disjoint open interval sequence $\{J_q\}_q$ in $(0,\infty)$ such that the Lebesgue measure of $(0,\infty)\setminus\cup_q J_q$ is zero, and the Leray-Hopf solution u can be modified on a set of Lebesgue measure zero so that its restriction to each $\mathbb{R}^3\times J_q$ becomes smooth. The first blow-up time T assumed in this paper may be considered as the right-side of an open interval J_q. From the arguments of this paper, we can see that the Leray-Hopf solution u can be extended smoothly over the right-side of J_q. So we get that u is smooth from the left-side of the open interval J_q. Since the Lebesgue measure of $(0,\infty)\setminus\cup_q J_q$ is zero, we get that for all $t > 0$, the Leray-Hopf solution $u(x,t)$ is smooth.

3. (L^∞, L^2)-DECOMPOSITION OF w

In this section we shall prove that w can be decomposed as the sum of a $L^\infty(0,\infty;L^m(\mathbb{R}^3))$ ($m\in[4,\infty]$) part and a $L^\infty(0,\infty;L^2(\mathbb{R}^3))\cap L^2(0,\infty;H^1(\mathbb{R}^3))$ part.

Let $\varphi\in C_0^\infty(\mathbb{R}^3,[0,1])$ be a radial symmetrical function satisfying

$$\varphi(\xi)=1\quad\forall|\xi|\le 1,\quad \varphi(\xi)=0\quad\forall|\xi|\ge 2,\quad \xi\cdot\nabla\varphi(\xi)\le 0\quad\forall\xi. \tag{3.1}$$

Like the Littlewood-Paley analysis, we define the operators

$$\Delta_{-1}f=\mathcal{F}^{-1}[\varphi(\xi)\mathcal{F}[f](\xi)],\quad \Delta_0 f=\mathcal{F}^{-1}[(1-\varphi(\xi))\mathcal{F}[f](\xi)].$$

Denote

$$\begin{aligned}\underline{w}(y,\tau)=\Delta_{-1}w(y,\tau)=\mathcal{F}^{-1}[\varphi]*w(y,\tau),\\ \overline{w}(y,\tau)=w(y,\tau)-\underline{w}(y,\tau)=\Delta_0 w(y,\tau)=\mathcal{F}^{-1}[1-\varphi]*w(y,\tau),\\ \tilde{\overline{w}}(y,\tau)=\mathcal{F}^{-1}[\sqrt{1-\varphi^2}]*w(y,\tau).\end{aligned}\tag{3.2}$$

Notice that

$$\|w(\tau)\|^2_{L^2(\mathbb{R}^3)}=\|\underline{w}(\tau)\|^2_{L^2(\mathbb{R}^3)}+\|\tilde{\overline{w}}(\tau)\|^2_{L^2(\mathbb{R}^3)}.$$

So (2.5) can be written as

$$\begin{aligned}\frac12\frac{d}{d\tau}\int_{\mathbb{R}^3}|\tilde{\overline{w}}(y,\tau)|^2dy\le-\int_{\mathbb{R}^3}|\nabla\tilde{\overline{w}}(y,\tau)|^2-\frac14|\tilde{\overline{w}}(y,\tau)|^2dy\\ -\int_{\mathbb{R}^3}|\nabla\underline{w}(y,\tau)|^2-\frac14|\underline{w}(y,\tau)|^2dy\\ -\frac12\frac{d}{d\tau}\int_{\mathbb{R}^3}|\underline{w}(y,\tau)|^2dy.\end{aligned}\tag{3.3}$$

Applying the operator Δ_{-1} to the first equation of (2.2), we have

$$\partial_\tau\Delta_{-1}w=\Delta\Delta_{-1}w-\frac12\Delta_{-1}(y\cdot\nabla w)-\frac12\Delta_{-1}w-\Delta_{-1}((w\cdot\nabla)w)-\nabla\Delta_{-1}q. \tag{3.4}$$

Multiplying (3.4) by $\Delta_{-1}w$ and integrating over $\mathbb{R}^3$ we get

$$\begin{aligned}\frac12\frac{d}{d\tau}\int_{\mathbb{R}^3}|\Delta_{-1}w|^2dy=-\int_{\mathbb{R}^3}|\nabla\Delta_{-1}w|^2dy-\frac12\int_{\mathbb{R}^3}|\Delta_{-1}w|^2dy\\ -\frac12\int_{\mathbb{R}^3}\Delta_{-1}(y\cdot\nabla w)\cdot\Delta_{-1}wdy-\int_{\mathbb{R}^3}\Delta_{-1}((w\cdot\nabla)w)\cdot\Delta_{-1}wdy,\end{aligned}\tag{3.5}$$

where div $w=0$ is used to cancel the term including q.

Because

$$\begin{aligned}&\int_{\mathbb{R}^3}y\cdot\nabla|\Delta_{-1}w|^2dy=2\int_{\mathbb{R}^3}y_j\Delta_{-1}w\cdot\partial_j\Delta_{-1}wdy\\ &=2\int_{\mathbb{R}^3}\xi_j\varphi\mathcal{F}[w]\cdot\partial_j\overline{(\varphi\mathcal{F}[w])}d\xi\\ &=-3\int_{\mathbb{R}^3}\varphi^2|\mathcal{F}[w]|^2dy,\end{aligned}$$

we have

$$\int_{\mathbb{R}^3} \partial_j\{y_j|\Delta_{-1}w|^2\}dy = 0.$$

So

$$\begin{aligned}
&\int_{\mathbb{R}^3} \partial_j(\Delta_{-1}(y_j w)\cdot \Delta_{-1}w)dy \\
&= \int_{\mathbb{R}^3} \partial_j\{\mathcal{F}^{-1}[\varphi]*(y_j w)\cdot \mathcal{F}^{-1}[\varphi]*w\}dy \\
&= (-1)\int_{\mathbb{R}^3} \partial_j\{\tilde{\varphi}_j * w\cdot \mathcal{F}^{-1}[\varphi]*w\}dy + \int_{\mathbb{R}^3} \partial_j\{y_j|\Delta_{-1}w|^2\}dy \\
&= 0
\end{aligned}$$

where $\tilde{\varphi}_j(y) = y_j\mathcal{F}^{-1}[\varphi](y)$.

Noting that

$$\begin{aligned}
&\int \Delta_{-1}(y\cdot\nabla w)\cdot\Delta_{-1}w dy \\
&= -\sum_{j=1}^{3}\int \Delta_{-1}(y_j w)\cdot \Delta_{-1}\partial_j w dy - 3\int |\Delta_{-1}w|^2 dy \\
&= -\sum_{j=1}^{3}\int \varphi(\xi)\mathcal{F}[y_j w]\cdot \overline{\varphi(\xi)\mathcal{F}[\partial_j w]}d\xi - 3\int |\Delta_{-1}w|^2 dy
\end{aligned}$$

and

$$\mathcal{F}[y_j w] = i\frac{\partial}{\partial \xi_j}\mathcal{F}[w], \quad \mathcal{F}[\partial_j w] = i\xi_j\mathcal{F}[w],$$

we have

$$\begin{aligned}
\int \Delta_{-1}(y\cdot\nabla w)\cdot\Delta_{-1}w dy &= -\sum_{j=1}^{3}\int \varphi^2(\xi)\xi_j\frac{\partial}{\partial\xi_j}\mathcal{F}[w]\cdot\overline{\mathcal{F}[w]}d\xi - 3\int|\Delta_{-1}w|^2 dy \\
&= -\sum_{j=1}^{3}\frac{1}{2}\int \varphi^2(\xi)\xi_j\frac{\partial}{\partial\xi_j}|\mathcal{F}[w]|^2 d\xi - 3\int|\Delta_{-1}w|^2 dy \\
&= \sum_{j=1}^{3}\frac{1}{2}\int \xi_j\frac{\partial}{\partial\xi_j}\varphi^2(\xi)|\mathcal{F}[w]|^2 d\xi + \frac{3}{2}\int\varphi^2(\xi)|\mathcal{F}[w]|^2 d\xi - 3\int|\Delta_{-1}w|^2 dy \\
&= \frac{1}{2}\int \xi\cdot\nabla\varphi^2(\xi)|\mathcal{F}[w]|^2 d\xi - \frac{3}{2}\int|\Delta_{-1}w|^2 dy.
\end{aligned} \tag{3.6}$$

From (3.5)-(3.6), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\int_{\mathbb{R}^3}|\Delta_{-1}w|^2 dy = -\int_{\mathbb{R}^3}|\nabla\Delta_{-1}w|^2 dy + \frac{1}{4}\int_{\mathbb{R}^3}|\Delta_{-1}w|^2 dy \\
&-\frac{1}{4}\int_{\mathbb{R}^3}\xi\cdot\nabla\varphi^2(\xi)|\mathcal{F}[w](\xi,\tau)|^2 d\xi - \int_{\mathbb{R}^3}\Delta_{-1}((w\cdot\nabla)w)\cdot\Delta_{-1}w dy.
\end{aligned} \tag{3.7}$$

From (3.3) and (3.7), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\int_{\mathbb{R}^3}|\tilde{\overline{w}}(y,\tau)|^2dy \le -\int_{\mathbb{R}^3}|\nabla\tilde{\overline{w}}(y,\tau)|^2-\frac{1}{4}|\tilde{\overline{w}}(y,\tau)|^2dy\\
&+\frac{1}{4}\int_{\mathbb{R}^3}\xi\cdot\nabla\varphi^2(\xi)|\mathcal{F}[w](\xi,\tau)|^2d\xi+\int_{\mathbb{R}^3}\Delta_{-1}((w\cdot\nabla)w)\cdot\Delta_{-1}wdy\\
&\le -\frac{3}{4}\int_{\mathbb{R}^3}|\nabla\tilde{\overline{w}}(y,\tau)|^2dy\\
&+\frac{1}{4}\int_{\mathbb{R}^3}\xi\cdot\nabla\varphi^2(\xi)|\mathcal{F}[w](\xi,\tau)|^2d\xi+\int_{\mathbb{R}^3}\Delta_{-1}((w\cdot\nabla)w)\cdot\Delta_{-1}wdy
\end{aligned}
\tag{3.8}
$$

where $|\xi||\mathcal{F}[\tilde{\overline{w}}]|^2\ge|\mathcal{F}[\tilde{\overline{w}}]|^2$ is used in the last step.

Let $\alpha\in(0,\frac{1}{8})$ and define

$$
\chi(\xi)=\begin{cases}|\xi|^{\frac{1}{2}+2\alpha}\varphi(\xi), & \forall|\xi|\le\frac{1}{2}+\alpha\\ (\frac{1}{2}+\alpha)^{\frac{1}{2}+2\alpha}\varphi(\xi), & \forall|\xi|\ge\frac{1}{2}+\alpha.\end{cases}
$$

Instead of φ by χ, we define the operator

$$
\tilde{\Delta}_{-1}f=\mathcal{F}^{-1}[\chi(\xi)\mathcal{F}[f](\xi)].
$$

Applying $\tilde{\Delta}_{-1}$ to (2.2), as (3.7) we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\int_{\mathbb{R}^3}|\tilde{\Delta}_{-1}w|^2dy\\
&=-\int_{\mathbb{R}^3}|\nabla\tilde{\Delta}_{-1}w|^2dy+\frac{1}{4}\int_{\mathbb{R}^3}|\tilde{\Delta}_{-1}w|^2dy\\
&-\frac{1}{4}\int_{\mathbb{R}^3}\xi\cdot\nabla\chi^2(\xi)|\mathcal{F}[w](\xi,\tau)|^2d\xi-\int_{\mathbb{R}^3}\tilde{\Delta}_{-1}((w\cdot\nabla)w)\cdot\tilde{\Delta}_{-1}wdy.
\end{aligned}
$$

Combining it with (3.8), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\int_{\mathbb{R}^3}|\tilde{\Delta}_{-1}w|^2+|\tilde{\overline{w}}|^2dy\\
&\le-\frac{3}{4}\int_{\mathbb{R}^3}|\nabla\tilde{\overline{w}}|^2dy-\alpha\int_{\mathbb{R}^3}|\tilde{\Delta}_{-1}w|^2dy\\
&+\int_{\mathbb{R}^3}\Delta_{-1}((w\cdot\nabla)w)\cdot\Delta_{-1}w-\tilde{\Delta}_{-1}((w\cdot\nabla)w)\cdot\tilde{\Delta}_{-1}wdy\\
&+\frac{1}{4}\int_{\mathbb{R}^3}\xi\cdot\nabla(\varphi^2-\chi^2)|\mathcal{F}[w]|^2d\xi-\int_{\mathbb{R}^3}|\nabla\tilde{\Delta}_{-1}w|^2dy+(\frac{1}{4}+\alpha)\int_{\mathbb{R}^3}|\tilde{\Delta}_{-1}w|^2dy.
\end{aligned}
$$

For $|\xi|\le 1$, the last term is written as

$$
A=\int\{\frac{1}{4}\xi\cdot\nabla(-\chi^2)-|\xi|^2\chi^2+(\frac{1}{4}+\alpha)\chi^2\}|\mathcal{F}[w]|^2d\xi
$$

and noting that $\varphi(\xi) = 1$ for $|\xi| \le 1$ as well as the definition of χ, $A \le 0$. For $|\xi| \in [1, 2]$, the last term is written as

$$B = \int \{\frac{1}{4}(1 - (\frac{1}{2} + \alpha)^{1+4\alpha})\xi \cdot \nabla\varphi^2 - (|\xi|^2 - (\frac{1}{4} + \alpha))(\frac{1}{2} + \alpha)^{1+4\alpha}\varphi^2\}|\mathcal{F}[w]|^2 d\xi,$$

and $B \le 0$. So we get

$$\begin{aligned} \frac{1}{2}\frac{d}{d\tau}\int_{\mathbb{R}^3} |\tilde{\Delta}_{-1}w|^2 + |\tilde{\tilde{w}}|^2 dy \le -\frac{3}{4}\int_{\mathbb{R}^3} |\nabla\tilde{\tilde{w}}|^2 dy - \alpha\int_{\mathbb{R}^3} |\tilde{\Delta}_{-1}w|^2 dy \\ + \int_{\mathbb{R}^3} \Delta_{-1}((w\cdot\nabla)w)\cdot\Delta_{-1}w - \tilde{\Delta}_{-1}((w\cdot\nabla)w)\cdot\tilde{\Delta}_{-1}w dy. \end{aligned} \tag{3.9}$$

Lemma 3.1. *(1) For any $m \in [4, \infty]$,*

$$\|\Delta_{-1}f\|_{L^m(\mathbb{R}^3)} \le C(\alpha)\|\tilde{\Delta}_{-1}f\|_{L^2(\mathbb{R}^3)}, \quad \forall f \in L^2(\mathbb{R}^3)$$

where the constant $C(\alpha) < \infty$ depends only on α.

(2) For all $\beta = (\beta_1, \beta_2, \beta_3)$ ($\beta_j \in \mathbb{N}$, $j = 1, 2, 3$)

$$\|D^\beta\Delta_0 w(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} \le \|D^\beta\tilde{\tilde{w}}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}$$

where $D^\beta = \partial_1^{\beta_1}\partial_2^{\beta_2}\partial_3^{\beta_3}$.

Proof. From Hausdorff-Young inequality

$$\begin{aligned} &\|\Delta_{-1}f\|_{L^m(\mathbb{R}^3)} \\ &\le (2\pi)^{3/m'}(\int_{\mathbb{R}^3} |\varphi(\xi)\mathcal{F}[f](\xi)|^{m'} d\xi)^{1/m'} \\ &\le (2\pi)^{3/m'}(\int_{\mathbb{R}^3} ||\xi|^{\frac{1}{2}+2\alpha}\varphi(\xi)\mathcal{F}[f](\xi)|^2)^{1/2}(\int_{|\xi|\le 2} |\xi|^{-(\frac{1}{2}+2\alpha)\frac{2m'}{2-m'}} d\xi)^{\frac{2-m'}{2m'}} \\ &\le C(\alpha)(\int_{\mathbb{R}^3} |\chi(\xi)\mathcal{F}[f](\xi)|^2)^{1/2} \end{aligned}$$

for $\alpha \in (0, \frac{1}{8})$. So we have (1).

To prove (2), we only need to consider the case $|\beta| = \sum_{1\le j\le 3}\beta_j = 0$. Since $0 \le \varphi \le 1$ and $1 - \varphi^2 = (1-\varphi)(1+\varphi) \ge (1-\varphi)^2$, in this case we have

$$\begin{aligned} &\int_{\mathbb{R}^3} |\Delta_0 w(y,\tau)|^2 dy = \int_{\mathbb{R}^3} (1-\varphi(\xi))^2 |\mathcal{F}[w](\xi,\tau)|^2 d\xi \\ &\le \int_{\mathbb{R}^3} (1-\varphi^2(\xi))|\mathcal{F}[w](\xi,\tau)|^2 d\xi = \int_{\mathbb{R}^3} |\tilde{\tilde{w}}(y,\tau)|^2 dy. \quad \square \end{aligned}$$

Now we estimate the last term in the right of (3.9). We only need to consider the integration for the first function in the last term, because for another function the proof is same. Notice that

$$\int \Delta_{-1}(w_j\partial_j w)\cdot\Delta_{-1}w dy = -\int \Delta_{-1}(w_j w)\cdot\Delta_{-1}(\partial_j w)dy$$
$$= -\int \Delta_{-1}(\Delta_{-1}w_j\Delta_{-1}w)\cdot\Delta_{-1}(\partial_j w)dy - \int \Delta_{-1}(\Delta_{-1}w_j\Delta_0 w)\cdot\Delta_{-1}\partial_j w dy$$
$$-\int \Delta_{-1}(\Delta_0 w_j\Delta_{-1}w)\cdot\Delta_{-1}(\partial_j w)dy - \int \Delta_{-1}(\Delta_0 w_j\Delta_0 w)\cdot\Delta_{-1}(\partial_j w)dy.$$

Because

$$|\int \Delta_{-1}(\Delta_{-1}w_j\Delta_{-1}w)\cdot\Delta_{-1}(\partial_j w)dy|$$
$$\le (\int |\Delta_{-1}w|^4 dy)^{1/2}(\int \varphi^2(\xi)|\xi_j\mathcal{F}[w](\xi,\tau)|^2 dx)^{1/2}$$
$$\le C\|\tilde{\Delta}_{-1}w\|^3_{L^2(\mathbb{R}^3)},\quad \text{(by Lemma 3.1 (1) and the definition of } \tilde{\Delta}_{-1})$$

and

$$|\int \Delta_{-1}(\Delta_{-1}w_j\Delta_0 w)\cdot\Delta_{-1}\partial_j w dy|$$
$$\le (\int |\Delta_{-1}w|^4 dy)^{1/2}(\int |\Delta_0 w|^2 dy)^{1/2}$$
$$\le C\|\tilde{\Delta}_{-1}w\|^2_{L^2(\mathbb{R}^3)}\|\tilde{\overline{w}}\|_{L^2(\mathbb{R}^3)}\quad \text{(by Lemma 3.1 (1)-(2))}$$

as well as

$$|\int \Delta_{-1}(\Delta_0 w_j\Delta_0 w)\cdot\Delta_{-1}(\partial_j w)dy|$$
$$\le \|\Delta_{-1}(\partial_j w)\|_{L^\infty(\mathbb{R}^3)}\|\Delta_0 w\|^2_{L^2(\mathbb{R}^3)}$$
$$\le C\|\tilde{\Delta}_{-1}w\|_{L^2(\mathbb{R}^3)}\|\tilde{\overline{w}}\|^2_{L^2(\mathbb{R}^3)}\quad \text{(by Lemma 3.1 (1)-(2))}$$

the last term in the right of (3.9) can be estimated by

$$C\{\|\tilde{\Delta}_{-1}w\|^3_{L^2(\mathbb{R}^3)} + \|\tilde{\Delta}_{-1}w\|^2_{L^2(\mathbb{R}^3)}\|\tilde{\overline{w}}\|_{L^2(\mathbb{R}^3)} + \|\tilde{\Delta}_{-1}w\|_{L^2(\mathbb{R}^3)}\|\tilde{\overline{w}}\|^2_{L^2(\mathbb{R}^3)}\}.$$

So we get

$$\frac{1}{2}\frac{d}{d\tau}\int_{\mathbb{R}^3}|\tilde{\Delta}_{-1}w(y,\tau)|^2+|\tilde{\overline{w}}(y,\tau)|^2 dy + (\frac{3}{4}-\alpha)\int_{\mathbb{R}^3}|\nabla\tilde{\overline{w}}(y,\tau)|^2 dy$$
$$\le -\alpha\int_{\mathbb{R}^3}|\tilde{\Delta}_{-1}w(y,\tau)|^2+|\tilde{\overline{w}}(y,\tau)|^2 dy \tag{3.10}$$
$$+C(\int_{\mathbb{R}^3}|\tilde{\Delta}_{-1}w(y,\tau)|^2+|\tilde{\overline{w}}(y,\tau)|^2 dy)^{3/2}$$

Proposition 3.2. *There is $\delta > 0$ such that if*

$$\int_{\mathbb{R}^3}|\tilde{\Delta}_{-1}w(y,0)|^2+|\tilde{\overline{w}}(y,0)|^2 dy \le \delta \tag{3.11}$$

then for all $\tau > 0$

$$\frac{d}{d\tau}\int_{\mathbb{R}^3}|\tilde{\Delta}_{-1}w(y,\tau)|^2+|\tilde{\overline{w}}(y,\tau)|^2 dy \le -\alpha\int_{\mathbb{R}^3}|\tilde{\Delta}_{-1}w(y,\tau)|^2+|\tilde{\overline{w}}(y,\tau)|^2 dy. \tag{3.12}$$

Moreover $w(y,\tau)=\underline{w}(y,\tau)+\overline{w}(y,\tau)$*, and for all* $m\in[4,\infty]$*,*

$$\begin{aligned}\|D^\beta\underline{w}(\tau)\|_{L^m(\mathbb{R}^3)}\le C(\beta)\delta,\quad\forall\tau>0,\quad\forall\beta,\\ \lim_{\tau\to\infty}\|\underline{w}(\tau)\|_{L^m(\mathbb{R}^3)}=0,\end{aligned}\tag{3.13}$$

$$\begin{aligned}\sup_{\tau\ge0}\int_{\mathbb{R}^3}|\overline{w}(y,\tau)|^2dy+\int_0^\infty d\tau\int_{\mathbb{R}^3}|\nabla\overline{w}(y,\tau)|^2dy\le C\delta,\\ \lim_{\tau\to\infty}\int_{\mathbb{R}^3}|\overline{w}(y,\tau)|^2dy=0.\end{aligned}\tag{3.14}$$

For example, we may take $\delta\le(\frac{\alpha}{2C})^2$. Proposition 3.2 follows from (3.10) and Lemma 3.1. Note that

$$\begin{aligned}\int_{\mathbb{R}^3}|\tilde{\Delta}_{-1}w(y,0)|^2+|\tilde{\overline{w}}(y,0)|^2dy=\int_{\mathbb{R}^3}(\chi^2(\xi)+1-\varphi^2(\xi))|\mathcal{F}[w](\xi,0)|^2d\xi\\ \le\int_{\mathbb{R}^3}|\mathcal{F}[w](\xi,0)|^2d\xi=\int_{\mathbb{R}^3}|w(y,0)|^2dy=\int_{\mathbb{R}^3}|u_0(x)|^2dx.\end{aligned}\tag{3.15}$$

So we have

Corollary 3.3. *There is* $\delta>0$ *such that if* $\|u_0\|_{L^2(\mathbb{R}^3)}\le\delta^{1/2}$*, then we have (3.12)-(3.14).*

Remark 3.4. Suppose ψ is a function satisfying

$$\psi\in C(\mathbb{R}^3,[0,1]),\quad\xi\cdot\nabla_\xi\psi(\xi)\in L^\infty(\mathbb{R}^3).\tag{3.16}$$

Since $\psi(\xi)\mathcal{F}[w](\xi,\tau)\in L^2(\mathbb{R}^3)$, we have $\mathcal{F}^{-1}[\psi]*w=\mathcal{F}^{-1}[\psi\mathcal{F}[w]]\in L^2(\mathbb{R}^3)$ and

$$\begin{aligned}&\int_{\mathbb{R}^3}|\mathcal{F}^{-1}[\psi]*w(y,\tau)|^2dy(T-t)^{\frac{3}{2}}\\ &=\int_{\mathbb{R}^3}|\mathcal{F}^{-1}[\psi]*w(\frac{\mu}{(T-t)^{1/2}},\tau)|^2d\mu\\ &=\int_{\mathbb{R}^3}|\int_{\mathbb{R}^3}\mathcal{F}^{-1}[\psi](\frac{\mu}{(T-t)^{1/2}}-z)w(z,\tau)dz|^2d\mu\\ &=\int_{\mathbb{R}^3}|\int_{\mathbb{R}^3}\mathcal{F}[\psi](\frac{\mu}{(T-t)^{1/2}}-z)(T-t)^{1/2}u((T-t)^{1/2}z,t)dz|^2d\mu\\ &=\int_{\mathbb{R}^3}|\int_{\mathbb{R}^3}\mathcal{F}^{-1}[\psi](\frac{\mu-x}{(T-t)^{1/2}})u(x,t)dx|^2d\mu(T-t)^{-2}\end{aligned}\tag{3.17}$$

where $\tau = (-)\ln(T-t)$. Note that
(3.18)

$$\begin{aligned}
&\partial_t\{(T-t)^{3/2}\psi((T-t)^{1/2}\xi)\mathcal{F}[u](\xi,t)\} \\
&= \partial_t\mathcal{F}[\int_{\mathbb{R}^3}\mathcal{F}^{-1}[\psi](\frac{\mu-x}{(T-t)^{1/2}})u(x,t)dx] \\
&= \mathcal{F}[\int_{\mathbb{R}^3}\mathcal{F}^{-1}[\psi](\frac{\mu-x}{(T-t)^{1/2}})\partial_t u(x,t) + \{\frac{\mu-x}{2(T-t)^{3/2}}\cdot\mathcal{F}^{-1}[\psi]'(\frac{\mu-x}{(T-t)^{1/2}})\}u(x,t)dx] \\
&= \mathcal{F}[\int_{\mathbb{R}^3}\mathcal{F}^{-1}[\psi](\frac{\mu-x}{(T-t)^{1/2}})\{\partial_\tau w(\frac{x}{(T-t)^{1/2}},\tau) + \frac{x}{2(T-t)^{1/2}}\cdot\nabla_y w(\frac{x}{(T-t)^{1/2}},\tau) \\
&\quad + \frac{1}{2}w(\frac{x}{(T-t)^{1/2}},\tau)\}(T-t)^{-\frac{3}{2}} \\
&\quad + \{\frac{\mu-x}{2(T-t)^{3/2}}\cdot\mathcal{F}^{-1}[\psi]'(\frac{\mu-x}{(T-t)^{1/2}})\}w(\frac{x}{(T-t)^{1/2}},\tau)(T-t)^{-\frac{1}{2}}dx]
\end{aligned}$$

and
(3.19)

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}|\int_{\mathbb{R}^3}\mathcal{F}^{-1}[\psi](\frac{\mu-x}{(T-t)^{1/2}})u(x,t)dx|^2 d\mu \\
&= \frac{d}{dt}\int_{\mathbb{R}^3}|(T-t)^{3/2}\psi((T-t)^{1/2}\xi)\mathcal{F}[u](\xi,t)|^2 d\xi \\
&= \frac{d}{dt}\int_{\mathbb{R}^3}\int_0^t\partial_h|(T-h)^{3/2}\psi((T-h)^{1/2}\xi)\mathcal{F}[u](\xi,h)|^2 dh d\xi \\
&= \frac{d}{dt}\int_0^t\int_{\mathbb{R}^3}\partial_h|(T-h)^{3/2}\psi((T-h)^{1/2}\xi)\mathcal{F}[u](\xi,h)|^2 d\xi dh \\
&= \int_{\mathbb{R}^3}\partial_t|(T-t)^{3/2}\psi((T-t)^{1/2}\xi)\mathcal{F}[u](\xi,t)|^2 d\xi \\
&= 2\int_{\mathbb{R}^3}\{(-)\frac{3}{2}(T-t)^{1/2}\psi((T-t)^{1/2}\xi)\mathcal{F}[u](\xi,t) - (T-t)\frac{\xi}{2}\cdot\psi'((T-t)^{1/2}\xi)\mathcal{F}[u](\xi,t) \\
&\quad + (T-t)^{3/2}\psi((T-t)^{1/2}\xi)\partial_t\mathcal{F}[u](\xi,t)\}\cdot(T-t)^{3/2}\psi((T-t)^{1/2}\xi)\overline{\mathcal{F}[u]}(\xi,t)d\xi
\end{aligned}$$

where noting that $\mathcal{F}[u](\xi,t)$, $\partial_t\mathcal{F}[u](\xi,t) \in L^2(\mathbb{R}^3)$ for $t < T$ and ψ satisfies (3.16), we have

$$\int_{\mathbb{R}^3}|\partial_t|(T-t)^{3/2}\psi((T-t)^{1/2}\xi)\mathcal{F}[u](\xi,t)|^2|d\xi < \infty$$

and Fubini theorem can be used.

From (3.17)-(3.19), we get

(3.20)

$$
\begin{aligned}
&(T-t)^{5/2}\frac{d}{d\tau}\int_{\mathbb{R}^3}|\mathcal{F}^{-1}[\psi]*w(y,\tau)-\frac{7}{2}(T-t)^{5/2}\int_{\mathbb{R}^3}|\mathcal{F}^{-1}[\psi]*w(y,\tau)|^2dy\\
&=\frac{d}{dt}\{(T-t)^{7/2}\int_{\mathbb{R}^3}|\mathcal{F}^{-1}[\psi]*w(y,\tau)|^2dy\}\\
&=\frac{d}{dt}\int_{\mathbb{R}^3}|\int_{\mathbb{R}^3}\mathcal{F}^{-1}[\psi](\frac{\mu-x}{(T-t)^{1/2}})u(x,t)dx|^2d\mu\\
&=2(T-t)^{5/2}\int_{\mathbb{R}^3}\{(-)\frac{3}{2}\psi(\xi)\mathcal{F}[w](\xi,\tau)-\frac{\xi}{2}\cdot\psi'(\xi)\mathcal{F}[w](\xi,\tau)\}\cdot\psi(\xi)\overline{\mathcal{F}[w]}(\xi,\tau)d\xi\\
&+2\int_{\mathbb{R}^3}\{\int_{\mathbb{R}^3}\mathcal{F}^{-1}[\psi](\frac{\mu-x}{(T-t)^{1/2}})\partial_t u(x,t)dx\}\cdot\{\int_{\mathbb{R}^3}\mathcal{F}^{-1}[\psi](\frac{\mu-x}{(T-t)^{1/2}})u(x,t)dx\}d\mu\\
&=(-3)(T-t)^{5/2}\int_{\mathbb{R}^3}|\mathcal{F}^{-1}[\psi]*w(y,\tau)|^2dy\\
&-\frac{(T-t)^{5/2}}{2}\int_{\mathbb{R}^3}(\xi\cdot\nabla_\xi\psi^2(\xi))|\mathcal{F}[w](\xi,\tau)|^2d\xi\\
&+2(T-t)^{5/2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathcal{F}^{-1}[\psi](y-z)\{\partial_\tau w(z,\tau)+\frac{z}{2}\cdot\nabla_z w(z,\tau)+\frac{1}{2}w(z,\tau)\}dz\\
&\cdot\{\mathcal{F}^{-1}[\psi]*w(y,\tau)\}dy
\end{aligned}
$$

So we have

$$
\begin{aligned}
&2\int_{\mathbb{R}^3}\mathcal{F}^{-1}[\psi]*\{\partial_\tau w+\frac{y}{2}\cdot\nabla_y w\}(y,\tau)\cdot\mathcal{F}^{-1}[\psi]*w(y,\tau)dy\\
&=\frac{d}{d\tau}\int_{\mathbb{R}^3}|\mathcal{F}^{-1}[\psi]*w(y,\tau)|^2dy-\frac{3}{2}\int_{\mathbb{R}^3}|\mathcal{F}^{-1}[\psi]*w(y,\tau)|^2dy\\
&+\frac{1}{2}\int_{\mathbb{R}^3}(\xi\cdot\nabla_\xi\psi^2(\xi))|\mathcal{F}[w](\xi,\tau)|^2d\xi.
\end{aligned}
\tag{3.21}
$$

Note that φ and χ satisfy (3.16), and we can use (3.21) to obtain (3.7) for φ and χ again. Furthermore, notice that $1-\varphi$ satisfies (3.16) and $\|\partial_t\nabla_x u(t)\|_{L^2(\mathbb{R}^3)}$ is bounded for $t<T$, we can prove the same equation as (3.21) for $(1-\varphi)$ and $\nabla_y w$ instead of ψ and w, which can be used to obtain (4.4) of section 4 from (4.1) too.

4. L^∞-estimate of $\overline{w}$

Applying the operator Δ_0 (see (3.2)) to (2.6), and integrating over $\mathbb{R}^3$ we have

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}\int_{\mathbb{R}^3}|\Delta_0\nabla w|^2dy = &-\int_{\mathbb{R}^3}|\nabla^2\Delta_0 w|^2dy - \int_{\mathbb{R}^3}|\Delta_0\nabla w|^2dy\\ &-\sum_{j=1}^{3}\frac{1}{2}\int_{\mathbb{R}^3}\Delta_0(y\cdot\nabla\partial_j w)\cdot\Delta_0\partial_j w dy\\ -\sum_{j=1}^{3}\int_{\mathbb{R}^3}\Delta_0((\partial_j w\cdot\nabla)w)\cdot\Delta_0\partial_j w &+ \Delta_0((w\cdot\nabla)\partial_j w)\cdot\Delta_0\partial_j w dy.\end{aligned} \tag{4.1}$$

Since the support set of $1-\varphi$ is not compact, we can not do the same thing as in (3.6) for the 3rd term in the right side of (4.1). But with more patient, by using $\Delta_0 f = f - \Delta_{-1}f$, we have

$$\begin{aligned}&\int\Delta_0((y\cdot\nabla)\partial_j w)\cdot\Delta_0\partial_j w dy\\ &= \int((y\cdot\nabla)\partial_j w)\cdot\partial_j w dy - \int((y\cdot\nabla)\partial_j w)\cdot\Delta_{-1}\partial_j w dy\\ &-\int\Delta_{-1}((y\cdot\nabla)\partial_j w)\cdot\partial_j w dy + \int\Delta_{-1}((y\cdot\nabla)\partial_j w)\cdot\Delta_{-1}(\partial_j w)dy.\end{aligned} \tag{4.2}$$

As in (2.4), we have

$$\int((y\cdot\nabla)\partial_j w)\cdot\partial_j w dy \geq -\frac{3}{2}\int|\partial_j w|^2dy.$$

On the other hand, as in (3.6) we have

$$\int\Delta_{-1}((y\cdot\nabla)\partial_j w)\cdot\Delta_{-1}(\partial_j w)dy = \frac{1}{2}\int\xi\cdot\nabla\varphi^2|\mathcal{F}[\partial_j w]|^2d\xi - \frac{3}{2}\int\varphi^2|\mathcal{F}[\partial_j w]|^2d\xi.$$

The remainder in the right of (4.2) is

$$\begin{aligned}&2\int\varphi\partial_k(\xi_k\mathcal{F}[\partial_j w])\cdot\overline{\mathcal{F}[\partial_j w]}d\xi\\ &= -\int 2(\xi\cdot\nabla\varphi)|\mathcal{F}[\partial_j w]|^2 + \varphi\xi\cdot\nabla|\mathcal{F}[\partial_j w]|^2d\xi\\ &= -\int(\xi\cdot\nabla\varphi)|\mathcal{F}[\partial_j w]|^2d\xi + 3\int\varphi|\mathcal{F}[\partial_j w]|^2d\xi.\end{aligned}$$

Then the right of (4.2) is larger than

$$\begin{aligned}-\frac{3}{2}\int|\partial_j w|^2dy - \frac{3}{2}\int\varphi^2|\mathcal{F}[\partial_j w]|^2d\xi + \frac{1}{2}\int\xi\cdot\nabla\varphi^2|\mathcal{F}[\partial_j w]|^2d\xi&\\ -\int(\xi\cdot\nabla\varphi)|\mathcal{F}[\partial_j w]|^2d\xi + 3\int\varphi|\mathcal{F}[\partial_j w]|^2d\xi&\\ = \frac{1}{2}\int\xi\cdot\nabla(1-\varphi(\xi))^2|\mathcal{F}[\partial_j w]|^2d\xi - \frac{3}{2}\int|\Delta_0\partial_j w|^2dy.&\end{aligned} \tag{4.3}$$

Since from (3.1)

$$\xi \cdot \nabla (1-\varphi(\xi))^2 = |\xi| \frac{d}{d|\xi|}(1-\varphi(\xi))^2 \geq 0$$

Instead of the 3rd term in the right side of (4.1) by (4.2)-(4.3), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}\int_{\mathbb{R}^3} |\Delta_0 \nabla w|^2 dy \leq -\int_{\mathbb{R}^3} |\nabla^2 \Delta_0 w|^2 dy - \frac{1}{4}\int_{\mathbb{R}^3} |\Delta_0 \nabla w|^2 dy \\ -\sum_{j=1}^{3}\int_{\mathbb{R}^3} \Delta_0((\partial_j w\cdot\nabla)w)\cdot\Delta_0\partial_j w + \Delta_0((w\cdot\nabla)\partial_j w)\cdot \Delta_0 \partial_j w dy.\end{aligned} \tag{4.4}$$

Decompose the last integration of the right side of (4.4) by $w = \underline{w} + \overline{w}$ and note that

$$\begin{aligned}|\int ((\overline{w}\cdot\nabla)\partial_j\overline{w})\cdot\partial_j\overline{w}dy| &\leq \|\nabla^2\overline{w}\|_{L^2(\mathbb{R}^3)}(\int |\overline{w}|^2|\nabla\overline{w}|^2dy)^{1/2} \\ &\leq C\|\nabla^2\overline{w}\|^{3/2}_{L^2(\mathbb{R}^3)}\|\nabla\overline{w}\|^{3/2}_{L^2(\mathbb{R}^3)},\end{aligned}$$

$$|\int ((\underline{w}\cdot\nabla)\partial_j\overline{w})\cdot\partial_j\overline{w}dy| \leq \|\underline{w}\|_{L^\infty(\mathbb{R}^3)}\|\nabla^2\overline{w}\|_{L^2(\mathbb{R}^3)}\|\nabla\overline{w}\|_{L^2(\mathbb{R}^3)},$$

$$|\int ((\overline{w}\cdot\nabla)\partial_j\underline{w})\cdot\partial_j\overline{w}dy| \leq C\|\underline{w}\|_{L^\infty(\mathbb{R}^3)}\|\overline{w}\|_{L^2(\mathbb{R}^3)}\|\nabla\overline{w}\|_{L^2(\mathbb{R}^3)},$$

and

$$|\int ((\underline{w}\cdot\nabla)\partial_j\underline{w})\cdot\partial_j\overline{w}dy| \leq C(\int |\underline{w}|^4dy)^{1/2}\|\nabla\overline{w}\|_{L^2(\mathbb{R}^3)}$$

as well as the same estimates for another one. Then by Proposition 3.2, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}\int_{\mathbb{R}^3}|\nabla\overline{w}(y,\tau)|^2dy \leq -\frac{1}{2}\int_{\mathbb{R}^3}|\nabla^2\overline{w}(y,\tau)|^2dy - \frac{1}{8}\int_{\mathbb{R}^3}|\nabla\overline{w}(y,\tau)|^2dy \\ +C\|\nabla\overline{w}(\tau)\|_{L^2(\mathbb{R}^3)}\{C\delta - \|\nabla\overline{w}(\tau)\|_{L^2(\mathbb{R}^3)} + C\|\nabla\overline{w}(\tau)\|^5_{L^2(\mathbb{R}^3)}\}\end{aligned} \tag{4.5}$$

Note that (see Remark 4.4) there is $\delta_1 > 0$ such that if for some $\tau_0 \geq 0$

$$\|\nabla\overline{w}(\tau_0)\|_{L^2(\mathbb{R}^3)} \leq \delta_1 \tag{4.6}$$

then

$$\|\nabla\overline{w}(\tau)\|_{L^2(\mathbb{R}^3)} \leq \delta_1, \quad \forall\tau\geq\tau_0.$$

From (3.14), (4.6) can be satisfied provided that (3.11) is satisfied. So we have

Lemma 4.1. *Suppose (3.11) is satisfied. Then there is $\delta_1 > 0$ ($\delta_1 \downarrow 0$ as $\delta \downarrow 0$) and $\tau_0 > 0$ such that*

$$\|\nabla\overline{w}(\tau)\|_{L^2(\mathbb{R}^3)} \leq \delta_1, \quad \forall\tau\geq\tau_0.$$

Estimate the last term in the right side of (2.7) by using $w = \underline{w} + \overline{w}$, and note that

$$\begin{aligned}|\int \partial_j\underline{w}_k\partial_j\underline{w}_l\partial_l\underline{w}_k dy| &\leq \|\nabla\underline{w}\|_{L^\infty(\mathbb{R}^3)}\int|\nabla\underline{w}|^2dy \\ &\leq C\delta\int|\nabla\underline{w}|^2dy,\end{aligned}$$

$$\begin{aligned}|\int \partial_j \underline{w}_k \partial_j \underline{w}_l \partial_l \overline{w}_k dy| &\leq (\int |\nabla \underline{w}|^4 dy)^{1/2} (\int |\nabla \overline{w}|^2 dy)^{1/2} \\ &\leq C\delta (\int |\nabla \overline{w}|^2 dy)^{1/2},\end{aligned}$$

and

$$\begin{aligned}|\int \partial_j \underline{w}_k \partial_j \overline{w}_l \partial_l \overline{w}_k dy| &\leq \|\nabla \underline{w}\|_{L^\infty(\mathbb{R}^3)} \int |\nabla \overline{w}|^2 dy \\ &\leq C\delta \int |\nabla \overline{w}|^2 dy,\end{aligned}$$

as well as

$$\begin{aligned}|\int \partial_j \overline{w}_k \partial_j \overline{w}_l \partial_l \overline{w}_k dy| &\leq C\|\nabla \overline{w}\|^{3/2}_{L^2(\mathbb{R}^3)} \|\nabla^2 \overline{w}\|^{3/2}_{L^2(\mathbb{R}^3)} \\ &\leq \|\nabla^2 \overline{w}\|^2_{L^2(\mathbb{R}^3)} + C\delta_1.\end{aligned}$$

So we have

$$\frac{d}{d\tau} \int_{\mathbb{R}^3} |\nabla w(y,\tau)|^2 dy \leq -\int_{\mathbb{R}^3} |\nabla^2 w(y,\tau)|^2 dy - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla w(y,\tau)|^2 dy + C\delta_1. \tag{4.7}$$

Lemma 4.2. *Suppose (3.11) is satisfied. Then there is $\delta_1 > 0$ ($\delta_1 \downarrow 0$ as $\delta \downarrow 0$) and $\tau_0 > 0$ such that for all $\tau \geq \tau_0$,*

$$\int_{\mathbb{R}^3} |\nabla w(y,\tau)|^2 dy \leq e^{-\frac{1}{2}(\tau-\tau_0)} \int_{\mathbb{R}^3} |\nabla w(y,\tau_0)|^2 dy + 2C\delta_1(1 - e^{-\frac{1}{2}(\tau-\tau_0)}). \tag{4.8}$$

Considering (2.8), and noting that div $\Delta w = 0$ implies

$$\int (\Delta w) \cdot ((w \cdot \nabla)\Delta w) dy = 0,$$

the last term in the right side of (2.8) can be written as the sum of the following terms

$$\int |\nabla^2 w|^2 |\nabla w| dy.$$

Since it can be estimated by

$$\begin{aligned}&(\int |\nabla w|^2 dy)^{1/2} (\int |\nabla^2 w|^4 dy)^{1/2} \\ &\leq C(\int |\nabla w|^2 dy)^{1/2} (\int |\Delta w|^2 dy)^{1/4} (\int |\nabla \Delta w|^2 dy)^{3/4}\end{aligned}$$

by (2.8) and Lemma 4.2 we have

Lemma 4.3. *Suppose (3.11) is satisfied. Then there is $\delta_1 > 0$ ($\delta_1 \downarrow 0$ as $\delta \downarrow 0$) and $\tau_0 > 0$ such that for all $\tau \geq \tau_0$,*

$$\int_{\mathbb{R}^3} |\Delta w(y,\tau)|^2 dy \leq e^{-(\frac{3}{2} - C\delta_1)(\tau-\tau_0)} \int_{\mathbb{R}^3} |\Delta w(y,\tau_0)|^2 dy$$

From Lemma 4.1-4.3 and Corollary 3.3, we proved the Theorem 1.1.

Remark 4.4. Suppose a nonnegative continuous function $h(\tau)$ satisfies

$$\frac{d}{d\tau}h(\tau) \leq F(h(\tau)) := C\delta - Bh(\tau) + h^5(\tau), \quad \forall \tau > 0,$$

where C, B and δ are positive constants. If δ is small enough so that

$$h_- := \frac{1}{2}(B - \sqrt{B^2 - 4C\delta}) \in (0, 1),$$

and if $h(0) < h_-$, then for all $\tau > 0$, $F(h(\tau)) \leq C\delta - Bh(\tau) + h^2(\tau)$ and

$$h(\tau) \in [0, h_-].$$

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