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Some Gronwall Type Inequalities and Applications

NOVA

Preface

As R. Bellman pointed out in 1953 in his book "*Stability Theory of Differential Equations*", McGraw Hill, New York, the Gronwall type integral inequalities of one variable for real functions play a very important role in the Qualitative Theory of Differential Equations.

The main aim of the present research monograph is to present some natural applications of Gronwall inequalities with nonlinear kernels of Lipschitz type to the problems of boundedness and convergence to zero at infinity of the solutions of certain Volterra integral equations. Stability, uniform stability, uniform asymptotic stability and global asymptotic stability properties for the trivial solution of certain differential system of equations are also investigated.

The work begins by presenting a number of classical facts in the domain of Gronwall type inequalities. We collected in a reorganized manner most of the above inequalities from the book "*Inequalities for Functions and Their Integrals and Derivatives*", Kluwer Academic Publishers, 1994, by D.S. Mitrinovic, J.E. Pečarić and A.M. Fink.

Chapter 2 contains some generalization of the Gronwall inequality for Lipschitzian type kernels and a systematic study of boundedness and convergence to zero properties for the solutions of those nonlinear inequations. These results are then employed in Chapter 3 to study the boundedness and convergence to zero properties of certain vector valued Volterra Integral Equations. Chapter 4 is entirely devoted to the study of stability, uniform stability, uniform asymptotic stability and global asymptotic stability properties for the trivial solution of certain differential system of equations.

The monograph ends with a large number of references about Gronwall inequalities that can be used by the interested reader to apply in a similar fashion to the one outlined in this work.

The book is intended for use in the fields of Integral and Differential Inequalities and the Qualitative Theory of Volterra Integral and Differential Equations.

The Author.
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Contents

Chapter 1

Integral Inequalities of Gronwall Type

1.1 Some Classical Facts

In the qualitative theory of differential and Volterra integral equations, the Gronwall type inequalities of one variable for the real functions play a very important role.

The first use of the Gronwall inequality to establish boundedness and stability is due to R. Bellman. For the ideas and the methods of R. Bellman, see [16] where further references are given.

In 1919, T.H. Gronwall [50] proved a remarkable inequality which has attracted and continues to attract considerable attention in the literature.

Theorem 1 ***(Gronwall).*** *Let x, Ψ and χ be real continuous functions defined in $[a,b]$, $\chi(t) \geq 0$ for $t \in [a,b]$. We suppose that on $[a,b]$ we have the inequality*

$$x(t) \leq \Psi(t) + \int_a^t \chi(s)\, x(s)\, ds. \tag{1.1}$$

Then

$$x(t) \leq \Psi(t) + \int_a^t \chi(s)\, \Psi(s) \exp\left[\int_s^t \chi(u)\, du\right] ds \tag{1.2}$$

in $[a,b]$ ([10, p. 25], [55, p. 9]).

Proof. Let us consider the function $y(t) := \int_a^t \chi(u)\, x(u)\, du,\ t \in [a,b]$. Then we have $y(a) = 0$ and

$$\begin{aligned} y'(t) &= \chi(t)\, x(t) \leq \chi(t)\, \Psi(t) + \chi(t) \int_a^b \chi(s)\, x(s)\, ds \\ &= \chi(t)\, \Psi(t) + \chi(t)\, y(t), \quad t \in (a,b). \end{aligned}$$

By multiplication with $\exp\left(-\int_a^t \chi(s)\, ds\right) > 0$, we obtain

$$\frac{d}{dt}\left(y(t) \exp\left(-\int_a^t \chi(s)\, ds\right)\right) \leq \Psi(t)\, \chi(t) \exp\left(-\int_a^t \chi(s)\, ds\right).$$

By integration on $[a,t]$, one gets

$$y(t) \exp\left(-\int_a^t \chi(s)\, ds\right) \leq \int_a^t \Psi(u)\, \chi(u) \exp\left(-\int_a^u \chi(s)\, ds\right) du$$

from where results

$$y(t) \leq \int_a^t \Psi(u)\, \chi(u) \exp\left(\int_u^t \chi(s)\, ds\right) du,\ t \in [a,b].$$

Since $x(t) \leq \Psi(t) + y(t)$, the theorem is thus proved. ■

Next, we shall present some important corollaries resulting from Theorem 1.

Corollary 2 *If* Ψ *is differentiable, then from* (1.1) *it follows that*

$$x(t) \leq \Psi(a)\left(\int_a^t \chi(u)\, du\right) + \int_a^t \exp\left(\int_s^t \chi(u)\, du\right) \Psi'(s)\, ds \qquad (1.3)$$

for all $t \in [a,b]$.

Proof. It is easy to see that

$$\begin{aligned} &-\int_a^t \Psi(s) \frac{d}{dt}\left(\exp\left(\int_s^t \chi(u)\, du\right)\right) ds \\ &\qquad = -\Psi(s) \exp\left(\int_s^t \chi(u)\, du\right)\Bigg|_a^b + \int_a^t \exp\left(\int_s^t \chi(u)\, du\right) \Psi'(s)\, ds \\ &\qquad = -\Psi(t) + \Psi(a) \exp\left(\int_a^t \chi(u)\, du\right) + \int_a^t \exp\left(\int_s^t \chi(u)\, du\right) \Psi'(s)\, ds \end{aligned}$$

for all $t \in [a,b]$.

Hence,

$$\Psi(t) + \int_a^t \Psi(u)\chi(u)\exp\left(\int_u^t \chi(s)\,ds\right)du$$
$$= \Psi(a)\exp\left(\int_a^t \chi(u)\,du\right) + \int_a^t \exp\left(\int_s^t \chi(u)\,du\right)\Psi'(s)\,ds, \quad t \in [a,b]$$

and the corollary is proved. ■

Corollary 3 *If* Ψ *is constant, then from*

$$x(t) \le \Psi + \int_a^t \chi(s)\,x(s)\,ds \tag{1.4}$$

it follows that

$$x(t) \le \Psi \exp\left(\int_a^t \chi(u)\,du\right). \tag{1.5}$$

Another well-known generalisation of Gronwall's inequality is the following result due to I. Bihari ([18], [10, p. 26]).

Theorem 4 *Let* $x : [a,b] \to \mathbb{R}_+$ *be a continuous function that satisfies the inequality:*

$$x(t) \le M + \int_a^t \Psi(s)\,\omega(x(s))\,ds, \quad t \in [a,b], \tag{1.6}$$

where $M \ge 0$, $\Psi : [a,b] \to \mathbb{R}_+$ *is continuous and* $w : \mathbb{R}_+ \to \mathbb{R}_+^*$ *is continuous and monotone-increasing.*
Then the estimation

$$x(t) \le \Phi^{-1}\left(\Phi(M) + \int_a^t \Psi(s)\,ds\right), \quad t \in [a,b] \tag{1.7}$$

holds, where $\Phi : \mathbb{R} \to \mathbb{R}$ *is given by*

$$\Phi(u) := \int_{u_0}^u \frac{ds}{\omega(s)}, \quad u \in \mathbb{R}. \tag{1.8}$$

Proof. Putting

$$y(t) := \int_a^t \omega(x(s))\,\Psi(s)\,ds, \quad t \in [a,b],$$

we have $y(a) = 0$, and by the relation (1.6), we obtain

$$y'(t) \le \omega(M + y(t))\,\Psi(t), \quad t \in [a,b].$$

By integration on $[a,t]$, we have

$$\int_0^{y(t)} \frac{ds}{\omega(M+s)} \le \int_a^t \Psi(s)\,ds + \Phi(M), \quad t \in [a,b]$$

that is,

$$\Phi(y(t) + M) \le \int_a^t \Psi(s)\,ds + \Phi(M), \quad t \in [a,b],$$

from where results the estimation (1.7). ■

Finally, we shall present another classical result which is important in the calitative theory of differential equations for monotone operators in Hilbert spaces ([10, p. 27], [19, Appendice]).

Theorem 5 *Let $x : [a,b] \to \mathbb{R}$ be a continuous function which satisfies the following relation:*

$$\frac{1}{2}x^2(t) \le \frac{1}{2}x_0^2 + \int_a^t \Psi(s)\,x(s)\,ds, \quad t \in [a,b], \tag{1.9}$$

where $x_0 \in \mathbb{R}$ and Ψ are nonnegative continuous in $[a,b]$. Then the estimation

$$|x(t)| \le |x_0| + \int_a^t \Psi(s)\,ds, \quad t \in [a,b] \tag{1.10}$$

holds.

Proof. Let y_ε be the function given by

$$y_\varepsilon(t) := \frac{1}{2}\left(x_0^2 + \varepsilon^2\right) + \int_a^t \Psi(s)\,x(s)\,ds, \quad t \in [a,b],$$

where $\varepsilon > 0$.

By the relation (1.9), we have

$$x^2(t) \le y_\varepsilon(t), \quad t \in [a,b]. \tag{1.11}$$

Since $y'_\varepsilon(t) = \Psi(t)|x(t)|$, $t \in [a,b]$, we obtain

$$y'_\varepsilon(t) \le \sqrt{2y_\varepsilon(a)} + \int_a^t \Psi(s)\,ds, \quad t \in [a,b].$$

By integration on the interval $[a,t]$, we can deduce that

$$\sqrt{2y_\varepsilon(t)} \le \sqrt{2y_\varepsilon(a)} + \int_a^t \Psi(s)\,ds, \quad t \in [a,b].$$

By relation (1.11), we obtain

$$|x(t)| \le |x_0| + \varepsilon + \int_a^t \Psi(s)\,ds, \quad t \in [a,b]$$

for every $\varepsilon > 0$, which implies (1.10) and the lemma is thus proved. ■

1.2 Other Inequalities of Gronwall Type

We will now present some other inequalities of Gronwall type that are known in the literature, by following the recent book of Mitrinović, Pečarić and Fink [85].

In this section, we give various generalisations of Gronwall's inequality involving an unknown function of a single variable.

A. Filatov [46] proved the following linear generalisation of Gronwall's inequality.

Theorem 6 *Let $x(t)$ be a continuous nonnegative function such that*

$$x(t) \le a + \int_{t_0}^t [b + cx(s)]\,ds, \quad \textit{for} \;\; t \ge t_0,$$

where $a \ge 0$, $b \ge 0$, $c > 0$. Then for $t \ge t_0$, $x(t)$ satisfies

$$x(t) \le \left(\frac{b}{c}\right)(\exp(c(t-t_0)) - 1) + a\exp c(t-t_0).$$

K.V. Zadiraka [134] (see also Filatov and Šarova [47, p. 15]) proved the following:

Theorem 7 *Let the continuous function $x(t)$ satisfy*

$$|x(t)| \leq |x(t_0)| \exp(-\alpha(t-t_0)) + \int_{t_0}^{t} (a|x(s)|+b) e^{-\alpha(t-s)} ds,$$

where a, b and α are positive constants. Then

$$|x(t)| \leq |x(t_0)| \exp(-\alpha(t-t_0)) + b(\alpha-a)^{-1}(1-\exp(-(\alpha-a)(t-t_0))).$$

In the book [16], R. Bellman cites the following result (see also Filatov and Šarova [47, pp. 10-11]):

Theorem 8 *Let $x(t)$ be a continuous function that satisfies*

$$x(t) \leq x(\tau) + \int_{\tau}^{t} a(s) x(s) ds,$$

for all t and τ in (a,b) where $a(t) \geq 0$ and continuous. Then

$$x(t_0) \exp\left(-\int_{t_0}^{t} a(s) ds\right) \leq x(t) \leq x(t_0) \exp\left(\int_{t_0}^{t} a(s) ds\right)$$

for all $t \geq t_0$.

The following two theorems were given in the book of Filatov and Šarova [47, pp. 8-9 and 18-20] and are due to G.I. Čandirov [25]:

Theorem 9 *Let $x(t)$ be continuous and nonnegative on $[0,h]$ and satisfy*

$$x(t) \leq a(t) + \int_{0}^{t} [a_1(s) x(s) + b(s)] ds,$$

where $a_1(t)$ and $b(t)$ are nonnegative integrable functions on the same interval with $a(t)$ bounded there. Then, on $[0,h]$

$$x(t) \leq \int_{0}^{t} b(s) ds + \sup_{0 \leq t \leq h} |a(t)| \exp\left(\int_{0}^{t} a_1(s) ds\right).$$

The second result is embodied in the following.

Theorem 10 *Let $x(t)$ be a nonnegative continuous function on $[0,\infty)$ such that*

$$x(t) \le ct^{\alpha} + mt^{\beta}\int_0^t \frac{x(s)}{s}ds,$$

where $c > 0$, $\alpha \ge 0$, $\beta \ge 0$. Then

$$x(t) \le ct^{\alpha}\left(1 + \sum_{n=1}^{\infty}\frac{m^n t^{n\beta}}{\alpha(\alpha+\beta)+\cdots+(\alpha+(n-1)\beta)}\right).$$

A more general result was given in Willett [125] and Harlamov [56]. Here we shall give an extended version due to Beesack [14, pp. 3-4].

Theorem 11 *Let x and k be continuous and a and b Riemann integrable functions on $J = [\alpha, \beta]$ with b and k nonnegative on J.*

(i) If

$$x(t) \le a(t) + b(t)\int_{\alpha}^{t} k(s)x(s)\,ds, \quad t \in J, \tag{1.12}$$

then

$$x(t) \le a(t) + b(t)\int_{\alpha}^{t} a(s)k(s)\exp\left(\int_s^t b(r)k(r)\,dr\right)ds, \quad t \in J. \tag{1.13}$$

Moreover, equality holds in (1.13) for a subinterval $J_1 = [\alpha, \beta_1]$ of J if equality holds in (1.12) for $t \in J_1$.

(ii) The result remains valid if $\le$ is replaced by $\ge$ in both (1.12) and (1.13).

(iii) Both (i) and (ii) remain valid if $\int_{\alpha}^{t}$ is replaced by $\int_t^{\beta}$ and $\int_s^t$ by $\int_t^s$ throughout.

Proof. (see [85, p. 357]) This proof is typical of those for inequalities of the Gronwall type. We set

$$U(t) = \int_{\alpha}^{t} k(s)x(s)\,ds \text{ so that } U(\alpha) = 0,$$

and

$$U'(t) = k(t) x(t).$$

Hence $U'(s) \leq a(s) k(s) + b(s) k(s) U(s)$. Multiplying by the integrating factor $\exp\left(\int_s^t b(r) k(r) dr\right)$ and integrating from α to t gives

$$U(t) \leq \int_\alpha^t a(s) k(s) \exp\left(\int_s^t b(r) k(r) dr\right) ds, \quad t \in J. \tag{1.14}$$

Since $b \geq 0$, substitution of (1.14) into (1.12) gives (1.13). The equality conditions are obvious and the proof of (ii) is similar or can be done by the change of variables $t \to -t$. ■

Remark 12 *B. Pachpatte [93] proved an analogous result on* $\mathbb{R}^+$ *and* $(-\infty, 0]$.

Remark 13 *Willett's paper [125] also contains a linear generalisation in which* $b(t) k(s)$ *is replaced by* $\sum_{i=1}^n b_i(t) k_i(s)$. *Such a result is also given by Š.T. Gamidov [48] (see also Filatov and Šarova [47, pp. 19-20]). Moreover, this result can be derived from the case of* $n = 1$ *by observing that* $\sum b_i k_i < \sup_{1\leq i\leq n} \{b_i\} \sum k_i$ *(note that the functions are positive).*

Š.T. Gamidov [48] also gave the following theorem:

Theorem 14 *If*

$$x(t) \leq a(t) + a_1(t) \int_{t_1}^t b_1(s) x(s) ds + a_2(t) \sum_{i=2}^n c_i \int_{t_1}^{t_i} b_i(s) x(s) ds,$$

for $t \in [a, b]$, *where* $a = t_0 < \cdots < t_n = b$, c_i *are constants and the functions appearing are all real, continuous and nonnegative, and if*

$$\sum_{i=2}^n c_i \int_{t_1}^{t_i} b_i(t) \left[a_2(t) + a_1(t) \int_{t_1}^t b_1(s) a_2(s) \left(\int_{t_1}^t a_1(r) b_1(r) dr\right) ds\right] dt < 1,$$

then

$$x(t) \leq K_1(t) + M K_2(t),$$

where

$$K_1(t) = a(t) + a_1(t)\int_{t_1}^{t} b_i(s)\,a(s)\exp\left(\int_s^t a_1(r)\,b_1(r)\,dr\right)ds,$$
$$K_2(t) = a_2(t) + a_1(t)\int_{t_1}^{t} b_1(s)\,a_2(s)\exp\left(\int_s^t a_1(r)\,b_1(r)\,dr\right)ds,$$

and

$$M = \left(\sum_{i=2}^{n} c_i \int_{t_1}^{t_i} b_i(s)\,K_1(s)\,ds\right)\left(1-\sum_{i=2}^{n} c_i\int_{t_1}^{t_i} b_i(s)\,K_2(s)\,ds\right)^{-1}.$$

H. Movljankulov and A. Filatov [87] proved the following result:

Theorem 15 *Let $x(t)$ be real, continuous, and nonnegative such that for $t > t_0$*

$$x(t) \le c + \int_{t_0}^{t} k(t,s)\,x(s)\,ds, \quad c > 0,$$

where $k(t,s)$ is a continuously differentiable function in t and continuous in s with $k(t,s) \ge 0$ for $t \ge s \ge t_0$. Then

$$x(t) \le c\exp\left(\int_{t_0}^{t}\left(k(s,s) + \int_{t_0}^{s}\frac{\partial k}{\partial s}(s,r)\,dr\right)ds\right).$$

In the same paper the following result appears:

Theorem 16 *Let $x(t)$ be real, continuous, and nonnegative on $[c,d]$ such that*

$$x(t) \le a(t) + b(t)\int_e^t k(t,s)\,x(s)\,ds,$$

where $a(t) \ge 0$, $b(t) \ge 0$, $k(t,s) \ge 0$ and are continuous functions for $c \le s \le t \le d$. Then

$$x(t) \le A(t)\exp\left(B(t)\int_c^t K(t,s)\,ds\right),$$

where $A(t) = \sup_{c\le s\le t} a(s)$, $B(t) = \sup_{c\le s\le t} b(s)$, $K(t,s) = \sup_{s\le\sigma\le t} k(\sigma,s)$.

Remark 17 *The inequality*

$$x(t) \le a(t) + \int_{t_0}^{t} (a_1(t) a_2(s) x(s) + b(t,s))\, ds$$

was considered by S.M. Sardar'ly [110]. Using the substitution $a(t)+\int_{t_0}^{t} b(t,s)$ *for* $a(t)$*, we see that this is the same as (1.12) so that this result follows from Theorem 11.*

S. Chu and F. Metcalf [28] proved the following linear generalisation of Gronwall's inequality:

Theorem 18 *Let* x *and* a *be real continuous functions on* $J = [\alpha, \beta]$ *and let* k *be a continuous nonnegative function on* $T : \alpha \le s \le t \le \beta$*. If*

$$x(t) \le a(t) + \int_{\alpha}^{t} k(t,s)\, x(s)\, ds, \quad t \in J, \tag{1.15}$$

then

$$x(t) \le a(t) + \int_{\alpha}^{t} R(t,s)\, a(s)\, ds, \quad t \in J, \tag{1.16}$$

where $R(t,s) = \sum_{i=1}^{\infty} K_i(t,s)$*, with* $(t,s) \in T$*, is the resolvent kernel of* $k(t,s)$ *and* $K_i(t,s)$ *are iterated kernels of* $k(t,s)$*.*

Remark 19 *P. Beesack [13] extended this result for the case when* $x, a \in L^2(J)$ *and* $k \in L^2(T)$*, and he noted that the result remains valid if* $\ge$ *is substituted for* $\le$ *in both (1.15) and (1.16).*

Remark 20 *If we put* $k(t,s) = b(t)\, k(s)$ *and* $k(t,s) = \sum_{i=1}^{n} b_i(t)\, k_i(s)$ *we get the results of D. Willett [125].*

G. Jones [63] extended Willet's result in the case of Riemann-Stieltjes integrals. For some analogous results, see Wright, Klasi, and Kennebeck [128], Schmaedeke and Sell [113], Herod [59], and B. Helton [57].

1.3 Nonlinear Generalisation

We can consider various nonlinear generalisations of Gronwall's inequality. The following theorem is proved in Perov [105]:

Theorem 21 *Let $u(t)$ be a nonnegative function that satisfies the integral inequality*

$$u(t) \leq c + \int_{t_0}^{t} (a(s)u(s) + b(s)u^{\alpha}(s))\,ds, \quad c \geq 0, \ \alpha \geq 0, \tag{1.17}$$

where $a(t)$ and $b(t)$ are continuous nonnegative functions for $t \geq t_0$. For $0 \leq \alpha < 1$ we have

$$u(t) \leq \left\{ c^{1-\alpha} \exp\left[(1-\alpha)\int_{t_0}^{t} a(s)\,ds\right] + (1-\alpha)\int_{t_0}^{t} b(s)\exp\left[(1-\alpha)\int_{s}^{t} a(r)\,dr\right] ds \right\}^{\frac{1}{1-\alpha}}; \tag{1.18}$$

for $\alpha = 1$,

$$u(t) \leq c\exp\left\{\int_{t_0}^{t} [a(s) + b(s)]\,ds\right\}; \tag{1.19}$$

and for $\alpha > 1$ with the additional hypothesis

$$c < \left\{\exp\left[(1-\alpha)\int_{t_0}^{t_0+h} a(s)\,ds\right]\right\}^{\frac{1}{\alpha-1}} \left\{(\alpha-1)\int_{t_0}^{t_0+h} b(s)\,ds\right\}^{-\frac{1}{\alpha-1}} \tag{1.20}$$

we also get for $t_0 \leq t \leq t_0 + h$, for $h > 0$,

$$u(t) \leq c\left\{\exp\left[(1-\alpha)\int_{t_0}^{t} a(s)\,ds\right] - c^{-1}(\alpha-1)\int_{t_0}^{t} b(s)\exp\left[(1-\alpha)\int_{s}^{t} a(r)\,dr\right] ds\right\}^{\frac{1}{\alpha-1}}. \tag{1.21}$$

Proof. (see [85, p. 361]) For $\alpha = 1$ we get the usual linear inequality so that (1.18) is valid. Assume now that $0 < \alpha < 1$. Denote by v a solution of the integral equation

$$v(t) = c + \int_{t_0}^{t} [a(s)v(s) + b(s)v^{\alpha}(s)]\,ds, \quad t \geq t_0.$$

In differential form this is the Bernoulli equation

$$v'(t) = a(t)\, v(t) + b(t)\, v^{\alpha}(t), \quad v(0) = c.$$

This is linear in the variable $v^{1-\alpha}$ so can readily be integrated to produce $v(t) =$ the right hand side of (1.18).

For $\alpha > 1$ we again get a Bernoulli equation and an analogous proof where we need the extra condition (1.20) if this condition is to hold on the bounded interval $t_0 \le t \le t_0 + h$. ■

Remark 22 *Inequality (1.17) is also considered in Willett [126] and Willet and Wong [127]. For a related result, see Ho [61].*

The following theorem is a modified version of a theorem proved in Gamidov [48] (see also [85, p. 361]):

Theorem 23 *If*

$$u(t) \le f(t) + c\int_0^t \phi(s)\, u^{\alpha}(s)\, ds,$$

where all functions are continuous and nonnegative on $[0,h]$*,* $0 < \alpha < 1$*,* $c \ge 0$*, then*

$$u(t) \le f(t) + c\xi_0^{\alpha}\left(\int_0^t \phi^{\frac{1}{1-\alpha}}(s)\, ds\right)^{1-\alpha},$$

where ξ_0 *is the unique root of* $\xi = a + b\xi^{\alpha}$*.*

Gamidov [48] also proved the following result:

Theorem 24 *If*

$$u(t) \le c_1 + c_2\int_0^t \phi(s)\, u^{\alpha}(s)\, ds + c_3\int_0^h \phi(s)\, u^{\alpha}(s)\, ds,$$

$c_1 \ge 0$*,* $c_2 \ge 0$*,* $c_3 > 0$*, and the functions* $u(t)$ *and* $\phi(t)$ *are continuous and nonnegative on* $[0,h]$*, then for* $0 < \alpha < 1$ *we have*

$$u(t) \le \left(\xi_0^{1-\alpha} + c_2(1-\alpha)\int_0^t \phi(s)\, ds\right)^{\frac{1}{1-\alpha}},$$

where ξ_0 *is the unique root of the equation*

$$\left[\frac{c_2+c_3}{c_3}\cdot\xi+\frac{c_1c_2}{c_3}\right]^{1-\alpha}-\xi^{1-\alpha}-c_2\left(1-\alpha\right)\int_0^h\phi\left(s\right)ds=0.$$

If $\alpha>1$ *and* $c_2\left(\alpha-1\right)\int_0^t\phi\left(s\right)ds<c_1^{1-\alpha}$, *there exists an interval* $[0,\delta]\subset[0,h]$ *where*

$$u\left(t\right)\leq\left(c_1^{1-\alpha}-c_2\left(\alpha-1\right)\int_0^t\phi\left(s\right)ds\right)^{\frac{1}{1-\alpha}}.$$

A related result was proved by B. Stachurska [115]:

Theorem 25 *Let the functions* x,a,b *and* k *be continuous and nonnegative of* $J=[\alpha,\beta]$, *and* n *be a positive integer* $(n\geq2)$ *and* $\frac{a}{b}$ *be a nondecreasing function. If*

$$x\left(t\right)\leq a\left(t\right)+b\left(t\right)\int_\alpha^t k\left(s\right)x^n\left(s\right)ds,\quad t\in J,\tag{1.22}$$

then

$$x\left(t\right)\leq a\left(t\right)\left\{1-\left(n-1\right)\int_\alpha^t k\left(s\right)b\left(s\right)a^{n-1}\left(s\right)ds\right\}^{\frac{1}{1-n}},\quad \alpha\leq t\leq\beta_n,\tag{1.23}$$

where

$$\beta_n=\sup\left\{t\in J:\left(n-1\right)\int_\alpha^t kba^{n-1}ds<1\right\}.$$

Remark 26 *(See [85, p. 363]) The inequality (1.22) was considered by P. Maroni [82], but without the assumption of the monotonicity of the ratio* $\frac{a}{b}$. *He obtained two estimates, one for* $n=2$ *and another for* $n\geq3$. *Both are more complicated than (1.23). For* $n=2$ *and* $\frac{a}{b}$ *nondecreasing, Starchurska's result can be better than Maroni's on long intervals.*

1.4 More Nonlinear Generalisations

One of the more important nonlinear generalisations of Gronwall's inequality is the well-known one of Bihari [18]. The result was proved seven years earlier by J.P. Lasalle [73].

Theorem 27 *Let $u(t)$ and $k(t)$ be positive continuous functions on $[c,d]$ and let a and b be nonnegative constants. Further, let $g(z)$ be a positive nondecreasing function for $z \geq 0$. If*

$$u(t) \leq a + b\int_c^t k(s)\, g(u(s))\, ds, \quad t \in [c,d],$$

then

$$u(t) \leq G^{-1}\left(G(a) + b\int_c^t k(s)\, ds\right), \quad c \leq t \leq d_1 \leq d,$$

where

$$G(\lambda) = \int_\xi^\lambda \frac{ds}{g(s)} \quad (\xi > 0,\ \lambda > 0)$$

and d_1 is defined such that

$$G(a) + b\int_c^t k(s)\, ds$$

belongs to the domain of G^{-1} for $t \in [c, d_1]$.

The following generalisation of the Bihari-Lasalle inequality was given by I. Györi [53]:

Theorem 28 *Suppose that $u(t)$ and $\beta(t)$ are continuous and nonnegative on $[t_0,\infty)$. Let $f(t)$, $g(u)$ and $\alpha(t)$ be differentiable functions with f nonnegative, g positive and nondecreasing, and $g\alpha$ nonnegative and nonincreasing. Suppose that*

$$u(t) \leq f(t) + \alpha(t)\int_{t_0}^t \beta(s)\, g(u(s))\, ds. \tag{1.24}$$

If

$$f'(t)\left\{\frac{1}{g(\eta(t))} - 1\right\} \leq 0 \quad on \quad [t_0,\infty) \tag{1.25}$$

for every nonnegative continuous function η, then

$$u(t) \leq G^{-1}\left\{G(f(t_0)) + \int_{t_0}^t [\alpha(s)\beta(s) + f'(s)]\, ds\right\}, \tag{1.26}$$

where

$$G(\delta) = \int_\varepsilon^\delta \frac{ds}{g(s)},\ \varepsilon > 0,\ \delta > 0, \tag{1.27}$$

and (1.26) holds for all values of t for which the function

$$\delta(t) = G[f(t_0)] + \int_{t_0}^{t} [\alpha(s)\beta(s) + f'(s)]\, ds$$

belongs to the domain of the inverse function G^{-1}.

Proof. (see [85, p. 364]) Let

$$V(t) = f(t) + \int_{t_0}^{t} \alpha(s)\beta(s)\, g[u(s)]\, ds.$$

Since g is nondecreasing and α is nonincreasing, we get from (1.24) that $g(u(t)) \leq g(V(t))$. From this we obtain

$$f'(t) + \alpha(t)\beta(t)\, g[u(t)] \leq \alpha(t)\beta(t)\, g[V(t)] + f'(t),$$

which maybe written as

$$\frac{V'(t)}{g[V(t)]} \leq \alpha(t)\beta(t) + \frac{f'(t)}{g[V(t)]}.$$

Using (1.25), we get

$$\frac{V'(t)}{g[V(t)]} \leq \alpha(t)\beta(t) + f'(t).$$

Upon integration we get

$$G[V(t)] \leq G[f(t_0)] + \int_{t_0}^{t} [\alpha(s)\beta(s) + f'(s)]\, ds.$$

If we suppose that $\delta(t)$ is in the domain of G^{-1} then we get the result (1.26) since $u(t) \leq V(t)$. ■

Remark 29 *Note that the special case $\alpha(t) \equiv 1$ already implies the general result since we substitute for β the product $\alpha\beta$ and observe that $\alpha(t)\beta(s) \leq \alpha(s)\beta(s)$ for $s \leq t$.*

K. Ahmedov, A. Jakubov and A. Veisov [2] proved the following theorem:

Theorem 30 *Let $u(t)$ be a continuous function on $[t_0, T]$ such that*

$$0 \leq u(t) \leq f(t) + \int_{t_0}^{\phi(t)} k(t,s)\, g(u(s))\, ds,$$

where

1) *$f(t)$ is continuous, nonnegative, and nonincreasing;*
2) *$\phi(t)$ is differentiable, $\phi'(t) \geq 0$, $\phi(t) \leq t$, $\phi(t_0) = t_0$;*
3) *$g(u)$ is positive and nondecreasing on $\mathbb{R}$; and*
4) *$k(t,s)$ is nonnegative and continuous on $[t_0, T] \times [t_0, T]$ with $\frac{\partial k}{\partial t}(t,s)$ nonnegative and continuous.*

Then for G defined by (1.27) we have

$$u(t) \leq f(t) - f(t_0) + G^{-1}\left\{ G(f(t_0)) + \int_{t_0}^{t} F(s)\, ds \right\},$$

where

$$F(t) = k(t, \phi(t))\, \phi'(t) + \int_{t_0}^{\phi(t)} \frac{\partial k}{\partial t}(t,s)\, ds.$$

C.E. Langenhop [71] proved the following result:

Theorem 31 *Let the functions $u(t)$ and $\alpha(t)$ be nonnegative and continuous on $[a,b]$, $g(u)$ be positive and nondecreasing for $u > 0$, and suppose that for every y in $[a,x]$*

$$u(y) \leq u(x) + \int_{y}^{x} \alpha(r)\, g(u(r))\, dr.$$

Then for every x in $[a,b]$ we have

$$u(x) \geq G^{-1}\left\{ G(u(a)) - \int_{a}^{x} \alpha(r)\, dr \right\},$$

where G is defined by (1.27) and we assume that the term in the $\{\ \}$ is in the domain of G^{-1}.

Proof. (see also [85, p. 365]) We define

$$R(y) = \int_y^x \alpha(r)\, g(u(r))\, dr$$

so that the inequality becomes

$$u(y) \le u(x) + R(y).$$

Since g is nondecreasing, we have

$$g[u(y)] \le g[u(x) + R(y)],$$

which may be written as

$$\frac{d(u(x) + R(y))}{g[u(x) + R(y)]} \ge -\alpha(y)\, dy.$$

An integration from y to x $(y \le x)$ yields

$$-G(u(x) + R(y)) + G(u(x)) \ge -\int_y^x \alpha(s)\, ds.$$

However, G is nondecreasing so $G(u(y)) \le G(u(x) + R(y))$. Combining the last two inequalities and rearranging, we arrive at

$$G(u(x)) > G(u(y)) - \int_y^x \alpha(s)\, ds.$$

Applying G^{-1} we get the result when y is set to a. ■

The following results are proved in Ahmedov, Jakubov and Veisov [2].

Theorem 32 *Suppose that*

$$u(t) < f(t) + \sum_{i=1}^{n} \alpha_i(t) \int_{t_0}^{t} \beta_i(s)\, g[u(s)]\, ds, \quad t \ge t_0,$$

where $u(t)$, $f(t)$ and $\beta_i(t)$ are positive and continuous on $[t_0, \infty)$, $\alpha_i(t) > 0$ while $\alpha_i'(t) \ge 0$; g is a nondecreasing function that satisfies $g(z) \ge z$ for $z > 0$. Then

$$u(t) \le f(t) - \bar{f} + G^{-1}\left[G(\bar{f}) + \ln\prod_{i=1}^{n} \alpha_i(t) - \ln\prod_{i=1}^{n} \alpha_i(t_0) + \int_{t_0}^{t} \sum_{i=1}^{n} \alpha_i(s)\, \beta_i(s)\, ds\right],$$

where $\bar{f} = \max f(t)$ and G is defined by (1.27).

Theorem 33 *Let the positive continuous function $u(t)$ satisfy*

$$u(t) \leq f(t) + \int_{t_0}^{\phi_1(t)} a_1(s) F_1(u(s)) \, ds + \int_{t_0}^{\phi_2(t)} a_2(s) F_2(u(s)) \, ds,$$

with the following conditions:

1) *$f(t)$ is a nonincreasing function on $[t_0, T]$;*

2) *the functions $a_1(t)$ and $a_2(t)$ are continuous and nonnegative on $[t_0, T]$;*

3) *$\phi_1(t)$ and $\phi_2(t)$ are continuously differentiable and nondecreasing functions with $\phi_i(t_0) = t_0$, $i = 1, 2$, and $\phi_1(t) \leq t$;*

4) *the functions $F_1(z)$ and $F_2(z)$ are continuous, nondecreasing and satisfy $F_2(z) > 0$ for all z and*

$$\frac{d}{dz}\left[\frac{F_1(z)}{F_2(z)}\right] = \frac{C}{F_2(z)}$$

for some constant C.

Then

$$u(t) \leq f(t) - f(t_0) + z(t),$$

where

$$G(z) = \frac{F_1(z)}{F_2(z)} = G(z_0) + \int_{z_0}^{z} \frac{C}{F_2(s)} ds,$$

G^{-1} is its inverse function and

$$z(t) = G^{-1}\left\{\exp\left(C\int_{t_0}^{\phi_1(t)} a_1(s)\,ds\right)\left[G(f(t_0)) \right.\right.$$
$$\left.\left. + C\int_{t_0}^{t} a_2(\phi_2(s))\,\phi_2'(s)\exp\left(-C\int_{t_0}^{\phi_1(s)} a_1(r)\,dr\right) ds\right]\right\}$$

is a continuous solution of the initial value problem

$$z'(t) = a_1(\phi_1(t))\,\phi_1'(t)\,F_1(z) + a_2(\phi_2(t))\,\phi_2'(t)\,F_2(z),$$
$$z(t_0) = f(t_0).$$

Theorem 34 *Let the continuous function $u(t)$ satisfy*

$$u(t) \leq f(t) + \sum_{i=1}^{n} a_i(t) \int_{t_0}^{\phi_i(t)} b_i(s) f(u(s)) \, ds$$

on $[t_0, T]$ with

1) the $a_i(t)$ bounded nonnegative nonincreasing functions;

2) the $b_i(t)$ continuous nonnegative functions;

3) the $\phi_i(t)$ continuous with $\phi_i'(t) > 0$, $\phi_i(t) \leq t$, $\phi_i(t_0) = t_0$;

4) $f(t)$ a continuous nonincreasing function;

5) $g(z)$ a nondecreasing positive function defined on $\mathbb{R}$.

Then

$$u(t) \leq f(t) - f(t_0) + z(t)$$

where G is defined by (1.27) and

$$z(t) = G^{-1}\left[G(f(t_0)) + \sum_{i=1}^{n} \int_{t_0}^{\phi_i(t)} a_i(s) b_i(s)\right]$$

is a continuous solution of the initial value problem $z(t_0) = f(t_0)$ and

$$z'(t) = \sum_{i=1}^{n} a_i(\phi_i(t)) b_i(\phi_i(t)) \phi_i'(t) g(z).$$

In the previous results, we have a nonlinearity in the unknown function only under the integral sign. In the next few results we allow the nonlinearity to appear everywhere.

The following theorem is due to Butler and Rogers [22].

Theorem 35 *Let the positive functions $u(t)$, $a(t)$ and $b(t)$ be bounded on $[c, d]$; $k(t, s)$ be a bounded nonnegative function for $c \leq s \leq t \leq d$; $u(t)$ is a measurable function and $k(\cdot, t)$ is a measurable function. Suppose that*

$f(u)$ is strictly increasing and $g(u)$ is nondecreasing. If $A(t) = \sup_{c \le s \le t} a(s)$, $B(t) = \sup_{c \le s \le t} b(s)$ and $K(t,s) = \sup_{s \le \sigma \le t} k(\sigma, s)$, then from

$$f(u(t)) \le a(t) + b(t) \int_c^t k(t,s)\, g(u(s))\, ds, \quad t \in [c,d],$$

follows

$$u(t) \le f^{-1}\left[G^{-1}\left\{G(A(t)) + B(t)\int_c^t K(t,s)\, ds\right\}\right], \quad t \in [c,d'],$$

where

$$G(u) = \int_\xi^u \frac{dw}{g(f^{-1}(w))} \quad (\xi > 0, \quad u > 0)$$

and

$$d' = \max\left[c \le r \le d : G\left(A(r) + B(r)\int_c^r K(r,s)\, ds\right) \le G(f(\infty))\right].$$

The following result can be found in Györi [53]:

Theorem 36 *Suppose*

$$F(u(t))\, s < f(a) + \alpha(t) \int_{t_0}^t \beta(s)\, g[u(s)]\, ds,$$

where the functions $f(t)$, $\alpha(t)$, $\beta(t)$ and $g(u)$ satisfy the conditions of Theorem 28 and the function $F(u)$ is monotone decreasing and positive for $u > 0$. Then on $[t_0, d']$

$$u(t) \le F^{-1}\left\{G^{-1}\left[G(f(t_0)) + \int_{t_0}^t [\alpha(s)\beta(s) + f'(s)]\, ds\right]\right\},$$

where

$$G(z) = \int_\varepsilon^t \frac{ds}{g[F^{-1}(s)]}, \quad z > \varepsilon \ge 0$$

and d' is defined such that the function $\delta(t)$ defined in Theorem 28 belongs to the domain of the definition of the function $F^{-1} \circ G^{-1}$.

The next result allows the integral to appear in the nonlinearity. It is due to Willett and Wong [127].

Theorem 37 *Let the functions x, a, b and k be continuous and nonnegative on $J = [\alpha, \beta]$, $1 \leq p < \infty$, and*

$$x(t) \leq a(t) + b(t) \left(\int_\alpha^t k(s) x^p(s) \, ds \right)^{\frac{1}{p}}, \quad t \in J.$$

Then

$$x(t) \leq a(t) + b(t) \frac{\left(\int_\alpha^t k(s) e(s) a^p(s) \, ds \right)^{\frac{1}{p}}}{1 - [1 - e(t)]^{\frac{1}{p}}}, \quad t \in J,$$

where

$$e(t) = \exp\left(- \int_\alpha^t k(s) b^p(s) \, ds \right).$$

Gollwitzer [49] replaces x and x^p by g and g^p respectively.

Generalisations of this result were given in Beesack [14, pp. 20-30]. Here we shall give some results obtained in Filatov and Šarova [47, pp. 34-37] and from Deo and Murdeshwar [35].

Theorem 38 *Suppose*

1) *$u(t)$, $f(t)$ and $F(t,s)$ are positive continuous functions on $\mathbb{R}$, and $s \leq t$;*

2) *$\frac{\partial F(t,s)}{\partial t}$ is nonnegative and continuous;*

3) *$g(u)$ is positive, continuous, additive and nondecreasing on $(0, \infty)$;*

4) *$h(z) > 0$ and in nondecreasing and continuous on $(0, \infty)$.*

If

$$u(t) \leq f(t) + h\left(\int_0^t F(t,s) g(u(s)) \, ds \right),$$

then, for $t \in I$, we have

$$u(t) \leq f(t) + h\left\{ G^{-1}\left[G\left(\int_0^t F(t,s) g(f(s)) \, ds \right) + \int_0^t \phi(s) \, ds \right] \right\},$$

where

$$G(u) = \int_{\varepsilon}^{u} \frac{ds}{g(h(s))}, \quad u > 0,\ \varepsilon > 0,$$

$$\phi(t) = F(t,t) + \int_0^t \frac{\partial F}{\partial t}(t,s)\,ds,$$

and

$$I = \left\{ t \in (0,\infty) : G(\infty) \geq G\left(\int_0^t F(t,s)\,g(f(s))\,ds \right) + \int_0^t \phi(s)\,ds \right\}.$$

Proof. (see [85, p. 371]) Using the additivity of the function g and the nondecreasing nature of $F(t,s)$ in t, we have

$$u(t) - f(t) \leq h(v(t)),$$

where

$$v(t) = \int_0^t F(t,s)\,g(u(s) - f(s))\,ds + \int_0^T F(T,s)\,g(f(s))\,ds,$$

$t \in (0,T)$ and $T \in (0,\infty)$. Since g is nondecreasing, we find that

$$g(u(t) - f(t)) \leq g(h(v(t))). \tag{1.28}$$

Multiplying this inequality by $\frac{\partial F(t,s)}{\partial t}$ and integrating from 0 to t, we arrive at

$$\int_0^t \frac{\partial F}{\partial t}(t,s)\,g(u(s) - f(s))\,ds \leq \int_0^t \frac{\partial F}{\partial t}(t,s)\,g(h(v(s)))\,ds.$$

On the other hand, if we multiply (1.28) by $F(t,t)$ and using this last inequality, we get

$$v'(t) \leq F(t,t)\,g(h(v(t))) + \int_0^t \frac{\partial F}{\partial t}(t,s)\,g(h(v(s)))\,ds,$$

that is,

$$\frac{d}{dt} G(v(T)) \leq F(t,t) + \int_0^t \frac{\partial F}{\partial t}(t,s)\,ds.$$

Now, by integrating from 0 to T we get

$$G(v(T)) - G(v(0)) \leq \int_0^T \phi(t)\,dt,$$

and since $u(T) - f(T) < h(v(T))$ we have

$$u(T) - f(T) \leq h\left\{G^{-1}\left(G\left[\int_0^T F(T,s)\,g(f(s))\,ds\right] + \int_0^T \phi(t)\,dt\right)\right\}.$$

Since T is arbitrary, we have the result. ■

A simple consequence of this result is:

Theorem 39 *Suppose that $J = (0, \infty)$ and*

1) $u(t)$, $f(t)$ and $F(t)$ are positive and continuous on J;

2) $g(u)$ is positive, continuous, additive and nondecreasing on J;

3) $h(z) > 0$, nondecreasing and continuous.

If

$$u(t) \leq f(t) + h\left(\int_0^t F(s)\,g(u(s))\,ds\right), \quad t \in J,$$

then for $t \in J_1$, we have

$$u(t) \leq f(t) + h\left\{G^{-1}\left[G\left(\int_0^t F(s)\,g(f(s))\,ds\right) + \int_0^t F(s)\,ds\right]\right\},$$

where G is defined as in Theorem 38 and

$$J_1 = \left\{t \in J : G(\infty) \geq G\left(\int_0^t F(s)\,g(f(s))\,ds\right) + \int_0^t F(s)\,ds\right\}.$$

Deo and Murdeshwar [35] also proved the following theorem.

Theorem 40 *Let the conditions 1), 2) and 3) of the previous theorem hold and let $g(u)$ be an even function on $\mathbb{R}$. If*

$$u(t) \geq f(t) - h\left(\int_0^t F(s)\,g(u(s))\,ds\right), \quad t \in (0, \infty),$$

then for $t \in J_1$ and g as defined in Theorem 39 we have

$$u(t) \geq f(t) - h\left\{G^{-1}\left[G\left(\int_0^t F(s)\,g(f(s))\,ds\right) + \int_0^t F(s)\,ds\right]\right\}.$$

Further generalisations of this result are given in Deo and Dhongade [36] and Beesack [14, pp. 65-86]. Beesack has also given corrections of some results from Deo and Dhongade [36]. Here we give only one result of P.R. Beesack (this result for $f(x) = x$, $h(u) = u$ and $a(t) = a$ becomes Theorem 39 in Deo and Dhongade).

Theorem 41 *Let x, a, k and k_1 be nonnegative continuous functions on $J = [\alpha, \beta)$, and let $a(t)$ be nondecreasing on J. Let g and h be continuous nondecreasing functions on $[0, \infty)$ such that g is positive, subadditive and submultiplicative on $[0, \infty)$ and $h(u) > 0$ for $u > 0$. Suppose f is a continuous strictly increasing function on $[0, \infty)$ with $f(u) \geq u$ for $u \geq 0$ and $f(0) = 0$. If*

$$f(x(t)) \leq a(t) + h\left(\int_\alpha^t k(s)\, g(x(s))\, ds\right) + \int_\alpha^t k_1(s)\, x(s)\, ds, \quad t \in J,$$

then

$$\begin{aligned} x(t) \leq \left(f^{-1} \circ F^{-1}\right) \Bigg\{ & \int_\alpha^t k_1(s)\, ds \\ & + F\left[a(t) + \left(h \circ G^{-1}\right)\left\{\int_\alpha^t k(s)\, g(E(s))\, ds\right.\right. \\ & \left.\left.+ \; G\left(\int_\alpha^t k(s)\, g(a(s)\, E(s))\, ds\right)\right\}\right]\Bigg\}, \quad \text{for} \quad \alpha \leq t \leq \beta_1, \end{aligned}$$

where

$$E(t) = \exp\left(\int_\alpha^t k_1 ds\right), \; F(u) = \int_{y_0}^u \frac{dy}{f^{-1}(y)}, \; y > 0, \; (y_0 > 0)$$

and

$$G(u) = \int_{u_0}^u \frac{dy}{g(h(y))}, \; u > 0, \; (u_0 > 0)$$

with

$$\begin{aligned} \beta_1 = \sup\Bigg\{t \in J : G & \left(\int_\alpha^t k(s)\, g(a(s)\, E(s))\, ds\right) \\ & + \int_\alpha^t k(s)\, g(E(s))\, ds \in G\left(R^+\right)\Bigg\}. \end{aligned}$$

If $a(t) = a$ then we may omit the requirement that g be subadditive and then for $\alpha \leq t \leq \beta_2$ we have

$$x(t) \leq \left(f^{-1} \circ F^{-1}\right) \left\{ \int_\alpha^t k_1(s)\, ds + F\left[a + \left(h \circ G_a^{-1}\right) \left(\int_\alpha^t k(s)\, g(E(s))\, ds \right) \right] \right\},$$

where

$$G_a(u) = \int_0^u \frac{dy}{g(|a + h(y)|)}, \ u > 0$$

and

$$\beta_2 = \sup \left\{ t \in J : \int_\alpha^t k(s)\, g(E(s))\, ds \in G_a\left(R^+\right) \right\}.$$

In previous sections we have given explicit estimates for unknown functions which satisfy integral inequalities. Several of these results may be given in terms of solutions of some differential or integral equation. The first one we give is due to B.N. Babkin [7].

Theorem 42 *Let $\phi(t, u)$ be continuous and nondecreasing in u on $[0, T] \times (-\delta, \delta)$ with $\delta \leq \infty$. If $v(t)$ is continuous and satisfies*

$$v(t) \leq u_0 + \int_0^T \phi(t, v(s))\, ds,$$

where u_0 is a constant, then

$$v(t) \leq u(t)$$

where $u(t)$ is the maximal solution of the problem

$$u'(t) = \phi(t, u), \ \ u(0) = u_0,$$

defined on $[0, T]$.

Proof. (see [85, p. 375]) Introduce the function

$$w(t) = u_0 + \int_0^t \phi(s, v(s))\, ds,$$

Then $v(t) \leq w(t)$ and

$$w'(t) = \phi(t, v(t)) \leq \phi(t, w(t)), \quad \text{with} \quad w(0) = u_0.$$

By Theorem 2 from 2. of Chapter XI, [85], we have $w(t) \leq u(t)$ so that the theorem is proved. ■

The following two theorems are generalisations of the preceding result.

Theorem 43 *Let $\phi(t, s, u)$ be continuous and nondecreasing in u for $0 \leq t$, $s \leq T$ and $|u| \leq \delta$. Let $u_0(t)$ be a continuous function on $[0, T]$. If $v(t)$ is a continuous function that satisfies the integral inequality (on $[0, T]$)*

$$v(t) < u_0(t) + \int_0^t \phi(t, s, v(s))\, ds \tag{1.29}$$

then

$$v(t) < u(t) \quad \text{on} \quad [0, T], \tag{1.30}$$

where $u(t)$ is a solution of the equation

$$u(t) = u_0(t) + \int_0^t \phi(t, s, u(s))\, ds \ \text{on} \ [0, T]. \tag{1.31}$$

Proof. (see [85, p. 375]) From (1.29) and (1.31) we get that (1.30) is valid at $t = 0$. By continuity of the functions involved, we get (1.30) holding on some nontrivial interval. If the result does not hold on $[0, T]$ then there is a t_0 such that $v(t) < u(t)$ on $[0, t_0)$ but $v(t_0) = u(t_0)$. From (1.29) and (1.31) we have

$$\begin{aligned} v(t_0) &= u_0(t_0) + \int_0^t \phi(t_0, s, v(s))\, ds \\ &\leq u_0(t_0) + \int_0^t \phi(t_0, s, u(s))\, ds = u(t_0). \end{aligned}$$

This contradiction proves the theorem. ■

In what follows, we shall say that the function $\phi(t, s, u)$ satisfies the condition (μ) if the equation

$$W(t) = u_0(t_0) + \lambda + \int_0^t \phi(t, s, W(s))\, ds$$

has a solution defined on $[0, T]$ for every constant $\lambda \in [0, \mu]$.

Theorem 44 *Let $\phi(t,s,u)$ be defined for $0 \le t,\ s \le T$, $|u| < \delta$, and be continuous and nondecreasing in u and satisfying condition (μ). If the continuous function $v(t)$ satisfies*

$$v(t) \le u_0(t_0) + \int_0^t \phi(t,s,v(s))\,ds \tag{1.32}$$

on $[0,T]$, then

$$v(t) \le u(t)$$

on $[0,T]$ where $u(t)$ satisfies (1.31) on the same interval.

Proof. (see [85, p. 375]) For every fixed n we denote by $W_n(t)$ a solution of the integral equation

$$W_n(t) = \frac{\varepsilon}{n} + u_0(t) + \int_0^t \phi(t,s,W_n(s))\,ds$$

defined on $[0,T]$. For ε sufficiently small, we may employ Theorem 43 to conclude that

$$u(t) < W_{n+1}(t) < W_n(t) < W_1(t)$$

as well as $v(t) < W_n(t)$. Letting n tend to ∞, we obtain the required result. ∎

Remark 45 *It can be shown (see Mamedov, Aširov and Atdaev [79, pp. 96-98]) that the condition of monotonicity of the function ϕ in u is sufficient for the validity of the theorem on integral inequalities but is not necessary.*

A generalisation of this result has a Fredholm term.

Theorem 46 *Let the functions $\phi(t,s,u)$ be continuous and nondecreasing in u for $0 \le t,\ s \le T$, $|u| < \delta$. Suppose that $u_0(t)$ is continuous on $[0,T]$ and either*

a) for every fixed continuous function $W_0(t)$ with values in $|u| < \delta$ on $[0,T]$ and every sufficiently small positive number λ we have

$$W(t) = u_0(t) + \lambda + \int_0^t \phi_1(t,s,W(s))\,ds + \int_0^T \phi(t,s,W_0(s))\,ds$$

has a continuous solution on $[0,T]$; or *b)*

$$|u(t)| + \int_0^T \max_{0\le t\le T} |\phi_1(t,s,\delta) + \phi_2(t,s,\delta)|\, ds \le \delta.$$

If a continuous function $v(t)$ *satisfies*

$$v(t) \le u_0(t) + \int_0^t \phi_1(t,s,v(s))\, ds + \int_0^T \phi_2(t,s,v(s))\, ds,$$

then

$$v(t) < u(t) \quad on \quad [0,T],$$

where

$$u(t) = u_0(t) + \int_0^t \phi_1(t,s,u(s))\, ds + \int_0^T \phi_2(t,s,u(s))\, ds.$$

Remark 47 *The previous theorem is given in Mamedov, Aširov and Atdaev [79]. The book [79] also contains the following result of Ja. D. Mamedov [78].*

Theorem 48 *Let the function* $\phi(t,s,u)$ *be continuous in* t *(in* $[0,\infty)$*) for almost all* $s \in [0,\infty)$ *and* u *with* $|u| < \infty$*. Suppose that for fixed* t *and every continuous function* $u(s)$ *on* $[0,\infty)$ *the function* $\phi(t,s,u(s))$ *is measurable in* s *on* $[0,\infty)$*. Further, let* ϕ *be nondecreasing in* u *and* $u_0(t)$ *be a continuous function on* $[0,\infty)$*.*
If the continuous function $v(t)$ *satisfies*

$$v(t) < u_0(t) + \int_t^\infty \phi(t,s,v(s))\, ds \quad on \quad [0,\infty), \tag{1.33}$$

then

$$v(t) < u(t) \quad on \quad [0,\infty), \tag{1.34}$$

where $u(t)$ *is a solution of*

$$u(t) = u_0(t) + \int_t^\infty \phi(t,s,u(s))\, ds \quad on \quad [0,\infty).$$

Remark 49 *Mamedov [78] (see also Mamedov, Aširov and Atdaev [79]) also considered (1.33) with "$\le$" instead of "$<$', as well as the inequality*

$$v(t) \le u_0(t) + \int_0^t \phi_1(t,s,v(s))\, ds + \int_t^\infty \phi_2(t,s,u(s))\, ds.$$

Theorem 42 was generalised in another way by. V. Lakshmikantham [68]. His results were extended by P.R. Beesack [14]. For this result, we shall use the following special notation and terminology. If the function $F_1(t,s,u)$ is defined on $I_1 \times J_1 \times K_1$ where each of the symbols represents an interval and if for every $(s,u) \in J_1 \times K_1$ the function $f(t) = F_1(t,s,u)$ is monotone on L_1, we say that $F_1(\cdot,s,u)$ is monotone. We say that F_1 and F_2 are monotone in the same sense of them being both nondecreasing or both nonincreasing, and monotone in the opposite sense if one is nondecreasing and the other nonincreasing.

Theorem 50 *Let $f(x)$ be continuous and strictly monotone on an interval I, and let $H(t,v)$ be continuous on $J \times K$ where $J = [\alpha,\beta]$ and K is an interval containing zero and H monotonic in v. Let $T_1 = \{(t,s) : \alpha \le s \le t \le \beta\}$ and assume that $W(t,s,u)$ is continuous and of one sign on $T_1 \times I$, monotone in u, and monotone in t. Suppose also that the functions x and a are continuous on J with $x(J) \subset I$ and*

$$a(t) + H(t,v) \in f(I) \quad \text{for} \quad t \in J \quad \text{and} \quad |v| \le b, \tag{1.35}$$

for some constant $b > 0$. Let

$$f(x(t)) \le a(t) + H\left(t, \int_\alpha^t W(t,s,x(s))\,ds\right), \; t \in J, \tag{1.36}$$

and let $r = r(t,T,\alpha)$ be the maximal (minimal) solution of the system

$$\begin{aligned} &r' = W\left(T,t,f^{-1}\left[a(t) + H(t,r)\right]\right), \\ &r(\alpha) = 0, \; \alpha \le t \le T \le \beta_1 \; (\beta_1 < \beta), \end{aligned} \tag{1.37}$$

if $W(t,s,\cdot)$ and f are monotonic in the same (opposite) sense, where $\beta_1 > a$ is chosen so that the maximal (minimal) solution exists for the indicated interval. Then, if $W(\cdot,s,u)$ and $H(t,\cdot)$ are monotonic in the same sense,

$$x(t) \le (\ge) f^{-1}\left[a(t) + H(t,\tilde{r}(t))\right], \; \alpha \le t \le \beta_1, \tag{1.38}$$

where $\tilde{r}(t) = r(t,t,\alpha)$ if

i) $H(t,\cdot)$ and $W(t,s,\cdot)$ are monotonic in the same sense and f is increasing;
if

ii) f *is decreasing and* $H(t,\cdot)$ *and* $W(t,s,\cdot)$ *are monotonic in the opposite sense (and the second reading of the other hypotheses), then the inequality is reversed in (1.38).*

Proof. (see [85, p. 379]) The function

$$F(T,t,r) = W\left(T,t,f^{-1}\left[a(t) + H(t,r)\right]\right)$$

is by (1.35) continuous on the compact set $T_1 \times [-b,b]$, so it is bounded there, say by the constant M. If follows from Lemma 1 in 2. of Chapter XI, [85], that there exists, independent of t, a $\alpha < \beta$ (in fact $\beta_1 > \alpha + \min\left(\beta - \alpha, bM^{-1}\right)$) such that the maximal (minimal) solution of the system (1.37) exists on $[\alpha,\beta]$. Now fix $T \in (\alpha,\beta]$ and let $t \in [\alpha,T]$. Then define

$$v(t,u) = \int_\alpha^t W(u,s,x(s))\,ds$$

and we have

$$v(t,t) = \int_\alpha^t W(t,s,x(s))\,ds \leq (\geq) \int_\alpha^t W(T,s,x(s))\,ds = v(t,T) \quad (1.39)$$

if $W(\cdot,s,u)$ is increasing (decreasing). Observe that (1.36) implies that $v(t,t) \in K$ for $t \in J$. Since K is an interval containing zero, it follows that $v(t,T) \in K$ in both cases of (1.39) regardless of the sign of W.

By (1.36) we obtain

$$x(t) \leq (\geq) f^{-1}\left[a(t) + H(t,v(t,t))\right], \quad (1.40)$$

if f is increasing (decreasing). Since $v'(t,T) = W(T,t,x(t))$ for $\alpha \leq t \leq T \leq \beta$, we have

$$v'(t,T) \leq (\geq) W\left(T,t,f^{-1}\left[a(t) + H(t,v(t,t))\right]\right), \quad (1.41)$$

if the functions $W(t,s,\cdot)$ and f are monotonic in the same (opposite) sense. On the other hand, by (1.39) we have

$$H(t,v(t,t)) \leq (\geq) H(t,v(t,T)),\ \alpha \leq t \leq T, \quad (1.42)$$

if (a) $H(t,\cdot)$ and $W(t,s,\cdot)$ are monotonic in the same ((b) opposite) sense. Thus

$$f^{-1}\left[a(t) + H(t,v(t,t))\right] \leq (\geq) f^{-1}\left[a(t) + H(t,v(t,T))\right],$$

on $\alpha \leq t \leq T$ if (a'): f is increasing and (a) or f is decreasing and (b) ((b') f is increasing and (b) or f is decreasing and (a)). This in turn implies that

$$W\left(T, t, f^{-1}\left[a(t) + H(t, v(t,t))\right]\right) \leq (\geq) W\left(T, t, f^{-1}\left[a(t) + H(t, v(t,T))\right]\right), \quad (1.43)$$

if (a"): $W(t,s,\cdot)$ is increasing and (a') or $W(t,s,\cdot)$ is decreasing and (b') ((b"): $W(t,s,\cdot)$ is increasing and (b') or $W(t,s,\cdot)$ is decreasing and (a')). Combining this with (1.41), we see that if $W(t,s,\cdot)$ and $H(t,\cdot)$ are monotonic in the same sense, then

$$v'(t,T) \leq (\geq) W\left(T, t, f^{-1}\left[a(t) + H(t, v(t,T))\right]\right), \ \alpha \leq t \leq T < \beta, \quad (1.44)$$

if $W(t,s,\cdot)$ and f are monotonic in the same (opposite) sense.

Since $v(\alpha, T) = 0$, Theorem 2 from 2. of Chapter XI, [85] shows that if $W(t,s,\cdot)$ and $H(t,\cdot)$ are monotonic in the same sense and if $r(t,T,\alpha)$ is the maximal or the minimal solution of (1.37) as specified, then

$$v(t,T) \leq (\geq) r(t,T,\alpha) \quad \text{for} \quad \alpha \leq t \leq T \leq \beta_1,$$

from which we get in particular that, this holds when $t = T$. Since T is an arbitrary element of $(\alpha, \beta_1]$, we have

$$v(t,t) \leq (\geq) \tilde{r}(t) \quad \text{on} \quad [\alpha, \beta_1] \quad (1.45)$$

provided that (A): $W(t,s,\cdot)$ and f are monotonic in the same sense ((B): $W(t,s,\cdot)$ and f are monotonic in the opposite sense).

As in the analysis of (1.39) and (1.42), we have on $[\alpha, \beta_1]$

$$H(t, v(t,t)) \leq (\geq) H(t, \tilde{r}(t))$$

if (A'): $H(t,\cdot)$ is increasing and (A) or $H(t,\cdot)$ is decreasing and (B), ((B'): $H(t,\cdot)$ is increasing and (B) or $H(t,\cdot)$ is decreasing and (A)). Now, if (A"): f is increasing and (A') or f is decreasing and (B') ((B"): f is increasing and (A') or f is decreasing and (B') ((B"): f is increasing and (B') or f is decreasing and (A')) then

$$f^{-1}\left[a(t) + H(t, v(t,t))\right] \leq (\geq) f^{-1}\left[a(t) + H(t, \tilde{r}(t))\right].$$

Analyzing the various cases, we see that if $W(\cdot, s, u)$ and $H(t, \cdot)$ are monotonic in the same sense and $W(t, s, \cdot)$ and $H(t, \cdot)$ are monotonic in the same (opposite) sense then

$$f^{-1}[a(t) + H(t, v(t,t))] \leq (\geq) f^{-1}[a(t) + H(t, \tilde{r}(t))] \tag{1.46}$$

on $[\alpha, \beta_1]$. The conclusion (1.38) now follows in cases (i) or (ii) from (1.40) and (1.35). ■

In the same way, one can prove:

Theorem 51 *Under the hypotheses of Theorem 50, suppose that*

$$f(x(t)) \geq a(t) + H\left(t, \int_\alpha^t W(t, s, x(s))\, ds\right), \; t \in J$$

and that $W(\cdot, s, u)$ and $H(t, \cdot)$ are monotonic in the opposite sense. Let $\tilde{r} = r(t, t, \alpha)$ where $r(t, T, \alpha)$ is the maximal (minimal) solution of problem (1.37) and suppose that $W(t, s, \cdot)$ and f are monotonic in the opposite (same) sense. Then

$$x(t) \leq (\geq) f^{-1}[a(t) + H(t, r(t))] \quad on \quad [\alpha, \beta_1]$$

provided that conditions (i) or (ii) of Theorem 50 hold.

Remark 52 *Similar results with $\int_t^\beta$ instead of $\int_\alpha^t$ can be obtained from the previous theorems, see P.R. Beesack [14, p. 52].*

Remark 53 *In case $K = [0, t_0]$, then $W > 0$ holds (since $v(t,t) \in K$), so in the condition (1.35) $|v| \leq b$ can be changed to $0 \leq v \leq b$.*

Remark 54 *The assumption that W has one sign can be replaced by the condition that $v(t, T) \in K$ for $\alpha \leq t \leq T \leq \beta$.*

The following theorem is a consequence of Theorem 50 (Lakshmikantham [70, Theorem 3.1 (ii)]):

Theorem 55 *Let x, a, b and c be continuous nonnegative functions on $J = [\alpha, \beta]$ and f and h be continuous nonnegative functions on $\mathbb{R}^+$, with f strictly* *increasing and h nondecreasing. In addition, suppose that $k(t,s)$ is continuous and nonnegative on $T=\{(t,s):\alpha\le s\le t\le\beta\}$, and $w(t,u)$ is continuous and nonnegative on $J\times\mathbb{R}^+$, with $w(t,\cdot)$ nondecreasing on $\mathbb{R}^+$. Define $C(t)=\max\limits_{\alpha\le s\le t}c(s)$, and $K(t,s)=\max\limits_{s\le\sigma\le t}k(\sigma,s)$, for $\alpha\le s\le t\le\beta$. If*

$$f(x(t))\le a(t)+b(t)h\left(c(t)+\int_\alpha^t k(t,s)w(s,x(s))ds\right),\ t\in J, \tag{1.47}$$

then

$$x(t)\le f^{-1}\left[a(t)+b(t)h(\tilde r_1(t,C(t)))\right],\quad t\in J_0, \tag{1.48}$$

with $\tilde r_1(t,c(\alpha))=r(t,t,c(\alpha))$, where $r=r(t,\beta,c(\alpha))$ is the maximal solution on $J=[\alpha,\beta_0]$ of

$$r'=K(\beta_0,t)w\left(t,f^{-1}\left[a(t)+b(t)h(r)\right]\right),\quad r(\alpha)=c(\alpha).$$

P.R. Beesack [17, pp. 56-65] showed that a sequence of well-known results can be simply obtained by using the previous results. Here we shall give one example. We consider the following inequality of Gollwitzer [49]:

$$x(t)\le a+g^{-1}\left(\int_\alpha^t k(t,s)g(x(s))ds\right),\ t\in J=[\alpha,\beta].$$

We use Theorem 50 with: $f(x)=x$, $H(t,v)=g^{-1}(v)$, $W(t,s,u)=k(s)g(u)$, $K=g(I)$ and I is an interval such that $x(J)\subset I$. The comparison equation is

$$r'=k(t)g\left(a+g^{-1}(r)\right),\ r(\alpha)=0. \tag{1.49}$$

By Theorem 50 we have

$$x(t)\le a+g^{-1}(r(t)),\quad \alpha\le t\le\beta_1, \tag{1.50}$$

where $r(t)$ is the unique solution of problem (1.49) on $[\alpha,\beta_1]$. If we define G by

$$G(u)=\int_0^u\frac{dr}{g\left[a+g^{-1}(r)\right]},\ u\in g(I)=K,$$

then by (1.50) we obtain

$$x(t)\le a+g^{-1}\left[G^{-1}\left(\int_\alpha^t kds\right)\right],\quad \alpha\le t\le\beta, \tag{1.51}$$

where $\beta_1 = \sup \left\{ t : \int_\alpha^t k ds \in G(K) \right\}$.

In fact, Beesack showed that this result is better than that of Gollwitzer and thus he formulated the following more general result (see [85, p. 382]):

Let x and k be continuous functions with $k \geq 0$ on $J = [\alpha, \beta]$, and let g be continuous and monotonic in the interval I such that $x(J) \subset J$ and g is nonzero on I except perhaps at an endpoint of I. Let h be continuous and monotone on an interval K such that $0 \in K$, and let a and b be constants such that $a + h(v) \in I^0$ for $v \in K$, $|v| \leq b$.

If g and h are monotone in the same sense and

$$x(t) \leq a + h\left(\int_\alpha^t k(t,s)\, g(x(s))\, ds\right), \ t \in J,$$

then

$$x(t) \leq a + h\left[G^{-1}\left(\int_\alpha^t k ds\right)\right], \ \ \alpha \leq t \leq \beta_1,$$

where

$$G(u) = \int_0^u \frac{dr}{g[a + h(r)]}, \ u \in K$$

and

$$\beta_1 = \sup\left\{ t \in J : \int_\alpha^t k ds \in G(K) \right\}.$$

It is interesting that there exists a sequence of integral inequalities with an unknown function of one variable in which we have several integrals.

B. Pachpatte [97] proved the following result:

Theorem 56 *Let $x(t)$, $a(t)$, $b(t)$, $c(t)$ and $d(t)$ be real, nonnegative and continuous functions defined on $\mathbb{R}^+$ such that for $t \in \mathbb{R}^+$,*

$$x(t) \leq a(t) + b(t)\left(\int_0^t c(s)\, x(s)\, ds + \int_0^t c(s)\, b(s)\left(\int_0^s d(u)\, x(u)\, du\right) ds\right).$$

Then on the same interval we have

$$x(t) \leq a(t) + b(t)\left(\int_0^t c(s)\left(a(s) + b(s)\exp\left(-\int_0^s b(r)\,(c(r) + d(r))\, dr\right)\right.\right.$$
$$\left.\left. \times \int_0^s a(r)\,(c(r) + d(r))\exp\left(-\int_0^t (b(u)\,(c(u) + d(u))\, du)\, dr\right) ds\right)\right).$$

The following three theorems from Bykov and Salpagarov [24] are given in the book Filatov and Šarova [47].

Theorem 57 *Let* $u(t)$*,* $v(t)$*,* $h(t,r)$ *and* $H(t,r,x)$ *be nonnegative functions for* $t \geq r \geq x \geq a$ *and* c_1*,* c_2*, and* c_3 *be nonnegative constants not all zero. If*

$$u(t) \leq c_1 + c_2 \int_a^t \left[v(s)\, u(s) + \int_a^s h(s,r)\, u(r)\, dr \right] ds + c_3 \int_a^t \int_a^r \int_a^s H(s,r,x)\, u(x)\, ds\, dr\, ds, \quad (1.52)$$

then for $t \geq a$

$$u(t) \leq c_1 \exp \left\{ c_2 \int_a^t \left[v(s) + \int_a^s h(s,r)\, dr \right] ds + c_3 \int_a^t \int_a^r \int_a^s H(s,r,x)\, dx\, dr\, ds \right\}. \quad (1.53)$$

Proof. (see [85, p. 384]) Let the right hand side of (1.52) be denoted by $b(t)$. Then $b(s) \leq b(t)$ for $s \leq t$ since all the terms are nonnegative. We have

$$\begin{aligned} \frac{b'(t)}{b(t)} &= c_2 v(t) \frac{u(t)}{b(t)} + c_2 \int_a^t \frac{h(t,r)\, u(r)}{b(t)} dr + c_3 \int_a^t \int_a^r \frac{H(t,r,x)\, u(x)}{b(t)} dx\, dr \\ &\leq c_2 v(t) + c_2 \int_a^t h(t,r)\, dr + c_3 \int_a^t \int_a^r H(t,r,x)\, dx\, dr. \end{aligned}$$

Integration from a to t yields

$$\begin{aligned} &\log b(t) - \log c_1 \\ &\quad \leq c_2 \int_0^b \left[v(s) + \int_a^s h(s,r)\, dr \right] ds + c_3 \int_a^t \int_a^s \int_a^r H(s,r,x)\, dx\, dr\, ds. \end{aligned}$$

Writing this in terms of $b(t)$ and using $u(t) \leq b(t)$ completes the proof. ■

Theorem 58 *Let the nonnegative function* $u(t)$ *defined on* $[t_0, \infty)$ *satisfy the inequality*

$$u(t) \leq c + \int_{t_0}^t k(t,s)\, u(s)\, ds + \int_{t_0}^t \int_{t_0}^s G(t,s,\sigma)\, u(\sigma)\, d\sigma ds,$$

where $k(t,s)$ and $G(t,s,\sigma)$ are continuously differentiable nonnegative functions for $t \geq s \geq \sigma \geq t_0$, and $c > 0$. Then

$$u(t) \leq c \exp\left\{\int_{t_0}^{t}\left[k(s,s) + \int_{t_0}^{s}\left(\frac{\partial k(s,\sigma)}{\partial s} + G(s,s,\sigma)\right)d\sigma \right.\right.$$
$$\left.\left. + \int_{t_0}^{t}\int_{t_0}^{s}\frac{\partial G(s,\sigma,r)}{\partial s}drd\sigma\right]ds\right\}.$$

Theorem 59 *Let the functions $u(t)$, $\sigma(t)$, $v(t)$ and $w(t,r)$ be nonnegative and continuous for $a \leq r \leq t$, and let c_1, c_2 and c_3 be nonnegative. If for $t \in I = [a,\infty)$*

$$u(t) \leq c_1 + \sigma(t)\left\{c_2 + c_3\int_a^t\left[v(s)u(s) + \int_a^s w(s,r)u(r)dr\right]ds\right\},$$

then for $t \in I$,

$$u(t) \leq c_1 + \sigma(t)\left\{c_2 \exp\left[c_3\int_a^t\left(v(s)\sigma(s) + \int_a^s w(s,r)\sigma(r)dr\right)ds\right]\right.$$
$$+ c_1c_3\int_a^t\left(v(s) + \int_a^s w(s,r)dr\right)$$
$$\left.\times \exp\left[c_3\int_s^t\left(v(\delta)\sigma(\delta) + \int_a^\delta w(\delta,r)\sigma(r)dr\right)d\delta\right]ds\right\}.$$

Other related results exist. Here we shall mention one which appears in E.H. Yang [132].

Theorem 60 *Let $x(t)$ be continuous and nonnegative on $I = [0,h)$ and let $p(t)$ be continuous, positive and nondecreasing on I. Suppose that $f_i(t,s)$, $i = 1,\ldots,n$, are continuous nonnegative functions on $I \times I$, and nondecreasing in t. If for $t \in I$*

$$x(t) \leq p(t) + \int_0^t f_1(t,t_1)\int_0^{t_1} f_2(t_1,t_2)\ldots\int_0^{t_{n-1}} f_n(t_{n-1},t_n)x(t_n)dt_n\ldots dt_1$$

then

$$x(t) \leq p(t)U(t),\quad t \in I,$$

where $U(t) = V_n(t,t)$ *and* $V_n(T,t)$ *is defined successively by*

$$V_1(T,t) = \exp\left\{\int_0^t \sum_{j=1}^t f_j(T,s)\,ds\right\}$$

$$V_k(T,t) = F_{n-k+1}(T,t)\left\{1 + \int_0^t f_{n-k+1}(T,s)\frac{V_{k-1}(T_1,s)}{F_{n-k+1}(T,s)}ds\right\},$$

where t, *and* T *are in* I *and* $k = 2,3,\ldots,n$ *and for* $i = 1,\ldots,n-1$

$$F_i(T,t) = \exp\left\{\int_0^t \left[\sum_{j=1}^{i-1} f_j(T,s) - f_i(T,s)\right] ds\right\}.$$

Remark 61 *In the special case of this theorem for* $n = 2$, *one can get the conclusion for* $t \in I$

$$x(t) \le p(t)\exp\left(-\int_0^t f_1(t,s)\,ds\right)$$
$$\times \left\{1 + \int_0^t f_1(t,s)\left\{\exp\int_0^s [2f_1(s,r) + f_2(s,r)]\,dr\right\}ds\right\}.$$

In the previous theorems we have had linear inequalities. We now turn to some nonlinear inequalities. B. Pachpatte in [91] and [93] proved the following two theorems.

Theorem 62 *Let* $x(t)$, $a(t)$ *and* $b(t)$ *be real nonnegative and continuous functions on* $I = [0,\infty)$ *such that*

$$x(t) \le x_0 + \int_0^t a(s)x(s)\,ds + \int_0^t a(s)\left(\int_0^s b(r)x^p(r)\,dr\right)ds,\ t \in I,$$

where x_0 *is a nonnegative constant and* $0 \le p \le 1$. *Then for* $t \in I$

$$x(t) \le x_0 + \int_0^t a(s)\exp\left(\int_0^s a(r)\,dr\right)$$
$$\times \left\{x_0^{1-p} + (1-p)\int_0^s b(r)\times\exp\left(-(1-p)\int_0^r a(u)\,du\right)dr\right\}^{\frac{1}{1-p}} ds.$$

Theorem 63 *Let $x(t)$, $a(t)$, $b(t)$ and $c(t)$ be nonnegative and continuous on $\mathbb{R}$ such that for $t \in I$*

$$x(t) \le x_0 + \int_0^t a(s)\left(x(s) + \int_0^s a(r)\left(\int_0^r b(u)x(u) + c(u)x^p(u)\,du\right)dr\right)ds,$$

where x_0 is a nonnegative constant and $0 \le p \le 1$. Then for $t \in I$

$$\begin{aligned} x(t) \le x_0 + \int_0^t a(s)\Bigg(x_0 + \int_0^s a(r)\exp\left(\int_0^r (a(u)+b(u))\,du\right) \\ \times\Bigg\{x_0^{1-p} + (1-p)\int_0^r c(u) \\ \times \exp\left(-(1-p)\int_0^r (a(v)+b(v))\,dv\right)du\Bigg\}^{\frac{1}{1-p}} dr\Bigg)\,ds \end{aligned}$$

E.H. Yang [131] proved the following result.

Theorem 64 *Let $u(t)$, $a(t)$, $f(t,s)$, $g_i(t,s)$ and $h_i(t,s)$, $i = 1,\dots,n$ be nonnegative continuous functions defined on $I = [0,h)$ and $I \times I$. Let $a(t)$ be nondecreasing and $f(t,s)$, $g_i(t,s)$ and $h_i(t,s)$ be nondecreasing in t. If $0 < p \le 1$ and*

$$u(t) \le a(t) + \int_0^t f(t,s)u(s)\,ds + \sum_{i=1}^n \int_0^t g_i(t,s)\left[\int_0^s h_i(s,r)u^p(r)\,dr\right]ds,$$

then

(A) for $0 < p < 1$ and $t \in I$ we have

$$u(t) \le \left\{[a(t)F(t)]^{1-p} + (1-p)\sum_{i=1}^n G_i(t)F(t)\int_0^t h_i(t,s)\,ds\right\}^{\frac{1}{1-p}}.$$

(B) for $p = 1$ and $t \in I$ we have

$$u(t) \le a(t)\exp\left\{\int_0^t \left[f(t,s) + \sum_{i=1}^n G_i(t)F(t)h_i(t,s)\right]ds\right\},$$

where

$$F(t) = \exp \int_0^t f(t,s)\,ds$$

and

$$G_i(t) = \int_0^t g_i(t,s)\,ds, \quad i = 1, \ldots, n.$$

The nonlinearity x^p in the previous results may be replaced by a more general nonlinearity and we give two results of Pachpatte [96] and [92].

Theorem 65 *Let $x(t)$, $a(t)$ and $b(t)$ be nonnegative continuous functions defined on $I = [a,b]$, and let $g(u)$ be a positive continuous strictly increasing subadditive function for $u > 0$ with $g(0) = 0$. If for $t \in I$*

$$x(t) \le a(t) + \int_a^t b(s) \left(x(s) + \int_a^s b(r)\, g(x(r))\,dr \right) ds,$$

then for $t \in I_0$ we have

$$\begin{aligned} x(t) \le{}& a(t) + \int_a^b b(s) \left(x(s) + \int_a^s b(r)\, g(a(r))\,dr \right) ds \\ &+ \int_a^t b(s)\, G^{-1} \left(G \left(\int_a^b b(r) \left(a(r) + b(u)\, g(a(u))\,du \right) dr \right) \right. \\ &\qquad\qquad \left. + \int_a^s b(r)\,dr \right) ds, \end{aligned}$$

where

$$G(u) = \int_{u_0}^u \frac{ds}{(s + g(s))}, \; u \ge u_0 > 0,$$

and

$$I_0 = \left\{ t \in I : G(\infty) \ge G \left(\int_a^b b(s) \left(a(s) + b(r)\, g(a(r))\,dr \right) ds + \int_a^t b(s)\,ds \right) \right\}.$$

Theorem 66 *Let $x(t)$, $a(t)$, $b(t)$, $c(t)$ and $k(t)$ be continuous on $I = [a,b]$ and $f(u)$ be a positive, continuous, strictly increasing, submultiplicative, and subadditive function for $u > 0$, with $f(0) = 0$. If for $t \in I$ we have*

$$x(t) \le a(t) + b(t) \left(\int_a^t c(s)\, f(x(s)) + b(s) \int_0^s k(r)\, f(x(r))\,dr \right) ds,$$

then we also have for $t \in I_0$

$$x(t) \leq a(t) + b(t) \left(d + \int_a^t c(s) f \left(b(s) G^{-1} \right.\right.$$
$$\left.\left. \times \left(G(d) + \int_a^s f(b(r)) (c(r) + k(r)) \, dr \right) \right) ds \right),$$

where

$$d = \int_a^b c(s) f \left(a(s) + b(s) \int_a^s k(r) f(a(r)) \, dr \right) ds,$$
$$G(u) = \int_{u_0}^u \frac{ds}{f(s)}, \; u \geq u_0 > 0,$$

and

$$I_0 = \left\{ t \in I : G(\infty) \geq G(d) + \int_a^b f(b(r)) (c(r) + k(r)) \, dr \right\}.$$

Theorem 67 *Let the functions* x, a, b, c, k *and* f *satisfy the hypotheses of the previous theorem. If for* $t \in I$ *we have*

$$x(t) \leq a(t) + b(t) f^{-1} \left(\int_a^t c(s) f(x(s)) \, ds \right.$$
$$\left. + \int_a^t c(s) f(b(s)) \left(\int_a^s k(r) f(x(r)) \, dr \right) ds \right),$$

then we also have for $t \in I$

$$x(t) \leq a(t) + b(t) f^{-1} \left(\int_a^t c(s) f(a(s)) + f(b(s)) \right.$$
$$\times \left(\exp \left(\int_a^b f(b(r)) (c(r) + k(r)) \, dr \right) \int_a^s f(a(r)) (c(r) - k(r)) \right.$$
$$\left.\left. \times \exp \left(- \int_a^r f(b(u)) (c(u) + k(u)) \, du \right) dr \right) \right) ds.$$

Theorem 68 *Let the functions be defined as in Theorem 66, while for* $a \leq s \leq t \leq b$ *we have*

$$x(t) \geq x(s) - b(t) f^{-1}\left(\int_s^t c(r) f(x(r))\, dr - \int_s^t c(r)\left(\int_r^t k(u) f(x(u))\, du\right) dr\right),$$

then for the same range of values we also have

$$x(t) \geq x(s)\left(f^{-1}(1 + f(b(t)) \int_s^t c(r) \exp\left(\int_r^t (c(u) f(b(t)) + k(u))\, du\right) dr\right)\right)^{-1}.$$

In Bykov and Salpagarov [24] the following theorem was proved:

Theorem 69 *Suppose that the functions* $u(t)$, $\alpha(t)$ *and* $\beta(t,s)$ *are nonnegative for* $0 < s < t < b$ *and* $\phi(s)$ *is positive, nondecreasing and continuous for* $s > 0$. *If*

$$\begin{aligned} u(t) &\leq c + \int_a^t \left\{\alpha(r)\phi(u(r)) + \int_a^r \beta(r,s)\phi(u(s))\, ds\right\} dr \\ &\equiv c + \int_a^t L\phi(u)\, dr = b(t), \end{aligned}$$

where c *is a positive number, then for* $t \in (a,b)$ *we have*

$$\int_a^{u(t)} \frac{dx}{\phi(x)} \leq \int_a^t \left[\alpha(r) + \int_a^r \beta(r,s)\, ds\right] dr = P(t).$$

Proof. (see [85, p. 390]) From the hypotheses, it follows that

$$\frac{b'(t)}{\phi(b(t))} = \alpha(t)\frac{\phi(u(t))}{\phi(b(t))} + \int_a^t \beta(t,s)\frac{\phi(u(s))}{\phi(b(t))} ds \leq \alpha(t) + \int_a^t \beta(t,s)\, ds.$$

By integration from a to t we obtain the desired result. ■

Remark 70 *For* $\beta \equiv 0$ *we get the Bihari inequality.*

Remark 71 *If the function $P(t)$ belongs to the domain of the definition of the inverse function H^{-1}, where*

$$H(h) = \int_0^h \frac{dx}{\phi(x)}$$

then $u(t) \leq H^{-1}[P(t)]$.

Remark 72 *Analogously, we can prove the corresponding theorem in which the integral $\int_a^t$ is replaced by the integral $\int_t^\infty$.*

The following generalisation of the previous theorem was given by H.M. Salpagarov [111]:

Theorem 73 *Suppose that:*

1) *$\phi(u)$ is a nonnegative, continuous nondecreasing function on $[0,\infty)$;*

2)
$$H(u) = \int_{u_0}^u \frac{ds}{\phi(s)} \quad \left(0 < u < \infty,\ u_0 \in (0,\infty) \quad \textit{is fixed}\right),$$
and H^{-1} is the inverse function of H;

3) *$u(t)$ is continuous on $[0,\infty)$; and*

4) *$M(t)$ is a nondecreasing nonnegative function.*

If

$$u(t) \leq M(t) + \int_a^t L\phi(u)\,dr,$$

where the operator L is defined in the previous theorem, then

$$u(t) \leq H^{-1}\{H[M(t) + P(t)]\}$$

where $P(t)$ is also defined as in Theorem 69.

Proof. (see [85, p. 391]) Let $T > a$ be fixed. Then for $t \in (a, T]$ we have

$$u(t) \leq M(T) + \int_0^t L\phi(u)\,dr,$$

since $M(T) \geq M(t)$. On the basis of in Theorem 69, we have

$$u(t) \leq H^{-1}\{H[M(T) + P(t)]\} \quad \text{for} \quad t \leq T.$$

Setting $t = T$ we get the conclusion. ■

For further results for functions of several variables, see the book [85, Chapter XIII] where further references are given.

Chapter 2

Inequalities for Kernels of $(L)-$Type

The first three sections of this chapter are devoted to the study of certain natural generalisations of Gronwall inequalities for real functions of one variable and kernels satisfying a Lipschitz type condition (see (2.1)). The fourth section contains the discrete version of the inequalities in Section 2.2. All the sections provide a large number of corollaries and consequences in connection with some well-known results that are important in the qualitative theory of differential equations.

In the fifth section of this chapter, some sufficient conditions of uniform boundedness for the nonnegative continuous solutions of Gronwall integral inequations are given while the last section contains some results referring to the uniform convergence of the nonnegative solutions of the above integral inequations.

2.1 Integral Inequalities

In this section we present some integral inequalities of Gronwall type and give estimates for the nonnegative continuous solutions of these integral inequations, [38] and [40].

Lemma 74 *Let* $A, B : [\alpha, \beta) \to \mathbb{R}_+,\ L : [\alpha, \beta) \times \mathbb{R}_+ \to \mathbb{R}_+$ *be continuous* *and*

$$0 \leq L(t,u) - L(t,v) \leq M(t,v)(u-v), \ t \in [\alpha,\beta), \ u \geq v \geq 0, \tag{2.1}$$

where M is nonnegative continuous on $[\alpha,\beta) \times \mathbb{R}_+$.

Then for every nonnegative continuous function $x : [\alpha,t] \to [0,\infty)$ satisfying the inequality

$$x(t) \leq A(t) + B(t) \int_\alpha^t L(s,x(s))\,ds, \ t \in [\alpha,\beta) \tag{2.2}$$

we have the estimation

$$x(t) \leq A(t) + B(t) \int_\alpha^t L(u,A(u)) \exp\left(\int_u^t M(s,A(s))B(s)\,ds \right) du \tag{2.3}$$

for all $t \in [\alpha,\beta)$.

Proof. Let us consider the mapping $y : [\alpha,\beta) \to \mathbb{R}_+$ given by $y(t) := \int_\alpha^t L(s,x(s))\,ds$. Then y is differentiable on (α,β), $y'(t) = L(t,x(t))$ if $t \in (\alpha,\beta)$ and $y(\alpha) = 0$.

By the relation (2.1), it follows that for any $t \in (\alpha,\beta)$

$$y'(t) \leq L(t, A(t) + B(t)y(t)) \leq L(t,A(t)) + M(t,A(t))B(t)y(t). \tag{2.4}$$

Putting

$$s(t) := y(t) \exp\left(-\int_\alpha^t M(s,A(s))B(s)\,ds \right), \quad t \in [\alpha,\beta),$$

then from (2.4) we obtain the following integral inequality:

$$s'(t) \leq L(t,A(t)) \exp\left(-\int_\alpha^t M(s,A(s))B(s)\,ds \right), \quad t \in (\alpha,\beta).$$

Integration on $[\alpha,\beta]$, reveals

$$z(t) \leq \int_\alpha^t L(u,A(u)) \exp\left(-\int_\alpha^u M(s,A(s))B(s)\,ds \right) du,$$

which implies that

$$y(t) \leq \int_\alpha^t L(u,A(u)) \exp\left(\int_u^t M(s,A(s))B(s)\,ds \right) du; \quad t \in [\alpha,\beta),$$

from where results the estimation (2.3).

The lemma is thus proved. ■

Now, we can give the following two corollaries that are obvious consequences of the above lemma.

Corollary 75 *Let us suppose that* $A, B : [\alpha, \beta) \to \mathbb{R}_+,\ G : [\alpha, \beta) \times \mathbb{R}_+ \to \mathbb{R}_+$ *are continuous and*

$$0 \le G(t, u) - G(t, v) \le N(t)(u - v),\ t \in [\alpha, \beta),\ u \ge v \ge 0, \tag{2.5}$$

where N *is nonnegative continuous on* $[\alpha, \beta)$.

If $x : [\alpha, \beta) \to [0, \infty)$ *is continuous and satisfies the inequality*

$$x(t) \le A(t) + B(t) \int_\alpha^t G(s, x(s))\, ds,\ t \in [\alpha, \beta); \tag{2.6}$$

then we have the estimate

$$x(t) \le A(t) + B(t) \int_\alpha^t G(u, A(u)) \exp\left(\int_u^t N(s) B(s)\, ds \right) du \tag{2.7}$$

for all $t \in [\alpha, \beta)$.

Corollary 76 *Let* $A, B, C : [\alpha, \beta) \to \mathbb{R}_+,\ H : \mathbb{R}_+ \to \mathbb{R}_+$ *be continuous and* H *satisfies the following condition of Lipschitz type:*

$$0 \le H(u) - H(v) \le M(u - v),\ M > 0,\ u \ge v \ge 0. \tag{2.8}$$

Then for every nonnegative continuous function verifying

$$x(t) \le A(t) + B(t) \int_\alpha^t C(s) H(x(s))\, ds,\ t \in [\alpha, \beta) \tag{2.9}$$

we have the bound

$$x(t) \le A(t) + B(t) \int_\alpha^t C(u) H(A(u)) \times \exp\left(M \int_u^t C(s) B(s)\, ds \right) du, \tag{2.10}$$

for all $t \in [\alpha, \beta)$.

Remark 77 *Putting $H:\mathbb{R}_+\to\mathbb{R}_+,\ H(X)=x$, one obtains Lemma 1 of [37] which gives a natural generalisation of the Gronwall inequality.*

A important consequence of Lemma 74 for differentiable kernels is the following result:

Lemma 78 *Let $A,B:[\alpha,\beta)\to\mathbb{R}_+,\ D:[\alpha,\beta)\times\mathbb{R}_+\to\mathbb{R}_+$ be continuous and*

$$\begin{aligned}&D \text{ is differentiable on domain } (\alpha,\beta)\times(0,\infty), \\ &\frac{\partial D(t,x)}{\partial x} \text{ is nonnegative on } (\alpha,\beta)\times(0,\infty), \text{ and there exists} \\ &\text{a continuous function } P:[\alpha,\beta)\times\mathbb{R}_+\to\mathbb{R}_+ \text{ such that} \\ &\frac{\partial D(t,u)}{\partial x}\le P(t,v) \text{ for any } t\in(\alpha,\beta) \text{ and } u\ge v>0.\end{aligned} \tag{2.11}$$

If $x:[\alpha,\beta)\to[0,\infty)$ is continuous and

$$x(t)\le A(t)+B(t)\int_\alpha^t D(s,x(s))\,ds,\ t\in[\alpha,\beta) \tag{2.12}$$

then we have the inequality

$$x(t)\le A(t)+B(t)\int_\alpha^t D(u,A(u)) \times\exp\left(\int_u^t P(s,A(s))\,B(s)\,ds\right)du; \tag{2.13}$$

for all $t\in[\alpha,\beta)$.

Proof. Applying Lagrange's theorem for the function D in the domain $\Delta=(\alpha,\beta)\times(0,\infty)$, for every $u>v>0$ and $t\in(\alpha,\beta)$, there exists a $\mu\in(v,u)$ such that

$$D(t,u)-D(t,v)=\frac{\partial D(t,\mu)}{\partial x}(u-v).$$

Since, by (2.11),

$$0\le\frac{\partial D(t,\mu)}{\partial x}\le P(t,v),$$

we obtain

$$0 \le D(t,u) - D(t,v) \le P(t,v)(u-v)$$

for every $u \ge v > 0$ and $t \in (\alpha, \beta)$.

The proof of the lemma follows now by a similar argument to that employed in Lemma 74. We omit the details. ■

In what follows, we give two corollaries that are important in applications:

Corollary 79 *Let $A : [\alpha,\beta) \to \mathbb{R}_+^*$, $B : [\alpha,\beta) \to \mathbb{R}_+$, $I : [\alpha,\beta) \times \mathbb{R}_+ \to \mathbb{R}_+$ be continuous on I and satisfies the assumption:*

$$\begin{aligned}&I \text{ is differentiable on domain } (\alpha,\beta)\times(0,\infty), \frac{\partial I(t,x)}{\partial x} \text{ is} \\ &\text{non-negative continuous on } (\alpha,\beta)\times(0,\infty) \text{ and we have} \\ &\frac{\partial I(t,u)}{\partial x} \le \frac{\partial I(t,v)}{\partial x} \text{ for any } u \ge v > 0 \text{ and } t \in (\alpha,\beta).\end{aligned} \tag{2.14}$$

If $x : [\alpha,\beta) \to [0,\infty)$ is continuous and satisfies the inequality

$$x(t) \le A(t) + B(t)\int_\alpha^t I(s, x(s))\,ds,\ t \in [\alpha,\beta), \tag{2.15}$$

then we have the estimate

$$x(t) \le A(t) + B(t)\int_\alpha^t I(u, A(u)) \exp\left(\int_u^t \frac{\partial I(s, A(s))}{\partial x} B(s)\,ds\right) du \tag{2.16}$$

for all $t \in [\alpha,\beta)$.

Corollary 80 *Let $A, B, C : [\alpha,\beta) \to \mathbb{R}_+$, $K : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous, $A(t) > 0$ for all $t \in [\alpha,\beta)$ and K satisfies the condition*

$$\begin{aligned}&K \text{ is monotone-increasing and differentiable in } (0,\infty) \text{ with} \\ &\frac{dK}{dx} : (0,\infty) \to \mathbb{R}_+ \text{ is monotone decreasing in } \mathbb{R}_+^*.\end{aligned} \tag{2.17}$$

If $x : [\alpha,\beta) \to [0,\infty)$ is continuous and

$$x(t) \le A(t) + B(t)\int_\alpha^t C(s) K(x(s))\,ds,\ t \in [\alpha,\beta), \tag{2.18}$$

then we have the bound

$$x(t) \le A(t) + B(t) \int_\alpha^t C(u) K(A(u)) \times \exp\left(\int_u^t \frac{dK}{dx}(A(s)) B(s) C(s) ds \right) du, \quad (2.19)$$

for all $t \in [\alpha, \beta)$.

The following natural consequences of the above corollaries hold.

Consequences

1. Let $A, B, C; r : [\alpha, \beta) \to \mathbb{R}_+$ be continuous and $A(t) > 0$, $r(t) \le 1$ for $t \in [\alpha, \beta)$. Then, for every nonnegative continuous function $x : [\alpha, \beta) \to [0, \infty)$ satisfying the integral inequality

$$x(t) \le A(t) + B(t) \int_\alpha^t C(s) x(s)^{r(s)} ds, \ t \in [\alpha, \infty) \quad (2.20)$$

we have the estimation

$$x(t) \le A(t) + B(t) \int_\alpha^t C(u) A(u)^{r(u)} \times \exp\left(\int_u^t \frac{r(s) B(s) C(c)}{A(s)^{1-r(s)}} ds \right) du, \quad (2.21)$$

for all $t \in [\alpha, \beta)$.
In particular, if r is constant, then

$$x(t) \le A(t) + B(t) \int_\alpha^t C(s) x(s)^r ds, \ t \in [\alpha, \beta) \quad (2.22)$$

implies

$$x(t) \le A(t) + B(t) \int_\alpha^t C(u) A(u)^r \times \exp\left(r \int_u^t \frac{B(s) C(s)}{A(s)^{1-r}} ds \right) du \quad (2.23)$$

in $[\alpha, \beta)$.

2. Let $A, B, C : [\alpha, \beta) \to \mathbb{R}_+$ be nonnegative and continuous on $[\alpha, \beta)$. If $x : [\alpha, \beta) \to \mathbb{R}_+$ is continuous and satisfies the relation

$$x(t) \leq A(t) + B(t) \int_\alpha^t C(s) \ln(x(s) + 1)\, ds, \ t \in [\alpha, \beta), \tag{2.24}$$

then we have the estimation

$$x(t) \leq A(t) + B(t) \int_\alpha^t C(s) \ln(A(s) + 1) \times \exp\left(\int_u^t \frac{C(u) B(u)}{A(u) + 1} du\right) ds \tag{2.25}$$

for all $t \in [\alpha, \beta)$.

3. Assume that A, B, C are nonnegative and continuous in $[\alpha, \beta)$. Then for every $x : [\alpha, \beta) \to \mathbb{R}_+$ a solution of the following integral inequation

$$x(t) \leq A(t) + B(t) \int_\alpha^t C(s) \arctan(x(s))\, ds, \ t \in [\alpha, \beta) \tag{2.26}$$

we have

$$x(t) \leq A(t) + B(t) \int_\alpha^t C(u) \arctan(A(u)) \exp \times \left(\int_u^t \frac{B(s) C(s)}{A(s)^2 + 1} ds\right) du \tag{2.27}$$

in the interval $[\alpha, \beta)$.

2.1.1 Some Generalisations

In this section we point out some generalisations of the results presented above.

The first result is embodied in the following theorem [41].

Theorem 81 *Let $A, B : [\alpha, \beta) \to [0, \infty)$, $L : [\alpha, \beta) \times [0, \infty) \to [0, \infty)$ be continuous. Further, let $\psi : [0, \infty) \to [0, \infty)$ be a continuous and strictly increasing mapping with $\psi(0) = 0$ that satisfies the assumption*

$$0 \leq L(t, u) - L(t, v) \leq M(t, v) \psi^{-1}(u - v) \text{ for all } u \geq v \geq 0, \tag{2.28}$$

where M is continuous on $[\alpha,\beta)\times[0,\infty)$ and ψ^{-1} is the inverse mapping of ψ.

Then for every nonnegative continuous function $x:[\alpha,\beta)\to[0,\infty)$ satisfying

$$x(t)\le A(t)+\psi\left(B(t)\int_{\alpha}^{t}L(s,x(s))\,ds\right),\quad t\in[\alpha,\beta),$$

we have the estimate

$$x(t)\le A(t)+\psi\left(B(t)\int_{\alpha}^{t}L(u,A(u))\exp\left(\int_{u}^{t}M(s,A(s))\,B(s)\,ds\right)du\right)$$

for all $t\in[\alpha,\beta)$.

Proof. Define the mapping $y:[\alpha,\beta)\to[0,\infty)$ by

$$y(t):=\int_{\alpha}^{t}L(s,x(s))\,ds.$$

Then y is differentiable on $[\alpha,\beta)$ and

$$y'(t)=L(t,x(t)),\ \text{for } t\in[\alpha,\beta).$$

By the use of the condition (2.28), it follows that:

$$\begin{aligned}y'(t)&=L(t,x(t))\le L(t,A(t)+\psi(B(t)\,y(t)))\\&\le L(t,A(t))+M(t,A(t))\,\psi^{-1}(\psi(B(t)\,y(t)))\\&=L(t,A(t))+M(t,A(t))\,B(t)\,y(t)\end{aligned}$$

for all $t\in[\alpha,\beta)$.

By a similar argument to the one in the proof of Lemma 74, we obtain

$$y(t)\le\int_{\alpha}^{t}L(u,A(u))\exp\left(\int_{u}^{t}M(s,A(s))\,B(s)\,ds\right)du,\quad t\in[\alpha,\beta).$$

On the other hand, we have:

$$y(t)\le A(t)+\psi(B(t)\,y(t)),\quad t\in[\alpha,\beta)$$

and since ψ is monotonic increasing on $[\alpha,\beta)$, we deduce that the desired inequality holds. ■

Remark 82 *Some general examples of mappings that satisfy the above conditions are:* $\psi : [0, \infty) \to [0, \infty)$, $\psi(x) := x^p$, $p > 0$.

The following two corollaries are natural consequences of the above theorem.

Corollary 83 *Let* A, B, ψ *be as above and* $G : [\alpha, \beta) \times [0, \infty) \to [0, \infty)$ *be a continuous mapping such that:*

$$0 \le G(t, u) - G(t, v) \le N(t)\, \psi^{-1}(u - v) \tag{2.29}$$

for all $t \in [\alpha, \beta)$, $u \ge v \ge 0$ *and* N *is continuous on* $[\alpha, \beta)$.
If $x : [\alpha, \beta) \to [0, \infty)$ *is continuous and satisfies the inequality*

$$x(t) \le A(t) + \psi\left(B(t)\int_\alpha^t L(s, x(s))\, ds\right), \quad t \in [\alpha, \beta),$$

then we have the bound:

$$x(t) \le A(t) + \psi\left(B(t)\int_\alpha^t G(u, A(u)) \exp\left(\int_u^t N(s)\, B(s)\, ds\right) du\right)$$

in the interval $[\alpha, \beta)$.

Corollary 84 *Let* A, B, ψ *be as above,* $C : [\alpha, \beta) \to [0, \infty)$ *and* $H : [0, \infty) \to [0, \infty)$ *be continuous and such that:*

$$0 \le H(u) - H(v) \le M\psi^{-1}(u - v)$$

for all $u \ge v \ge 0$ *and* M *is a constant with* $M > 0$.
The for every nonnegative continuous function x *satisfying*

$$x(t) \le A(t) + \psi\left(B(t)\int_\alpha^t C(s)\, H(x(s))\, ds\right), \quad t \in [\alpha, \beta),$$

we have the evaluation:

$$x(t) \le A(t) + \psi\left(B(t)\int_\alpha^t C(u)\, H(A(u)) \times \exp\left(M\int_u^t C(s)\, B(s)\, ds\right) du\right)$$

for all $t \in [\alpha, \beta)$.

The second generalisation of Lemma 74 is the following:

Theorem 85 *Let A, B, L and ψ be as in Theorem 81 and suppose, in addition, that*

$$\psi^{-1}(ab) \leq \psi^{-1}(a)\,\psi^{-1}(b) \quad \text{for all } a, b \in [0, \infty).$$

If $x : [\alpha, \beta) \to [0, \infty)$ is continuous and

$$x(t) \leq A(t) + B(t)\,\psi\left(\int_\alpha^t L(s, x(s))\,ds\right), \quad t \in [\alpha, \beta),$$

then we have the evaluation:

$$x(t) \leq A(t) + B(t)\,\psi\left(\int_\alpha^t L(u, A(u)) \times \exp\left(M(s, A(s))\,\psi^{-1}(B(s))\,ds\right) du\right)$$

for all $t \in [\alpha, \beta)$.

Proof. If y is as defined in Theorem 81, we have:

$$\begin{aligned} y'(t) = L(t, x(t)) &\leq L(t, A(t)) + B(t)\,\psi(y(t)) \\ &\leq L(t, A(t)) + M(t, A(t))\,\psi^{-1}(B(t)\,\psi(y(t))) \\ &\leq L(t, A(t)) + M(t, A(t))\,\psi^{-1}B(t)\,y(t), \end{aligned}$$

for all $t \in [\alpha, \beta)$. Thus we obtain

$$y(t) \leq \int_\alpha^t L(u, A(u)) \exp\left(\int_u^t M(s, A(s))\,\psi^{-1}(B(s))\,ds\right) du$$

and since

$$x(t) \leq A(t) + B(t)\,\psi(y(t)), \quad t \in [\alpha, \beta),$$

the proof is completed. ■

Remark 86 *As examples of functions ψ satisfying the conditions of Theorem 85, we may give the mappings $\psi : [0, \infty) \to [0, \infty)$, $\psi(x) = x^p$, $p > 0$.*

Remark 87 *If $p = 1$, we also recapture Lemma 74.*

Corollary 88 *Let A, B, ψ be as in the above theorem and G satisfies the condition (2.29). Then for every nonnegative continuous function $x : [\alpha, \beta) \to [0, \infty)$ verifying*

$$x(t) \leq A(t) + B(t)\,\psi\left(\int_\alpha^t G(s, x(s))\,ds\right), \quad t \in [\alpha, \beta),$$

we have the bound:

$$x(t) \leq A(t) + B(t)\,\psi\left(\int_\alpha^t G(u, A(u)) \times \exp\left(\int_u^t N(s)\,\psi^{-1}(B(s))\,ds\right) du\right)$$

for all $t \in [\alpha, \beta)$.

Finally, we have:

Corollary 89 *Let A, B, ψ be as above, C and H be as in Corollaries 83 and 84 of Theorem 81. If x is a nonnegative continuous solution of the integral inequality:*

$$x(t) \leq A(t) + B(t)\,\psi\left(\int_\alpha^t C(s)\,H(x(s))\,ds\right), \quad t \in [\alpha, \beta),$$

then we have

$$x(t) \leq A(t) + B(t)\,\psi\left(\int_\alpha^t C(s)\,H(A(u)) \times \exp\left(M \int_u^t C(s)\,\psi^{-1}(B(s))\,ds\right) du\right)$$

in the interval $[\alpha, \beta)$.

2.1.2 Further Generalisations

In paper [14], P.R. Beesack proved the following comparison theorem.

Theorem 90 *Let σ and k be continuous and of one sign on the interval $J=[\alpha,\beta]$ and let g be continuous monotone and non-zero on an interval I containing the point v_0. Suppose that either g is nondecreasing and $k\geq 0$ or g is nonincreasing and $k\leq 0$. If $\sigma\geq 0$ $(\delta\leq 0)$ let the maximal (minimal) solution, $v(t)$, of*

$$\frac{dv}{dt}=k(t)\,g(v(t))+\sigma(t)\,,\ v(\alpha)=v_0 \tag{2.30}$$

exists on the interval $[\alpha,\beta_1)$, and let

$$\begin{aligned}u_1&=\sup\left\{u\in J:v_0+\int_\alpha^u\sigma(s)\,ds\in I\right\};\\ u_2&=\sup\left\{u\in J:G\left(v_0+\int_\alpha^u\sigma(s)\,ds+\int_a^t k(s)\,ds\right)\in G(I)\,,\right.\\ &\qquad\alpha\leq t\leq u\}\,,\end{aligned}$$

where

$$G(u):=\int_{u_0}^u\frac{dy}{g(y)},\ \ u\in I,\ (u_0\in I)\,.$$

Let $\beta_2=\min(u_1,u_2)$ and $\bar\beta_1=\min(\beta_1,\beta_2)$. Then for $\alpha\leq t\leq\bar\beta_1$, we have

$$v(t)\leq G^{-1}\left[\int_a^t k(s)\,ds+G\left(v_0+\int_\alpha^u\sigma(s)\,ds\right)\right]\quad \text{if}\ \ \sigma\geq 0 \tag{2.31}$$

or

$$v(t)\geq G^{-1}\left[\int_a^t k(s)\,ds+G\left(v_0+\int_\alpha^u\sigma(s)\,ds\right)\right]\quad \text{if}\ \ \sigma\leq 0 \tag{2.32}$$

Moreover, if $\sigma\geq 0$ and k,g have the same sign ($\sigma\leq 0$ and k,g have the opposite sign), then $\beta_1\geq\beta_2$ and we also have:

$$G^{-1}\left(\int_a^t k(s)\,ds+G\left(v_0+\int_\alpha^u\sigma(s)\,ds\right)\right)\leq(\geq)\,v(t)\,,\ \alpha\leq t\leq\beta_1. \tag{2.33}$$

Now we can give the following generalisation of the Gronwall inequality [103].

Theorem 91 *Let $A, B : [\alpha, \beta) \to \mathbb{R}_+$, $L : [\alpha, \beta) \times \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and the condition (2.1) is valid. If $x : [\alpha, \beta) \to \mathbb{R}_+$ is a nonnegative continuous functions which satisfies the inequality:*

$$x(t) \leq A(t) + B(t) g\left(\int_\alpha^t L(s, x(s))\, ds\right),\ t \in [\alpha, \beta), \tag{2.34}$$

where $g : [0, \infty) \to \mathbb{R}_+$ is a nondecreasing continuous function, then we have the estimation:

$$x(t) \leq A(t) + B(t) g \circ \Gamma^{-1}\left(\int_\alpha^t M(s, A(s)) B(s)\, ds + \Gamma\left(\int_\alpha^t L(s, A(s))\, ds\right)\right),\ t \in [\alpha, \beta), \tag{2.35}$$

where

$$\Gamma(u) := \int_{u_0}^u \frac{dy}{g(y)},\ u_0 > 0.$$

Proof. Let us consider the mapping

$$y(t) := \int_\alpha^t L(s, x(s))\, ds.$$

Then y is differentiable on $[\alpha, \beta)$ and

$$\dot{y}(t) = L(t, x(t)),\ t \in [\alpha, \beta).$$

By the condition (2.1), we obtain

$$\begin{aligned} \dot{y}(t) &\leq L(t, A(t) + B(t) g(y(t))) \\ &\leq L(t, A(t)) + M(t, A(t)) B(t) g(y(t)),\ t \in [\alpha, \beta), \end{aligned}$$

i.e.,

$$\dot{y}(t) \leq L(t, A(t)) + M(t, A(t)) B(t) g(y(t)),\ t \in [\alpha, \beta). \tag{2.36}$$

By a similar argument to that in the proof of the Beesack theorem of comparison, we deduce

$$y(t) \leq \Gamma^{-1}\left[\int_a^t M(s, A(s)) B(s)\, ds + \Gamma\left(\int_\alpha^t L(s, A(s))\, ds\right)\right] \tag{2.37}$$

for all $t \in [\alpha, \beta)$, and now

$$x(t) \leq A(t) + B(t)\, g \circ \Gamma^{-1} \left[\int_\alpha^t M(s, A(s))\, B(s)\, ds + \Gamma\left(\int_\alpha^t L(s, A(s))\, ds \right) \right],$$

for all $t \in [\alpha, \beta)$. ■

Corollary 92 *Let $A, B : [\alpha, \beta) \to \mathbb{R}_+$, $G : [\alpha, \beta) \times \mathbb{R}_+ \to \mathbb{R}_+$, be continuous and G satisfies the condition (2.5).*
Then for every nonnegative continuous function x verifying the integral inequality

$$x(t) \leq A(t) + B(t)\, g\left(\int_\alpha^t G(s, x(s))\, ds \right),\ t \in [\alpha, \beta) \tag{2.38}$$

we have the estimation

$$x(t) \leq A(t) + B(t)\, g \circ \Gamma^{-1} \left(\int_\alpha^t N(s)\, B(s)\, ds + \Gamma\left(\int_\alpha^t G(s, A(s))\, ds \right) \right) \tag{2.39}$$

for all $t \in [\alpha, \beta)$.

Corollary 93 *Let $A, B, C : [\alpha, \beta) \to \mathbb{R}_+$, $H : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and H satisfies the condition of Lipschitz type (2.8). Then for every nonnegative continuous solution of*

$$x(t) \leq A(t) + B(t)\, g\left(\int_\alpha^t C(s)\, H(x(s))\, ds \right),\ t \in [\alpha, \beta) \tag{2.40}$$

then we have

$$x(t) \leq A(t) + B(t)\, g \circ \Gamma^{-1} \left[M \int_\alpha^t C(s)\, B(s)\, ds + \Gamma\left(\int_\alpha^t C(s)\, H(A(s))\, ds \right) \right] \tag{2.41}$$

for all $t \in [\alpha, \beta)$.

Another result for differentiable kernels is embodied in the following theorem [103].

Theorem 94 *Let* $A, B : [\alpha, \beta) \to \mathbb{R}_+$, $D : [\alpha, \beta) \times \mathbb{R}_+ \to \mathbb{R}_+$, *be continuous and* D *satisfies the condition (2.11).*
If $x : [\alpha, \beta) \to [0, \infty)$ *is continuous and*

$$x(t) \leq A(t) + B(t)\, g\left(\int_\alpha^t D(s, x(s))\, ds\right),\ t \in [\alpha, \beta), \tag{2.42}$$

where g *is a nonnegative nondecreasing continuous function on the interval* $[0, \infty)$*, then the following estimation holds:*

$$x(t) \leq A(t) + B(t)\, g \circ \Gamma^{-1}\left(\int_\alpha^t P(s, A(s))\, B(s)\, ds \right.$$
$$\left. + \Gamma\left(\int_\alpha^t D(s, A(s))\, ds\right)\right), \tag{2.43}$$

where

$$\Gamma(u) := \int_{u_0}^u \frac{dy}{g(y)},\ u_0 > 0.$$

Proof. Applying Lagrange's theorem for the mapping D in $[\alpha, \beta) \times [0, \infty)$, then for every $u > v \geq 0$ and $t \in [\alpha, \beta)$, there exists $\mu \in (v, u)$ such that:

$$D(t, u) - D(t, v) = \frac{\partial D(t, \mu)}{\partial x}(u - v).$$

Since $0 \leq \frac{\partial D(t,\mu)}{\partial x} \leq P(t, v)$, we obtain

$$0 \leq D(t, u) - D(t, v) \leq P(t, v)(u - v) \tag{2.44}$$

for all $u \geq v \geq 0$ and $t \in [\alpha, \beta)$.

Applying Theorem 91 for $L = D$ and $M = P$, we obtain the evaluation (2.43). The theorem is thus proved. ■

Further on, we shall give some corollaries that are important in applications.

Corollary 95 *Let $A, B : [\alpha, \beta) \to \mathbb{R}_+$, $I : [\alpha, \beta) \times \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and I satisfies the relation (2.14). Then for every nonnegative continuous function x satisfying*

$$x(t) \leq A(t) + B(t) g\left(\int_\alpha^t I(s, x(s))\, ds\right), \; t \in [\alpha, \beta), \tag{2.45}$$

we have the estimation

$$x(t) \leq A(t) + B(t) g \circ \Gamma^{-1}\left[\int_\alpha^t \frac{\partial I(s, A(s))}{\partial x} D(s)\, ds + \Gamma\left(\int_\alpha^t I(s, A(s))\, ds\right)\right] \tag{2.46}$$

for all $t \in [\alpha, \beta)$.

Corollary 96 *Let $A, B, C : [\alpha, \beta) \to \mathbb{R}_+$, $K : [\alpha, \beta) \times \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and I satisfies the relation (2.14). If $x : [\alpha, \beta) \to [0, \infty)$ is continuous and*

$$x(t) \leq A(t) + B(t) g\left(\int_\alpha^t C(s) K(x(s))\, ds\right), \; t \in [\alpha, \beta), \tag{2.47}$$

then we have the bound

$$x(t) \leq A(t) + B(t) g \circ \Gamma^{-1}\left[\int_\alpha^t C(s) \frac{dK}{dx}(A(s)) B(s)\, ds + \Gamma\left(\int_\alpha^t C(s) K(A(s))\, ds\right)\right] \tag{2.48}$$

for all $t \in [\alpha, \beta)$.

In the following we mention some particular cases of interest [103].

Proposition 97 *Let $A, B, C, r : [\alpha, \beta) \to \mathbb{R}_+$ be continuous and $A(t) > 0$, $r(t) \leq 1$ for $t \in [\alpha, \beta)$. Then for every nonnegative continuous function $x : [\alpha, \beta) \to [0, \infty)$ satisfying*

$$x(t) \leq A(t) + B(t) g\left(\int_\alpha^t C(s) x(s)^{r(s)}\, ds\right), \; t \in [\alpha, \beta), \tag{2.49}$$

we have the estimation

$$x(t) \le A(t) + B(t)\, g \circ \Gamma^{-1} \left[\int_{\alpha}^{t} \frac{r(s)\, C(s)\, B(s)}{A(s)^{1-r(s)}} ds + \Gamma \left(\int_{\alpha}^{t} C(s)\, A(s)^{r(s)}\, ds \right) \right] \quad (2.50)$$

for all $t \in [\alpha, \beta)$.
In particular, if r *is constant, then*

$$x(t) \le A(t) + B(t)\, g \left(\int_{\alpha}^{t} C(s)\, x(s)^{r}\, ds \right), \ t \in [\alpha, \beta), \quad (2.51)$$

implies

$$x(t) \le A(t) + B(t)\, g \circ \Gamma^{-1} \left[r \int_{\alpha}^{t} \frac{C(s)\, B(s)}{A(s)^{1-r}} ds + \Gamma \left(\int_{\alpha}^{t} C(s)\, A(s)^{r}\, ds \right) \right], \ t \in [\alpha, \beta). \quad (2.52)$$

The proof follows by Corollary 95 on putting $I(t, x) = C(t)\, x^{r(t)}$, $t \in [\alpha, \beta)$, $x \in [0, \infty)$.

Proposition 98 *Let* $A, B, C : [\alpha, \beta) \to \mathbb{R}$ *be continuous and nonnegative on* $[\alpha, \beta)$. *If* $x : [\alpha, \beta) \to \mathbb{R}_+$ *is continuous and satisfies the relation*

$$x(t) \le A(t) + B(t)\, g \left(\int_{\alpha}^{t} C(s) \ln(x(s) + 1)\, ds \right), \ t \in [\alpha, \beta), \quad (2.53)$$

then

$$x(t) \le A(t) + B(t)\, g \circ \Gamma^{-1} \left[r \int_{\alpha}^{t} \frac{C(s)\, B(s)}{A(s) + 1} ds + \Gamma \left(\int_{\alpha}^{t} C(s) \ln(A(s) + 1)\, ds \right) \right] \quad (2.54)$$

for all $t \in [\alpha, \beta)$.

The proof is evident by Corollary 96 on putting $K(x) = \ln(x+1)$ where $x \in \mathbb{R}_+$.

Finally, we have the following.

Proposition 99 *Assume that A, B, C are nonnegative and continuous in $[\alpha, \beta)$. Then for every $x : [\alpha, \beta) \to \mathbb{R}_+$ a solution of the integral inequation*

$$x(t) \leq A(t) + B(t)\, g\left(\int_\alpha^t C(s) \arctan(x(s))\, ds\right),\ t \in [\alpha, \beta), \tag{2.55}$$

we have the estimation

$$x(t) \leq A(t) + B(t)\, g \circ \Gamma^{-1}\left[r \int_\alpha^t \frac{C(s)\, B(s)}{A^2(s)+1} ds + \Gamma\left(\int_\alpha^t C(s) \arctan(A(s))\, ds\right)\right] \tag{2.56}$$

for all $t \in [\alpha, \beta)$.

The proof follows by Corollary 96 on putting $K(x) := \arctan(x)$, $x \in \mathbb{R}_+$.

2.1.3 The Discrete Version

We give now a discrete version.

In paper [101], B.G. Pachpatte proved the following discrete inequality of Gronwall type.

Lemma 100 *Let $x, f, g, h : \mathbb{N} \to [0, \infty)$ be such that*

$$x(n) \leq f(n) + g(n) \sum_{s=0}^{n-1} h(s)\, x(s),\quad n \geq 1$$

then

$$x(n) \leq f(n) + g(n) \sum_{s=0}^{n-1} h(s)\, f(s) \prod_{\tau=s+1}^{n-1} (h(\tau)\, g(\tau) + 1)$$

for all $n \in \mathbb{N}^*$.

Further on, we shall give the discrete analogue of the nonlinear integral inequality:

$$x(t) \leq A(t) + B(t) \int_{\alpha}^{t} L(s, x(s))\, ds, \quad t \in [\alpha, \beta) \subset \mathbb{R}, \tag{2.57}$$

where all the involved functions are continuous and nonnegative as defined and the mapping L satisfies the condition

$$\begin{aligned} &0 \leq L(t,u) - L(t,v) \leq M(t,v)(u-v) \quad \text{for all } \ t \in [\alpha, \beta) \\ &u \geq v \geq 0 \ \text{ and } M \text{ is nonnegative continuous on } [\alpha, \beta) \times \mathbb{R}_{+}. \end{aligned} \tag{2.58}$$

The following theorem holds [39].

Theorem 101 *Let* $(A(n))_{n\in\mathbb{N}}$, $(B(n))_{n\in\mathbb{N}}$ *be nonnegative sequences and* $L : \mathbb{N} \times \mathbb{R}_{+} \to \mathbb{R}_{+}$ *be a mapping with the property*

$$\begin{aligned} 0 \leq L(n,x) - L(n,y) \leq M(n,y)(x-y) \ \textit{ for } n \in \mathbb{N},\ x \geq y \geq 0 \\ \textit{and } M \textit{ is nonnegative on } \mathbb{N} \times \mathbb{R}_{+}. \end{aligned} \tag{2.59}$$

If $x(n) \geq 0$ $(n \in \mathbb{N})$ *and :*

$$x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} L(s, x(s)), \ \textit{for } n \geq 1,$$

then the following estimation

$$x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} L(s, A(s)) \prod_{\tau=s+1}^{n-1} (M(\tau, A(\tau)) B(\tau) + 1)$$

holds for all $n \geq 1$.

Proof. Put $y(n) := \sum_{s=0}^{n-1} L(s, x(s))$ for $n \geq 1$ and $y(0) := 0$. Then we have:

$$\begin{aligned} y(n+1) - y(n) = L(n, x(n)) &\leq L(n, A(n) + B(n) y(n)) \\ &\leq L(n, A(n) + M(n, A(n)) B(n) y(n)) \end{aligned}$$

for $n \geq 0$, i.e.,

$$y(n+1) \leq L(n, A(n) + [M(n, A(n)) B(n) + 1] y(n)), \ n \geq 0.$$

Using the notation:

$$\alpha(n) := L(n, A(n)), \ \beta(n) := M(n, A(n)) B(n) + 1 > 0 \text{ for } n \geq 0,$$

and

$$z(n) := \frac{y(n)}{\prod_{\tau=0}^{n-1} \beta(\tau)} \text{ for } n \geq 1 \text{ and } z(0) = 0,$$

we may write

$$\left(\prod_{\tau=0}^{n} \beta(\tau)\right) z(n+1) \leq \alpha(n) + \left(\prod_{\tau=0}^{n} \beta(\tau)\right) z(n) \ \text{ for } n \geq 0$$

and since

$$\prod_{\tau=0}^{n} \beta(\tau) > 0 \ \ (n \geq 0)$$

we have

$$z(n+1) - z(n) \leq \frac{\alpha(n)}{\prod_{\tau=0}^{n} \beta(\tau)}.$$

Summing these inequalities, we deduce

$$z(n) \leq \frac{\sum_{s=0}^{n-1} \alpha(s)}{\prod_{\tau=0}^{s} \beta(\tau)}$$

which implies:

$$y(n) \leq \sum_{s=0}^{n-1} L(s, A(s)) \prod_{\tau=s+1}^{n-1} (M(\tau, A(\tau)) B(\tau) + 1)$$

for all $n \geq 1$ and the theorem is proved. ■

Corollary 102 *Let $A, B : \mathbb{N} \to \mathbb{R}$ be nonnegative sequences and $G : \mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}_+$ be a mapping satisfying the condition:*

$$0 \leq G(n,x) - G(n,y) \leq N(n)(x-y) \ \text{ for } n \in \mathbb{N}, \tag{2.60}$$
$$x \leq y \leq 0 \ \text{ and } N(n) \ \text{ is nonnegative for } n \in \mathbb{N}.$$

If $x(n) \geq 0 \quad (n \in \mathbb{N})$ *and*

$$x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} G(s, x(s)), \text{ for } n \geq 1,$$

then the following evaluation:

$$x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} G(s, A(s)) \prod_{\tau=s+1}^{n-1} (N(\tau) B(\tau) + 1)$$

holds, for $n \geq 1$.

The statement follows directly from the above theorem.

Corollary 103 *Let* $A, B, C : \mathbb{N} \to \mathbb{R}$ *be nonnegative sequences and* $H : \mathbb{R}_+ \to \mathbb{R}_+$ *be a function satisfying the following Lipschitz type condition:*

$$0 \leq H(x) - H(y) \leq M(x - y) \text{ where } M > 0, \, x \geq y \geq 0. \tag{2.61}$$

Then for any $(x(n))_{n \in \mathbb{N}}$ *a nonnegative sequence verifying the condition*

$$x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} C(s) H(x(s)), \; n \geq 1,$$

we have

$$x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} C(s) H(A(s)) \prod_{\tau=s+1}^{n-1} (MC(\tau) B(\tau) + 1)$$

for all $n \geq 1$.

The proof is obvious and we omit the details.

Remark 104 *If in the previous corollary we put:*

$A(n) = f(n), \; B(n) = g(n), \; C(n) = h(n) \; (n \in \mathbb{N})$ *and* $H(x) = x, \; (x \in \mathbb{R});$

we obtain the result of B.G. Pachpatte [101], see Lemma 100.

Another result of this subsection is the following [39].

Theorem 105 *Let $(A(n))_{n\in\mathbb{N}}$, $(B(n))_{n\in\mathbb{N}}$ be nonnegative sequences and $D:\mathbb{N}\times\mathbb{R}_+\to\mathbb{R}_+$ be a mapping satisfying the condition:*

$$\begin{aligned}&\textit{for any } n\in\mathbb{N},\ D(n,\cdot)\ \textit{ is differentiable on } \mathbb{R}_+,\\&\frac{dD(n,t)}{dt}\ \textit{is nonnegative}\\&\textit{for } n\in\mathbb{N},\ t\in\mathbb{R}_+ \textit{ and there exists a function}\\&P:\mathbb{N}\times\mathbb{R}_+\to\mathbb{R}_+\ \textit{such that } \frac{dD(n,u)}{Dt}\le P(n,v)\\&\textit{for each } n\in\mathbb{N} \textit{ and } u\ge v\ge 0.\end{aligned}\tag{2.62}$$

If $x(n)$ is nonnegative and verifies the inequality

$$x(n)\le A(n)+B(n)\sum_{s=0}^{n-1}D(s,x(s)),\ n\le 1,$$

then the following estimation is valid:

$$x(n)\le A(n)+B(n)\sum_{s=0}^{n-1}D(s,A(s))\prod_{\tau=s+1}^{n-1}(P(\tau,A(\tau))B(\tau)+1)$$

for all $n\ge 1$.

Proof. Let $n\in\mathbb{N}$. Then by Lagrange's theorem, for any $u\ge v\ge 0$ there exists $\mu_n\in(v,u)$ such that:

$$D(n,u)-D(n,v)=\frac{dD(u,\mu_n)}{dt}(u-v),$$

since

$$0\le\frac{dD(n,\mu_n)}{dt}\le P(n,v)\ \text{ for } n\ge 1,$$

we obtain

$$0\le D(n,u)-D(n,v)\le P(n,v)(u-v)\ \text{ for } u\ge v\ge 0$$

and $n\in\mathbb{N}$.

Applying Theorem 101 for $L(n,x)=D(n,x)$, $M(n,x)=P(n,x)$, $n\in\mathbb{N}$ and $x\in\mathbb{R}_+$, the proof is completed. ■

Corollary 106 *Let $A(n)$, $B(n)$ be nonnegative for $n \geq 1$ and $I : \mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}_+$ be such that:*

$$I(n,\cdot) \text{ is differentiable on } \mathbb{R}_+ \text{ for } n \in \mathbb{N}, \ \frac{dI(n,t)}{dt} \text{ is nonnegative} \tag{2.63}$$
$$\text{on } \mathbb{N} \times \mathbb{R}_+ \text{and } \frac{dI(n,u)}{dt} \leq \frac{dI(n,v)}{dt} \text{ for all } u \geq v \geq 0 \text{ and } n \in \mathbb{N}.$$

If $x(n) \geq 0$ $(n \in \mathbb{N})$ satisfies the inequality:

$$x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} I(s, x(s)), \ n \geq 1,$$

then we have

$$x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} I(s, A(s)) \prod_{\tau=s+1}^{n-1} \left(\frac{dI(\tau, A(\tau))}{dt} B(\tau) + 1 \right)$$

for all $n \geq 1$.

Finally, we have

Corollary 107 *Let $(A(n))_{n\in\mathbb{N}}$, $(B(n))_{n\in\mathbb{N}}$, $(C(n))_{n\in\mathbb{N}}$, be nonnegative sequences and $K : \mathbb{R}_+ \to \mathbb{R}_+$ be a mapping with the property:*

$$K \text{ is momotonic decreasing and differentiable on } \mathbb{R}_+ \text{ with} \tag{2.64}$$
$$\text{the derivative } \frac{dK}{dt} \text{ monotonic nonincreasing on } \mathbb{R}_+.$$

If $x(n)$ is nonnegative and

$$x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} C(s) K(x(s)), \ n \geq 1,$$

then

$$x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} C(s) K(A(s)) \prod_{\tau=s+1}^{n-1} \left(\frac{dK}{dt} A(\tau) B(\tau) C(\tau) + 1 \right)$$

for all $n \geq 1$.

Now, we can give some natural consequences of the above corollaries [39].

Consequences

1. Let $(A(n))_{n\in\mathbb{N}}$, $(B(n))_{n\in\mathbb{N}}$, $(C(n))_{n\in\mathbb{N}}$, $(r(n))_{n\in\mathbb{N}}$ and $(x(n))_{n\in\mathbb{N}}$ be nonnegative sequences and $A(n)>0$, $r(n)\leq 1$ for $n\geq 1$. If

$$x(n)\leq A(n)+B(n)\sum_{s=0}^{n-1}C(s)\left[x(s)\right]^{r(s)},\ n\geq 1, \tag{2.65}$$

then

$$x(n)\leq A(n)+B(n)\sum_{s=0}^{n-1}C(s)\,A(s)^{r(s)}\prod_{\tau=s+1}^{n-1}\left(\frac{r(\tau)\,B(\tau)\,C(\tau)}{\left[A(\tau)\right]^{1-r(\tau)}+1}\right)$$

for $n\geq 1$.

In particular, if $r\in[0,1]$, then

$$x(n)\leq A(n)+B(n)\sum_{s=0}^{n-1}C(s)\left[x(s)\right]^{r},\ n\geq 1, \tag{2.66}$$

implies

$$x(n)\leq A(n)+B(n)\sum_{s=0}^{n-1}C(s)\,A^{r}(s)\prod_{\tau=s+1}^{n-1}\left(\frac{r(\tau)\,B(\tau)\,C(\tau)}{A^{1-r}(\tau)+1}\right)$$

for all $n\geq 1$.

2. If $(A(n))_{n\in\mathbb{N}}$, $(B(n))_{n\in\mathbb{N}}$, $(C(n))_{n\in\mathbb{N}}$ and $(x(n))_{n\in\mathbb{N}}$ are nonnegative sequences and

$$x(n)\leq A(n)+B(n)\sum_{s=0}^{n-1}C(s)\ln\left(x(s)+1\right),\ n\geq 1, \tag{2.67}$$

then we have the estimation:

$$x(n)\leq A(n)+B(n)\sum_{s=0}^{n-1}C(s)\ln\left(A(s)+1\right)\prod_{\tau=s+1}^{n-1}\left(\frac{B(\tau)\,C(\tau)}{A(\tau)+1}+1\right)$$

for all $n\geq 1$.

3. Assume that $A, B, C, x : \mathbb{N} \to \mathbb{R}$ are as above and

$$x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} C(s) \arctan(x(s)), \; n \geq 1, \tag{2.68}$$

then the following inequality is also true:

$$x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} C(s) \arctan(A(s)) \prod_{\tau=s+1}^{n-1} \left(\frac{B(\tau) C(\tau)}{A^2(\tau) + 1} + 1 \right)$$

for all $n \geq 1$.

2.2 Boundedness Conditions

The main purpose of this section is to give some sufficient conditions of uniform boundedness for the nonnegative solutions defined in the interval $[\alpha, \infty)$ of the Gronwall integral inequations (2.2), (2.6), (2.9), (2.12), (2.15) and (2.18).

Theorem 108 *If the kernel L of the integral inequation* (2.2) *satisfies the relation* (2.1) *in $[\alpha, \infty)$ and the following conditions:*

$$A(t) \leq M_1, \; B(t) \leq M_2, \; t \in [\alpha, \infty) \tag{2.69}$$

$$\int_{\alpha}^{\infty} M(s, A(s))\, ds, \; \int_{\alpha}^{\infty} L(s, M_1)\, ds < \infty \tag{2.70}$$

hold, then there exists a constant $M > 0$ such that for every nonnegative continuous solution defined in $[\alpha, \infty)$ of (2.2) *we have $x(t) \leq M$ for any $t \in (\alpha, \infty)$ i.e., the nonnegative solutions of integral inequation* (2.2) *are uniformly bounded in $[\alpha, \infty)$.*

Proof. Let $x \in C([\alpha, \infty); \mathbb{R}_+)$ be a solution of (2.2). Applying Lemma 74, we have the following estimation:

$$x(t) \leq A(t) + B(t) \int_{\alpha}^{t} L(u, A(u)) \exp\left(\int_{u}^{t} M(s, A(s)) B(s)\, ds \right) du \tag{2.71}$$

for any $t \in [\alpha, \infty)$.

If the conditions (2.69) and (2.70) are satisfied, we obtain

$$x(t) \leq M_1 + M_2 \int_\alpha^\infty L(s, M_1)\, ds \exp\left(M_2 \int_\alpha^\infty M(s, A(s))\, ds \right)$$

in $[\alpha, \infty)$ and the theorem is proved. ■

Let us now suppose that the kernel G of the integral inequation (2.6) satisfies the relation (2.5) on $[\alpha, \infty)$ and the conditions (2.69) and

$$\int_\alpha^\infty N(s)\, ds, \int_\alpha^\infty G(s, M_1)\, ds < \infty \tag{2.72}$$

hold. Then the nonnegative continuous solutions of (2.6) are uniformly bounded in $[\alpha, \infty)$.

If we suppose that the kernel H of integral inequation (2.9) verifies the relation (2.8) on $[\alpha, \infty)$, and (2.69),

$$\int_\alpha^\infty C(s)\, ds < \infty \tag{2.73}$$

are true, then the nonnegative continuous solutions defined in $[\alpha, \infty)$ of (2.9) are also uniformly bounded in $[\alpha, \infty)$.

Further, we shall suppose that $A(t) > 0$ for all $t \in [\alpha, \infty)$.

If the kernel D of integral inequation (2.12) has the property (2.11) in $[\alpha, \infty)$ and the conditions (2.69) and

$$\int_\alpha^\infty D(s, M_1)\, ds, \int_\alpha^\infty P(s, A(s))\, ds < \infty \tag{2.74}$$

hold, then the nonnegative continuous solutions of (2.12) are uniformly bounded in $[\alpha, \infty)$.

If we assume that the kernel I of integral inequation (2.15) satisfies the relation (2.14) in $[\alpha, \infty)$ and (2.69) and

$$\int_\alpha^\infty I(s, M_1)\, ds, \int_\alpha^\infty \frac{\partial I(s, A(s))}{\partial x} ds < \infty \tag{2.75}$$

hold, then the nonnegative continuous solution of (2.15) is uniformly bounded in $[\alpha, \infty)$.

In particular, if the kernel K of integral inequation (2.18) verifies the relation (2.17) in $[\alpha,\infty)$ and the conditions (2.69) and

$$\int_{\alpha}^{\infty} C(s)\,ds,\ \int_{\alpha}^{\infty} \frac{dK}{dx}(A(s))\,C(s) < \infty \tag{2.76}$$

are valid, then the solutions of (2.18) are also uniformly bounded in the interval $[\alpha,\infty)$.

Finally, if we consider the integral inequation (2.20) defined in $[\alpha,\infty)$ and assume that the relations (2.69) and

$$\int_{\alpha}^{\infty} C(s)\,M^{r(s)}ds,\ \int_{\alpha}^{\infty} \frac{r(s)\,C(s)}{A(s)^{1-r(s)}} < \infty \tag{2.77}$$

hold, then the nonnegative solutions of (2.20) are uniformly bounded in $[\alpha,\infty)$.

Now we shall prove another theorem which gives sufficient conditions of uniform boundedness for the solutions of the above integral inequations.

Theorem 109 *If the kernel L of integral inequation* (2.2) *defined in* $[\alpha,\infty)$ *satisfies the relation* (2.1), *and the conditions*

$$A(t) \le M_1,\ \lim_{t\to\infty} B(t) = 0,\ t\in[\alpha,\infty), \tag{2.78}$$

$$\int_{\alpha}^{\infty} L(s,M_1)\,ds,\ \int_{\alpha}^{\infty} M(s,A(s))\,B(s) < \infty \ \ \textit{or} \tag{2.79}$$

$$M(t,A(t))\,B(t) \le \frac{k}{t},\ B(t)\,t^k \le l < \infty,\ \alpha,k>0, \tag{2.80}$$

$$t\in[\alpha,\infty)\ \ \textit{and}\ \int_{\alpha}^{\infty} \frac{L(s,M_1)}{s^k}ds < \infty$$

hold, then the nonnegative continuous solutions of (2.2) *are uniformly bounded in* $[\alpha,\infty)$.

Proof. Let $x\in C([\alpha,\infty);\mathbb{R}_+)$ be a solution of (2.2). Applying Lemma 74, we obtain the estimation (2.71). If the conditions (2.78) and (2.79) or (2.80) are satisfied, we have

$$x(t) \le M_1 + B(t)\int_{\alpha}^{\infty} L(s,M_1)\,ds\exp\left(\int_{\alpha}^{\infty} M(s,A(s))\,B(s)\,ds\right)$$

or

$$x(t) \leq M_1 + l \int_{\alpha}^{\infty} \frac{L(s, M_1)}{s^k} ds,\ t \in [\alpha, \infty).$$

Since B is continuous in $[\alpha, \infty)$ and $\lim_{t \to \infty} B(t) = 0$, it results that B is bounded in $[\alpha, \infty)$; and the theorem is proved. ■

Let us now suppose that the kernel G of the integral inequation (2.6) satisfies the relation (2.5) in the interval $[\alpha, \infty)$ and the following conditions (2.78) and

$$\int_{\alpha}^{\infty} G(s, M_1) ds,\quad \int_{\alpha}^{\infty} N(s) B(s) < \infty \qquad \text{or} \tag{2.81}$$

$$N(t) B(t) \leq \frac{k}{t},\quad B(t) t^k \leq l < \infty,\ \alpha, k > 0,\ t \in [\alpha, \infty) \tag{2.82}$$

hold, then the nonnegative continuous solutions of (2.6) are uniformly bounded in $[\alpha, \infty)$.

If we assume that the kernel H of integral inequation (2.9) verifies the relation (2.8) in $[\alpha, \infty)$ and the following conditions (2.78) and

$$\int_{\alpha}^{\infty} C(s) < \infty \qquad \text{or} \tag{2.83}$$

$$MC(t) B(t) \leq \frac{k}{t},\ B(t) t^k \leq l < \infty,\ \alpha, k > 0,\ t \in [\alpha, \infty) \tag{2.84}$$

$$\text{and } \int_{\alpha}^{\infty} \frac{C(s)}{s^k} ds < \infty$$

are valid, then the nonnegative continuous solutions of (2.9) are also uniformly bounded in $[\alpha, \infty)$.

Further, we shall suppose that $A(t) > 0$ for all $t \in [\alpha, \infty)$. In that assumption, and if the kernel D of the integral inequation (2.12) has the property (2.11) in $[\alpha, \infty)$ and the following conditions (2.78) and

$$\int_{\alpha}^{\infty} D(s, M_1) ds,\ \int_{\alpha}^{\infty} P(s, A(s)) ds < \infty \ \text{ or} \tag{2.85}$$

$$P(t, A(t)) B(t) \leq \frac{k}{t},\ B(t) t^k \leq l < \infty,\ \alpha, k > 0 \text{ and} \tag{2.86}$$

$$\int_{\alpha}^{\infty} \frac{D(s, M_1)}{s^k} ds < \infty$$

are true, then the nonnegative continuous solutions of (2.12) are uniformly bounded in $[\alpha, \infty)$.

If we assume that the kernel I of (2.15) satisfies the relation (2.14) in $[\alpha, \infty)$ and (2.78),

$$\int_{\alpha}^{\infty} I(s, M_1)\, ds, \int_{\alpha}^{\infty} \frac{\partial I(s, A(s))}{\partial x} B(s)\, ds < \infty \text{ or } \tag{2.87}$$

$$\frac{\partial I(t, A(t))}{\partial x} B(t) \leq \frac{k}{t},\ B(t)\, t^k \leq l < \infty,\ \alpha, k > 0 \tag{2.88}$$

$$\text{and } \int_{\alpha}^{\infty} \frac{I(s, M_1)}{s^k} ds < \infty$$

hold, then the nonnegative continuous solutions of (2.15) are uniformly bounded.

In particular, if the kernel K of integral inequation (2.18) verifies the relation (2.17) in $[\alpha, \infty)$ and the following conditions: (2.78) and

$$\int_{\alpha}^{\infty} C(s)\, ds, \int_{\alpha}^{\infty} \frac{dK}{dx}(A(s))\, C(s)\, B(s)\, ds < \infty \text{ or } \tag{2.89}$$

$$\frac{dK}{dx}(A(t))\, C(t)\, B(t) \leq \frac{k}{t},\ B(t)\, t^k \leq l < \infty,\ \alpha, k > 0 \tag{2.90}$$

$$\text{and } \int_{\alpha}^{\infty} \frac{C(s)}{s^k} ds < \infty$$

hold, then the nonnegative continuous solutions of (2.18) are also uniformly bounded in $[\alpha, \infty)$.

Consequences

1. Let us consider the integral inequation (2.20) of Section 2.1 defined for $t \in [\alpha, \infty)$. If the following conditions are satisfied: (2.78) and

$$\int_{\alpha}^{\infty} C(s)\, M_1^{r(s)} ds, \int_{\alpha}^{\infty} \frac{r(s)\, C(s)\, B(s)}{A(s)^{1-r(s)}} ds < \infty \quad \text{or} \tag{2.91}$$

$$\frac{r(t)\, C(t)\, B(t)}{A(t)^{1-r(t)}} \leq \frac{k}{t},\ B(t)\, t^k \leq l < \infty,\ \alpha, k > 0 \text{ and} \tag{2.92}$$

$$\int_{\alpha}^{\infty} \frac{C(s)\, M_1^{r(s)}}{s^k} ds < \infty;$$

then the nonnegative continuous solutions of (2.20) are uniformly bounded in $[\alpha, \infty)$.

2. Let us now consider the integral inequations (2.24) of Section 2.1 defined in $[\alpha, \infty)$. If the following conditions: (2.78) and

$$\frac{C(t)\, B(t)}{A(t)+1} \leq \frac{k}{t},\ B(t)\, t^k \leq l < \infty,\ t \in [\alpha, \infty) \tag{2.93}$$

$$\text{and} \quad \int_{\alpha}^{\infty} \frac{C(s)}{s^k} ds < \infty$$

are valid, then the nonnegative continuous solutions of (2.24) are bounded in $[\alpha, \infty)$.

3. Finally, if we consider the integral inequation (2.26) defined in $[\alpha, \infty)$ and if we assume that the following conditions are satisfied: (2.78) and

$$\frac{C(t)\, B(t)}{A^2(t)+1} \leq \frac{k}{t},\ B(t)\, t^k \leq l < \infty,\ \alpha, k > 0, \tag{2.94}$$

$$\text{and} \quad \int_{\alpha}^{\infty} \frac{C(s)}{s^k} ds < \infty$$

then the nonnegative continuous solutions of (2.26) are also uniformly bounded in $[\alpha, \infty)$.

We can now state another result.

Theorem 110 *If the kernel L of integral inequation* (2.2) *satisfies the relation* (2.1) *in* $[\alpha,\infty)$ *and the conditions*

$$A(t) \le M, \;\; t \in [\alpha,\infty); \tag{2.95}$$

there exists a function $U:[\alpha,\infty)\to\mathbb{R}_+^*$ *differentiable in* (α,∞) (2.96)

such that $B(t) \le \dfrac{1}{U(t)}$, $t\in[\alpha,\infty)$ *and* $\lim\limits_{t\to\infty} U(t)=\infty$,

$$\lim_{t\to\infty}\frac{L(t,M)}{U'(t)} = l<\infty, \quad \int_\alpha^\infty M(s,A(s))\,B(s)\,ds<\infty \tag{2.97}$$

or

$$\lim_{t\to\infty}\frac{\exp\left(\int_\alpha^t M(s,A(s))\,B(s)\,ds\right)}{U(t)} = l<\infty, \;\; \text{and} \tag{2.98}$$
$$\int_\alpha^\infty \frac{L(s,M_1)\,ds}{\exp\left(\int_\alpha^s M(u,A(u))\,B(u)\,du\right)}<\infty$$

or

$$M(t,A(t))\,B(t) \le \frac{k}{t}, \;\; k,\alpha>0, t \in [\alpha,\infty); \tag{2.99}$$

and

$$\lim_{t\to\infty}\frac{t^k\int_\alpha^t \frac{L(s,M_1)}{s^k}ds}{U(t)} = l<\infty$$

hold, then the nonnegative continuous solutions of (2.2) *are uniformly bounded in* $[\alpha,\infty)$.

Proof. Let $x\in C([\alpha,\infty);\mathbb{R}_+)$ be a solution of (2.2). Applying Lemma 74, we obtain the estimation (2.71). If the conditions (2.95), (2.96) and (2.97) or (2.98) or (2.99) are satisfied, we have:

$$x(t) \le M_1 + \frac{\int_\alpha^t L(s,M_1)}{U(t)}\exp\left(\int_\alpha^\infty M(s,A(s))\,B(s)\,ds\right)$$

or

$$x(t) \le M_1 + \frac{\exp\left(\int_\alpha^t M(s,A(s))\,B(s)\,ds\right)}{U(t)}\int_\alpha^\infty \frac{L(s,M_1)\,ds}{\exp\left(\int_\alpha^s M(u,A(u))\,B(u)\,du\right)}$$

or

$$x(t) \le M_1 + \frac{t^k \int_\alpha^t \frac{L(s,M_1)}{s^k} ds}{U(t)} \text{ for all } t \in [\alpha, \infty).$$

Since

$$\lim_{t\to\infty} \frac{\int_\alpha^t L(s,M_1)\,ds}{U(t)} = \lim_{t\to\infty} \frac{L(s,M_1)}{U'(t)} = l < \infty,$$
$$\lim_{t\to\infty} \frac{\exp\left(\int_\alpha^t M(s,A(s))\,B(s)\,ds\right)}{U(t)} = l < \infty,$$
$$\lim_{t\to\infty} \frac{t^k \int_\alpha^t \frac{L(s,M_1)}{s^k} ds}{U(t)} = l < \infty$$

and the functions are continuous in $[\alpha, \infty)$, then they are bounded in $[\alpha, \infty)$ and the theorem is thus proved. ■

Let us now suppose that the kernel (2.5) of integral inequation (2.6) satisfies the relation (2.5) in $[\alpha, \infty)$. If the following conditions: (2.95), (2.96) and, either,

$$\lim_{t\to\infty} \frac{G(t,M_1)}{U'(t)} = l < \infty, \quad \int_\alpha^\infty N(s)\,B(s)\,ds < \infty \tag{2.100}$$

or

$$\lim_{t\to\infty} \frac{\exp\left(\int_\alpha^t N(s)\,B(s)\,ds\right)}{U(t)} = l < \infty, \tag{2.101}$$
$$\int_\alpha^\infty \frac{G(s,M_1)}{\exp\left(\int_\alpha^s N(u)\,B(u)\,du\right)} ds < \infty,$$

or

$$N(t)\,B(t) \le \frac{k}{t}, \; k, \alpha > 0 \text{ and} \tag{2.102}$$
$$\lim_{t\to\infty} \frac{t^k \int_\alpha^t \frac{C(s)}{s^k} ds}{U(t)} = l < \infty$$

hold, then the nonnegative continuous solutions of integral inequation (2.6) are uniformly bounded in $[\alpha, \infty)$.

If we assume that the kernel H of integral inequation (2.9) verifies the relation (2.8) in the interval $[\alpha, \infty)$ and the following assertion: (2.95), (2.96) and, either,

$$\lim_{t\to\infty} \frac{C(t)}{U'(t)} = l < \infty, \quad \int_\alpha^\infty C(s) B(s)\, ds < \infty \tag{2.103}$$

or

$$\lim_{t\to\infty} \frac{\exp\left(M \int_\alpha^t C(s) B(s)\, ds\right)}{U(t)} = l < \infty, \tag{2.104}$$

$$\int_\alpha^\infty \frac{C(s)}{\exp\left(M \int_\alpha^s C(u) B(u)\, du\right)} ds < \infty$$

or

$$MC(t) B(t) \leq \frac{k}{t}, \; k, \alpha > 0 \text{ and} \tag{2.105}$$

$$\lim_{t\to\infty} \frac{t^k \int_\alpha^t \frac{C(s)}{s^k} ds}{U(t)} = l < \infty$$

are valid, then the nonnegative continuous solutions of (2.9) are uniformly bounded in $[\alpha, \infty)$.

Further, we shall suppose that $A(t) > 0$ for all $t \in [\alpha, \infty)$. In that assumption, if the kernel D of integral inequation (2.12) satisfies the relation (2.11) in $[\alpha, \infty)$ and the following conditions: (2.95), (2.96) and, either,

$$\lim_{t\to\infty} \frac{D(t, M_1)}{U'(t)} = l < \infty, \quad \int_\alpha^\infty P(s, A(s)) B(s)\, ds < \infty \tag{2.106}$$

or

$$\lim_{t\to\infty} \frac{\exp\left(\int_\alpha^t P(s, A(s)) B(s)\, ds\right)}{U(t)} = l < \infty, \tag{2.107}$$

$$\int_\alpha^\infty \frac{D(s, M_1)\, ds}{\exp\left(\int_\alpha^s P(u, A(u)) B(u)\, du\right)} < \infty$$

or

$$P(t,A(t))B(t)\leq \frac{k}{t},\ k,\alpha>0,\ t\in[\alpha,\infty) \text{ and} \tag{2.108}$$
$$\lim_{t\to\infty}\frac{t^k\int_\alpha^t \frac{D(s,M_1)}{s^k}ds}{U(t)}=l<\infty$$

hold, then the nonnegative continuous solutions of (2.12) are uniformly bounded in $[\alpha,\infty)$.

If we assume that the kernel I of (2.15) satisfies the relation (2.14) in $[\alpha,\infty)$ and (2.95), (2.96) and, either,

$$\lim_{t\to\infty}\frac{I(t,M_1)}{U'(t)}=l<\infty,\ \int_\alpha^\infty \frac{\partial I}{\partial x}(s,A(s))B(s)\,ds<\infty \tag{2.109}$$

or

$$\lim_{t\to\infty}\frac{\exp\left(\int_\alpha^t \frac{\partial I}{\partial x}(s,A(s))B(s)\,ds\right)}{U(t)}=l<\infty \quad\text{and} \tag{2.110}$$
$$\int_\alpha^\infty \frac{I(s,M_1)\,ds}{\exp\left(\int_\alpha^s \frac{\partial I}{\partial x}(u,A(u))B(u)\,du\right)}<\infty$$

or

$$\frac{\partial I}{\partial x}(t,A(t))B(t)\leq\frac{k}{t},\ k,\alpha>0,\ t\in[\alpha,\infty) \quad\text{and} \tag{2.111}$$
$$\lim_{t\to\infty}\frac{t^k\int_\alpha^t \frac{I(s,M_1)}{s^k}ds}{U(t)}=l<\infty$$

are true, then the nonnegative continuous solutions of (2.15) are uniformly bounded in $[\alpha,\infty)$.

In particular, if the kernel K of integral inequation (2.18) verifies the relation (2.17) in $[\alpha,\infty)$ and the following conditions (2.95), (2.96) and, either,

$$\lim_{t\to\infty}\frac{C(t)}{U'(t)}=l<\infty,\quad \int_\alpha^\infty \frac{dK}{dx}(A(a))C(s)B(s)\,ds<\infty \tag{2.112}$$

or

$$\lim_{t\to\infty}\frac{\exp\left(\int_\alpha^\infty \frac{dK}{dx}(A(s))B(s)C(s)\,ds\right)}{U(t)}=l<\infty \ \text{ and} \tag{2.113}$$
$$\int_\alpha^\infty \frac{C(s)\,ds}{\exp\left(\int_\alpha^s \frac{dK}{dx}(A(u))B(u)C(u)\,du\right)}<\infty$$

or

$$\frac{dK}{dx}\left(A\left(t\right)\right)B\left(t\right)C\left(t\right)\leq\frac{k}{t},\ k,\alpha>0,\ t\in\left[\alpha,\infty\right)\ \text{ and } \tag{2.114}$$

$$\lim_{t\to\infty}\frac{t^{k}\int_{\alpha}^{t}\frac{C(s)}{s^{k}}ds}{U\left(t\right)}=l<\infty$$

hold, then the nonnegative continuous solutions of (2.18) are uniformly bounded in $[\alpha,\infty)$.

Consequences

1. Let us consider the integral inequation (2.20) of Section 2.1. If the following conditions are satisfied: (2.95), (2.96) and, either,

$$\lim_{t\to\infty}\frac{C\left(t\right)M^{r(t)}}{U'\left(t\right)}=l<\infty,\quad\int_{\alpha}^{\infty}\frac{r\left(s\right)C\left(s\right)B\left(s\right)}{A\left(s\right)^{1-r(s)}}ds<\infty \tag{2.115}$$

or

$$\lim_{t\to\infty}\frac{\exp\left(\int_{\alpha}^{t}\frac{r(s)C(s)B(s)}{A(s)^{1-r(s)}}ds\right)}{U\left(t\right)}=l<\infty\ \text{ and } \tag{2.116}$$

$$\int_{\alpha}^{\infty}\frac{C\left(s\right)}{\exp\left(\int_{\alpha}^{s}\frac{r(u)C(u)B(u)}{A(u)^{1-r(u)}}du\right)}ds<\infty$$

or

$$\frac{r\left(t\right)C\left(t\right)B\left(t\right)}{A\left(t\right)^{1-r(t)}}\leq\frac{k}{t},\ k,\alpha>0,\ t\in\left[\alpha,\infty\right)\ \text{ and } \tag{2.117}$$

$$\lim_{t\to\infty}\frac{t^{k}\int_{\alpha}^{t}\frac{C(s)}{s^{k}}ds}{U\left(t\right)}=l<\infty,$$

hold, then the nonnegative continuous solutions of (2.20) are uniformly bounded in $[\alpha,\infty)$.

2. Finally, we shall consider the integral inequation (2.24) of Section 2.1. If we assume that the conditions (2.95), (2.96) and, either,

$$\lim_{t\to\infty}\frac{C\left(t\right)}{U'\left(t\right)}=l<\infty,\quad\int_{\alpha}^{\infty}\frac{C\left(s\right)B\left(s\right)}{A\left(s\right)+1}ds<\infty \tag{2.118}$$

or

$$\lim_{t\to\infty}\frac{\exp\left(\int_\alpha^t \frac{C(s)B(s)}{A(s)+1}ds\right)}{U(t)} = l < \infty, \tag{2.119}$$
$$\int_\alpha^\infty \frac{C(s)}{\exp\left(\int_\alpha^s \frac{C(u)B(u)}{A(u)+1}du\right)}ds < \infty$$

or

$$\frac{C(t)B(t)}{A(t)+1} \leq \frac{k}{t},\ k,\alpha > 0,\ t \in [\alpha,\infty) \text{ and} \tag{2.120}$$
$$\lim_{t\to\infty}\frac{t^k\int_\alpha^t \frac{C(s)}{s^k}ds}{U(t)} = l < \infty$$

hold, then the nonnegative continuous solutions of (2.24) are uniformly bounded in $[\alpha,\infty)$.

Now we shall prove another result which is embodied in the following theorem.

Theorem 111 *If the kernel L of integral inequation* (2.2) *satisfies the relation* (2.1) *in* $[\alpha,\infty)$ *and the following conditions*

$$\lim_{t\to\infty} A(t) = 0, \quad B(t) \leq M_1, \quad t \in [\alpha,\infty), \tag{2.121}$$
$$\int_\alpha^\infty M(s,A(s))\,ds, \quad \int_\alpha^\infty L(s,A(s))\,ds < \infty \tag{2.122}$$

hold, then the nonnegative continuous solutions of (2.2) *are uniformly bounded in* $[\alpha,\infty)$.

Proof. Let $x \in C([\alpha,\infty);\mathbb{R}_+)$ be a solution of (2.2). Applying Lemma 74, we have the estimation (2.71). If (2.121) and (2.122) hold, then we have the evaluation

$$x(t) \leq A(t) + M_1\int_\alpha^\infty L(s,A(s))\,ds \exp\left(M_1\int_\alpha^\infty M(s,A(s))\,ds\right)$$

for all $t \in [\alpha,\infty)$.

Since A is continuous in $[\alpha, \infty)$ and $\lim_{t \to \infty} A(t) = 0$, it follows that A is bounded on $[\alpha, \infty)$ and the theorem is proved. ■

Let us now suppose that the kernel G of integral inequation (2.6) satisfies the relation (2.5) in $[\alpha, \infty)$ and the following conditions: (2.121) and

$$\int_\alpha^\infty N(s)\, ds, \quad \int_\alpha^\infty G(s, A(s))\, ds < \infty \tag{2.123}$$

hold. Then the nonnegative continuous solutions of (2.6) are uniformly bounded in $[\alpha, \infty)$.

If the kernel H of integral inequation (2.9) satisfies the relation (2.8) in $[\alpha, \infty)$ and the following assertion: (2.121) and

$$\int_\alpha^\infty C(s)\, ds < \infty \tag{2.124}$$

are true, then the nonnegative continuous solutions of (2.9) are uniformly bounded in $[\alpha, \infty)$.

In what follows, we shall suppose that $A(t) > 0$ for all $t \in [\alpha, \infty)$. In that assumption, if the kernel D of (2.12) satisfies the relation (2.11) in $[\alpha, \infty)$ and the following conditions: (2.121) and (2.125) where

$$\int_\alpha^\infty P(s, A(s))\, ds, \quad \int_\alpha^\infty D(s, A(s))\, ds < \infty \tag{2.125}$$

hold, then the nonnegative continuous solutions of (2.12) are uniformly bounded in $[\alpha, \infty)$.

If we assume that the kernel I of the integral inequation (2.15) verifies the relation (2.14) and the following assertions: (2.121) and

$$\int_\alpha^\infty \frac{\partial I}{\partial x}(s, A(s))\, ds, \quad \int_\alpha^\infty I(s, A(s))\, ds < \infty \tag{2.126}$$

are valid, then the nonnegative continuous solutions of (2.15) are uniformly bounded in $[\alpha, \infty)$.

In particular, if we suppose that the kernel K of (2.18) satisfies the condition (2.17) in $[\alpha, \infty)$ and the following conditions: (2.121) and

$$\int_\alpha^\infty \frac{dK}{dx}(A(s))\, C(s)\, ds, \quad \int_\alpha^\infty C(s)\, K(A(s))\, ds < \infty \tag{2.127}$$

holds, then the nonnegative continuous solutions of (2.18) are also uniformly bounded in the interval $[\alpha, \infty)$.

Consequences

1. Let us consider the integral inequation (2.20) of Section 2.1. If the following conditions are satisfied: (2.121) and

$$\int_{\alpha}^{\infty} \frac{r(s) C(s)}{A(s)^{1-r(s)}} ds, \quad \int_{\alpha}^{\infty} C(s) A(s)^{r(s)} ds < \infty \tag{2.128}$$

   then the nonnegative continuous solutions of (2.20) are uniformly bounded in $[\alpha, \infty)$.

2. Finally, if we consider the integral inequation (2.24) of Section 2.1 and suppose that the following assertions are satisfied (2.121) and

$$\int_{\alpha}^{\infty} \frac{C(s)}{A(s)+1} ds, \quad \int_{\alpha}^{\infty} C(s) \ln(A(s)+1) ds < \infty \tag{2.129}$$

   then the nonnegative continuous solutions of (2.24) are also uniformly bounded in $[\alpha, \infty)$.

The following theorem also holds.

Theorem 112 *If the kernel L of integral inequation* (2.2) *satisfies the relation* (2.1) *on* $[\alpha, \infty)$ *and the following conditions*

$$\lim_{t\to\infty} A(t) = 0, \; B(t) t^k \le l < \infty, \; \alpha, k > 0, \; t \in [\alpha, \infty) \tag{2.130}$$

$$M(t, A(t)) B(t) \le \frac{k}{t}, \quad \int_{\alpha}^{\infty} \frac{L(s, A(s))}{s^k} ds < \infty \tag{2.131}$$

hold, then the nonnegative continuous solutions of (2.2) *are uniformly bounded in* $[\alpha, \infty)$.

Proof. Let $x \in C([\alpha, \infty); \mathbb{R}_+)$ be a solution of (2.2). Applying Lemma 74, we have the estimation (2.71).

If the conditions (2.130), (2.131) are satisfied, we have the estimation

$$x(t) \le A(t) + l \int_{\alpha}^{\infty} \frac{L(s, A(s))}{s^k} ds,\ t \in [\alpha, \infty).$$

Since A is continuous and $\lim_{t\to\infty} A(t) = 0$, it results that A is bounded in $[\alpha, \infty)$ and the theorem is proved. ■

Let us now suppose that the kernel G of integral inequation (2.6) satisfies the relation (2.5) in $[\alpha, \infty)$ and the following conditions (2.130) and

$$N(t) B(t) \le \frac{k}{t},\ t \in [\alpha, \infty),\ \int_{\alpha}^{\infty} \frac{G(s, A(s))}{s^k} ds < \infty \tag{2.132}$$

hold, then the nonnegative continuous solutions of (2.6) are uniformly bounded in $[\alpha, \infty)$.

If the kernel H of the integral inequation (2.9) verifies the relation (2.8) in $[\alpha, \infty)$ and the assertions (2.130) and

$$M\, C(t) B(t) \le \frac{k}{t},\ t \in [\alpha, \infty),\ \int_{\alpha}^{\infty} \frac{C(s) H(A(s))}{s^k} ds < \infty \tag{2.133}$$

are true, then the nonnegative continuous solutions of (2.9) are uniformly bounded in $[\alpha, \infty)$.

Further, we shall suppose that $A(t) > 0$ for all $t \in [\alpha, \infty)$. In that assumption, if the kernel D of (2.12) satisfies the relation (2.11) in $[\alpha, \infty)$ and the following conditions: (2.130) and

$$P(t, A(t)) B(t) \le \frac{k}{t},\ t \in [\alpha, \infty),\ \int_{\alpha}^{\infty} \frac{D(s, A(s))}{s^k} ds < \infty \tag{2.134}$$

hold, then the nonnegative solutions of (2.12) are uniformly bounded in $[\alpha, \infty)$.

If we assume the kernel I of integral inequation (2.15) verifies the relation (2.14) in $[\alpha, \infty)$ and the assertions (2.130) and

$$\frac{\partial I}{\partial x}(t, A(t)) B(t) \le \frac{k}{t},\ t \in [\alpha, \infty),\ \int_{\alpha}^{\infty} \frac{I(s, A(s))}{s^k} ds < \infty \tag{2.135}$$

are valid, then the nonnegative continuous solutions on (2.15) are uniformly bounded in $[\alpha, \infty)$.

In particular, if we suppose that the kernel K of integral inequation (2.18) satisfies the condition (2.17) in $[\alpha, \infty)$, and the following conditions: (2.130) and

$$\frac{dK}{dx}(A(t))\,B(t)\,C(t) \leq \frac{k}{t},\ t \in [\alpha, \infty), \tag{2.136}$$
$$\int_{\alpha}^{\infty} \frac{C(s)\,K(A(s))}{s^k}ds < \infty$$

hold, then the nonnegative continuous solutions of (2.18) are also uniformly bounded in $[\alpha, \infty)$.

Consequences

1. Let us consider the integral inequation (2.20) of Section 2.1. If the following conditions are satisfied: (2.130) and
$$\frac{r(t)\,C(t)\,B(t)}{A(t)^{1-r(t)}} \leq \frac{k}{t},\ t \in [\alpha, \infty),\ \int_{\alpha}^{\infty} C(s)\,A(s)^{r(s)}\,ds < \infty \tag{2.137}$$
then the nonnegative continuous solutions of (2.20) are uniformly bounded in $[\alpha, \infty)$.

2. Finally, we consider the integral inequation (2.24) of Section 2.1. If the conditions (2.130) and
$$\frac{C(t)\,B(t)}{A(t)+1} \leq \frac{k}{t},\ t \in [\alpha, \infty),\ \int_{\alpha}^{\infty} \frac{C(s)\ln(A(s)+1)}{s^k}ds < \infty \tag{2.138}$$
are satisfied, then the nonnegative continuous solutions of (2.24) are also uniformly bounded in $[\alpha, \infty)$.

Now we shall prove the last theorem of this section.

Theorem 113 *If the kernel L of integral inequation* (2.2) *satisfies the relation* (2.1) *in* $[\alpha, \infty)$ *and the following conditions:*

$$\lim_{t\to\infty} A(t) = 0, \tag{2.139}$$

there exists a function $U : [\alpha, \infty) \to \mathbb{R}_+^*$ *differentiable on* (2.140)

(α, ∞) *such that* $B(t) \leq \dfrac{1}{U(t)}$, $t \in [\alpha, \infty)$ *and* $\lim_{t\to\infty} U(t) = \infty$,

$$\lim_{t\to\infty} \frac{L(t, A(t))}{U'(t)} = l < \infty,\ \int_{\alpha}^{\infty} M(s, A(s))\,B(s)\,ds < \infty \tag{2.141}$$

or

$$\lim_{t\to\infty}\frac{\exp\left(\int_\alpha^t M\left(s,A\left(s\right)\right)B\left(s\right)ds\right)}{U\left(t\right)}=l<\infty \tag{2.142}$$

$$\int_\alpha^\infty \frac{L\left(s,A\left(s\right)\right)}{\exp\left(\int_\alpha^s M\left(u,A\left(u\right)\right)B\left(u\right)du\right)}ds<\infty$$

or

$$M\left(t,A\left(t\right)\right)B\left(t\right)\le\frac{k}{t},\ t\in\left[\alpha,\infty\right),\ \alpha,k>0\quad \textit{and} \tag{2.143}$$

$$\lim_{t\to\infty}\frac{t^k\int_\alpha^t\frac{L(s,A(s))}{s^k}ds}{U\left(t\right)}=l<\infty$$

hold, then the nonnegative continuous solutions of (2.2) *are uniformly bounded in* $[\alpha,\infty)$.

Proof. Let $x\in([\alpha,\infty);\mathbb{R}_+)$ be a solution of (2.2). Applying Lemma 74, we have the estimation (2.3). If the conditions (2.139), (2.140) and, either, (2.141) or (2.142) or (2.143) are satisfied, then we have either the evaluation

$$x\left(t\right)\le A\left(t\right)+\frac{\int_\alpha^t L\left(u,A\left(u\right)\right)}{U\left(t\right)}du\exp\left(\int_\alpha^\infty M\left(s,A\left(s\right)\right)B\left(s\right)ds\right)$$

or

$$x\left(t\right)\le A\left(t\right)+\frac{\exp\left(\int_\alpha^t M\left(s,A\left(s\right)\right)B\left(s\right)ds\right)}{U\left(t\right)} \times\int_\alpha^\infty\frac{L\left(s,A\left(s\right)\right)ds}{\exp\left(\int_\alpha^s M\left(u,A\left(u\right)\right)B\left(u\right)du\right)}$$

or

$$x\left(t\right)\le A\left(t\right)+\frac{t^k\int_\alpha^t\frac{L(s,A(s))}{s^k}ds}{U\left(t\right)},\quad t\in[\alpha,\infty).$$

Since $\lim_{t\to\infty} A(t) = 0$ and

$$\lim_{t\to\infty} \frac{\int_\alpha^t L(u, A(u))\,du}{U(t)} = \lim_{t\to\infty} \frac{L(t, A(t))}{U'(t)} = l < \infty,$$

$$\lim_{t\to\infty} \frac{\exp\left(\int_\alpha^t L(s, A(s))\,B(s)\,ds\right)}{U(t)} = l < \infty,$$

and

$$\lim_{t\to\infty} \frac{t^k \int_\alpha^t \frac{L(s,A(s))}{s^k} ds}{U(t)} = l < \infty$$

and the functions are continuous on $[\alpha, \infty)$, it follows that they are bounded in $[\alpha, \infty)$ and the theorem is proved. ∎

Observation

If we suppose that the kernels G, H, D, I, K of the integral inequations (2.6), (2.9), (2.12), (2.15), (2.18) satisfy the conditions (2.5), (2.8), (2.11), (2.14), (2.17) in the interval $[\alpha, \infty)$, by the above theorem, we can deduce a large number of corollaries which may be useful in applications. We omit the details.

2.3 Convergence to Zero Conditions

The main purposes of this section are to give some conditions of convergence to zero at infinity for the solutions of Gronwall integral inequations defined in the interval $[\alpha, \infty)$ with kernels which satisfy the relation (2.1) in that interval.

Theorem 114 *If the kernel L of integral inequation* (2.2) *satisfies the relation* (2.1) *in* $[\alpha, \infty)$ *and the following conditions*

$$\lim_{t\to\infty} A(t) = 0, \quad \lim_{t\to\infty} B(t) = 0 \tag{2.144}$$

$$\int_\alpha^\infty L(s, A(s))\,ds, \quad \int_\alpha^\infty M(s, A(s))\,B(s)\,ds < \infty \tag{2.145}$$

hold, then for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > \alpha$ such that for any $x \in C([\alpha, \infty); \mathbb{R}_+)$ a solution of (2.2), *we have $x(t) < \varepsilon$ if $t > \delta(\varepsilon)$, i.e., the* *nonnegative continuous solutions of* (2.2) *are uniformly convergent to zero at infinity.*

Proof. Let $\varepsilon > 0$. If $x \in C([\alpha, \infty); \mathbb{R}_+)$ is a solution of (2.2), applying Lemma 74, we have the estimation (2.71) of Section 2.2. If the conditions (2.144) and (2.145) are satisfied, then we have

$$x(t) \leq A(t) + B(t) \int_\alpha^\infty L(s, A(s))\, ds \exp\left(\int_s^t M(s, A(s))\, B(s)\, ds\right)$$

for all $t \in [\alpha, \infty)$ and there exists $\delta_1(\varepsilon) > \alpha$, $\delta_2(\varepsilon) > \alpha$ such that

$$A(t) < \frac{\varepsilon}{2} \text{ if } t > \delta_1(\varepsilon)$$

and

$$B(t) \int_\alpha^\infty L(s, A(s))\, ds \exp\left(\int_\alpha^\infty M(s, A(s))\, B(s)\, ds\right) < \frac{\varepsilon}{2} \text{ if } t > \delta_2(\varepsilon)$$

from where results

$$x(t) < \varepsilon \text{ if } t > \delta(\varepsilon) = \max(\delta_1(\varepsilon), \delta_2(\varepsilon)).$$

Hence the theorem is proved. ■

Let us now suppose that the kernel G of integral inequation (2.6) satisfies the relation (2.5) in $[\alpha, \infty)$ and the following conditions: (2.144) and

$$\int_\alpha^\infty G(s, A(s))\, ds, \int_\alpha^\infty N(s)\, B(s)\, ds < \infty \tag{2.146}$$

hold, then the nonnegative continuous solutions of (2.6) are uniformly convergent to zero at infinity.

If the kernel H of (2.9) verifies the relation (2.8) in $[\alpha, \infty)$ and the following assertions: (2.144) and

$$\int_\alpha^\infty C(s)\, H(A(s))\, ds, \int_\alpha^\infty C(s)\, B(s)\, ds < \infty \tag{2.147}$$

are true, then the nonnegative continuous solutions of (2.9) are uniformly convergent to zero at infinity.

In what follows, we shall suppose that $A(t) > 0$ for all $t \in [\alpha, \infty)$. In that assumption, if the kernel D of integral inequation (2.12) satisfies the relation (2.11) in $[\alpha, \infty)$ and the following conditions: (2.144) and

$$\int_\alpha^\infty D(s, A(s))\, ds,\ \int_\alpha^\infty P(s) A(s) B(s)\, ds < \infty \tag{2.148}$$

hold, then the nonnegative continuous solutions of (2.12) are uniformly convergent to zero at infinity.

If we assume that the kernel I of integral inequation (2.15) verifies the assertion: (2.144) and

$$\int_\alpha^\infty I(s, A(s))\, ds,\ \int_\alpha^\infty \frac{\partial I}{\partial x}(s, A(s)) B(s)\, ds < \infty \tag{2.149}$$

are valid, then the nonnegative continuous solutions of (2.15) are uniformly convergent to zero at infinity.

In particular, if the kernel K of (2.18) verifies the relation (2.17) in $[\alpha, \infty)$ and the condition: (2.144) and

$$\int_\alpha^\infty C(s) K(A(s))\, ds,\ \int_\alpha^\infty \frac{dK}{dx}(A(s)) B(s) C(s)\, ds < \infty \tag{2.150}$$

hold, then the nonnegative continuous solutions of (2.18) are uniformly convergent to zero at infinity.

Consequences

1. Let us consider the integral inequation (2.20) of Section 2.1. If the following conditions are satisfied: (2.144) and

$$\int_\alpha^\infty C(s)\, ds,\ \int_\alpha^\infty \frac{r(s) B(s) C(s)}{A(s)^{1-r(s)}} ds < \infty, \tag{2.151}$$

then the nonnegative continuous solutions of (2.20) are uniformly convergent to zero at infinity.

2. Finally, if we consider the integral inequation (2.24) of Section 2.1 and the following conditions are satisfied: (2.144) and

$$\int_\alpha^\infty C(s) \ln(A(s) + 1)\, ds,\ \int_\alpha^\infty \frac{B(s) C(s)}{A(s) + 1} ds < \infty, \tag{2.152}$$

then the nonnegative continuous solutions of (2.24) are uniformly convergent to zero at infinity.

Now we shall prove another result which is embodied in the following theorem.

Theorem 115 *If the kernel L of integral inequation* (2.2) *satisfies the relation* (2.1) *in* $[\alpha, \infty)$ *and the following conditions*

$$\lim_{t\to\infty} A(t) = 0, \quad \lim_{t\to\infty} t^k B(t) = 0, \ k > 0; \tag{2.153}$$

$$M(t, A(t)) B(t) \le \frac{k}{t}, \ t \in [\alpha, \infty), \ \alpha > 0, \ \int_\alpha^\infty \frac{L(s, A(s))}{s^k} ds \tag{2.154}$$

hold, then the nonnegative continuous solutions of (2.2) *are uniformly convergent to zero at infinity.*

Proof. Let $\varepsilon > 0$. If $x \in C([\alpha, \infty); \mathbb{R}_+)$ is a solution of (2.2), applying Lemma 74, we have the estimation (2.71).

If the conditions (2.153), (2.154) are satisfied, we have the evaluation

$$x(t) \le A(t) + B(t)\, t^k \int_\alpha^\infty \frac{L(s, A(s))}{s^k} ds, \ \ t \in [\alpha, \infty)$$

and there exists $\delta_1(\varepsilon) > \alpha$, $\delta_2(\varepsilon) > \alpha$ such that

$$A(t) < \frac{\varepsilon}{2} \text{ if } \ t > \delta_1(\varepsilon)$$

and

$$B(t)\, t^k \int_\alpha^\infty \frac{L(s, A(s))}{s^k} ds < \frac{\varepsilon}{2} \text{ if } \ t > \delta_2(\varepsilon)$$

from where results $x(t) < \varepsilon$ if $t > \max(\delta_1(\varepsilon), \delta_2(\varepsilon)) = \delta(\varepsilon)$ and the theorem is thus proved. ■

Remark 116 *If we assume that the kernel L of integral inequation* (2.2) *satisfies the relations* (2.5) *or* (2.11) *or* (2.14) *in* $[\alpha, \infty)$, *we can deduce a large number of corollaries of the above theorem. We omit the details.*

The following theorem also holds.

Theorem 117 *If the kernel L of integral inequation* (2.2) *satisfies the relation* (2.1) *in* $[\alpha,\infty)$ *and the following conditions:*

$$\lim_{t\to\infty} A(t)=0; \tag{2.155}$$

there exists a function $U:[\alpha,\infty)\to\mathbb{R}_+^*$ *differentiable on* (2.156)

$$(\alpha,\infty) \text{ such that } B(t)\le\frac{1}{U(t)},\ t\in[\alpha,\infty) \text{ and } \lim_{t\to\infty}U(t)=\infty;$$

$$\lim_{t\to\infty}\frac{L(t,A(t))}{U'(t)}=0,\ \int_\alpha^\infty M(s,A(s))\,B(s)\,ds<\infty \tag{2.157}$$

or

$$\lim_{t\to\infty}\frac{\exp\left(\int_\alpha^t M(s,A(s))\,B(s)\,ds\right)}{U(t)}=0, \tag{2.158}$$

$$\int_\alpha^\infty \frac{L(s,A(s))}{\exp\left(\int_\alpha^s M(u,A(u))\,B(u)\,du\right)}ds<\infty$$

or

$$M(t,A(t))\,B(t)\le\frac{k}{t},\ t\in[\alpha,\infty)\,,\ \alpha,k>0, \tag{2.159}$$

$$\lim_{t\to\infty}\frac{t^k\int_\alpha^t\frac{L(u,A(u))}{u^k}du}{U(t)}=0$$

hold, then the nonnegative continuous solutions of (2.2) *are uniformly convergent to zero at infinity.*

Proof. Let $\varepsilon>0$. If $x\in C([\alpha,\infty);\mathbb{R}_+)$ is a solution of (2.2), applying Lemma 74, we have the estimation (2.71).

If the conditions (2.155), (2.156) and either (2.157) or (2.158) or (2.159) are satisfied, then we have

$$x(t)\le A(t)+\frac{\int_\alpha^t L(u,A(u))\,du}{U(t)}\int_\alpha^\infty M(s,A(s))\,B(s)\,ds,$$

or

$$x(t) \le A(t) + \frac{\exp\left(\int_\alpha^t M(s, A(s)) B(s)\, ds\right)}{U(t)} \times \int_\alpha^\infty \frac{L(s, A(s))\, ds}{\exp\left(\int_\alpha^s M(u, A(u)) B(u)\, du\right)}$$

or

$$x(t) \le A(t) + \frac{t^k \int_\alpha^t \frac{L(s,A(s))}{s^k} ds}{U(t)} \quad \text{for all } t \in [\alpha, \infty)$$

and there exists $\delta_1(\varepsilon) > \alpha$, $\delta_2(\varepsilon) > \alpha$ such that

$$A(t) < \frac{\varepsilon}{2} \text{ if } \; t > \delta_1(\varepsilon)$$

and, either,

$$\frac{\int_\alpha^t L(s, A(s))\, ds}{U(t)} \int_\alpha^\infty M(s, A(s)) B(s)\, ds < \frac{\varepsilon}{2} \text{ if } \; t > \delta_2(\varepsilon)$$

or

$$\frac{\exp\left(\int_\alpha^t M(s, A(s)) B(s)\, ds\right)}{U(t)} \times \int_\alpha^\infty \frac{L(s, A(s))\, ds}{\exp\left(\int_\alpha^s M(u, A(u)) B(u)\, du\right)} < \frac{\varepsilon}{2}$$

or

$$\frac{t^k \int_\alpha^t \frac{L(s,A(s))}{s^k} ds}{U(t)} < \frac{\varepsilon}{2} \text{ if } \; t > \delta_2(\varepsilon)$$

from where results $x(t) < \varepsilon$ if $t > \delta(\varepsilon) = \max(\delta_1(\varepsilon), \delta_2(\varepsilon))$ and the theorem is thus proved. ■

Let us now suppose that the kernel G of the integral inequation (2.6) satisfies the relation (2.5) in $[\alpha, \infty)$ and the following conditions: (2.155), (2.156) and, either,

$$\lim_{t\to\infty} \frac{G(t, A(t))}{U'(t)} = 0, \quad \int_\alpha^\infty N(s) B(s)\, ds < \infty \tag{2.160}$$

or

$$\lim_{t\to\infty} \frac{\exp\left(\int_\alpha^t N(s) B(s)\, ds\right)}{U(t)} = 0, \quad \int_\alpha^\infty \frac{G(s, A(s)\, ds}{\exp\left(\int_\alpha^s N(u) B(u)\, du\right)} < \infty \tag{2.161}$$

or

$$N(t)\,B(t) \le \frac{k}{t},\ t \in [\alpha,\infty),\ \alpha, k > 0, \tag{2.162}$$

$$\text{and } \lim_{t\to\infty} \frac{t^k \int_\alpha^t \frac{G(u,A(u))}{u^k}du}{U'(t)} = 0$$

hold, then the nonnegative continuous solutions of (2.6) are uniformly convergent to zero at infinity.

If the kernel H of integral inequation (2.9) verifies the assertions: (2.155), (2.156) and, either,

$$\lim_{t\to\infty} \frac{C(t)\,H(A(t))}{U'(t)} = 0,\ \int_\alpha^\infty C(s)\,B(s)\,ds < \infty \tag{2.163}$$

or

$$\lim_{t\to\infty} \frac{\exp\left(\int_\alpha^t C(s)\,B(s)\,ds\right)}{U(t)} = 0, \tag{2.164}$$

$$\int_\alpha^\infty \frac{C(s)\,H(A(s))\,ds}{\exp\left(M\int_\alpha^s C(u)\,B(u)\,du\right)}$$

or

$$M\,C(t)\,B(t) \le \frac{k}{t},\ t \in [\alpha,\infty),\ \alpha, k > 0, \tag{2.165}$$

$$\text{and } \lim_{t\to\infty} \frac{t^k \int_\alpha^t \frac{C(s)H(A(s))}{s^k}ds}{U(t)} = 0,$$

hold, then the nonnegative continuous solutions of (2.9) are uniformly convergent to zero at infinity.

Further, we shall suppose that $A(t) > 0$ for all $t \in [\alpha,\infty)$. In that assumption, if the kernel $D(\cdot,\cdot)$ of integral inequation (2.12) satisfies the relation (2.11) in $[\alpha,\infty)$ and the following conditions: (2.155), (2.156) and, either,

$$\lim_{t\to\infty} \frac{D(t,A(t))}{U'(t)} = 0,\ \int_\alpha^\infty P(s,A(s))\,B(s)\,ds < \infty \tag{2.166}$$

or

$$\lim_{t\to\infty}\frac{\exp\left(\int_\alpha^t P(s,A(s))\,B(s)\,ds\right)}{U(t)}=0, \tag{2.167}$$

$$\int_\alpha^\infty \frac{D(s,A(s))\,ds}{\exp\left(\int_\alpha^s P(u,A(u))\,B(u)\,du\right)}<\infty$$

or

$$P(t,A(t))\,B(t)\le\frac{k}{t},\ t\in[\alpha,\infty),\ \alpha,k>0, \tag{2.168}$$

$$\text{and}\quad \lim_{t\to\infty}\frac{t^k\int_\alpha^t\frac{D(u,A(u))}{u^k}du}{U(t)}=0$$

hold, then the nonnegative continuous solutions of (2.12) are uniformly convergent to zero at infinity.

Now, if we assume that the kernel I of integral inequation (2.15) satisfies the relation (2.14) in $[\alpha,\infty)$ and the following assertions: (2.155), (2.156) and, either,

$$\lim_{t\to\infty}\frac{I(t,A(t))}{U'(t)}=0,\ \int_\alpha^\infty\frac{\partial I}{\partial x}(s,A(s))\,B(s)\,ds<\infty \tag{2.169}$$

or

$$\lim_{t\to\infty}\frac{\exp\left(\int_\alpha^t\frac{\partial I}{\partial x}(s,A(s))\,B(s)\,ds\right)}{U(t)}=0, \tag{2.170}$$

$$\int_\alpha^\infty\frac{I(s,A(s))\,ds}{\exp\left(\int_\alpha^s\frac{\partial I}{\partial x}(u,A(u))\,B(u)\,du\right)}<\infty$$

or

$$\frac{\partial I}{\partial x}(t,A(t))\,B(t)\le\frac{k}{t},\ t\in[\alpha,\infty),\ \alpha,k>0, \tag{2.171}$$

$$\text{and}\quad \lim_{t\to\infty}\frac{t^k\int_\alpha^t\frac{I(u,A(u))}{u^k}du}{U(t)}=0$$

are valid, then the nonnegative continuous solutions of (2.15) are uniformly convergent to zero at infinity.

In particular, if the kernel K of (2.18) verifies the relation (2.17) in $[\alpha, \infty)$ and the following conditions: (2.155), (2.156) and, either,

$$\lim_{t\to\infty} \frac{C(t)\,K(A(t))}{U'(t)} = 0,\ \int_\alpha^\infty \frac{dK}{dx}(A(s))\,B(s)\,C(s)\,ds < \infty \tag{2.172}$$

or

$$\lim_{t\to\infty} \frac{\exp\left(\int_\alpha^t K'(A(s))\,B(s)\,C(s)\,ds\right)}{U(t)} = 0 \ \text{ and} \tag{2.173}$$

$$\int_\alpha^\infty \frac{C(s)\,K(A(s))}{\exp\left(\int_\alpha^t \frac{dK}{dx}(A(u))\,B(u)\,C(u)\,du\right)} < \infty$$

or

$$\frac{dK}{dx}(A(t))\,B(t)\,C(t) \le \frac{k}{t},\ \alpha, k > 0,\ t \in [\alpha, \infty), \tag{2.174}$$

$$\text{and} \quad \lim_{t\to\infty} \frac{t^k \int_\alpha^t \frac{C(s)K(A(s))}{s^k} ds}{U(t)} = 0$$

hold, then the nonnegative continuous solutions of (2.18) are uniformly convergent to zero at infinity.

Chapter 3

Applications to Integral Equations

In this chapter we apply the results established in the second chapter to obtain estimates for the solutions of Volterra integral equations in Banach spaces and to get uniform boundedness and uniform convergence to zero at infinity conditions for the solutions of these equations.

In Sections 3.1 and 3.2 we point out some natural applications of Lemma 74, Lemma 78 and their consequences to obtain estimates for the solutions of the general Volterra integral equations and Volterra equations with degenerate kernels , respectively.

The last sections are devoted to the qualitative study of some aspects for the solutions of the above equations by using the results established in Sections 3.1 and 3.2. Results of uniform boundedness, uniform convergence to zero at infinity, and asymptotic equivalence at infinity, for the continuous solutions of a large class of equations are given. The discrete case embodied in the last section is also analysed.

3.1 Solution Estimates

The purpose of this section is to provide some estimates for the solutions of general Volterra integral equations in Banach spaces.

Let us consider the following integral equation:

$$x(t) = g(t) + \int_{\alpha}^{t} V(t, s, x(s))\, ds,\ t \in [\alpha, \beta), \tag{3.1}$$

where $V : [\alpha, \beta)^2 \times X \to X$, $g : [\alpha, \beta) \to X$ are continuous and X is a Banach space over the real or complex number field.

Further, we shall suppose that the integral equation (3.1) has solutions in $C([\alpha, \beta); X)$ and in that assumption, by using the results established in Section 2.1 of Chapter 2, we can give the following lemma.

Lemma 118 *If the kernel V of integral equation (3.1) satisfies the relation:*

$$\|V(t,s,x)\| \le B(t) L(s, \|x\|); \ t,s \in [\alpha,\beta), \ x \in X \text{ and } L \tag{3.2}$$

verifies the condition (2.1) of Section 2.1 of Chapter 2, and B is nonnegative continuous in $[\alpha, \beta)$;

then for every $x \in C([\alpha, \beta); X)$ a solution of (3.1) we have the estimate

$$\|x(t)\| \le \|g(t)\| + B(t) \int_\alpha^t L(u, \|g(u)\|) \times \exp\left(\int_u^t M(s, \|g(s)\|) B(s)\, ds\right) du \tag{3.3}$$

for all $t \in [\alpha, \beta)$.

Proof. Let $x : [\alpha, \beta) \to X$ be a continuous solution of (3.1). Then we have

$$\|x(t)\| \le \|g(t)\| + \int_\alpha^t \|V(t,s,x)\|\, ds, \ t \in [\alpha, \beta).$$

Since the condition (3.2) is satisfied, we obtain

$$\|x(t)\| \le \|g(t)\| + B(t) \int_\alpha^t L(s, \|x(s)\|)\, ds, \ t \in [\alpha, \beta).$$

Applying Lemma 74, the estimation (3.3) holds and the lemma is thus proved. ■

Now we can give the following two corollaries which are obvious by the above lemma.

Corollary 119 *If the kernel V of integral equation (3.1) satisfies the relation:*

$$\|V(t,s,x)\| \le B(t) G(s, \|x\|); \ t,s \in [\alpha,\beta), \ x \in X, \ G \tag{3.4}$$

verifies the condition (2.5) and B is nonnegative continuous in $[\alpha, \beta)$;

then for every continuous solution of (3.1) we have the bound:

$$\|x(t)\| \leq \|g(t)\| + B(t) \int_\alpha^t G(u, \|g(u)\|) \times \exp\left(\int_u^t N(s) B(s) \, ds \right) du, \quad (3.5)$$

for all $t \in [\alpha, \beta)$.

Corollary 120 *If the kernel* V *of integral equation (3.1) satisfies the relation*

$$\|V(t,s,x)\| \leq B(t) C(s) H(\|x\|); \; t,s \in [\alpha,\beta), \; x \in X, \; H \quad (3.6)$$

verifies the condition (2.8) and B, C *are nonnegative continuous in* $[\alpha, \beta)$;

then for every $x \in C([\alpha,\beta); X)$ *a solution of (3.1) we have*

$$\|x(t)\| \leq \|g(t)\| + B(t) \int_\alpha^t C(u) H(\|g(u)\|) \times \exp\left(M \int_u^t C(s) B(s) \, ds \right) du \quad (3.7)$$

for all $t \in [\alpha, \beta)$.

In what follows, we suppose that $\|g(t)\| > 0$ for all $t \in [\alpha, \beta)$. In that assumption, we can state the following corollaries.

Corollary 121 *If the kernel* V *of integral equation (3.1) satisfies the relation:*

$$\|V(t,s,x)\| \leq B(t) D(s, \|x\|); \; t,s \in [\alpha,\beta), \; x \in X, \; D \quad (3.8)$$

verifies the condition (2.11) and B *is nonnegative continuous in* $[\alpha, \beta)$;

then for every $x \in C([\alpha,\beta); X)$ *a solution of (3.1) we have*

$$\|x(t)\| \leq \|g(t)\| + B(t) \int_\alpha^t D(u, \|g(u)\|) \times \exp\left(\int_u^t P(s, \|g(s)\|) B(s) \, ds \right) du \quad (3.9)$$

for all $t \in [\alpha, \beta)$.

Corollary 122 *If we assume that the kernel V of integral equation (3.1) satisfies the relation*

$$\|V(t,s,x)\| \le B(t)\, I(s,\|x\|)\,;\ t,s\in[\alpha,\beta)\,,\ x\in X,\ I \tag{3.10}$$

verifies the condition (2.14) and B is nonnegative continuous in $[\alpha,\beta)$;

then for every continuous solution of (3.1) we have the bound

$$\|x(t)\| \le \|g(t)\| + B(t)\int_\alpha^t I(u,\|g(u)\|) \times \exp\left(\int_u^t \frac{\partial I}{\partial x}(s,\|g(s)\|)\, B(s)\, ds\right) du \tag{3.11}$$

for all $t\in[\alpha,\beta)$.

Also,

Corollary 123 *Let us suppose that the kernel V satisfies the relation*

$$\|V(t,s,x)\| \le B(t)\, C(s)\, K(\|x\|)\,;\ t,s\in[\alpha,\beta)\,,\ x\in X,\ K \tag{3.12}$$

verifies the condition (2.14) and B, C are nonnegative continuous in $[\alpha,\beta)$;

then for every $x\in C([\alpha,\beta);X)$ a solution of (3.1), we have the inequality

$$\|x(t)\| \le \|g(t)\| + B(t)\int_\alpha^t C(u)\, K(\|g(u)\|) \times \exp\left(\int_u^t \frac{dK}{dx}(\|g(s)\|)\, B(s)\, C(s)\, ds\right) du \tag{3.13}$$

for all $t\in[\alpha,\beta)$.

By Corollaries 122 and 123 we can deduce the following consequences.

Consequences

1. If we suppose that the kernel V of integral equation (3.1) satisfies the relation

$$\|V(t,s,x)\| \le B(t)\,C(s)\,\|x\|^{r(s)}\,;\ t,s\in[\alpha,\beta),\ x\in X, \tag{3.14}$$

B, C, r are nonnegative continuous in $[\alpha,\beta)$ and $0\le r(t)\le 1$ for all $t\in[\alpha,\beta)$;

then for every $x\in C([\alpha,\beta);X)$ a solution of (3.1) we have the estimation

$$\|x(t)\| \le \|g(t)\| + B(t)\int_\alpha^t C(u)\,\|g(u)\|^{r(u)} \times \exp\left(\int_u^t \frac{r(s)\,B(s)\,C(s)\,ds}{\|g(s)\|^{r(s)}}\right)du \tag{3.15}$$

for all $t\in[\alpha,\beta)$.

2. If we assume that the kernel V satisfies the relation

$$\|V(t,s,x)\| \le B(t)\,C(s)\ln(\|x\|+1)\,;\ t,s\in[\alpha,\beta),\ x\in X, \tag{3.16}$$

and B, C are nonnegative continuous in $[\alpha,\beta)$;

then for every $x:[\alpha,\beta)\to X$ a continuous solution of (3.1) we have the estimation:

$$\|x(t)\| \le \|g(t)\| + B(t)\int_\alpha^t C(u)\ln(\|g(u)\|+1) \times \exp\left(\int_u^t \frac{C(s)\,B(s)\,ds}{\|g(s)\|+1}\right)du \tag{3.17}$$

for all $t\in[\alpha,\beta)$.

Further, we shall give another lemma concerning the estimation of the solution of equation (3.1) assuming that the kernel (3.1) satisfies the following Lipschitz type condition.

Lemma 124 *Let us suppose that the kernel* V *satisfies the relation*

$$\begin{aligned}&\|V(t,s,x)-V(t,s,y)\|\leq B(t)\,L(s,\|x-y\|)\,; \qquad (3.18)\\ &t,s\in[\alpha,\beta)\,,\ x,y\in X,\ L \text{ verifies the condition (2.1)}\\ &\text{and } B \text{ is nonnegative continuous in } [\alpha,\beta)\,;\end{aligned}$$

then for every continuous solution of (3.1) we have the estimate:

$$\begin{aligned}&\|x(t)-g(t)\|\\ &\quad\leq k(t)+B(t)\int_\alpha^t L(u,k(u))\exp\left(\int_u^t M(s,k(s))\,B(s)\,ds\right)du, \qquad (3.19)\end{aligned}$$

where $k(t):=\int_\alpha^t \|V(t,s,g(s))\|\,ds$ *and* $t\in[\alpha,\beta)$.

Proof. Let $x\in C([\alpha,\beta);X)$ be a solution of (3.1). Then we have

$$\begin{aligned}&\|x(t)-g(t)\|\\ &\leq \int_\alpha^t \|V(t,s,x(s))\|\,ds\\ &\leq \int_\alpha^t \|V(t,s,g(s))\|\,ds+B(t)\int_\alpha^t L(s,\|x(s)-g(s)\|)\,ds\\ &= k(t)+B(t)\int_\alpha^t L(s,\|x(s)-g(s)\|)\,ds\end{aligned}$$

for all $t\in[\alpha,\beta)$.

Applying Lemma 74, the inequality (3.19) holds and the lemma is thus proved. ■

The following corollaries may be useful in applications.

Corollary 125 *If the kernel* V *satisfies the relation*

$$\begin{aligned}&\|V(t,s,x)-V(t,s,y)\|\leq B(t)\,G(s,\|x-y\|)\,; \qquad (3.20)\\ &t,s\in[\alpha,\beta)\,,\ x,y\in X,\ G \text{ verifies the}\\ &\text{condition (2.5) of Section 2.1 of Chapter 2}\\ &\text{and } B \text{ is nonnegative continuous in } [\alpha,\beta)\,;\end{aligned}$$

then for every continuous solution of (3.1) we have

$$\|x(t)-g(t)\| \leq k(t)+B(t)\int_{\alpha}^{t} G(u,k(u)) \times \exp\left(\int_{u}^{t} N(s)\,B(s)\,ds\right) du, \quad (3.21)$$

for all $t\in[\alpha,\beta)$.

Corollary 126 *If the kernel* V *satisfies the relation*

$$\|V(t,s,x)-V(t,s,y)\| \leq B(t)\,C(s)\,H(\|x-y\|); \quad (3.22)$$

$t,s\in[\alpha,\beta)$, $x,y\in X$, H *verifies the condition (2.8) and* B,C *are nonnegative continuous in* $[\alpha,\beta)$;

then for every $x\in C([\alpha,\beta);X)$ *a solution of (3.1) we have the bound*

$$\|x(t)-g(t)\| \leq k(t)+B(t)\int_{\alpha}^{t} C(u)\,H(k(u))\exp\left(M\int_{u}^{t} C(s)\,B(s)\,ds\right) du, \quad (3.23)$$

for all $t\in[\alpha,\beta)$.

Further, we shall assume that $\|g(t)\|>0$ for all $t\in[\alpha,\beta)$.
With that assumption, the following three corollaries hold.

Corollary 127 *If we assume that the kernel* V *satisfies the relation*

$$\|V(t,s,x)-V(t,s,y)\| \leq B(t)\,D(s,\|x-y\|); \quad (3.24)$$

$t,s\in[\alpha,\beta)$, $x,y\in X$, *and* D *verifies the condition (2.11) and* B *is nonnegative continuous in* $[\alpha,\beta)$;

then for every continuous solution of (3.1) we have

$$\|x(t)-g(t)\| \leq k(t)+B(t)\int_{\alpha}^{t} D(u,k(u))\exp\left(\int_{u}^{t} P(s,k(s))\,B(s)\,ds\right) du, \quad (3.25)$$

for all $t\in[\alpha,\beta)$.

Corollary 128 *Let us suppose that the kernel V satisfies the relation:*

$$\begin{aligned} &\|V(t,s,x)-V(t,s,y)\| \le B(t)\, I(s,\|x-y\|)\,; \\ &t,s\in[\alpha,\beta)\,,\ x,y\in X,\ \text{and } I \text{ verifies the condition (2.14)} \\ &\text{and } B \text{ is nonnegative continuous in } [\alpha,\beta)\,; \end{aligned} \tag{3.26}$$

Then, for every continuous solution of (3.1) we have the estimation

$$\begin{aligned} &\|x(t)-g(t)\| \\ &\quad\le k(t)+B(t)\int_\alpha^t I(u,k(u))\exp\left(\int_u^t \frac{\partial I}{\partial x}(s,k(s))\,B(s)\,ds\right)du, \end{aligned} \tag{3.27}$$

for all $t\in[\alpha,\beta)$.

Moreover,

Corollary 129 *If the kernel V of integral equation (3.1) satisfies the relation*

$$\begin{aligned} &\|V(t,s,x)-V(t,s,y)\| \le B(t)\,C(s)\,K(\|x-y\|)\,; \\ &t,s\in[\alpha,\beta)\,,\ x,y\in X,\ K \text{ verifies the relation (2.17)} \\ &\text{and } B,C \text{ are nonnegative continuous in } [\alpha,\beta)\,; \end{aligned} \tag{3.28}$$

then for every continuous solution of (3.1) we have:

$$\begin{aligned} \|x(t)-g(t)\| &\le k(t)+B(t)\int_\alpha^t C(u)\,K(k(u)) \\ &\qquad\times\exp\left(\int_u^t \frac{dK}{dx}(k(s))\,B(s)\,C(s)\,ds\right)du, \end{aligned} \tag{3.29}$$

for all $t\in[\alpha,\beta)$.

By Corollaries 128 and 129 we can deduce the following consequences.

Consequences

1. Let us suppose that the kernel V satisfies the relation

$$\begin{aligned} &\|V(t,s,x)-V(t,s,y)\| \le B(t)\,C(s)\,\|x-y\|^{r(s)}\,; \\ &t,s\in[\alpha,\beta)\,,\ x,y\in X,\ B,C,r \text{ are nonnegative} \\ &\text{continuous in } [\alpha,\beta) \text{ and } 0<r(t)\le 1 \text{ for all } t\in[\alpha,\beta)\,; \end{aligned} \tag{3.30}$$

then for every $x : [\alpha, \beta) \to X$ a continuous solution of (3.1) we have

$$\|x(t) - g(t)\| \leq k(t) + B(t) \int_\alpha^t C(u) k(u)^{r(u)} \times \exp\left(\int_u^t \frac{r(s) B(s) C(s)}{k(s)^{1-r(s)}} ds \right) du, \quad (3.31)$$

for all $t \in [\alpha, \beta)$.

2. Finally, if the kernel V satisfies the relation

$$\|V(t,s,x) - V(t,s,y)\| \leq B(t) C(s) \ln(\|x-y\| + 1); \quad (3.32)$$
$$t, s \in [\alpha, \beta), \ x, y \in X, \ B, C \in C([\alpha, \beta); \mathbb{R}_+);$$

then for every $x \in C([\alpha, \beta); X)$ a solution of (3.1) we have

$$\|x(t) - g(t)\| \leq k(t) + B(t) \int_\alpha^t C(u) \ln(k(u) + 1) \times \exp\left(\int_u^t \frac{B(s) C(s)}{k(s) + 1} ds \right) du, \quad (3.33)$$

for all $t \in [\alpha, \beta)$.

3.2 The Case of Degenerate Kernels

Further, we consider the Volterra integral equations with degenerate kernel given by:

$$x(t) = g(t) + B(t) \int_\alpha^t U(s, x(s)) ds, \ t \in [\alpha, \beta), \quad (3.34)$$

where $g : [\alpha, \beta) \to X$, $B : [\alpha, \beta) \to X$, $U : [\alpha, \beta) \times X \to X$ are continuous and X is a Banach space over the real or complex number field.

In what follows, we assume that the integral equation (3.34) has solutions in $C([\alpha, \beta); X)$ and in that assumption we give two estimation lemmas for the solution of the above integral equation.

1. The first result is embodied in the following lemma.

Lemma 130 *If the kernel U of integral equation (3.34) satisfies the relation*

$$\begin{array}{l} \|U(t,x)\| \leq L(t,\|x\|)\,,\ t \in [\alpha,\beta) \ \ \textit{and} \\ L \textit{ verifies the conditon (2.1);} \end{array} \tag{3.35}$$

then for every continuous solution of (3.34), we have

$$\begin{aligned} &\|x(t) - g(t)\| \\ &\quad \leq |B(t)| \int_{\alpha}^{t} L(s,\|g(s)\|) \exp\left(\int_{s}^{t} |B(u)|\, M(u,\|g(u)\|)\, du\right) ds, \end{aligned} \tag{3.36}$$

for all $t \in [\alpha,\beta)$.

Proof. Let $x \in C([\alpha,\beta);X)$ be a solution of (3.34). Putting

$$y : [\alpha,\beta) \to X, y(t) := \int_{\alpha}^{t} U(s,x(s))\, ds,$$

we have $y(\alpha) = 0$ and

$$y(t) = U(t,g(t) + B(t)\,y(t))\,, t \in [\alpha,\beta)\,.$$

Hence

$$\begin{aligned} \|y(t)\| &= \|U(t,g(t) + B(t)\,y(t))\| \leq L(t,\|g(t) + B(t)\,y(t)\|) \\ &\leq L(t,\|g(t)\| + |B(t)|\,\|y(t)\|) \\ &\leq L(t,\|g(t)\|) + |B(t)|\, M(t,\|g(t)\|)\,\|y(t)\| \end{aligned}$$
for all $t \in [\alpha,\beta)$.

By integration and since

$$\|y(t)\| \leq \int_{\alpha}^{t} \|\dot{y}(s)\|\, ds,$$

we obtain by simple computation that

$$\|y(t)\| \leq \int_{\alpha}^{t} L(s,\|g(s)\|)\, ds + \int_{\alpha}^{t} |B(s)|\, M(s,\|g(s)\|)\,\|y(s)\|\, ds. \tag{3.37}$$

Applying Corollary 2 of the Introduction, we obtain

$$\|y(t)\| \leq L(s, \|g(s)\|) \times \exp\left(\int_s^t |B(u)| M(u, \|g(u)\|) \, du\right) ds, \ t \in [\alpha, \beta)$$

from where results (3.36) and the lemma is thus proved. ■

The following two corollaries are obvious by the above lemma.

Corollary 131 *If we assume that the kernel U satisfies*

$$\|U(t,x)\| \leq G(t, \|x\|), \ t \in [\alpha, \beta), \tag{3.38}$$

$x \in X$ and G verifies the relation (2.5);

then for every $x \in C([\alpha, \beta); X)$ a solution of (3.34) we have the estimate

$$\|x(t) - g(t)\| \leq |B(t)| \int_\alpha^t G(s, \|g(s)\|) \times \exp\left(\int_s^t |B(u)| N(u) \, du\right) ds, \tag{3.39}$$

for all $t \in [\alpha, \beta)$.

Corollary 132 *Let us suppose that the kernel U satisfies the relation*

$$\|U(t,x)\| \leq C(t) H(\|x\|), \ t \in [\alpha, \beta), \tag{3.40}$$

$x \in X$ and H verifies the relation (2.8);

then for every $x : [\alpha, \beta) \to X$ a continuous solution of (3.34) we have

$$\|x(t) - g(t)\| \leq |B(t)| \int_\alpha^t C(s) H(\|g(s)\|) \exp\left(M \int_s^t |B(u)| C(u) \, du\right) ds, \tag{3.41}$$

for all $t \in [\alpha, \beta)$.

Further, we shall suppose that $\|g(t)\| > 0$ for all $t \in [\alpha, \beta)$. With this assumption we can deduce the following three corollaries which can be useful in applications.

Corollary 133 *If the kernel U satisfies the relation*

$$\left\|U\left(t,x\right)\right\| \leq D\left(t,\left\|x\right\|\right),\ t\in\left[\alpha,\beta\right), \tag{3.42}$$

$x \in X$ and D verifies the relation (2.11);

then for every $x \in C\left(\left[\alpha,\beta\right);X\right)$ a solution of (3.34), we have the estimate

$$\begin{aligned}&\left\|x\left(t\right)-g\left(t\right)\right\| \\ &\leq \left|B\left(t\right)\right| \int_{\alpha}^{t} D\left(s,\left\|g\left(s\right)\right\|\right) \exp\left(\int_{s}^{t}\left|B\left(u\right)\right| P\left(u,\left\|g\left(u\right)\right\|\right) du\right) ds,\end{aligned} \tag{3.43}$$

for all $t \in \left[\alpha,\beta\right)$.

Corollary 134 *Let us suppose the kernel U satisfies the relation*

$$\left\|U\left(t,x\right)\right\| \leq I\left(t,\left\|x\right\|\right),\ t\in\left[\alpha,\beta\right), \tag{3.44}$$

$x \in X$ and I verifies the condition (2.14);

then for every $x : \left[\alpha,\beta\right) \to X$ a continuous solution of U, we have

$$\begin{aligned}&\left\|x\left(t\right)-g\left(t\right)\right\| \\ &\leq \left|B\left(t\right)\right| \int_{\alpha}^{t} I\left(s,\left\|g\left(s\right)\right\|\right) \exp\left(\int_{s}^{t}\left|B\left(u\right)\right| \frac{\partial I}{\partial x}\left(u,\left\|g\left(u\right)\right\|\right) du\right) ds,\end{aligned} \tag{3.45}$$

for all $t \in \left[\alpha,\beta\right)$.

Finally, we have

Corollary 135 *If we assume that the kernel U satisfies the relation:*

$$\left\|U\left(t,x\right)\right\| \leq C\left(t\right) K\left(\left\|x\right\|\right),\ t\in\left[\alpha,\beta\right),\ x\in X \tag{3.46}$$

and K verifies the relation (2.17);

then for every $x \in C\left(\left[\alpha,\beta\right);X\right)$ a solution of (3.34) we have the estimate

$$\begin{aligned}\left\|x\left(t\right)-g\left(t\right)\right\| \leq \left|B\left(t\right)\right| &\int_{\alpha}^{t} C\left(s\right) K\left(\left\|g\left(s\right)\right\|\right) \\ &\times \exp\left(\int_{s}^{t}\left|B\left(u\right)\right| C\left(u\right) \frac{dK}{dx}\left(\left\|g\left(u\right)\right\|\right) du\right) ds,\end{aligned} \tag{3.47}$$

for all $t \in \left[\alpha,\beta\right)$.

Consequences

1. If the kernel U of integral equation (3.34) satisfies the assumption:

$$\|U(t,x)\| \leq C(t)\|x\|^{r(t)}, \ t\in[\alpha,\beta), \ x\in X \text{ and } r,C \\ \text{are nonnegative continuous in } [\alpha,\beta) \text{ and } 0<r(t)\leq 1 \tag{3.48}$$

then for every $x\in C([\alpha,\beta);X)$ a solution of (3.34) we have

$$\|x(t)-g(t)\| \leq |B(t)|\int_\alpha^t C(s)\|g(s)\|^{r(s)} \\ \times\exp\left(\int_s^t \frac{r(u)C(u)|B(u)|}{\|g(u)\|^{1-r(u)}}du\right)ds, \tag{3.49}$$

in the interval $[\alpha,\beta)$.

2. Finally, if we assume that the kernel U verifies the condition

$$\|U(t,x)\| \leq C(t)\ln(\|x\|+1), \ t\in[\alpha,\beta), \ x\in X \\ \text{and } C \text{ is nonnegative continuous in } [\alpha,\beta). \tag{3.50}$$

then for every $x\in C([\alpha,\beta);X)$ a solution of (3.34) we have the estimate

$$\|x(t)-g(t)\| \leq |B(t)|\int_\alpha^t C(s)\ln(\|g(s)\|+1) \\ \times\exp\left(\int_s^t \frac{C(u)|B(u)|}{\|g(u)\|+1}du\right)ds, \tag{3.51}$$

for all $t\in[\alpha,\beta)$.

2. Now we shall present the second lemma of this section.

Lemma 136 *If the kernel U of integral equation (3.34) satisfies the relation*

$$\|U(t,x+y)-U(t,x)\| \leq S(t,\|x\|)\|y\|, \ t\in[\alpha,\beta), \\ x,y\in X \text{ where } S \text{ is nonnegative continuous in } [\alpha,\beta)\times\mathbb{R}_+, \tag{3.52}$$

then for every $x \in C([\alpha,\beta);X)$ a solution of (3.34) we have the following estimate:

$$\begin{aligned}&\|x(t)-g(t)\| \\ &\quad\le \int_s^t |B(t)|\,\|U(s,g(s))\|\exp\left(\int_s^t S(u,\|g(u)\|)\,|B(u)|\,du\right)ds, \quad (3.53)\end{aligned}$$

for all $t \in [\alpha,\beta)$.

Proof. Let $x \in C([\alpha,\beta);X)$ be a solution of (3.34). Putting

$$y(t) := \int_\alpha^t U(s,x(s))\,ds,$$

we have $y(\alpha)=0$ and

$$y'(t) = U(t,g(t)+B(t)\,y(t)).$$

Since

$$\begin{aligned}\|y(t)\| &= \|U(t,g(t)+B(t)\,y(t))\| \\ &\le \|U(t,g(t))\| + S(t,\|g(t)\|)\,|B(t)|\,\|y(t)\|\end{aligned}$$

for all $t \in [\alpha,\beta)$, from where results

$$\|y(t)\| \le \int_\alpha^t \|U(s,g(s))\|\,ds + \int_\alpha^t S(s,\|g(s)\|)\,|B(s)|\,\|y(s)\|\,ds.$$

Applying Corollary 2 of the Introduction, we obtain

$$\|y(t)\| \le \int_\alpha^t \|U(s,g(s))\|\exp\left(\int_s^t S(u,\|g(u)\|)\,|B(u)|\,du\right)ds$$

for all $t \in [\alpha,\beta)$ from where results the estimation (3.53) and the theorem is proved. ■

Corollary 137 *If the kernel U of (3.34) satisfies the relation:*

$$\begin{aligned}&\|U(t,x+y)-U(t,x)\| \le C(t)\,R(\|x\|)\,\|y\|, \; t \in [\alpha,\beta), \quad (3.54)\\ &x,y \in X \text{ and } C:[\alpha,\beta)\to\mathbb{R}_+,\; R:\mathbb{R}_+\to\mathbb{R}_+ \\ &\text{are continuous,}\end{aligned}$$

then for every $x \in C([\alpha,\beta);X)$ a solution of (3.34) we have

$$\begin{aligned}&\|x(t)-g(t)\|\\&\leq |B(t)|\int_{\alpha}^{t}\|U(s,g(s))\|\exp\left(\int_{s}^{t}C(u)\,R(\|g(u)\|)\,|B(u)|\,du\right)ds, \quad (3.55)\end{aligned}$$

for all $t\in[\alpha,\beta)$.

Corollary 138 *If the kernel U of (3.34) satisfies the relation:*

$$\begin{aligned}&\|U(t,x+y)-U(t,x)\|\leq T(t)\,\|y\|\,,\ t\in[\alpha,\beta)\,, \quad (3.56)\\&x,y\in X \text{ and } T \text{ is a nonnegative continuous function}\\&\text{in } [\alpha,\beta)\,,\end{aligned}$$

then for every $x \in C([\alpha,\beta);X)$ a solution of (3.34) we have the estimate

$$\begin{aligned}&\|x(t)-g(t)\|\\&\qquad\leq |B(t)|\int_{s}^{t}\|U(s,g(s))\|\exp\left(\int_{s}^{t}T(u)\,|B(u)|\,du\right)ds, \quad (3.57)\end{aligned}$$

for all $t\in[\alpha,\beta)$.

3.3 Boundedness Conditions

In this section we point out some boundedness conditions for the solutions of the following Volterra integral equation:

$$x(t)=g(t)+\int_{\alpha}^{t}V(t,s,x(s))\,ds,\ t\in[\alpha,\beta)\,, \quad (3.58)$$

where $V:[\alpha,\beta)^2\times X\to X,\ g:[\alpha,\beta)\to X$ are continuous in $[\alpha,\beta)$ and X is a Banach space.

We suppose that the integral equation (3.58) has solutions in $C([\alpha,\beta);X)$ and in that assumption, by using Lemma 118, and the results established in Section 2.2 of Chapter 2, we can state the following theorems.

Theorem 139 *If the kernel V of integral equation (3.58) satisfies the relation (3.2) and the following conditions:*

$$\|g(t)\| \le M_1,\ B(t) \le M_2,\ t \in [\alpha, \infty); \tag{3.59}$$

$$\int_\alpha^\infty M(s, \|g(s)\|)\, ds,\ \int_\alpha^\infty L(s, M_1)\, ds < \infty, \tag{3.60}$$

hold, then there exists a constant $\tilde{M} > 0$ such that for every solution $x \in C([\alpha, \beta); X)$ of (3.58) we have $\|x(t)\| \le \tilde{M}$ in $[\alpha, \beta)$. That is, the continuous solutions of (3.58) are uniformly bounded in $[\alpha, \beta)$.

The proof follows by Lemma 118 and by Theorem 108. We omit the details.

Now, using Theorem 109, we can mention another result.

Theorem 140 *Let us suppose that the kernel V of (3.58) verifies the relation (3.2) and the following conditions*

$$\|g(t)\| \le M_1,\ \lim_{t\to\infty} B(t) = 0,\ M_1 > 0,\ t \in [\alpha, \infty) \tag{3.61}$$

$$\int_\alpha^\infty L(s, M_1)\, ds,\ \int_\alpha^\infty M(s, \|g(s)\|)\, B(s)\, ds < \infty, \quad \text{or} \tag{3.62}$$

$$M(t, \|g(t)\|)\, B(t) \le \frac{k}{t},\ B(t)\, t^k \le l < \infty, \tag{3.63}$$

$$\int_\alpha^\infty \frac{L(s, M_1)}{s^k} ds < \infty,\ k, \alpha > 0,\ t \in [\alpha, \infty)$$

hold, then the continuous solutions of (3.58) are uniformly bounded in $[\alpha, \beta)$.

Another result is embodied in the following theorem.

Theorem 141 *If we assume that the kernel V verifies the relation (3.2) and the following conditions*

$$\|g(t)\| \le M_1,\ t \in [\alpha, \infty) \tag{3.64}$$

there exists a function $U : [\alpha, \infty) \to \mathbb{R}_+^*$ differentiable (3.65)

in (α, ∞) such that $B(t) \le \dfrac{1}{U(t)}$ and $\lim_{t\to\infty} U(t) = \infty$;

$$\lim_{t\to\infty} \frac{L(t, M_1)}{U'(t)} = l < \infty,\ \int_\alpha^\infty M(s, \|g(s)\|)\, B(s)\, ds < \infty, \tag{3.66}$$

or

$$\lim_{t\to\infty} \frac{\exp\left(\int_\alpha^t M(s, \|g(s)\|) B(s)\, ds\right)}{U(t)} = l < \infty, \tag{3.67}$$

$$\int_\alpha^\infty \frac{L(s, M_1)}{\exp\left(\int_\alpha^s M(u, \|g(u)\|) B(u)\, du\right)} ds < \infty,$$

or

$$M(t, \|g(t)\|) B(t) \le \frac{k}{t},\ k, \alpha > 0,\ t \in [\alpha, \infty) \tag{3.68}$$

$$\text{and} \quad \lim_{t\to\infty} \frac{t^k \int_\alpha^\infty \frac{L(s,M_1)}{s^k} ds}{U(t)} = l < \infty$$

are true, then the continuous solutions of (3.58) are uniformly bounded in $[\alpha, \beta)$.

The proof follows by Lemma 118 and by Theorem 110 of Chapter 2. By using Theorem 111, we may deduce the following result as well.

Theorem 142 *Let us assume that the kernel* V *verifies the relation (3.2) and the conditions*

$$\lim_{t\to\infty} \|g(t)\| = 0,\ B(t) \le M_1,\ t \in [\alpha, \infty) \tag{3.69}$$

$$\int_\alpha^\infty L(s, \|g(s)\|)\, ds,\ \int_\alpha^\infty M(s, \|g(s)\|)\, ds < \infty, \tag{3.70}$$

are valid. Then the continuous solutions of (3.58) are uniformly bounded in $[\alpha, \beta)$.

Another result is embodied in the following theorem.

Theorem 143 *If we assume that the kernel* V *satisfies the same relation (3.2) and the conditions*

$$\lim_{t\to\infty} \|g(t)\| = 0,\ B(t)\, t^k \le l < \infty,\ k > 0,\ t \in [\alpha, \infty) \tag{3.71}$$

$$M(t, \|g(t)\|) B(t) \le \frac{k}{t},\ t \in [\alpha, \infty),\ \int_\alpha^\infty \frac{L(s, \|g(s)\|)}{s^k} ds < \infty, \tag{3.72}$$

hold, then the continuous solutions of (3.58) are uniformly bounded in $[\alpha, \beta)$.

The proof follows by Lemma 118 and by Theorem 112.
Finally, we have

Theorem 144 *If the kernel V satisfies the relation (3.2) and the conditions*

$$\lim_{t\to\infty} \|g(t)\| = 0, \tag{3.73}$$

there exists a function $U : [\alpha,\infty) \to \mathbb{R}_+^*$ differentiable (3.74)

in $[\alpha,\infty)$ such that $B(t) \leq \frac{1}{U(t)}$ and $\lim_{t\to\infty} U(t) = \infty$;

$$\lim_{t\to\infty} \frac{\exp\left(\int_\alpha^t M(s,\|g(s)\|)\, B(s)\, ds\right)}{U(t)} = l < \infty, \tag{3.75}$$

$$\int_\alpha^\infty \frac{L(s,\|g(s)\|)}{\exp\left(\int_\alpha^s M(u,\|g(u)\|)\, B(u)\, du\right)} ds < \infty,$$

or

$$M(t,\|g(t)\|)\, B(t) \leq \frac{k}{t},\ k,\alpha > 0,\ t \in [\alpha,\infty) \tag{3.76}$$

$$\text{and } \lim_{t\to\infty} \frac{t^k \int_\alpha^\infty \frac{L(s,\|g(s)\|)}{s^k} ds}{U(t)} = l < \infty$$

hold, then the solutions of (3.58) are uniformly bounded in $[\alpha,\beta)$.

The proof is evident by Theorem 113.

3.4 Convergence to Zero Conditions

In what follows, we give some convergence to zero at infinity conditions for solutions of Volterra integral equations with kernels satisfying the relation (3.2) of Section 3.1.

Further, we suppose that the integral equation (3.58) of Section 1 has solutions in $C([\alpha,\beta);X)$ and in this assumption, by using Lemma 118 and the results established in Section 2.3 of Chapter 2, we may state the following theorems.

Theorem 145 *If the kernel* V *of integral equation (3.58) verifies the relation (3.2) and the following conditions:*

$$\lim_{t\to\infty} \|g(t)\| = 0, \ \lim_{t\to\infty} B(t) = 0; \tag{3.77}$$

$$\int_{\alpha}^{\infty} L(s, \|g(s)\|)\, ds, \ \int_{\alpha}^{\infty} M(s, \|g(s)\|)\, B(s)\, ds < \infty, \tag{3.78}$$

hold, then for every $\varepsilon > 0$ *there exists a* $\delta(\varepsilon) > \alpha$ *such that for every* $x \in C([\alpha, \beta); X)$ *a solution of (3.58) we have* $\|x(t)\| < \varepsilon$ *if* $t > \delta(\varepsilon)$ *i.e. the continuous solutions of (3.58) are uniformly convergent to zero at infinity.*

The proof follows by Lemma 118 and by Theorem 114.

Now, by using Theorem 115 of the previous chapter, we can point out another result which is embodied in the following theorem.

Theorem 146 *Let us suppose that the kernel* V *of integral equation (3.58) satisfies the relation (3.2) and the following conditions:*

$$\lim_{t\to\infty} \|g(t)\| = 0, \ \lim_{t\to\infty} t^k B(t) = 0, \ k > 0, \tag{3.79}$$

$$M(t, \|g(t)\|)\, B(t) \le \frac{k}{t}, \ t \in [\alpha, \infty), \ \alpha > 0, \tag{3.80}$$

$$\int_{\alpha}^{\infty} \frac{L(s, \|g(s)\|)}{s^k} ds < \infty$$

hold, then the solutions of (3.58) are uniformly convergent to zero at infinity.

Finally, we have:

Theorem 147 *If we assume that the kernel* V *satisfies the conditions*

$$\lim_{t\to\infty} \|g(t)\| = 0, \tag{3.81}$$

there exists a function $U : [\alpha, \infty) \to \mathbb{R}_+^*$ *differentiable* (3.82)

in (α, ∞) *such that* $B(t) \le \dfrac{1}{U(t)}$, $t > \alpha$ *and* $\lim\limits_{t\to\infty} U(t) = \infty$;

$$\lim_{t\to\infty} \frac{L(t, \|g(t)\|)}{U'(t)} = 0, \ \int_{\alpha}^{\infty} M(s, \|g(s)\|)\, B(s)\, ds < \infty, \tag{3.83}$$

or

$$\lim_{t\to\infty}\frac{\exp\left(\int_\alpha^t M\left(s,\|g\left(s\right)\|\right)B\left(s\right)ds\right)}{U\left(t\right)}=0 \quad and \tag{3.84}$$

$$\int_\alpha^\infty \frac{L\left(s,\|g\left(s\right)\|\right)}{\exp\left(\int_\alpha^s M\left(u,\|g\left(u\right)\|\right)B\left(u\right)du\right)}ds<\infty$$

or

$$M\left(t,\|g\left(t\right)\|\right)B\left(t\right)\le \frac{k}{t},\ k,\alpha>0,\ t\in[\alpha,\infty) \tag{3.85}$$

$$and\ \lim_{t\to\infty}\frac{t^k\int_\alpha^t \frac{L(u,\|g(u)\|)}{u^k}du}{U\left(t\right)}=0$$

then the continuous solutions of (3.58) are uniformly convergent to zero at infinity.

The proof follows by Theorem 117.

3.5 Boundedness Conditions for the Difference $x-g$

In what follows, we consider the integral equation

$$x\left(t\right)=g\left(t\right)+\int_\alpha^t V\left(t,s,x\left(s\right)\right)ds,\ t\in[\alpha,\beta)\,, \tag{V}$$

where $g:[\alpha,\beta)\to X,\ V:[\alpha,\beta)^2\times X\to X$ are continuous, X is a real or complex Banach space. Using Lemma 124, we point out some sufficient conditions of boundedness for the difference $x-g$ in the normed linear space $\left(BC\left([\alpha,\beta);X\right),\|\cdot\|_\infty\right)$ where $BC\left([\alpha,\beta);X\right)$ is the linear space of all bounded continuous functions defined in $[\alpha,\beta)$ and $\|f\|_\infty:=\sup_{t\in[\alpha,\beta)}\|f\left(t\right)\|$.

Theorem 148 *If the kernel V of integral equation (V) satisfies the relation (3.18) and the following conditions*

$$\int_{\alpha}^{\infty} \|V(t,s,g(s))\|\,ds \le M_1,\ B(t) \le M_2,\ t\in[\alpha,\beta), \tag{3.86}$$

$$\int_{\alpha}^{\infty} M\left(s, \int_{\alpha}^{s} \|V(t,u,g(u))\|\,du\right) ds,\ \int_{\alpha}^{\infty} L(s,M_1)\,ds < \infty \tag{3.87}$$

hold, then there exists a $\tilde{M}>0$ such that for every $x\in C([\alpha,\beta);X)$ a solution of (V) we have $x-g\in BC([\alpha,\beta);X)$ and in addition $\|x-g\|_{\infty}\le \tilde{M}$.

Proof. Let $x\in C([\alpha,\beta);X)$ be a solution of (V). By using Lemma 124, we obtain the estimate

$$\|x(t)-g(t)\| \le k(t)+B(t)\int_{\alpha}^{t} L(u,k(u))\exp\left(\int_{u}^{t} M(s,k(s))\,B(s)\,ds\right)du,$$

where

$$k(t) := \int_{\alpha}^{t} \|V(t,s,g(s))\|\,ds,\ t\in[\alpha,\beta).$$

The proof of the theorem follows by an argument similar to that in the proof of Theorem 108. We omit the details. ■

Theorem 149 *Let us assume that the kernel V of integral equation (V) verifies the relation (3.18) and the following conditions:*

$$\int_{\alpha}^{t} \|V(t,s,g(s))\|\,ds \le M_1,\ \lim_{t\to\infty} B(t)=0,\ M_1>0,\ t\in[\alpha,\infty), \tag{3.88}$$

$$\int_{\alpha}^{\infty} L(s,M_1)\,ds,\ \int_{\alpha}^{\infty} M\left(s,\int_{\alpha}^{s}\|V(s,u,g(u))\|\,du\right)B(s)\,ds<\infty \tag{3.89}$$

or

$$M\left(t,\int_{\alpha}^{t}\|V(t,s,g(s))\|\,ds\right)B(t)\le \frac{k}{t},\ B(t)\,t^{k}\le 1,\ \alpha,k>0, \tag{3.90}$$

$$t\in[\alpha,\infty),\ \text{and}\ \int_{\alpha}^{\infty}\frac{L(s,M_1)}{s^{k}}ds<\infty$$

hold, then there exists an $\tilde{M} > 0$ *such that for every* $x \in C([\alpha, \beta); X)$ *a solution of (V), we have* $x - g \in BC([\alpha, \beta); X)$ *and in addition* $\|x - g\|_\infty \leq \tilde{M}$*.*

The proof follows by Theorem 109 of Chapter 2.

Now, using Theorem 110, we can deduce the following result.

Theorem 150 *If we suppose that the kernel* V *satisfies (3.18) and the following conditions:*

$$\int_\alpha^t \|V(t, s, g(s))\| \, ds \leq M_1, \ t \in [\alpha, \infty); \tag{3.91}$$

$$\text{there exists a function } U : [\alpha, \infty) \to \mathbb{R}_+^* \text{ differentiable} \tag{3.92}$$

$$\text{in } (\alpha, \infty) \text{ such that } B(t) \leq \frac{1}{U(t)}, \quad \text{and } \lim_{t \to \infty} U(t) = \infty;$$

$$\lim_{t \to \infty} \frac{L(t, M_1)}{U'(t)} = l < \infty, \tag{3.93}$$

$$\int_\alpha^\infty M\left(s, \int_\alpha^s \|V(s, u, g(u))\| \, du\right) B(s) \, ds < \infty$$

or

$$\lim_{t \to \infty} \frac{\exp\left(\int_\alpha^t M\left(s, \int_\alpha^s \|V(s, u, g(u))\| \, du\right) B(s) \, ds\right)}{U(t)} = l < \infty \tag{3.94}$$

and

$$\int_\alpha^\infty \frac{L(s, M_1) \, ds}{\exp\left(\int_\alpha^s M\left(u, \int_\alpha^u \|V(u, z, g(z))\| \, dz\right) B(u) \, du\right)} < \infty$$

or

$$M\left(t, \int_\alpha^t \|V(t, u, g(u))\| \, du\right) B(t) \leq \frac{k}{t}, \ \alpha, k > 0, \ t \in [\alpha, \infty); \tag{3.95}$$

$$\text{and } \lim_{t \to \infty} \frac{t^k \int_\alpha^t \frac{L(s, M_1)}{s^k} ds}{U(t)} = l < \infty$$

hold, then there exists an $M > 0$ *such that for every* $x \in C([\alpha, \beta); X)$ *a solution of (V), we have* $x - g \in BC([\alpha, \beta); X)$ *and in addition* $\|x - g\|_\infty \leq M$*.*

Another result is embodied in the following theorem.

Theorem 151 *Let us suppose that the kernel V satisfies (3.18) and the following conditions:*

$$\lim_{t\to\infty}\int_{\alpha}^{t}\|V(t,s,g(s))\|\,ds=0,\ B(t)\leq M_1,\ t\in[\alpha,\infty); \tag{3.96}$$

$$\int_{\alpha}^{\infty}M\left(s,\int_{\alpha}^{s}\|V(s,u,g(u))\|\,du\right)ds, \tag{3.97}$$

$$\int_{\alpha}^{\infty}L\left(s,\int_{\alpha}^{s}\|V(s,u,g(u))\|\,du\right)ds<\infty$$

are valid. Then there exists an $\tilde{M}>0$ such that for every $x\in C([\alpha,\beta);X)$ a solution of (V), we have $x-g\in BC([\alpha,\beta);X)$ and in addition $\|x-g\|_{\infty}\leq\tilde{M}$.

The proof follows by using a similar argument to that of Theorem 111 of the first chapter and we omit the details.

Now, by using Theorem 112, we can give another result.

Theorem 152 *If the kernel V satisfies the relation (3.18) and the following assertions:*

$$\lim_{t\to\infty}\int_{\alpha}^{t}\|V(t,s,g(s))\|\,ds=0,\ B(t)\,t^{k}\leq l<\infty,\ \alpha,k>0\quad \textit{and} \tag{3.98}$$

$$M\left(t,\int_{\alpha}^{t}\|V(t,s,g(s))\|\,ds\right)B(t)\leq\frac{k}{t},\ t\in[\alpha,\infty)\quad \textit{and} \tag{3.99}$$

$$\int_{\alpha}^{\infty}\frac{L\left(s,\int_{\alpha}^{s}\|V(s,u,g(u))\|\,du\right)}{s^{k}}ds<\infty$$

are true, then there exists an $\tilde{M}>0$ such that for every $x\in C([\alpha,\beta);X)$ a solution of (V), we have $x-g\in BC([\alpha,\beta);X)$ and $\|x-g\|_{\infty}\leq\tilde{M}$.

Finally, we have

Theorem 153 *If we assume that the kernel V satisfies the assumption (3.18) and the conditions*

$$\lim_{t\to\infty}\int_{\alpha}^{t}\|V(t,s,g(s))\|\,ds=0; \tag{3.100}$$

there exists a function $U:[\alpha,\infty)\to\mathbb{R}_{+}^{*}$ differentiable (3.101)

in (α,∞) such that $B(t)\leq\frac{1}{U(t)}$, $t>\alpha$ and $\lim_{t\to\infty}U(t)=\infty$;

$$\lim_{t\to\infty}\frac{L\left(t,\int_{\alpha}^{t}\|V(t,s,g(s))\|\,ds\right)}{U'(t)}=l<\infty \tag{3.102}$$

$$\text{and}\quad\int_{\alpha}^{\infty}M\left(s,\int_{\alpha}^{s}\|V(s,u,g(u))\|\,du\right)ds<\infty$$

or

$$\lim_{t\to\infty}\frac{\exp\left(\int_{\alpha}^{t}M\left(s,\int_{\alpha}^{s}\|V(s,u,g(u))\|\,du\right)B(s)\,ds\right)}{U(t)}=l<\infty \tag{3.103}$$

$$\text{and}\quad\int_{\alpha}^{\infty}\frac{L\left(s,\int_{\alpha}^{s}\|V(s,u,g(u))\|\,du\right)}{\exp\left(\int_{\alpha}^{s}M\left(u,\int_{\alpha}^{u}\|V(u,\tau,g(\tau))\|\,d\tau\right)B(u)\,du\right)}ds<\infty$$

or

$$M\left(t,\int_{\alpha}^{t}\|V(t,s,g(s))\|\,ds\right)B(t)\leq\frac{k}{t},\ t\in[\alpha,\infty),\ \alpha,k>0 \tag{3.104}$$

$$\text{and}\quad\lim_{t\to\infty}\frac{t^{k}\int_{\alpha}^{t}\frac{L\left(s,\int_{\alpha}^{s}\|V(s,u,g(u))\|du\right)}{s^{k}}ds}{U(t)}=l<\infty$$

hold, then there exists an $\tilde{M}>0$ such that for every $x\in C([\alpha,\beta);X)$ a solution of (V), we have $x-g\in BC([\alpha,\beta);X)$ and $\|x-g\|_{\infty}\leq\tilde{M}$.

The proof is obvious by Theorem 113.

Remark 154 *If we assume that the kernel V satisfies the relations (3.20), (3.22), (3.24), (3.26) and (3.28) or the particular conditions (3.30), (3.32) of Section 3.1, we can obtain a large number of corollaries and consequences for the above theorems. We omit the details.*

3.6 Asymptotic Equivalence Conditions

We assume that the integral equation (V) has solutions in $C([\alpha,\beta);X)$ and in this assumption, by using Lemma 124, we give some theorems of asymptotic equivalence as follows.

Theorem 155 *Let us suppose that the kernel V satisfies the relation (3.18) and the following conditions:*

$$\lim_{t\to\infty}\int_\alpha^t \|V(t,s,g(s))\|\,ds = 0,\ \lim_{t\to\infty} B(t) = 0, \tag{3.105}$$

$$\int_\alpha^\infty L\left(s,\int_\alpha^s \|V(s,u,g(u))\|\,du\right)ds, \tag{3.106}$$

$$\int_\alpha^\infty M\left(s,\int_\alpha^s \|V(s,u,g(u))\|\,du\right)B(s)\,ds < \infty$$

hold. Then for every $\varepsilon > 0$, there exists a $\delta(\varepsilon) > \alpha$ such that for every $x \in C([\alpha,\beta);X)$ a solution of (V), we have $\|x(t)-g(t)\| < \varepsilon$ for all $t > \delta(\varepsilon)$, i.e. the continuous solutions of (V) are uniformly asymptotically equivalent with g.

Proof. Let $x \in C([\alpha,\beta);X)$ be a solution of (V). Applying Lemma 124, we obtain the bound

$$\begin{aligned}&\|x(t)-g(t)\|\\ &\quad\le k(t) + B(t)\int_\alpha^t L(u,k(u))\exp\left(\int_u^t M(s,k(s))\,B(s)\,ds\right)du,\end{aligned}$$

where

$$k(t) := \int_\alpha^t \|V(t,s,g(s))\|\,ds,\ t\in[\alpha,\beta).$$

The proof of this theorem follows by an argument similar to that in the proof of Theorem 114. We omit the details. ■

Now, using Lemma 124 and Theorem 115, we can state the following results.

Theorem 156 *If the kernel V satisfies the relation (3.18) and the following conditions:*

$$\lim_{t\to\infty}\int_{\alpha}^{t}\|V(t,s,g(s))\|\,ds=0,\ \lim_{t\to\infty}t^{k}B(t)=0, \tag{3.107}$$

$$M\left(t,\int_{\alpha}^{t}\|V(t,s,g(s))\|\,ds\right)B(t)\le\frac{k}{t},\ t\in[\alpha,\infty),\ \alpha>0\ \ \text{and} \tag{3.108}$$

$$\int_{\alpha}^{\infty}\frac{L\left(s,\int_{\alpha}^{s}\|V(s,u,g(u))\|\,du\right)}{s^{k}}ds<\infty$$

hold, then the continuous solutions of (V) are uniformly asymptotically equivalent with g.

Finally, we have

Theorem 157 *Let us suppose that the kernel V of integral equation (V) verifies the relation (3.18) and the following conditions:*

$$\lim_{t\to\infty}\int_{\alpha}^{t}\|V(t,s,g(s))\|\,ds=0, \tag{3.109}$$

there exists a function $U:[\alpha,\infty)\to\mathbb{R}_{+}^{*}$ *differentiable* (3.110)

in (α,∞) *such that* $B(t)\le\dfrac{1}{U(t)},\ t\in[\alpha,\infty)$ *and* $\lim_{t\to\infty}U(t)=\infty$;

$$\lim_{t\to\infty}\frac{L\left(t,\int_{\alpha}^{t}\|V(t,s,g(s))\|\,ds\right)}{U'(t)}=0 \tag{3.111}$$

and $$\int_{\alpha}^{\infty}M\left(s,\int_{\alpha}^{s}\|V(s,u,g(u))\|\,du\right)B(s)\,ds<\infty$$

or

$$\lim_{t\to\infty}\frac{\exp\left(\int_{\alpha}^{t}M\left(s,\int_{\alpha}^{s}\|V(s,u,g(u))\|\,du\right)B(s)\,ds\right)}{U(t)}=0 \tag{3.112}$$

and $$\int_{\alpha}^{\infty}\frac{L\left(s,\int_{\alpha}^{s}\|V(s,u,g(u))\|\,du\right)}{\exp\left(\int_{\alpha}^{s}M\left(u,\int_{\alpha}^{u}\|V(u,\tau,g(\tau))\|\,d\tau\right)B(u)\,du\right)}ds<\infty$$

or

$$M\left(t,\int_\alpha^t \|V(t,s,g(s))\|\,ds\right)B(t)\leq\frac{k}{t},\ t\in[\alpha,\infty),\ \alpha,k>0 \tag{3.113}$$

and $$\lim_{t\to\infty}\frac{t^k\int_\alpha^t\frac{L\left(u,\int_\alpha^u\|V(u,\tau,g(\tau))\|d\tau\right)}{u^k}du}{U(t)}=0$$

hold, then the continuous solutions of (V) are uniformly asymptotically equivalent with g.

The proof follows by Lemma 124 and Theorem 117.

Remark 158 *If we assume that the kernel V of integral equation (V) satisfies the relations (3.20), (3.22), (3.24), (3.26) and (3.28) or the particular conditions (3.30), (3.32) of Section 3.1. we can obtain a large number of corollaries and consequences for the above theorems. We omit the details.*

3.7 The Case of Degenerate Kernels

In what follows, we consider the Volterra integral equation with degenerate kernel given by:

$$x(t)=g(t)+B(t)\int_\alpha^t U(s,x(s))\,ds,\ t\in[\alpha,\infty), \tag{3.114}$$

where $g:[\alpha,\beta)\to X$, $U:[\alpha,\beta)\times X\to X$, $B:[\alpha,\beta)\to K$ are continuous and X is a Banach space over the real or complex number field.

We assume that the equation (3.114) has solutions in $C([\alpha,\beta);X)$ and by using Lemma 130, we point out some sufficient conditions of boundedness for the difference $x-g$ in the normed linear space $(BC([\alpha,\beta);X),\|\cdot\|_\infty)$.

Theorem 159 *If the kernel U satisfies the relation (3.35) of Section 3.2, and the following conditions:*

$$|B(t)|\leq M_1,\ t\in[\alpha,\infty),\ M_1>0; \tag{3.115}$$

$$\int_\alpha^\infty L(s,\|g(s)\|)\,ds,\ \int_\alpha^\infty M(s,\|g(s)\|)\,ds<\infty \tag{3.116}$$

hold, then there exists an $\tilde{M}>0$ such that for every $x\in C([\alpha,\beta);X)$ a solution of (V) we have $x-g\in BC([\alpha,\beta);X)$ and $\|x-g\|_\infty\leq\tilde{M}$.

Proof. Let $x \in C([\alpha,\beta);X)$ be a solution of (3.114). Applying Lemma 130, we have the estimate

$$\|x(t)-g(t)\| \leq |B(t)| \int_\alpha^t L(s,\|g(s)\|) \times \exp\left(M_1 \int_s^t |B(u)| M(u,\|g(u)\|)\,du\right) ds, \quad (3.117)$$

for all $t \in [\alpha,\beta)$.

If the conditions (3.115) and (3.116) are satisfied, then we have

$$\|x(t)-g(t)\| \leq M_1 \int_\alpha^\infty L(s,\|g(s)\|)\,ds \exp\left(M_1 \int_\alpha^\infty M(s,\|g(s)\|)\,ds\right),$$

for all $t \in [\alpha,\beta)$, and the theorem is proved. ■

Theorem 160 *If the kernel U satisfies the relation (3.35) and the conditions*

$$|B(t)|\,t^k \leq l,\ k,\alpha > 0,\ t \in [\alpha,\infty); \quad (3.118)$$

$$M(t,\|g(t)\|)\,|B(t)| \leq \frac{k}{t},\ \int_\alpha^\infty \frac{L(s,\|g(s)\|)}{s^k}ds < \infty \quad (3.119)$$

hold, then there exists an $\tilde{M} > 0$ such that for every $x \in C([\alpha,\beta);X)$ a solution of (3.114) we have $x-g \in BC([\alpha,\beta);X)$ and $\|x-g\|_\infty \leq \tilde{M}$.

Proof. Let $x \in C([\alpha,\beta);X)$ be a solution of (3.114). Applying Lemma 130, we have the inequality (3.117).

If the conditions (3.118), (3.119) are satisfied, we obtain

$$\|x(t)-g(t)\| \leq l \int_\alpha^\infty \frac{L(s,\|g(s)\|)}{s^k}ds,\ t \in [\alpha,\infty)$$

and the theorem is proved. ■

We now prove another theorem which gives some sufficient conditions of boundedness for the difference $x-g$ in $BC([\alpha,\beta);X)$.

Theorem 161 *Let us suppose that the kernel U of integral equation (3.114) verifies the relation (3.35) and the following conditions:*

$$\text{there exists a function } T : [\alpha, \infty) \to \mathbb{R}_+^* \text{ differentiable} \quad (3.120)$$
$$\text{in } (\alpha, \infty) \text{ such that } |B(t)| \leq \frac{1}{T(t)}, \ \lim_{t\to\infty} T(t) = \infty;$$

$$\lim_{t\to\infty} \frac{L(t, \|g(t)\|)}{T'(t)} = l < \infty, \ \int_\alpha^\infty M(s, \|g(s)\|) B(s) \, ds < \infty \quad (3.121)$$

or

$$\lim_{t\to\infty} \frac{\exp\left(\int_\alpha^t M(s, \|g(s)\|) |B(t)| \, ds\right)}{T(t)} = l < \infty, \quad (3.122)$$

$$\int_\alpha^\infty \frac{L(s, \|g(s)\|) B(t) \, ds}{\exp\left(\int_\alpha^s M(u, \|g(u)\|) |B(u)| \, du\right)} < \infty$$

or

$$M(t, \|g(t)\|) B(t) \leq \frac{k}{t}, \ t \in [\alpha, \infty) \quad (3.123)$$

$$\text{and} \quad \lim_{t\to\infty} \frac{t^k \int_\alpha^t \frac{L(s,\|g(s)\|)}{s^k} ds}{T(t)} = l < \infty$$

hold, then there exists an $\tilde{M} > 0$ such that for every $x \in C([\alpha, \beta); X)$ a solution of (3.114) we have $x - g \in BC([\alpha, \beta); X)$ and $\|x - g\|_\infty \leq \tilde{M}$.

Proof. Let $x \in C([\alpha, \beta); X)$ be a solution of (3.114). Applying Lemma 130, we have the inequality (3.117).

If the conditions (3.120), and either (3.121) or (3.122) or (3.123) are satisfied, then we have either

$$\|x(t) - g(t)\| \leq \frac{\int_\alpha^t L(s, \|g(s)\|) \, ds}{T(t)} \exp\left(\int_\alpha^\infty M(s, \|g(s)\|) |B(s)| \, ds\right)$$

or

$$\|x(t) - g(t)\|$$
$$\leq \frac{\exp\left(\int_\alpha^t M(s, \|g(s)\|) |B(s)| \, ds\right)}{T(t)} \int_\alpha^\infty \frac{L(s, \|g(s)\|)}{\exp\left(\int_\alpha^s M(u, \|g(u)\|) |B(u)| \, du\right)}$$

or

$$\|x(t)-g(t)\| \leq \frac{t^k \int_\alpha^t \frac{L(s,\|g(s)\|)}{s^k} ds}{T(t)} \quad \text{for all } t \in [\alpha, \infty).$$

Since

$$\lim_{t\to\infty} \frac{\int_\alpha^t L(s,\|g(s)\|)\,ds}{T(t)} = \lim_{t\to\infty} \frac{L(t,\|g(t)\|)}{T'(t)} = l < \infty,$$

$$\lim_{t\to\infty} \frac{\exp\left(\int_\alpha^t M(s,\|g(s)\|)\,|B(s)|\,ds\right)}{T(t)} = l < \infty,$$

$$\lim_{t\to\infty} \frac{t^k \int_\alpha^t \frac{L(s,\|g(s)\|)}{s^k} ds}{T(t)} = l < \infty$$

and the functions are continuous in $[\alpha, \beta)$, it follows that these functions are bounded in $[\alpha, \beta)$ and the theorem is thus proved. ■

Now, by using Lemma 136, we can give the following three theorems.

Theorem 162 *Let us suppose that the kernel U of integral equation (3.114) satisfies the relation (3.52) and the following conditions:*

$$|B(t)| \leq M_1,\ t \in [\alpha, \infty), \tag{3.124}$$

$$\int_\alpha^\infty \|U(s, g(s))\|\,ds,\ \int_\alpha^\infty S(u, \|g(u)\|)\,du < \infty \tag{3.125}$$

hold, then there exists an $\tilde{M} > 0$ such that for every $x \in C([\alpha, \beta); X)$ a solution of (3.114) we have $x - g \in BC([\alpha, \beta); X)$ and $\|x - g\|_\infty \leq \tilde{M}$.

Proof. Let $x \in C([\alpha, \beta); X)$ be a solution of (3.34). Applying Lemma 136, we have the estimate

$$\begin{aligned} &\|x(t)-g(t)\| \\ &\quad \leq |B(t)| \int_\alpha^t \|U(s,g(s))\| \exp\left(\int_s^t S(u,\|g(u)\|)\,|B(u)|\,du\right) ds \end{aligned} \tag{3.126}$$

for all $t \in [\alpha, \beta)$.

If the conditions (3.124) and (3.125) are satisfied, then we have

$$\|x(t) - g(t)\| \leq M_1 \int_\alpha^\infty \|U(s, g(s))\| \exp\left(M_1 \int_\alpha^s S(u, \|g(u)\|)\, du\right) ds$$

for all $t \in [\alpha, \beta)$, and the theorem is thus proved. ■

The following theorem also holds.

Theorem 163 *If the kernel U satisfies the relation (3.52) and the following conditions:*

$$|B(t)|\, t^k \leq l,\ k > 0,\ t \in [\alpha, \infty); \tag{3.127}$$

$$S(t, \|g(t)\|)\, |B(t)| \leq \frac{k}{t} \text{ and } \int_\alpha^\infty \frac{\|U(s, g(s))\|}{s^k} ds < \infty \tag{3.128}$$

hold, then there exists an $\tilde{M} > 0$ such that for every $x \in C([\alpha, \beta); X)$ a solution of (3.114) we have $x - g \in BC([\alpha, \beta); X)$ and $\|x - g\|_\infty \leq \tilde{M}$.

Proof. Let $x \in C([\alpha, \beta); X)$ be a solution of (3.114). Applying Lemma 136, we have the estimate (3.126).

If the conditions (3.127) and (3.128) are satisfied, then we have

$$\|x(t) - g(t)\| \leq l \int_\alpha^\infty \frac{\|U(s, g(s))\|}{s^k} ds,\ t \in [\alpha, \infty);$$

and the theorem is proved. ■

Finally, we have

Theorem 164 *If the kernel U of integral equation (3.114) satisfies the relation (3.52) and the following conditions:*

$$\text{there exists a function } T : [\alpha, \infty) \to \mathbb{R}_+^* \text{ differentiable} \tag{3.129}$$

$$\text{in } (\alpha, \infty) \text{ such that } |B(t)| \leq \frac{1}{T(t)},\ \lim_{t \to \infty} T(t) = \infty;$$

$$\lim_{t \to \infty} \frac{U\,\|(t, g(t))\|}{T'(t)} = l < \infty \text{ and } \int_\alpha^\infty S(s, \|g(s)\|)\, |B(s)|\, ds < \infty \tag{3.130}$$

or

$$\lim_{t\to\infty}\frac{\exp\left(\int_\alpha^t S\left(s,\|g\left(s\right)\|\right)|B\left(s\right)|\,ds\right)}{T\left(t\right)}=l<\infty \tag{3.131}$$

$$and\quad \int_\alpha^\infty \frac{U\left\|\left(s,g\left(s\right)\right)\right\|ds}{\exp\left(\int_\alpha^s S\left(u,\|g\left(u\right)\|\right)|B\left(u\right)|\,du\right)}<\infty$$

or

$$S\left(t,\|g\left(t\right)\|\right)|B\left(t\right)|\leq\frac{k}{t},\ t\in[\alpha,\infty) \tag{3.132}$$

$$and\ \lim_{t\to\infty}\frac{t^k\int_\alpha^t\frac{U\|(s,g(s))\|ds}{s^k}ds}{T\left(t\right)}=l<\infty$$

hold, then there exists an $M\tilde{M}0$ *such that for every* $x\in C\left([\alpha,\beta);X\right)$ *a solution of (3.114) we have* $x-g\in BC\left([\alpha,\beta);X\right)$ *and* $\|x-g\|_\infty\leq\tilde{M}$.

The proof of the theorem follows by an argument similar to that in the proof of the above theorems. We omit the details.

Remark 165 *If we assume that the kernel* U *satisfies the relations (3.38), (3.40), (3.42), (3.44) and (3.46) or (3.54), (3.56) of Section 3.2, we can obtain a large number of Corollaries of the above theorems. We omit the details.*

3.8 Asymptotic Equivalence Conditions

It is the purpose of this section to establish some theorems of asymptotic equivalence between the continuous solutions of Volterra integral equations with degenerate kernels and the perturbation g.

Theorem 166 *If the kernel* U *of integral equation (3.114) satisfies the relation (3.35) and the following conditions:*

$$\lim_{t\to\infty}|B\left(t\right)|=0, \tag{3.133}$$

$$\int_\alpha^\infty L\left(s,\|g\left(s\right)\|\right)ds,\ \int_\alpha^\infty |B\left(s\right)|\,M\left(s,\|g\left(s\right)\|\right)ds<\infty \tag{3.134}$$

are valid, then the continuous solutions of (3.114) are uniformly equivalent with g.

Proof. Let $\varepsilon > 0$ and $x \in C([\alpha, \beta); X)$ be a solution of (3.114).
If the conditions (3.133) and (3.134) are satisfied, we have:

$$\|x(t) - g(t)\| \leq |B(t)| \int_{\alpha}^{\infty} L(s, \|g(s)\|)\, ds \times \exp\left(\int_{\alpha}^{\infty} |B(s)|\, M(s, \|g(s)\|)\, ds\right)$$

and there exists a $\delta(\varepsilon) > \alpha$ such that

$$|B(t)| \int_{\alpha}^{\infty} L(s, \|g(s)\|)\, ds \exp\left(\int_{\alpha}^{\infty} |B(s)|\, M(s, \|g(s)\|)\, ds\right) < \varepsilon$$

for all $t > \delta(\varepsilon)$.
It results that

$$\|x(t) - g(t)\| < \varepsilon \ \text{ if } \ t > \delta(\varepsilon)$$

and the theorem is thus proved. ■

Theorem 167 *Let us suppose that the kernel* U *satisfies the relation (3.35) and the following conditions:*

$$\lim_{t \to \infty} t^k |B(t)| = 0,\ k > 0,\ t \in [\alpha, \infty) \tag{3.135}$$

$$M(t, \|g(t)\|)\, |B(t)| \leq \frac{k}{t},\ k, \alpha > 0,\ \int_{\alpha}^{\infty} \frac{L(s, \|g(s)\|)}{s^k} ds < \infty \tag{3.136}$$

hold. Then the continuous solutions of (3.114) are uniformly equivalent with g.

Proof. Let $\varepsilon > 0$ and $x \in C([\alpha, \beta); X)$ be a solution of (3.114).
If the conditions (3.135) and (3.136) are satisfied, then we have:

$$\|x(t) - g(t)\| \leq |B(t)|\, t^k \int_{\alpha}^{\infty} \frac{L(s, \|g(s)\|)}{s^k} ds,\ t \in [\alpha, \infty)$$

and there exists a $\delta(\varepsilon) > \alpha$ such that

$$|B(t)|\, t^k \int_{\alpha}^{\infty} \frac{L(s, g(s))}{s^k} ds < \varepsilon \text{ if } t > \delta(\varepsilon).$$

This proves the theorem. ■

Now, we can give another result which is embodied in the following theorem.

Theorem 168 *If we suppose that the kernel U verifies the relation (3.35) and the following assertions*

$$\begin{aligned} &\text{there exists a function } T:[\alpha,\infty)\to\mathbb{R}_+^* \text{ differentiable} \\ &\text{in } (\alpha,\infty) \text{ and } B(t)\le\frac{1}{T(t)},\ \lim_{t\to\infty}T(t)=\infty; \end{aligned} \tag{3.137}$$

$$\lim_{t\to\infty}\frac{L(t,\|g(t)\|)}{T'(t)}=0,\ \int_\alpha^\infty M(s,\|g(s)\|)\,|B(s)|\,ds<\infty \tag{3.138}$$

or

$$\lim_{t\to\infty}\frac{\exp\left(\int_\alpha^t M(s,\|g(s)\|)\,|B(s)|\,ds\right)}{T(t)}=0 \tag{3.139}$$

$$\text{and}\quad \int_\alpha^\infty\frac{L(s,\|g(s)\|)\,|B(s)|}{\exp\left(\int_\alpha^s M(u,\|g(u)\|)\,|B(u)|\,du\right)}ds<\infty$$

or

$$M(t,\|g(t)\|)\,|B(t)|\le\frac{k}{t},\ t\in[\alpha,\infty) \tag{3.140}$$

$$\text{and}\quad \lim_{t\to\infty}\frac{t^k\int_\alpha^t\frac{L(s,\|g(s)\|)}{s^k}ds}{T(t)}=0,$$

are valid, then the continuous solutions of (3.114) are uniformly equivalent with g.

The proof follows by a similar argument to that in the proof of the above theorems. We omit the details.

Further, by using Lemma 136, we can give the following three theorems.

Theorem 169 *If we assume that the kernel U satisfies (3.52) and the following assertions*

$$\lim_{t\to\infty} |B(t)| = 0 \tag{3.141}$$

$$\int_\alpha^\infty \|U(s, g(s))\| \, ds, \; \int_\alpha^\infty S(s, \|g(s)\|) \, |B(s)| \, ds < \infty \tag{3.142}$$

are valid, then the continuous solutions of (3.114) are uniformly equivalent with g.

Proof. Let $\varepsilon > 0$ and $x \in C([\alpha, \beta); X)$ be a solution of (3.114).
If the conditions (3.141) and (3.142) are satisfied, then we have:

$$\|x(t) - g(t)\| \\ \leq |B(t)| \int_\alpha^\infty \|U(s, g(s))\| \, ds \exp\left(\int_\alpha^\infty S(u, \|g(u)\|) \, |B(u)| \, du\right)$$

and there exists a $\delta(\varepsilon) > \alpha$ such that

$$|B(t)| \int_\alpha^\infty \|U(s, g(s))\| \, ds \exp\left(\int_\alpha^\infty S(u, \|g(u)\|) \, |B(u)| \, du\right) < \varepsilon$$

if $t > \delta(\varepsilon)$.
The theorem is thus proved. ■

The second results are embodied in the following theorem.

Theorem 170 *Let us suppose that the kernel U of integral equation (3.114) verifies (3.52) and the following conditions:*

$$\lim_{t\to\infty} t^k |B(t)| = 0, \; k > 0 \tag{3.143}$$

$$S(t, \|g(t)\|) \, |B(t)| \leq \frac{k}{t}, \; \int_\alpha^\infty \frac{\|U(s, g(s))\|}{s^k} ds < \infty \tag{3.144}$$

hold. Then the continuous solutions of (3.114) are uniformly asymptotic equivalent with g.

Proof. Let $\varepsilon > 0$ and $x \in C([\alpha, \beta); X)$ be a solution of (3.114).
If the conditions (3.143) and (3.144) are satisfied, we have:

$$\|x(t) - g(t)\| \leq |B(t)| \, t^k \int_\alpha^\infty \frac{\|U(s, g(s))\|}{s^k} ds,$$

and there exists a $\delta(\varepsilon) > \alpha$ such that

$$|B(t)|\, t^k \int_\alpha^\infty \frac{\|U(s, g(s))\|}{s^k} ds < \varepsilon \text{ if } t > \delta(\varepsilon).$$

and the theorem is thus proved. ■

Finally, we have

Theorem 171 *If the kernel U of integral equation (3.114) verifies the relation (3.52) and the following conditions:*

$$\text{there exists a function } T : [\alpha, \infty) \to \mathbb{R}_+^* \text{ differentiable} \tag{3.145}$$

$$\text{in } (\alpha, \infty) \text{ such that } |B(t)| \leq \frac{1}{T(t)}, \ \lim_{t\to\infty} T(t) = \infty;$$

$$\lim_{t\to\infty} \frac{\|U(t, g(t))\|}{T'(t)} = 0, \ \int_\alpha^\infty S(s, \|g(s)\|)\, |B(s)|\, ds < \infty \tag{3.146}$$

or

$$\lim_{t\to\infty} \frac{\exp\left(\int_\alpha^t S(s, \|g(s)\|)\, |B(s)|\, ds\right)}{T(t)} = 0 \tag{3.147}$$

$$\text{and} \quad \int_\alpha^\infty \frac{\|U(s, g(s))\|}{\exp\left(\int_\alpha^s S(u, \|g(u)\|)\, |B(u)|\, du\right)} ds < \infty$$

or

$$S(t, \|g(t)\|)\, |B(t)| \leq \frac{k}{t}, \ t \in [\alpha, \infty) \tag{3.148}$$

$$\text{and} \quad \lim_{t\to\infty} \frac{t^k \int_\alpha^t \frac{\|U(s,g(s))\|}{s^k} ds}{T(t)} = 0,$$

hold, then the solutions of (3.114) are uniformly equivalent with g.

The proof follows by an argument similar to that in the proof of the above theorems. We omit the details.

Remark 172 *If we assume that the kernel U satisfies the relations (3.38), (3.40), (3.42), (3.44) and (3.46) or (3.54), (3.56) of Section 3.2, we can obtain a large number of Corollaries for the above theorems. We omit the details.*

3.9 A Pair of Volterra Integral Equations

3.9.1 Estimation Theorems

Further on, we consider the following two Volterra integral equations in Banach spaces:

$$x_0(t) = \lambda_0(t) + \int_\alpha^t V_0(t, s, x_0(s))\, ds,\ t \in [\alpha, \beta) \tag{3.149}$$

$$x(t) = \lambda(t) + \int_\alpha^t V(t, s, x(s))\, ds,\ t \in [\alpha, \beta) \tag{3.150}$$

where $\lambda_0,\ \lambda : [\alpha, \beta) \to X;\ V_0, V : [\alpha, \beta)^2 \times X \to X$ are continuous and X is a real or complex Banach space.

We assume that the above equations have solutions in $C([\alpha, \beta) : X)$, which denotes the set of all vector-valued continuous mappings which are defined in $[\alpha, \beta)$.

The following theorem holds [42]

Theorem 173 *Suppose that the kernel V satisfies the relation:*

$$\|V(t, s, x) - V(t, s, y)\| \le B(t) L(s, \|x - y\|)\,;\ s, t \in [\alpha, \beta) \tag{3.151}$$

and $x, y \in X$; where $B : [\alpha, \beta) \to \mathbb{R}_+,\ L : [\alpha, \beta) \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and L verifies the condition (2.1).
Then for every x_0 a continuous solution of (3.149) and x a continuous solution of (3.150), we have the estimation

$$\begin{aligned} &\|x(t) - x_0(t)\| \\ &\quad \le \tilde{\lambda}(t) + \tilde{V}(t, x_0) + B(t) \int_\alpha^t L\left(u, \tilde{\lambda}(u) + \tilde{V}(u, x_0)\right) \\ &\qquad \times \exp\left(\int_u^t M\left(s, \tilde{\lambda}(s) + \tilde{V}(s, x_0)\right) B(s)\, ds\right) du \end{aligned} \tag{3.152}$$

for all $t \in [\alpha, \beta)$ where $\tilde{\lambda}(t) := \left\|\tilde{\lambda}(t) - \lambda_0(t)\right\|$; and

$$\tilde{V}(t, x_0) := \int_\alpha^t \|V(t, s, x_0(s)) - V_0(t, s, x_0(s))\|\, ds,\ t \in [\alpha, \beta).$$

Proof. Let x_0 be a solution of (3.149) and x be a solution of (3.150). It is easy to see that;

$$x(t) - x_0(t) = \lambda(t) - \lambda_0(t) + \int_\alpha^t V(t,s,x(s)) - V_0(t,s,x_0(s))\,ds$$

for all $t \in [\alpha, \beta)$, and by passing at norms, we obtain

$$\begin{aligned}\|x(t) - x_0(t)\| &\le \|\lambda(t) - \lambda_0(t)\| + \int_\alpha^t \|V(t,s,x_0(s)) - V_0(t,s,x_0(s))\|\,ds \\ &\quad + \int_\alpha^t \|V(t,s,x(s)) - V_0(t,s,x_0(s))\|\,ds \\ &\le \tilde{\lambda}(t) + \tilde{V}(t,x_0) + B(t)\int_\alpha^t L(s,\|x(s) - x_0(s)\|)\,ds\end{aligned}$$

for all $t \in [\alpha, \beta)$.

Applying Lemma 118 for $A(t) := \tilde{\lambda}(t) + \tilde{V}(t,x_0)$, we deduce the estimation (3.152). ■

The following corollaries are important in applications:

Corollary 174 *Suppose that the kernel V satisfies the condition:*

$$\|V(t,s,x) - V(t,s,y)\| \le B(t)\,G(s,\|x-y\|)\,; \quad s,t \in [\alpha,\beta) \tag{3.153}$$

and $x, y \in X$, where $B : [\alpha,\beta) \to \mathbb{R}_+$, $G : [\alpha,\beta) \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and G verifies the condition (2.5).
Then for every x_0 a continuous solution of (3.149) and x a continuous solution of (3.150), we have the estimation:

$$\begin{aligned}&\|x(t) - x_0(t)\| \\ &\quad \le \tilde{\lambda}(t) + \tilde{V}(t,x_0) + B(t)\int_\alpha^t G\left(u, \tilde{\lambda}(u) + \tilde{V}(u,x_0)\right) \\ &\qquad \times \exp\left(\int_u^t N(s)\,B(s)\,ds\right)du\end{aligned} \tag{3.154}$$

for all $t \in [\alpha, \beta)$.

Corollary 175 *If we assume that the kernel V satisfies the condition:*

$$\|V(t,s,x) - V(t,s,y)\| \le B(t)\,C(s)\,H(\|x-y\|)\,; \quad s,t \in [\alpha,\beta) \tag{3.155}$$

for $x, y \in X$, where $B, C : [\alpha, \beta) \to \mathbb{R}_+$, $H : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and H verifies the relation (2.8), then for every x_0 a continuous solution of (3.149) and x a solution of (3.150), we have the estimation:

$$\begin{aligned} &\|x(t) - x_0(t)\| \\ &\quad \le \tilde{\lambda}(t) + \tilde{V}(t, x_0) + B(t) \int_\alpha^t C(u) H\left(\tilde{\lambda}(u) + \tilde{V}(u, x_0)\right) \\ &\qquad \times \exp\left(M \int_u^t C(s) B(s)\, ds\right) du \quad (3.156) \end{aligned}$$

for all $t \in [\alpha, \beta)$.

Further on, we assume that $\tilde{\lambda}(t) > 0$ for all $t \in [\alpha, \beta)$.

Theorem 176 *Suppose that the kernel V satisfies the condition:*

$$\|V(t, s, x) - V(t, s, y)\| \le B(t) D(s, \|x - y\|); \quad s, t \in [\alpha, \beta) \qquad (3.157)$$

and $x, y \in X$ where $B : [\alpha, \beta) \to \mathbb{R}_+$, $D : [\alpha, \beta) \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and D verifies the condition (2.11).
Then for every x_0 a continuous solution of (3.149) and x a continuous solution of (3.150), we have the estimation:

$$\begin{aligned} &\|x(t) - x_0(t)\| \\ &\quad \le \tilde{\lambda}(t) + \tilde{V}(t, x_0) + B(t) \int_\alpha^t D\left(u, \tilde{\lambda}(u) + \tilde{V}(u, x_0)\right) \\ &\qquad \times \exp\left(\int_u^t P\left(s, \tilde{\lambda}(s) + \tilde{V}(s, x_0)\right) B(s)\, ds\right) du \quad (3.158) \end{aligned}$$

for all $t \in [\alpha, \beta)$.

The proof follows by an argument similar to that in the proof of Theorem 173. We omit the details.

The following two corollaries hold.

Corollary 177 *If we suppose that the kernel V verifies the relation:*

$$\|V(t, s, x) - V(t, s, y)\| \le B(t) I(s, \|x - y\|); \quad s, t \in [\alpha, \beta) \qquad (3.159)$$

and $x, y \in X$ where $B : [\alpha, \beta) \to \mathbb{R}_+$, $I : [\alpha, \beta) \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and I verifies the condition (2.14), then for every x_0 a solution of (3.149) and x a solution of (3.150) we have

$$\begin{aligned} &\|x(t) - x_0(t)\| \\ &\leq \tilde{\lambda}(t) + \tilde{V}(t, x_0) + B(t) \int_\alpha^t I\left(u, \tilde{\lambda}(u) + \tilde{V}(u, x_0)\right) \\ &\times \exp\left(\int_u^t \frac{\partial I}{\partial x}\left(s, \tilde{\lambda}(s) + \tilde{V}(s, x_0)\right) B(s)\, ds\right) du \end{aligned} \quad (3.160)$$

for all $t \in [\alpha, \beta)$.

Finally, we have:

Corollary 178 *If we suppose that the kernel V verifies the relation:*

$$\begin{aligned} &\|V(t, s, x) - V(t, s, y)\| \leq B(t)\, C(s)\, K(\|x - y\|); \\ &s, t \in [\alpha, \beta),\ x, y \in X, \end{aligned} \quad (3.161)$$

where $B, C : [\alpha, \beta) \to \mathbb{R}_+$, $K : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and K verifies the condition (2.17), then for every x_0 a solution of (3.149) and x a solution of (3.150), we have the estimation:

$$\begin{aligned} &\|x(t) - x_0(t)\| \\ &\leq \tilde{\lambda}(t) + \tilde{V}(t, x_0) + B(t) \int_\alpha^t C(u)\, K\left(\tilde{\lambda}(u) + \tilde{V}(u, x_0)\right) \\ &\times \exp\left(\int_u^t \frac{dK}{dx}\left(\tilde{\lambda}(s) + \tilde{V}(s, x_0)\right) C(s)\, B(s)\, ds\right) du \end{aligned} \quad (3.162)$$

for all $t \in [\alpha, \beta)$.

3.9.2 Boundedness Conditions

In what follows, we consider the following two Volterra integral equations:

$$x_0(t) = \lambda_0(t) + \int_\alpha^t V_0(t, s, x_0(s))\, ds,\ t \in [\alpha, \infty) \quad (3.163)$$

$$x(t) = \lambda(t) + \int_\alpha^t V(t, s, x(s))\, ds,\ t \in [\alpha, \infty) \quad (3.164)$$

where $\lambda_0, \lambda : [\alpha, \beta) \to X$, V_0, $V : [\alpha, \beta)^2 \times X \to X$ are continuous and X is a real or complex Banach space.

We suppose that the above equations have solutions in $C([\alpha, \infty); X)$ and in this assumption we give some sufficient conditions of boundedness for the difference $x - x_0$, where x_0 is a continuous solution of (3.163) and x is a similar solution of (3.164).

Theorem 179 *If the kernel V if integral equation (3.164) satisfies the condition (3.151) and the following conditions hold:*

$$\tilde{\lambda}(t) \leq M_1 < \infty, \;\; B(t) \leq M_2 < \infty \;\; \text{for all } t \in [\alpha, \infty); \tag{3.165}$$

$$\tilde{V}(t, y) \leq M_3(y) < \infty, \; \int_\alpha^\infty L(u, M_3(y))\, du \leq M_4(y) < \infty; \tag{3.166}$$

$$\int_\alpha^\infty M\left(s, \tilde{\lambda}(s) + \tilde{V}(s, y)\right) ds \leq M_5(y) < \infty \tag{3.167}$$

for all $y \in C([\alpha, \infty); X)$;

then for every x_0 a continuous solution of (3.163) there exists $\tilde{M}(x_0) < \infty$ such that for every continuous solution x of (3.164) we have the estimation

$$\|x(t) - x_0(t)\| \leq \tilde{M}(x_0), \;\; t \in [\alpha, \infty). \tag{3.168}$$

Proof. Let x_0 be a solution of (3.163) and x be a solution of (3.164). Then we have the estimation:

$$\begin{aligned} \|x(t) - x_0(t)\| &\leq \tilde{\lambda}(t) + \tilde{V}(t, x_0) + B(t) \int_\alpha^t L\left(u, \tilde{\lambda}(u) + \tilde{V}(u, x_0)\right) \\ &\quad \times \exp\left(\int_u^t M\left(s, \tilde{\lambda}(s) + \tilde{V}(s, x_0)\right) B(s)\, ds\right) du \\ &\leq M_1 + M_3(x_0) + M_2 M_4(x_0) \exp M_2 M_5(x_0) = \tilde{M}(x_0) \end{aligned}$$

for all $t \in [\alpha, \infty)$, and the theorem is proved. ■

Corollary 180 *If the following conditions: (3.165) and*

$$\tilde{V}(t, y) \leq M_3 < \infty, \; \int_\alpha^\infty L(u, M_3)\, du \leq M_4 < \infty; \tag{3.169}$$

$$\int_\alpha^\infty M\left(s, \tilde{\lambda}(s) + \tilde{V}(s, y)\right) ds \leq M_5 < \infty \;\; \text{for all } y \in C([\alpha, \infty); X); \tag{3.170}$$

are true, then there exists $\tilde{M} < \infty$ such that for every x_0 a solution of (3.163) and x a solution of (3.164), we have

$$\|x(t) - x_0(t)\| \le \tilde{M}, \text{ for all } t \in [\alpha, \infty). \tag{3.171}$$

The proof follows by an argument similar to that in the proof of the above theorem. We omit the details.

Remark 181 *If the equation (3.163) has a bounded continuous solution on $[\alpha, \infty)$ and the conditions (3.165) - (3.167) hold, then the continuous solutions of (3.164) are uniformly bounded.*

Indeed, of x_0 is a bounded continuous solution of (3.163), then for every x a continuous solution of (3.164) we have:

$$\|x(t)\| \le \|x(t) - x_0(t)\| + \|x_0(t)\| \le \tilde{M}(x_0) + \tilde{M}, \ t \in [\alpha, \infty).$$

and the remark is proved.

Theorem 182 *Suppose that V satisfies the relation (3.151) and the following conditions hold:*

$$\tilde{\lambda}(t) \le M_1 < \infty, \ \lim_{t\to\infty} B(t) = 0; \tag{3.172}$$

$$\tilde{V}(t, y) \le M_2(y) < \infty; \tag{3.173}$$

$$\int_\alpha^\infty L(u, M(y))\, du \le M_3 < \infty; \tag{3.174}$$

$$\int_\alpha^\infty M\left(s, \tilde{\lambda}(s) + \tilde{V}(s, y)\right) B(s)\, ds \le M_4(y) < \infty \tag{3.175}$$

for all $y \in C([\alpha, \infty); X)$;

or

$$M\left(t, \tilde{\lambda}(t) + \tilde{V}(t, y)\right) B(t) \le \frac{k(y)}{t}, \ k(y) > 0, \ t \in [\alpha, \infty), \ \alpha > 0; \tag{3.176}$$

$$B(t)\, t^{k(y)} \le l(y) < \infty, \ t \in [\alpha, \infty), \tag{3.177}$$

$$\int_\alpha^\infty \frac{L(s, M_1 + M_2(y))}{s^{k(y)}} ds \le M_5(y) < \infty \ \text{ for all } y \in C([\alpha, \infty); X); \tag{3.178}$$

where M_2 is as given in (3.173);

then for every x_0 a continuous solution of (3.163) there exists $\tilde{M}(x_0) < \infty$ such that for every x a solution of (3.164) we have the estimation (2.147).

Proof. Let x_0 be a solution of (3.163) and X be a solution of (3.164). We have the estimation:

$$\|x(t)-x_0(t)\| \le \tilde{\lambda}(t)+\tilde{V}(t,x_0)+B(t)\int_\alpha^t L\left(u,\tilde{\lambda}(u)+\tilde{V}(u,x_0)\right) \times \exp\left(\int_u^t M\left(s,\tilde{\lambda}(s)+\tilde{V}(s,x_0)\right)B(s)\,ds\right)du.$$

If the conditions (3.172), (3.173), (3.174) and (3.175) are satisfied, we have:

$$\|x(t)-x_0(t)\| \le M_1+M_2(x_0)+B(t)\,M_3(x_0)\exp M_4(x_0)\,,\ \ t\in[\alpha,\infty)\,.$$

Since B is continuous on $[\alpha,\infty)$ and $\lim_{t\to\infty} B(t)=0$, then it is bounded on $[\alpha,\infty)$, which implies the estimation (3.168).

Let us now suppose that the conditions (3.172), (3.176), (3.177) and (3.178) are satisfied. Then we have:

$$\|x(t)-x_0(t)\| \le M_1+M_2(x_0)+B(t)\ t^{k(x_0)}\int_\alpha^t \frac{L(s,M_1+M_2(x_0))}{s^{k(x_0)}}ds \le M_1+M_2(x_0)+l(x_0)\,M_5(x_0)=\tilde{M}(x_0)\,;$$

and the estimation (3.168) is valid with $\tilde{M}(x_0)$ given as above.

The proof is thus complete. ■

Corollary 183 *If the following conditions: (3.172) and*

$$\tilde{V}(t,y)\le M_2<\infty \tag{3.179}$$

$$\int_\alpha^\infty L(u,M_2)\,du \le M_3<\infty; \tag{3.180}$$

$$\int_\alpha^\infty M\left(s,\tilde{\lambda}(s)+\tilde{V}(s,y)\right)B(s)\,ds\le M_4<\infty \tag{3.181}$$

for all $y\in C([\alpha,\infty);X)$;

or

$$M\left(t,\tilde{\lambda}(t)+\tilde{V}(t,y)\right)B(t)\le\frac{k}{t},\ k>0,\ \ t\in[\alpha,\infty)\,; \tag{3.182}$$

$$B(t)\,t^k\le l<\infty,\ t\in[\alpha,\infty)\,, \tag{3.183}$$

$$\int_\alpha^\infty \frac{L(s,M_1+M_2(y))}{s^k}ds\le M_5<\infty\ \ \text{for all } y\in C([\alpha,\infty);X)\,; \tag{3.184}$$

are valid, then there exists a $\tilde{M} < \infty$ *such that for every* x_0 *a solution of (3.163) and* x *a solution of (3.164) the estimation (3.168) holds.*

Another result is embodied in the following theorem.

Theorem 184 *Suppose that the kernel* V *satisfies the relation (3.151) and the following conditions hold:*

$$\tilde{\lambda}(t) \le M_1,\ \tilde{V}(\cdot, y) \le M_2(y) < \infty \tag{3.185}$$

there exists a function $U : [\alpha, \infty) \to \mathbb{R}_+^*$ *differentiable in* (α, ∞) *such that*

$$B(t) \le \frac{1}{U(t)},\ t \in [\alpha, \infty) \text{ and } \lim_{t \to \alpha} U(t) = \infty; \tag{3.186}$$

$$\lim_{t \to \infty} \frac{L(t, M_1 + M_2(y))}{U'(t)} = l(y) < \infty \tag{3.187}$$

and

$$\int_\alpha^\infty M\left(s, \tilde{\lambda}(s) + \tilde{V}(s, y)\right) B(s)\, ds \le M_3(y) < \infty$$
$$\text{for all } y \in C([\alpha, \infty); X);$$

or

$$M\left(t, \tilde{\lambda}(t) + \tilde{V}(t, y)\right) B\left(t^3\right) \le \frac{k(y)}{t},\quad t \in [\alpha, \infty);$$
$$\lim_{t \to \alpha} \frac{t^{k(y)} \int_\alpha^\infty \frac{L(s, M_1 + M_2(y))}{s^k} ds}{U(t)} = \tilde{l}(y) < \infty \tag{3.188}$$
$$\text{for all } y \in C([\alpha, \infty); X);$$

then for every x_0 *a solution of (3.163) there exists* $\tilde{M}(x_0) < \infty$ *such that for every* x *a solution of (3.164), we have the estimation (3.168).*

Proof. Let x_0 be a solution of (3.163) and x be a solution of (3.164). We have the estimation

$$\|x(t) - x_0(t)\| \le \tilde{\lambda}(t) + \tilde{V}(t, x_0) + B(t) \int_\alpha^t L\left(u, \tilde{\lambda}(u) + \tilde{V}(u, x_0)\right)$$
$$\times \exp\left(\int_u^t M\left(s, \tilde{\lambda}(s) + \tilde{V}(s, x_0)\right) B(s)\, ds\right) du.$$

If the conditions (3.185), (3.186) and (3.187) or (3.188) are satisfied, we have:

$$\|x(t) - x_0(t)\| \leq M_1 + M_2(x_0) + \int_\alpha^t \frac{L(s, M_1 + M_2(y))}{U(t)} ds \exp M_3(x_0)$$

respectively

$$\|x(t) - x_0(t)\| \leq M_1 + M_2(x_0) + \frac{t^{k(x)} \int_\alpha^t \frac{L(s,M_1+M_2(x_0))}{s^{k(x)}} ds}{U(t)}$$

for all $t \in [\alpha, \infty)$.

Since

$$\lim_{t\to\infty} \frac{\int_\alpha^t L(s, M_1 + M_2(x_0))}{U(t)} ds = \lim_{t\to\infty} \frac{L(t, M_1 + M_2(x))}{U'(t)} = l(x_0) < \infty$$

and

$$\lim_{t\to\infty} \frac{t^{k(x_0)} \int_\alpha^t \frac{L(s,M_1+M_2(x_0))}{s^{k(x_0)}} ds}{U(t)} = \tilde{l}(x_0) < \infty$$

and they are continuous in $[\alpha, \infty)$, then there exists $\tilde{M}(x_0) < \infty$ such that $\|x(t) - x_0(t)\| \leq \tilde{M}(x_0)$ for all $t \in [\alpha, \infty)$.

The theorem is thus proved. ■

Corollary 185 *If the conditions:*

$$\tilde{\lambda}(t) \leq M_1, \; \tilde{V}(t, y) \leq M_2 < \infty \tag{3.189}$$

and

$$\lim_{t\to\infty} \frac{L(t, M_1 + M_2)}{U(t)} = l(y) < \infty; \tag{3.190}$$

$$\int_\alpha^\infty M\left(s, \tilde{\lambda}(s) + \tilde{V}(x, y)\right) B(s)\, ds \leq M_3 < \infty$$

for all $y \in C([\alpha, \infty); X)$;

or

$$M\left(t, \tilde{\lambda}(t) + \tilde{V}(t, y)\right) B(t) \leq \frac{k}{t}, \quad t \in [\alpha, \infty); \tag{3.191}$$

and

$$\lim_{t\to\alpha} \frac{t^k \int_\alpha^t \frac{L(s,M_1+M_2(y))}{s^k} ds}{U(t)} = \tilde{l} < \infty \quad \text{for all } y \in C([\alpha, \infty); X);$$

are valid, then there exists a constant $\tilde{M} < \infty$ *such that for every* x_0 *a solution of (3.163) and* x *a solution of (3.164) we have the estimation (3.171).*

Now we can prove another result which gives a sufficient condition of boundedness for the difference $x - x_0$.

Theorem 186 *Suppose that the kernel V satisfies the relation (3.151) and the following conditions hold:*

$$\lim_{t\to\infty} \tilde{\lambda}(t) = 0, \ \lim_{t\to\infty} \tilde{V}(t,y) = 0; \tag{3.192}$$

$$B(t) \le M_1, \quad t \in [\alpha,\infty); \tag{3.193}$$

$$\int_\alpha^\infty L\left(s, \tilde{\lambda}(s) + \tilde{V}(x,y)\right) ds \le M_2(y) < \infty$$

and

$$\int_\alpha^\infty M\left(s, \tilde{\lambda}(s) + \tilde{V}(x,y)\right) ds \le M_3 < \infty$$

$$\textit{for all } y \in C([\alpha,\infty); X); \tag{3.194}$$

then for every x_0 a solution of (3.163) there exists $\tilde{M}(x_0) < \infty$ such that for every x a solution of (3.164) we have the estimation (3.168).

Proof. Let x_0 be a solution of (3.163) and x be a solution of (3.164). We have the estimation:

$$\|x(t) - x_0(t)\| \le \tilde{\lambda}(t) + \tilde{V}(t,x_0) + B(t) \int_\alpha^t L\left(u, \tilde{\lambda}(u) + \tilde{V}(u,x_0)\right)$$
$$\times \exp\left(\int_\alpha^t M\left(s, \tilde{\lambda}(s) + \tilde{V}(s,x_0)\right) B(s)\, ds\right) du.$$

If the conditions (3.192), (3.193) and (3.194) are satisfied, we have:

$$\|x(t) - x_0(t)\| \le \tilde{\lambda}(t) + \tilde{V}(t,x_0) + M_1 M_2(x_0) \exp M_1 M_3(x_0),$$

for all $t \in [\alpha,\infty)$.

Since $\lim_{t\to\infty} \tilde{\lambda}(t) = 0$ and $\lim_{t\to\infty} \tilde{V}(t,x_0) = 0$ for all x_0 and they are continuous in $[\alpha,\infty)$, it follows that there exists $\tilde{M}(x_0) < \infty$ such that the estimation (3.168) is valid. ■

Corollary 187 *If the conditions:*

$$\lim_{t\to\infty} \tilde{\lambda}(t) = 0,\ \lim_{t\to\infty} \tilde{V}(t,y) = 0 \ \text{ uniformly in rapport with } y; \tag{3.195}$$

and

$$\int_\alpha^\infty L\left(s, \tilde{\lambda}(s) + \tilde{V}(s,y)\right) ds \le M_2 < \infty \tag{3.196}$$

and

$$\int_\alpha^\infty M\left(s, \tilde{\lambda}(s) + \tilde{V}(x,y)\right) ds \le M_3 < \infty \ \text{ for all } y \in C([\alpha,\infty);X);$$

are valid, then there exists $\tilde{M} < \infty$ *such that for every* x_0 *a solution of (3.163) and* x *a solution of (3.164) we have the estimation (3.168).*

Finally, we have:

Theorem 188 *Suppose that the kernel* V *satisfies the relation (3.151) and the following conditions hold:*

$$\lim_{t\to\infty} \tilde{\lambda}(t) = 0,\ \lim_{t\to\infty} \tilde{V}(t,y) = 0 \tag{3.197}$$

$$M\left(t, \tilde{\lambda}(t) + \tilde{V}(t,y)\right) B(t) \le \frac{k(y)}{t},\ \ t \in [\alpha,\infty); \tag{3.198}$$

$$B(t)\, t^{k(y)} \le l(y) < \infty;$$

$$\int_\alpha^\infty \frac{L\left(s, \tilde{\lambda}(s) + \tilde{V}(s,y)\right)}{s^{k(y)}} ds \le M_1 \ \text{ for all } y \in C([\alpha,\infty);X); \tag{3.199}$$

then for every x_0 *a solution of (3.163) there exists* $\tilde{M}(x_0) < \infty$ *such that for every* x *a solution of (3.164) we have the estimation (3.168).*

The proof follows by an argument similar to that in the proof of the above theorems. We omit the details.

Corollary 189 *If the following conditions hold:*

$$\lim_{t\to\infty} \tilde{\lambda}(t) = 0,\ \lim_{t\to\infty} \tilde{V}(t,y) = 0 \ \text{ uniformly in rapport with } y; \tag{3.200}$$

and

$$\int_{\alpha}^{\infty} \frac{L\left(s, \tilde{\lambda}(s) + \tilde{V}(s,y)\right)}{s^{k(y)}} ds \le M_1 \text{ for all } y \in C([\alpha,\infty);X); \quad (3.201)$$

then there exists $\tilde{M} < \infty$ *such that for every* x_0 *a solution of (3.163) and* x *a solution of (3.164) we have the estimation (3.171).*

Remark 190 *If we assume that the kernel* V *satisfies the relations (3.153), (3.155), (3.157), (3.159) and (3.161), we can obtain a great number of corollaries and consequences for the above theorems. We omit the details.*

3.10 The Case of Discrete Equations

Further on, we consider the following discrete equation in Banach spaces:

$$x(n) = y(n) + \sum_{s=0}^{n-1} V(n,s,x(s)), \quad n \ge 1, \quad (3.202)$$

where $x, y : \mathbb{N} \to X$, $V : \mathbb{N}^2 \times X \to X$ and X is a real or complex Banach space.

The following result holds [39].

Lemma 191 *Let us suppose that the kernel* V *of the discrete equation (3.202) satisfies*

$$\|V(n,s,x)\| \le B(n) L(s, \|x\|) \quad (3.203)$$

for all $n, s \in \mathbb{N}$ *and* $x \in X$*, where* $B(n)$ *is nonnegative and* $L : \mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}_+$ *verifies the assumption (2.59). Then for any solution* $x(\cdot)$ *of (3.202), the following estimation*

$$x(n) \le \|y(n)\| + B(n) \sum_{s=0}^{n-1} L(s, \|y(s)\|) \prod_{\tau=s+1}^{n-1} (M(\tau, \|y(\tau)\|) B(\tau) + 1) \quad (3.204)$$

holds for all $n \ge 1$*.*

Proof. Let x be a solution of (3.202). Then we have:

$$\begin{aligned}\|x(n)\| &\le \|y(n)\| + \left\|\sum_{s=0}^{n-1} V(n,s,x(s))\right\| \\ &\le \|y(n)\| + B(n)\sum_{s=0}^{n-1} L(s,\|x(s)\|),\ n \ge 1.\end{aligned}$$

Applying Theorem 101, the bound (3.204) is obtained. ■

If the kernel V satisfies the condition:

$$\|V(n,s,x)\| \le B(n)\,G(s,\|x\|) \quad \text{for all } n,s \in \mathbb{N} \text{ and } x \in X \tag{3.205}$$

where $(B(n))_{n\in\mathbb{N}}$ is nonnegative and $G : \mathbb{N}\times\mathbb{R}_+ \to \mathbb{R}_+$ verifies the assumption (2.60), then for any $x(\cdot)$ a solution of (3.202) we have the bound:

$$\|x(n)\| \le \|y(n)\| + B(n)\sum_{s=0}^{n-1} L(s,\|y(s)\|) \times \prod_{\tau=s+1}^{n-1}\left(N(\|y(\tau)\|)\,B(\tau)+1\right)$$

for all $n \ge 1$.

Now, if we assume that V fulfills the assumption

$$\|V(n,s,x)\| \le B(n)\,C(s)\,H(\|x\|) \quad \text{for all } n,s \in \mathbb{N} \text{ and } x \in X; \tag{3.206}$$

where $B(n)$, $C(n)$ are nonnegative and H satisfies (2.61), then we have the evaluation

$$\|x(n)\| \le \|y(n)\| + B(n)\sum_{s=0}^{n-1} C(s)\,H(\|y(s)\|) \times \prod_{\tau=s+1}^{n-1}\left(MC(\tau)\,B(\tau)+1\right)$$

for all $n \ge 1$.

The following lemma also holds.

Lemma 192 *Let us suppose that the kernel V of the equation (3.202) verifies:*

$$\|V(n,s,x)\| \leq B(n) D(s,\|x\|) \quad \text{for all } n,s\in\mathbb{N} \text{ and } x\in X; \tag{3.207}$$

where $B(n)$ is nonnegative and $D:\mathbb{N}\times\mathbb{R}_+\to\mathbb{R}_+$ satisfies the condition (2.62). Then for any solution $(x(n))_{n\in\mathbb{N}}$ of (3.202) we have:

$$\|x(n)\| \leq \|y(n)\| + B(n)\sum_{s=0}^{n-1} D(s,\|y(s)\|) \times \prod_{\tau=s+1}^{n-1} \left(P(\tau,\|y(\tau)\|)B(\tau)+1\right)$$

for all $n\geq 1$.

The proof is obvious from Theorem 105 and we omit the details.

If V verifies the condition:

$$\|V(n,s,x)\| \leq B(n) I(s,\|x\|) \quad \text{for all } n,s\in\mathbb{N} \text{ and } x\in X; \tag{3.208}$$

then the solutions of (3.202) satisfy the inequality:

$$\|x(n)\| \leq \|y(n)\| + B(n)\sum_{s=0}^{n-1} I(s,\|y(s)\|) \prod_{\tau=s+1}^{n-1} \left(\frac{dI(\tau,\|y(\tau)\|)}{dt}B(\tau)+1\right)$$

for all $n\geq 1$.

Finally, if we assume that V fulfills the condition

$$\|V(n,s,x)\| \leq B(n) C(s) K(\|x\|) \quad \text{for all } n,s\in\mathbb{N} \text{ and } x\in X; \tag{3.209}$$

then the evaluation

$$\|x(n)\| \leq \|y(n)\| + B(n)\sum_{s=0}^{n-1} C(s) K(\|y(s)\|) \times \prod_{\tau=s+1}^{n-1} \left(\frac{dK(\|y(\tau)\|)}{dt}B(\tau)c(\tau)+1\right)$$

is also true for $n\geq 1$ and $x(\cdot)$ a solution of (3.202).

Now, by the use of the above results, we may give the following boundedness theorems [39].

Theorem 193 *Assume that the kernel V of (3.202) satisfies the assertion (3.203). If the following conditions hold:*

(i) $\|y(n)\| \leq M_1 < \infty,\ B(n) \leq M_2 < \infty$ *for all* $n \geq 1$;

(ii) $\sum_{n=0}^{\infty} L(n, M_1) < \infty,\ \prod_{n=0}^{\infty} (M(n, \|y(n)\|) M_2 + 1) < \infty$;
or

(iii) $\|y(n)\| \leq M_1 < \infty,\ n \in \mathbb{N},\ \lim_{n\to\infty} B(n) = 0$;

(iv) $\sum_{n=0}^{\infty} L(n, M_1) < \infty,\ \prod_{n=0}^{\infty} (M(n, \|y(n)\|) B(n) + 1) < \infty$ *or*

(iv') $M(n, \|y(n)\|) B(n) \leq \frac{1}{n}\ (n \geq 1),\ nB(n) \leq M_3 < \infty\ \ (n \in \mathbb{N})$ *and* $\sum_{n=0}^{\infty} \frac{L(n,M_1)}{n+1} < \infty$;
or

(v) $\|y(n)\| \leq M_1 < \infty,\ nB(n) \leq M_4 < \infty$ *for* $n \geq 1$;

(vi) $\lim_{n\to\infty} \|y(n)\| < \infty,\ B(n) \leq M_2 < \infty$ *for* $n \geq 1$
or

(vi') $M(n, \|y(n)\|) B(n) \leq \frac{1}{n}\ (n \geq 1)$ *and* $\sum_{n=0}^{\infty} \frac{L(n,M_1)}{n+1} < \infty$;

or

(vii) $\lim_{n\to\infty} \|y(n)\| = 0,\ B(n) \leq M_2 < \infty$ *for* $n \geq 1$;

(viii) $\sum_{n=0}^{\infty} (n, \|y(n)\|) < \infty,\ \prod_{n=0}^{\infty} (M(n, \|y(n)\|) M_2 + 1) < \infty$;
or

(ix) $\lim_{n\to\infty} \|y(n)\| = 0,\ nB(n) \leq M_4 < \infty$ *for* $n \geq 1$;

(x) $\lim_{n\to\infty} L(n, \|y(n)\|) < \infty,\ \prod_{n=0}^{\infty} (M(n, \|y(n)\|) B(n) + 1) < \infty$
or

(x') $M(n, \|y(n)\|) B(n) \leq \frac{1}{n}\ (n \geq 1)$ *and* $\sum_{n=0}^{\infty} \frac{L(n,\|y(n)\|)}{n+1} < \infty$;

then there exists a constant $0 < \tilde{M} < \infty$ *such that for any* $x(\cdot)$ *a solution of (3.202) we have the evaluation:*

$$\|x(n)\| \leq \tilde{M} \text{ for all } n \geq 1, \tag{3.210}$$

i.e., the solutions of (3.202) are uniformly bounded.

Proof. Let $x(\cdot)$ be a solution of (3.202). By the use of Lemma 191, we conclude that

$$\|x(n)\| \leq \|y(n)\| + B(n) \sum_{s=0}^{n-1} L(s, \|y(s)\|) \times \prod_{\tau=s+1}^{n-1} (M(\tau, \|y(\tau)\|) B(\tau) + 1) \tag{3.211}$$

for all $n \geq 1$.

If the conditions (i) and (ii) are satisfied, we get:

$$\|x(n)\| \leq M_1 + M_2 \sum_{s=0}^{\infty} L(s, \|y(s)\|) \times \prod_{s=0}^{\infty} (M(s, \|y(s)\|) M_1 + 1) = \tilde{M} < \infty$$

and the estimation (3.210) holds.

Let us suppose that (iii), (iv) or (iv') hold. Then we have the bound:

$$\|x(n)\| \leq M_1 + B(n) \sum_{n=0}^{\infty} L(n, M_1) \prod_{n=0}^{\infty} (M(n, \|y(n)\|) B(n) + 1)$$

or

$$\|x(n)\| \leq M_1 + nB(n) \sum_{n=0}^{\infty} \frac{M(s, M_1)}{s+1} \leq M_1 + M_3 \sum_{n=0}^{\infty} \frac{M(s, M_1)}{s+1}$$

for all $n \in \mathbb{N}$.

Since $\lim_{n\to\infty} B(n) = 0$, then there exists a constant $\tilde{M}$ such that the estimation (3.210) holds.

Now, let us suppose that (v), (vi) or (vi') are valid. Then by (3.211) we have

$$\|x(n)\| \leq M_1 + nB(n)\frac{\sum\limits_{s=0}^{n-1} L(s, \|y(s)\|)}{n}\prod_{n=0}^{\infty}(M(n, \|y(n)\|)B(n)+1)$$
$$\leq M_1 + M_4\frac{\sum\limits_{s=0}^{n-1} L(s, \|y(s)\|)}{n}, \quad n \geq 1,$$

respectively

$$\|x(n)\| \leq M_1 + nB(n)\sum_{s=0}^{n-1}\frac{L(s, M_1)}{s+1} \leq M_1 + M_4\sum_{s=0}^{\infty}\frac{L(s, M_1)}{s+1}.$$

Since

$$\lim_{n\to\infty}\frac{\sum\limits_{s=0}^{n-1} L(s, \|y(s)\|)}{n} = \lim_{n\to\infty} L(n, \|y(n)\|)$$

then there exists $\tilde{M} < \infty$ such that $\|x(n)\| \leq \tilde{M}$ for all $n \geq 1$ and the assertion is proved.

If (vii) and (viii) are valid, then we also have:

$$\|x(n)\| \leq \|y(n)\| + M_2\sum_{n=0}^{\infty} L(n, \|y(n)\|)\prod_{n=0}^{\infty}(M(n, \|y(n)\|)M_2+1)$$

and since $\lim\limits_{n\to\infty}\|y(n)\| = 0$, then there exists a constant $\tilde{M} < \infty$ such that (3.210) holds.

Finally, if we assume that (ix) and (x) or (x') hold, then we have the bound

$$\|x(n)\| \leq \|y(n)\| + nB(n)\frac{\sum\limits_{s=0}^{n-1} L(s, \|y(s)\|)}{n}$$
$$\times\prod_{n=0}^{\infty}(M(n, \|y(n)\|)B(n)+1)$$

respectively

$$\|x(n)\| \leq \|y(n)\| + nB(n)\frac{\sum_{s=0}^{n-1} L(s, \|y(s)\|)}{n}$$
$$\leq \|y(n)\| + M_4 \frac{\sum_{s=0}^{\infty} L(s, \|y(s)\|)}{n}$$

for all $n \geq 1$.

Since $\lim_{n\to\infty} \|y(n)\| = 0$ and $\lim_{n\to\infty} L(n\|y(n)\|) < \infty$, then there exists a constant $\tilde{M} < \infty$ such that (3.210) holds.

The proof is thus completed. ■

Remark 194 *If we assume that the kernel V of the discrete equation (3.202) satisfies one of the assertions: (3.205), (3.206), (3.207) (3.208) or (3.209) then, by the above theorem, we may formulate a great number of sufficient conditions for uniform boundedness for the solutions of this equation, but we omit the details.*

Lastly, we shall give some convergence results for the solutions of (3.202). Namely, we have the following theorem.

Theorem 195 *Assume that the kernel V of (3.202) satisfies the assertion (3.203). If the following conditions hold:*

(i) $\lim_{n\to\infty} \|y(n)\| = \lim_{n\to\infty} B(n) = 0$;

(ii) $\sum_{n=0}^{\infty} L(n, \|y(n)\|)$, $\prod_{n=0}^{\infty} (M(n, \|y(n)\|) B(n) + 1) < \infty$;
or

(iii) $\lim_{n\to\infty} \|y(n)\| = \lim_{n\to\infty} nB(n) = 0$;

(vi) $M(n, \|y(n)\|) B(n) \leq \frac{1}{n}$ *for* $n \geq 1$ *and* $\sum_{n=0}^{\infty} \frac{L(n,\|y(n)\|)}{n+1} < \infty$

or

(vii) $\lim_{n\to\infty} \|y(n)\| = 0$, $nB(n) \leq M_1$, *for* $n \geq 1$;

(vi) $\lim_{n\to\infty} L(n, \|y(n)\|) = 0, \ \prod_{n=0}^{\infty} (M(n, \|y(n)\|) B(n) + 1) < \infty;$

or

(vii) $\lim_{n\to\infty} \|y(n)\| = \lim_{n\to\infty} nB(n) = 0;$

(viii) $\lim_{n\to\infty} L(n, \|y(n)\|), \ \prod_{n=0}^{\infty} (M(n, \|y(n)\|) B(n) + 1) < \infty;$

then for any $\varepsilon > 0$ *there exists a natural number* $n(\varepsilon) > 0$ *such that for any* $x(\cdot)$ *a solution of (3.202) we have*

$$\|x(n)\| < \varepsilon \quad \text{for} \quad n \geq n(\varepsilon), \ n \in \mathbb{N}; \tag{3.212}$$

i.e., the solutions of (3.202) are uniformly convergent to zero as n tends to infinity.

Proof. Let $x(\cdot)$ be a solution of (3.202). By the use of Lemma 191, we have the estimation (3.211).

If (i) and (ii) hold, then

$$\|x(n)\| \leq \|y(n)\| + B(n) \sum_{n=0}^{\infty} L(n, \|y(n)\|) \prod_{n=0}^{\infty} (M(n, \|y(n)\|) B(n) + 1)$$

for all $n \geq 1$. Since $\lim_{n\to\infty} \|y(n)\| = \lim_{n\to\infty} B(n) = 0$, the assertion (3.212) is proved.

If (iii) and (iv) are valid, then we have the estimation:

$$\begin{aligned} \|x(n)\| &\leq \|y(n)\| + nB(n) \sum_{n=0}^{n-1} \frac{L(s, \|y(s)\|)}{s+1} \\ &\leq \|y(n)\| + nB(n) \sum_{s=0}^{\infty} \frac{L(s, \|y(s)\|)}{s+1} \end{aligned}$$

for $n \geq 1$ and the proof goes likewise.

Now, assume that (v) and (vi) are valid. Then

$$\begin{aligned}\|x(n)\| &\leq \|y(n)\| + nB(n)\frac{\sum_{s=0}^{n-1} L(s,\|y(s)\|)}{n}\prod_{n=0}^{\infty}\left(M(n,\|y(n)\|)B(n)+1\right)\\ &\leq \|y(n)\| + M_1\frac{\sum_{s=0}^{n-1} L(s,\|y(s)\|)}{n}\prod_{n=0}^{\infty}\left(M(n,\|y(n)\|)B(n)+1\right)\end{aligned}$$

for $n \geq 1$. Since

$$\lim_{n\to\infty}\frac{\sum_{s=0}^{n-1} L(s,\|y(s)\|)}{n} = \lim_{n\to\infty} L(n,\|y(n)\|) = 0,$$

then (3.212) also holds.

Finally, if we suppose that (vii) and (viii) hold, then the following inequality is also valid

$$\|x(n)\| \leq \|y(n)\| + nB(n)\frac{\sum_{s=0}^{n-1} L(s,\|y(s)\|)}{n}\prod_{n=0}^{\infty}\left(M(n,\|y(n)\|)B(n)+1\right)$$

and since

$$\lim_{n\to\infty}\frac{\sum_{s=0}^{n-1} L(s,\|y(s)\|)}{n} = \lim_{n\to\infty} L(n,\|y(n)\|)$$

the proof is thus completed. ∎

Chapter 4

Applications to Differential Equations

In this chapter we apply the results established in Sections 3.1 and 3.2 of Chapter 3 to obtain estimation results for the solutions of differential equations in $\mathbb{R}^n$, and to get sufficient conditions of boundedness and stability for the solutions of these equations.

In Section 3.1, we point out some applications of Lemmas 130 and 136 and their consequences to obtain estimates for the solutions of the Cauchy problem $(f; t_0, x_0)$ associated to general system (f).

In the second paragraph we present some applications of the above lemmas to obtain estimation results for the solutions of the Cauchy problem associated to a differential system of equations by the first approximation.

The last sections are devoted to qualitative study of some aspects for the trivial solutions of differential equations in $\mathbb{R}^n$ by using the results established in Sections 4.1 and 4.2.

Some results of uniform stability, uniform asymptotic stability, global exponential stability and global asymptotic stability for the trivial solution of a general differential system or differential system of equations by the first approximation are also given.

4.1 Estimates for the General Case

Let us consider the system of differential equations given by the following relation

$$\frac{dx}{dt} = f(t, x), \ t \in [\alpha, \beta); \tag{4.1}$$

where $f : [\alpha, \beta) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in $[\alpha, \beta) \times \mathbb{R}^n$.

In what follows, we shall assume that the Cauchy problem

$$\begin{cases} \frac{dx}{dt} = f(t, x), \ t \in [\alpha, \beta); \\ \\ x(t_0) = x_0 \in \mathbb{R}^n \end{cases} \tag{4.2}$$

has a unique solution in $[\alpha, \beta)$, for every $t_0 \in [\alpha, \beta)$ and $x_0 \in \mathbb{R}^n$. We shall denote this solution by $x(\cdot, t_0, x_0)$.

Lemma 196 *If the function f satisfies the relation*

$$\|f(t, x)\| \leq L(t, \|x\|), \ t \in [\alpha, \beta), \ x \in \mathbb{R}^n \tag{4.3}$$

and L verifies the condition (2.1) of Section 2.1 of Chapter 2,

then we have the estimate

$$\|x(t, t_0, x_0) - x_0\| \leq \int_{t_0}^{t} L(s, \|x_0\|) \exp\left(\int_s^t M(u, \|x_0\|)\, du\right) ds \tag{4.4}$$

for all $t \in [t_0, \beta)$.

Proof. Let $t_0 \in [\alpha, \beta)$, $x_0 \in \mathbb{R}^n$ and $x(\cdot, t_0, x_0)$ be the solution of the Cauchy problem $(f; t_0, x_0)$. Then we have

$$x(t, t_0, x_0) = x_0 + \int_{t_0}^{t} f(s, x(s, t_0, x_0))\, ds, \ t \in [t_0, \beta).$$

Applying Lemma 130 of Section 3.2, Chapter 3, we obtain the bound (4.4) and the lemma is proved. ■

Now, let us suppose that the function f verifies the relation

$$\|f(t, x)\| \leq G(t, \|x\|), \ t \in [\alpha, \beta), \ x \in \mathbb{R}^n \tag{4.5}$$

and G satisfies the condition (2.5);

then the following inequality

$$\|x(t,t_0,x_0)-x_0\| \le \int_{t_0}^{t} G(s,\|x_0\|)\ \exp\left(\int_s^t N(u)\,du\right) ds \tag{4.6}$$

holds for all $t \in [t_0,\beta)$.

If we assume that the function f satisfies the following particular relation

$$\begin{aligned}&\|f(t,x)\| \le C(t)\,H(\|x\|),\ t\in[\alpha,\beta),\ x\in\mathbb{R}^n \\ &\text{and } H \text{ verifies the property (2.8) of Chapter 2} \\ &\text{and } C \text{ is nonnegative continuous in } [\alpha,\beta),\end{aligned} \tag{4.7}$$

then we also have the inequality

$$\|x(t,t_0,x_0)-x_0\| \le H(\|x_0\|)\int_{t_0}^{t} C(s)\ \exp\left(M\int_s^t C(u)\,du\right) ds \tag{4.8}$$

for all $t \in [t_0,\beta)$.

Further, we shall suppose that $x_0 \neq 0$. With this assumption, and if f satisfies the relation:

$$\begin{aligned}&\|f(t,x)\| \le D(t,\|x\|),\ t\in[\alpha,\beta),\ x\in\mathbb{R}^n \\ &\text{and } D \text{ verifies the condition (2.11) of Chapter 2,}\end{aligned} \tag{4.9}$$

then we obtain

$$\|x(t,t_0,x_0)-x_0\| \le \int_{t_0}^{t} D(s,\|x_0\|)\ \exp\left(\int_s^t P(u,\|x_0\|)\,du\right) ds \tag{4.10}$$

for any $t \in [t_0,\beta)$.

Now, if we assume that the function f has the property

$$\begin{aligned}&\|f(t,x)\| \le I(t,\|x\|),\ t\in[\alpha,\beta),\ x\in\mathbb{R}^n \\ &\text{and } I \text{ satisfies the condition (2.14),}\end{aligned} \tag{4.11}$$

then we have the bound

$$\|x(t,t_0,x_0)-x_0\| \le \int_{t_0}^{t} I(s,\|x_0\|)\ \exp\left(\int_s^t \frac{\partial I}{\partial x}(u,\|x_0\|)\,du\right) ds \tag{4.12}$$

for all $t \in [t_0,\beta)$.

In particular, if the function f satisfies the assumption

$$\|f(t,x)\| \leq C(t)\, K(\|x\|)\,,\ t \in [\alpha,\beta)\,,\ x \in \mathbb{R}^n \tag{4.13}$$

and K verifies the condition (2.17) of Chapter 2,

then the following estimate:

$$\|x(t,t_0,x_0) - x_0\| \leq K(\|x_0\|) \int_{t_0}^{t} C(s)\ \exp\left(\frac{dK}{dx}(\|x_0\|) \int_s^t C(u)\,du\right) ds \tag{4.14}$$

holds, for all $t \in [t_0,\beta)$.

Consequences

1. Let us suppose that the function f fulfils the condition

$$\|f(t,x)\| \leq C(t)\, \|x\|^{r(t)}\,,\ t \in [\alpha,\beta)\,,\ x \in \mathbb{R}^n \tag{4.15}$$

and C, r are nonnegative continuous in $[\alpha,\beta)$ and $0 < r(t) \leq 1$ for all $t \in [\alpha,\beta)$;

then we have

$$\|x(t,t_0,x_0) - x_0\| \leq \int_{t_0}^{t} C(s)\, \|x_0\|^{r(s)}\ \exp\left(\int_s^t \frac{r(u)\, C(u)\,du}{\|x_0\|^{1-r(u)}}\right) ds \tag{4.16}$$

for all $t \in [t_0,\beta)$ and $x_0 \neq 0$.

2. Finally, if we assume that the function f verifies the relation:

$$\|f(t,x)\| \leq C(t) \ln(\|x\|+1)\,,\ t \in [\alpha,\beta)\,,\ x \in \mathbb{R}^n \tag{4.17}$$

and C are nonnegative continuous in $[\alpha,\beta)$,

then the following estimation

$$\|x(t,t_0,x_0) - x_0\| \leq \ln(\|x_0\|+1) \int_{t_0}^{t} C(s)\ \exp\left(\frac{1}{\|x_0\|+1} \int_s^t C(u)\,du\right) ds \quad \text{for all } t \in [\alpha,\beta)\,, \tag{4.18}$$

also holds.

Now, using Lemma 124, we can prove the following result.

Lemma 197 *If the function f satisfies the relation*

$$\|f(t,x)-f(t,y)\| \le L(s,\|x-y\|),\ s\in[\alpha,\beta),\ x,y\in\mathbb{R}^n \tag{4.19}$$

and the function L verifies the condition (2.1)
of Chapter 2,

then we have:

$$\begin{aligned}\|x(t,t_0,x_0)-x_0\| \\ \le \int_{t_0}^{t}\|f(s,x_0)\|\,ds + \int_{t_0}^{t} L\left(u, \int_{t_0}^{u}\|f(s,x_0)\|\,ds\right) \\ \times \exp\left(\int_{u}^{t} M\left(s, \int_{t_0}^{t}\|f(\tau,x)\|\,d\tau\right)ds\right)du,\end{aligned} \tag{4.20}$$

for all $t\in[t_0,\beta)$.

The proof follows by Lemma 124 of Chapter 3. We omit the details.

Let us now suppose that the function f verifies the relation

$$\|f(t,x)-f(t,y)\| \le G(t,\|x-y\|),\ t\in[\alpha,\beta),\ x,y\in\mathbb{R}^n \tag{4.21}$$

and G satisfies the condition (2.5) of Chapter 2.

Then the following estimation

$$\begin{aligned}\|x(t,t_0,x_0)-x_0\| \le \int_{t_0}^{t}\|f(s,x_0)\|\,ds + \int_{t_0}^{t} G\left(s, \int_{t_0}^{s}\|f(u,x_0)\|\,du\right) \\ \exp\left(\int_{u}^{s} N(u)\,du\right)ds,\end{aligned} \tag{4.22}$$

holds, for all $t\in[t_0,\beta)$.

If the function f satisfies the property

$$\|f(t,x)-f(t,y)\| \le C(t)\,H(\|x-y\|),\ t\in[\alpha,\beta),\ x,y\in\mathbb{R}^n, \tag{4.23}$$

H verifies the condition (2.8), and C is nonnegative continuous in $[\alpha,\beta)$, then we have

$$\begin{aligned} \|x(t,t_0,x_0)-x_0\| \leq & \int_{t_0}^{t} \|f(s,x_0)\|\,ds \\ & + \int_{t_0}^{t} C(s)\,H\left(\int_{t_0}^{s} \|f(u,x_0)\|\,du\right) \\ & \times \exp\left(M\int_{s}^{t} C(u)\,du\right) ds, \quad (4.24) \end{aligned}$$

for any $t \in [t_0, \beta)$.

Further, we shall suppose that $x_0 \neq 0$. With this assumption, and if the function f verifies the relation

$$\|f(t,x)-f(t,y)\| \leq D(s,\|x-y\|),\ t \in [\alpha,\beta),\ x,y \in \mathbb{R}^n \quad (4.25)$$

and D satisfies the condition (2.11) of Chapter 2,

then we have

$$\begin{aligned} & \|x(t,t_0,x_0)-x_0\| \\ & \leq \int_{t_0}^{t} \|f(s,x_0)\|\,ds + \int_{t_0}^{t} D\left(s, \int_{t_0}^{s} \|f(u,x_0)\|\,du\right) \\ & \times \exp\left(\int_{t_0}^{t} P\left(u, \int_{s}^{u} \|f(\tau,x_0)\|\,d\tau\right) du\right) ds, \quad (4.26) \end{aligned}$$

for all $t \in [t_0, \beta)$.

Let us now suppose that the mapping f satisfies the following relation of Lipschitz type:

$$\|f(t,x)-f(t,y)\| \leq I(t,\|x-y\|),\ t \in [\alpha,\beta),\ x,y \in \mathbb{R}^n \quad (4.27)$$

and I satisfies the property (2.14) of Chapter 2.

Then the following estimate

$$\begin{aligned} & \|x(t,t_0,x_0)-x_0\| \\ & \leq \int_{t_0}^{t} \|f(s,x_0)\|\,ds + \int_{t_0}^{t} I\left(s, \int_{t_0}^{s} \|f(u,x_0)\|\,du\right) \\ & \times \exp\left(\int_{s}^{t} \frac{\partial I}{\partial x}\left(u, \int_{s}^{t} \|f(\tau,x_0)\|\,d\tau\right) du\right) ds, \quad (4.28) \end{aligned}$$

holds, for any $t \in [t_0, \beta)$.

In particular, if we assume that the function f satisfies the relation:

$$\|f(t,x) - f(t,y)\| \leq C(t) K(\|x-y\|), \ t \in [\alpha, \beta), \ x, y \in \mathbb{R}^n \tag{4.29}$$

and K verifies the property (2.17),

then we have the evaluation

$$\begin{aligned} &\|x(t,t_0,x_0) - x_0\| \\ &\quad \leq \int_{t_0}^{t} \|f(s,x_0)\| \, ds + \int_{t_0}^{t} C(s) K\left(\int_{t_0}^{s} \|f(u,x_0)\| \, du\right) \\ &\qquad \times \exp\left(\int_{s}^{t} \frac{dK}{dx}\left(\int_{t_0}^{u} \|f(\tau,x_0)\| \, d\tau\right) C(u) \, du\right) ds, \end{aligned} \tag{4.30}$$

in the interval $[t, \beta)$.

By the above considerations, we can deduce the following two consequences.

Consequences

1. If the function f satisfies the condition

$$\|f(t,x) - f(t,y)\| \leq C(t) \|x-y\|^{r(t)}, \ t \in [\alpha, \beta), \tag{4.31}$$

$x, y \in \mathbb{R}^n$ and r, C are nonnegative continuous in $[\alpha, \beta)$ and $0 < r(t) \leq 1$ in $[\alpha, \beta)$;

then we have:

$$\begin{aligned} &\|x(t,t_0,x_0) - x_0\| \\ &\quad \leq \int_{t_0}^{t} \|f(s,x_0)\| \, ds + \int_{t_0}^{t} C(s) \left(\int_{t_0}^{s} \|f(u,x_0)\| \, du\right)^{r(u)} \\ &\qquad \times \exp\left(\int_{s}^{t} \frac{r(u) C(u)}{\left(\int_{t_0}^{t} \|f(\tau,x_0)\| \, d\tau\right)^{1-r(u)}} du\right) ds, \end{aligned} \tag{4.32}$$

for all $t \in [t_0, \beta)$.

2. Finally, if the mapping f verifies the relation

$$\begin{gathered}\|f(t,x)-f(t,y)\| \leq C(t)\ln(\|x-y\|+1),\ t\in[\alpha,\beta), \\ x,y\in\mathbb{R}^n \text{ and } C \text{ is nonnegative continuous in } [\alpha,\beta),\end{gathered} \tag{4.33}$$

then the following inequality

$$\begin{aligned}&\|x(t,t_0,x_0)-x_0\| \\ &\leq \int_{t_0}^{t}\|f(s,x_0)\|\,ds+\int_{t_0}^{t}C(s)\ln\left[\left(\int_{t_0}^{s}\|f(u,x_0)\|\,du\right)+1\right] \\ &\qquad\times\exp\left(\int_{s}^{t}\frac{C(u)\,du}{\int_{t_0}^{u}\|f(\tau,x_0)\|\,d\tau+1}\right)ds,\end{aligned} \tag{4.34}$$

holds in the interval $[t_0,\beta)$.

Now, by using Lemma 136 of Chapter 3, we can deduce the following result.

Lemma 198 *Let us suppose that the function f satisfies the relation:*

$$\begin{gathered}\|f(t,x+y)-f(t,x)\| \leq S(t,\|x\|)\|y\|,\ t\in[\alpha,\beta), \\ x,y\in\mathbb{R}^n \text{ where } S:[\alpha,\beta)\times\mathbb{R}_+\to\mathbb{R}_+ \text{ are continuous} \\ \text{in } [\alpha,\beta)\times\mathbb{R}_+,\end{gathered} \tag{4.35}$$

then we have the estimate

$$\|x(t,t_0,x_0)-x_0\| \leq \int_{t_0}^{t}\|f(s,x_0)\|\exp\left(\int_{s}^{t}S(u,\|x_0\|)\,du\right)ds \tag{4.36}$$

for all $t\in[t_0,\beta)$.

It is easy to see also that, if the mapping f verifies the property

$$\begin{gathered}\|f(t,x+y)-f(t,x)\| \leq C(t)R(\|x\|)\|y\|,\ t\in[\alpha,\beta), \\ x,y\in\mathbb{R}^n \text{ and } C:[\alpha,\beta)\times\mathbb{R}_+,\ R:\mathbb{R}_+\to\mathbb{R}_+ \\ \text{are continuous,}\end{gathered} \tag{4.37}$$

then we have

$$\begin{aligned}&\|x(t,t_0,x_0)-x_0\| \\ &\qquad\leq \int_{t_0}^{t}\|f(s,x_0)\|\exp(R(\|x_0\|))\left(\int_{s}^{t}C(u)\,du\right)ds\end{aligned} \tag{4.38}$$

for all $t\in[t_0,\beta)$.

4.2 Differential Equations by First Approximation

Let us consider the non-homogeneous system of differential equations

$$\frac{dx}{dt} = A(t)\,x + f(t,x)\,,\ t \in [\alpha,\beta)\,, \tag{4.39}$$

where $A : [\alpha,\beta) \to B(\mathbb{R}^n)\,,\ f : [\alpha,\beta) \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous.

If $C(t,t_0)$ denotes the fundamental matrix of solutions of the corresponding homogeneous system, then the solutions of the Cauchy problem:

$$\begin{cases} \frac{dx}{dt} = A(t)\,x + f(t,x)\,,\ t \in [\alpha,\beta)\,, \\ \\ x(t_0) = x_0,\ t_0 \in [\alpha,\beta)\,,\ x_0 \in \mathbb{R}^n \end{cases} \tag{4.40}$$

are given by the following integral equation of Volterra type

$$x(t,t_0,x_0) = C(t,t_0)\,x_0 + C(t,t_0)\int_{t_0}^{t} C(t_0,s)\,f(s,x(s,t_0,x_0))\,ds \tag{4.41}$$

for all $t \in [\alpha,\beta)$.

Further, we shall establish the following result for the solutions of differential equations by the first approximation (4.39).

Lemma 199 *Let us suppose that the function f satisfies the relation (4.3). If $x(\cdot,t_0,x_0)$ is the solution of the Cauchy problem $(A,f;t_0,x_0)$, then we have the estimate*

$$\begin{aligned} &\|x(t,t_0,x_0) - C(t,t_0)\,x_0\| \\ &\qquad \le \|C(t,t_0)\| \int_{t_0}^{t} \|C(t_0,s)\|\, L(s, \|C(s,t_0)\,x_0\|) \\ &\qquad \times \exp\left(\int_{s}^{t} \|C(t_0,u)\|\,\|C(u,t_0)\|\, M(u, \|C(u,t_0)\,x_0\|)\,du\right) ds \end{aligned} \tag{4.42}$$

for all $t \in [t_0,\beta)$.

Proof. If $x(\cdot,t_0,x_0)$ is the solution of $(A,f;t,x)$, then $x(\cdot,t_0,x_0)$ satisfies the relation (4.41).

Putting $y : [\alpha, \beta) \to \mathbb{R}^n$,

$$y(t) := \int_{t_0}^{t} C(t_0, s) f(s, x(s, t_0, x_0)) \, ds$$

we have

$$\begin{aligned} y(t_0) = 0 \quad \text{and} \quad y'(t) &= C(t_0, t) f(t, x(t, t_0, x_0)) \\ &= C(t_0, t) f(t, C(t, t_0) x_0 + C(t, t_0) y(t)) . \end{aligned}$$

Hence

$$\begin{aligned} \|y'(t)\| &= \|C(t_0, t) f(t, C(t, t_0) x_0 + C(t, t_0) y(t))\| \\ &\le \|C(t_0, t)\| L(t, \|C(t, t_0) x_0\|) + \|C(t, t_0)\| \|y(t)\| \\ &\le \|C(t_0, t)\| L(t, \|C(t, t_0) x_0\|) \\ &\quad + \|C(t_0, t)\| M(t, \|C(t, t_0) x_0\|) \|C(t, t_0)\| \|y(t)\| . \end{aligned}$$

By integration and since

$$\|y(t)\| \le \int_{\alpha}^{t} \|y'(s)\| \, ds,$$

we obtain

$$\begin{aligned} \|y(t)\| \le & \int_{t_0}^{t} \|C(t_0, s)\| L(s, \|C(s, t_0) x_0\|) \, ds \\ & + \int_{t_0}^{t} \|C(t_0, s)\| \|C(s, t_0)\| M(s, \|C(s, t_0) x_0\|) \|y(s)\| \, ds \end{aligned}$$

for $t \in [t_0, \beta)$.

Applying Corollary 2 of the Introduction we obtain, by simple computation, that

$$\begin{aligned} \|y(t)\| \le & \int_{t_0}^{t} \|C(t_0, t)\| L(s, \|C(s, t_0) x_0\|) \\ & \times \exp\left(\int_{s}^{t} \|C(t_0, u)\| \|C(u, t_0)\| M(u, \|C(u, t_0) x_0\|) \, du \right) ds \end{aligned}$$

from where results (4.42) and the lemma is proved. ■

Let us now suppose that the function f verifies the relation (4.5). If $x(\cdot, t_0, x_0)$ is the solution of the Cauchy problem $(A, f; t_0, x_0)$, then we have

$$\begin{aligned} &\|x(t,t_0,x_0) - C(t,t_0)x_0\| \\ &\qquad \le \|C(t,t_0)\| \int_{t_0}^{t} \|C(t_0,s)\| \, G(s, \|C(s,t_0)x_0\|) \\ &\qquad\qquad \times \exp\left(\int_s^t \|C(t_0,u)\| \, \|C(u,t_0)\| \, N(u)\, du\right) ds, \quad (4.43) \end{aligned}$$

for all $t \in [t_0, \beta)$.

If we assume that the mapping f has the property (4.7), then we have the estimate

$$\begin{aligned} &\|x(t,t_0,x_0) - C(t,t_0)x_0\| \\ &\qquad \le \|C(t,t_0)\| \int_{t_0}^{t} \|C(t_0,s)\| \, C(s)\, H(\|C(s,t_0)x_0\|) \\ &\qquad\qquad \times \exp\left(M \int_s^t \|C(t_0,u)\| \, \|C(u,t_0)\| \, C(u)\, du\right) ds, \quad (4.44) \end{aligned}$$

for all $t \in [t_0, \beta)$.

Further, we shall suppose that $x_0 \neq 0$. In this assumption, and if the function f satisfies the relation (4.9), then we have the bound

$$\begin{aligned} &\|x(t,t_0,x_0) - C(t,t_0)x_0\| \\ &\qquad \le \|C(t,t_0)\| \int_{t_0}^{t} \|C(t_0,s)\| \, D(s, \|C(s,t_0)x_0\|) \\ &\qquad \times \exp\left(\int_s^t \|C(t_0,u)\| \, \|C(u,t_0)\| \, P(u, \|C(u,t_0)x_0\|)\, du\right) ds, \quad (4.45) \end{aligned}$$

in the interval $[t_0, \beta)$.

If the function f verifies the property (4.11), then the following evaluation

$$\begin{aligned} &\|x(t,t_0,x_0) - C(t,t_0)x_0\| \\ &\qquad \le \|C(t,t_0)\| \int_{t_0}^{t} \|C(t_0,s)\| \, I(s, \|C(s,t_0)x_0\|) \\ &\times \exp\left(\int_s^t \|C(t_0,u)\| \, \|C(u,t_0)\| \frac{\partial I}{\partial x}(u, \|C(u,t_0)x_0\|)\, du\right) ds, \quad (4.46) \end{aligned}$$

holds in $[t_0, \beta)$.

In particular, if we assume that the mapping f satisfies the relation (2.17), then we have

$$\begin{aligned}
&\|x(t,t_0,x_0) - C(t,t_0)x_0\| \\
&\qquad \le \|C(t,t_0)\| \int_{t_0}^{t} \|C(t_0,s)\| \, C(s) \, K(\|C(s,t_0)x_0\|) \\
&\times \exp\left(\int_s^t \|C(t_0,u)\| \, \|C(u,t_0)\| \, C(u) \, \frac{dK}{dx}(\|C(u,t_0)x_0\|) \, du \right) ds. \quad (4.47)
\end{aligned}$$

By the above considerations, we can deduce the following two consequences.

Consequences

1. If the function f satisfies the relation (4.15), then we have

$$\begin{aligned}
&\|x(t,t_0,x_0) - C(t,t_0)x_0\| \\
&\qquad \le \|C(t,t_0)\| \int_{t_0}^{t} \|C(t_0,s)\| \, C(s) \, \|C(s,t_0)x_0\|^{r(s)} \\
&\qquad \times \exp\left(\int_s^t \frac{\|C(t_0,u)\| \, \|C(u,t_0)\| \, r(u) \, C(u)}{\|C(u,t_0)x_0\|^{1-r(u)}} du \right) ds \quad (4.48)
\end{aligned}$$

   for all $t \in [t_0, \beta)$.

2. Finally, if the function f verifies the relation (4.17), then we have the estimate

$$\begin{aligned}
&\|x(t,t_0,x_0) - C(t,t_0)x_0\| \\
&\qquad \le \|C(t,t_0)\| \int_{t_0}^{t} \|C(t_0,s)\| \, C(s) \ln(\|C(s,t_0)x_0\| + 1) \\
&\qquad \times \exp\left(\int_s^t \frac{\|C(t_0,u)\| \, \|C(u,t_0)\| \, C(u)}{\|C(u,t_0)x_0\| + 1} du \right) ds \quad (4.49)
\end{aligned}$$

   in the interval $[t_0, \beta)$.

We may also state the following result.

Lemma 200 *Let us suppose that the function f satisfies the relation (4.19). If $x(\cdot, t_0, x_0)$ is the solution of the Cauchy problem $(A, f; t_0, x_0)$, then we have the estimate*

$$\begin{aligned} &\|x(t,t_0,x_0) - C(t,t_0)x_0\| \\ &\qquad \leq k(t) + \|C(t,t_0)\| \int_{t_0}^{t} \|C(t_0,s)\| L(s,k(s)) \\ &\qquad \times \exp\left(\int_s^t M(u,k(u)) \|C(t_0,u)\| \|C(u,t_0)\| du\right) ds, \quad (4.50) \end{aligned}$$

where

$$k(t) := \int_{t_0}^{t} \|C(t,s) f(s, C(s,t_0)x_0)\| ds$$

and $t \in [t_0, \beta)$.

Proof. If $x(\cdot, t_0, x_0)$ is the solution of $(A, f; t_0, x_0)$, then we have

$$x(t,t_0,x_0) = C(t,t_0)x_0 + \int_{t_0}^{t} C(t,s) f(s, x(s,t_0,x_0)) ds$$

for all $t \in [\alpha, \beta)$.

Putting

$$V(t,s,x) = C(t,s) f(s,x),$$

we have

$$\begin{aligned} \|V(t,s,x) - V(t,s,y)\| &\leq \|C(t,t_0)\| \|C(t_0,s)\| \|f(s,x) - f(s,y)\| \\ &\leq \|C(t,t_0)\| \|C(t_0,s)\| L(s, \|x-y\|). \end{aligned}$$

The proof follows by an argument similar to that in the proof of Lemma 124. We omit the details. ■

Now, if we suppose that the mapping f verifies the relation (4.21), then

$$\begin{aligned} &\|x(t,t_0,x_0) - C(t,t_0)x_0\| \\ &\qquad \leq k(t) + \|C(t,t_0)\| \int_{t_0}^{t} \|C(t_0,s)\| G(s,k(s)) \\ &\qquad \times \exp\left(\int_s^t \|C(t_0,u)\| \|C(u,t_0)\| N(u) du\right) ds \quad (4.51) \end{aligned}$$

holds, for all $t \in [t_0, \beta)$.

Let us now assume that the function f satisfies the property (4.23). Then we have the estimate

$$\begin{aligned} &\|x(t,t_0,x_0) - C(t,t_0)x_0\| \\ &\quad \le k(t) + \|C(t,t_0)\| \int_{t_0}^{t} \|C(t_0,s)\| C(s) H(k(s)) \\ &\quad \times \exp\left(M \int_s^t \|C(t_0,u)\| \|C(u,t_0)\| C(u)\, du\right) ds \end{aligned} \tag{4.52}$$

for all $t \in [t_0, \beta)$.

Further, we shall suppose that $x_0 \neq 0$. In this assumption, and if f satisfies the relation (4.25), then we have the estimate

$$\begin{aligned} &\|x(t,t_0,x_0) - C(t,t_0)x_0\| \\ &\quad \le k(t) + \|C(t,t_0)\| \int_{t_0}^{t} \|C(t_0,s)\| D(s,k(s)) \\ &\quad \times \exp\left(\int_s^t P(u,k(u)) \|C(t_0,u)\| \|C(u,t_0)\|\, du\right) ds \end{aligned} \tag{4.53}$$

in the interval $[t_0, \beta)$.

Now, if we suppose that the mapping f has the property (4.27), then we have the evaluation:

$$\begin{aligned} &\|x(t,t_0,x_0) - C(t,t_0)x_0\| \\ &\quad \le k(t) + \|C(t,t_0)\| \int_{t_0}^{t} \|C(t_0,s)\| I(s,k(s)) \\ &\quad \times \exp\left(\int_s^t \frac{\partial I}{\partial x}(u,k(u)) \|C(t_0,u)\| \|C(u,t_0)\|\, du\right) ds \end{aligned} \tag{4.54}$$

for all $t \in [t_0, \beta)$.

In particular, if the function f verifies the relation (4.29), then we have

$$\begin{aligned} &\|x(t,t_0,x_0) - C(t,t_0)x_0\| \\ &\quad \le k(t) + \|C(t,t_0)\| \int_{t_0}^{t} \|C(t_0,s)\| C(s) K(k(s)) \\ &\quad \times \exp\left(\int_s^t \frac{dK}{dx}(k(u)) C(u) \|C(t_0,u)\| \|C(u,t_0)\|\, du\right) ds \end{aligned} \tag{4.55}$$

in the interval $[t_0, \beta)$.

Finally, we have another result which is embodied in the following lemma.

Lemma 201 *Let us suppose that the function f satisfies the relation (4.35). If $x(\cdot, t_0, x_0)$ is the solution of the Cauchy problem $(A, f; t_0, x_0)$, then we have the estimation*

$$\begin{aligned} &\|x(t, t_0, x_0) - C(t, t_0)x_0\| \\ &\qquad \leq \|C(t, t_0)\| \int_{t_0}^{t} \|C(t_0, s) f(s, C(s, t_0)x_0)\| \\ &\times \exp\left(\int_s^t S(u, \|C(u, t_0)x_0\|)\|C(u, t_0)\|\|C(t_0, u)\| \, du\right) ds \quad (4.56) \end{aligned}$$

for all $t \in [t_0, \beta)$.

The proof follows by an argument similar to that in the proof of Lemma 136. We omit the details.

Let us now suppose that the mapping f verifies the relation (4.37) of Section 4.1. Then we have

$$\begin{aligned} &\|x(t, t_0, x_0) - C(t, t_0)x_0\| \\ &\qquad \leq \|C(t, t_0)\| \int_{t_0}^{t} \|C(t_0, s) f(s, C(s, t_0)x_0)\| \\ &\times \exp\left(\int_s^t C(u) R(\|C(u, t_0)x_0\|)\|C(u, t_0)\|\|C(t_0, u)\| \, du\right) ds \quad (4.57) \end{aligned}$$

for any $t \in [t_0, \beta)$.

4.3 Boundedness Conditions

Let us consider the system of differential equations given in the following from

$$\frac{dx}{dt} = f(t, x), \; t \in [\alpha, \infty); \tag{4.58}$$

where $f : [\alpha, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in $[\alpha, \infty) \times \mathbb{R}^n$.

In what follows, we shall assume that the Cauchy problem:

$$\begin{cases} \frac{dx}{dt} = f(t,x), \ t \in [\alpha, \infty); \\ \\ x(t_0) = x_0 \in \mathbb{R}^n \end{cases} \tag{$f; t_0, x_0$}$$

has a unique solution $x(\cdot, t_0, x_0)$ in $[\alpha, \infty)$ where $t_0 \geq \alpha$ and $x_0 \in \mathbb{R}^n$.

By using the lemmas established in Section 4.1 of this chapter, we can formulate the following theorems of boundedness.

Theorem 202 *If the function f satisfies the relation (4.3) and the following conditions:*

$$\int_{t_0}^{\infty} L(s, \|x_0\|)\, ds \leq M_1(t_0, x_0) < \infty, \tag{4.59}$$

$$\int_{t_0}^{\infty} M(s, \|x_0\|)\, ds \leq M_2(t_0, x_0) < \infty \tag{4.60}$$

hold, then there exists an $\tilde{M}(t_0, x_0) > 0$ such that

$$\|x(t, t_0, x_0) - x_0\| \leq \tilde{M}(t_0, x_0)$$

for all $t \geq t_0$.

The proof follows by Lemma 196. We omit the details.

Another result is embodied in the following theorem.

Theorem 203 *If the function f satisfies the relation (4.19) and the following conditions:*

$$\int_{t_0}^{\infty} \|f(s, x_0)\|\, ds \leq M_1(t_0, x_0) < \infty, \tag{4.61}$$

$$\int_{t_0}^{\infty} L(s, M_1(t_0, x_0))\, ds \leq M_2(t_0, x_0) < \infty, \tag{4.62}$$

$$\int_{t_0}^{\infty} M\left(s, \int_{t_0}^{s} \|f(\tau, x_0)\|\, d\tau\right) ds \leq M_3(t_0, x_0) < \infty \tag{4.63}$$

hold, then there exists an $\tilde{M}(t_0, x_0) > 0$ such that

$$\|x(t, t_0, x_0) - x_0\| \leq \tilde{M}(t_0, x_0)$$

for all $t \geq t_0$.

The proof of this theorem follows by Lemma 197.

Finally we have

Theorem 204 *Let us suppose that the function f satisfies the relation (4.35) and the following conditions:*

$$\int_{t_0}^{\infty} \|f(s, x_0)\| \, ds \leq M_1(t_0, x_0) < \infty, \tag{4.64}$$

$$\int_{t_0}^{\infty} S(s, \|x_0\|) \, ds \leq M_2(t_0, x_0) < \infty, \tag{4.65}$$

hold, then there exists an $\tilde{M}(t_0, x_0) > 0$ such that

$$\|x(t, t_0, x_0) - x_0\| \leq \tilde{M}(t_0, x_0)$$

for all $t \geq t_0$.

4.4 The Case of Non-Homogeneous Systems

Let us consider the non-homogeneous system of differential equation:

$$\frac{dx}{dt} = A(t)\,x + f(t, x), \; t \in [\alpha, \infty), \tag{4.66}$$

where $A : [\alpha, \infty) \to B(\mathbb{R}^n)$, $f : [\alpha, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous.

In what follows, we assume that the Cauchy problem

$$\begin{cases} \frac{dx}{dt} = A(t)\,x + f(t, x), \; t \in [\alpha, \infty), \\ \\ x(t_0) = x_0, \; x_0 \in \mathbb{R}^n \end{cases} \tag{4.67}$$

has a unique solution defined in $[\alpha, \infty)$.

By using the results established in Section 4.2 of the present chapter, we can formulate the following theorems.

Theorem 205 *Let us suppose that the mapping f satisfies the relation (4.3). If the trivial solution $x \equiv 0$ of the corresponding homogeneous system is stable, i.e., $\|C(t, t_0)\| \leq \mu(t_0)$ for all $t \geq t_0$, and the following conditions:*

$$\int_{t_0}^{\infty} \|C(t_0, s)\| \, L(s, \mu(t_0)\|x_0\|) \, ds \leq M_1(t_0, x_0) < \infty, \tag{4.68}$$

$$\int_{t_0}^{\infty} \|C(t_0, s)\| \, M(s, \|C(s, t_0)\, x_0\|) \, ds \leq M_2(t_0, x_0) < \infty \tag{4.69}$$

hold, then there exists an $\tilde{M}(t_0, x_0) > 0$ *such that*

$$\|x(t, t_0, x_0) - C(t, t_0) x_0\| \leq \tilde{M}(t_0, x_0)$$

for all $t \geq t_0$.

The proof follows by Lemma 199. We omit the details.

Another result is embodied in the following theorem.

Theorem 206 *Let us suppose that the mapping* f *satisfies the relation (4.19). If the trivial solution* $x \equiv 0$ *of the corresponding homogeneous system is stable and the following conditions*

$$\lim_{t \to \infty} \int_{t_0}^{t} \|C(t, s) f(s, C(s, t_0) x_0)\| \, ds \leq M_1(t_0, x_0) < \infty, \tag{4.70}$$

$$\int_{t_0}^{\infty} \|C(t_0, s)\| \, L(s, M_1(t_0, x_0)) \, ds \leq M_2(t_0, x_0) < \infty, \tag{4.71}$$

$$\int_{t_0}^{\infty} M(s, k(s)) \, \|C(t_0, s)\| \, ds \leq M_3(t_0, x_0) < \infty \tag{4.72}$$

hold, then there exists an $\tilde{M}(t_0, x_0) > 0$ *such that*

$$\|x(t, t_0, x_0) - C(t, t_0) x_0\| \leq \tilde{M}(t_0, x_0)$$

for all $t \geq t_0$.

The proof is obvious by Lemma 200.

Finally, we have

Theorem 207 *Let us suppose that the function* f *satisfies the relation (4.35). If the trivial solution* $x \equiv 0$ *of the corresponding homogeneous system is stable and the following conditions*

$$\int_{t_0}^{\infty} \|C(t_0, s) f(s, C(s, t_0) x_0)\| \, ds \leq M_1(t_0, x_0) < \infty, \tag{4.73}$$

$$\int_{t_0}^{\infty} S(s, \|C(s, t_0) x_0\|) \, \|C(t_0, s)\| \, ds \leq M_2(t_0, x_0) < \infty \tag{4.74}$$

hold, then there exists an $\tilde{M}(t_0, x_0) > 0$ *such that*

$$\|x(t, t_0, x_0) - C(t, t_0) x_0\| \leq \tilde{M}(t_0, x_0)$$

for all $t \geq t_0$.

The proof follows by Lemma 201, and we omit the details.

4.5 Theorems of Uniform Stability

Let us consider the system of differential equations

$$\frac{dx}{dt} = f(t, x), \ t \in [\alpha, \infty); \tag{4.75}$$

where f is continuous in $[\alpha, \infty) \times \mathbb{R}^n$ and $f(t, 0) \equiv 0$.

The main purpose of this section is to give some theorems of uniform stability for the trivial solution of the above equation.

Theorem 208 *If the mapping f satisfies the relation (4.3), $L(t, 0) \equiv 0$ for all $t \in [\alpha, \infty)$ and the following condition*

$$\text{there exists a } \delta_0 > 0 \text{ such that } \int_\alpha^\infty M(s, \delta)\, ds \le \tilde{M} \text{ for all } 0 \le \delta \le \delta_0, \tag{4.76}$$

holds, then the trivial solution $x \equiv 0$ of (4.1) is uniformly stable.

Proof. Firstly, we observe that $L(t, u) \le M(t, 0)\, u$ for all $t \in [\alpha, \infty)$ and $u \ge 0$, which implies that

$$\int_\alpha^\infty L(s, u)\, ds \le u \int_\alpha^\infty M(s, 0)\, ds \le \tilde{M} u.$$

Let $\varepsilon > 0$ and let $x(\cdot, t_0, x_0)$ be the solution of the Cauchy problem $(f; t_0, x_0)$ where x_0 is such that

$$\|x_0\| < \delta(\varepsilon) = \min\left(\frac{\varepsilon}{2}, \delta_0, \frac{\varepsilon}{2\tilde{M} \exp \tilde{M}}\right).$$

Then we have, by Lemma 196, that

$$\begin{aligned} \|x(t, t_0, x_0)\| &\le \|x_0\| + \|x(t, t_0, x_0) - x_0\| \\ &\le \|x_0\| + \int_{t_0}^t L(s, \|x_0\|) \exp\left(\int_s^t M(u, \|x_0\|)\, du\right) ds \\ &< \frac{\varepsilon}{2} + \|x_0\| \tilde{M} \exp \tilde{M} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for all $t \ge t_0$. This implies the fact that the trivial solution of (4.1) is uniformly stable. ∎

Corollary 209 *If the mapping f satisfies the relation (4.5), $G(t,0) \equiv 0$ for all $t \in [\alpha, \infty)$ and the condition*

$$\int_{\alpha}^{\infty} N(s)\, ds < \infty \tag{4.77}$$

holds, then the trivial solution $x \equiv 0$ of (4.1) is uniformly stable.

Corollary 210 *If the function f satisfies the relation (4.7), $H(0) \equiv 0$, and the condition*

$$\int_{\alpha}^{\infty} C(s)\, ds < \infty \tag{4.78}$$

holds, then the trivial solution of (4.1) is uniformly stable.

Another result is embodied in the following theorem.

Theorem 211 *Let us suppose that the function f verifies the relation (4.9), $D(t,0) \equiv 0$ for all $t \in [\alpha, \infty)$ and the condition*

$$\text{there exists a } \delta_0 > 0 \text{ such that } \int_{\alpha}^{\infty} D(s, \delta_0)\, ds < M_1 \tag{4.79}$$

$$\text{and } \int_{\alpha}^{\infty} P(s, \delta)\, ds < M_2 \ \text{ for all } 0 < \delta \leq \delta_0.$$

holds. Then the trivial solution $x \equiv 0$ of (4.1) is uniformly stable.

Proof. Let $\varepsilon > 0$. If $D(t,0) \equiv 0$ and D is continuous, then there exists a $\delta_1(\varepsilon) > \delta_0$ such that

$$\int_{\alpha}^{\infty} D(s, \|x_0\|)\, ds < \frac{\varepsilon}{2 \exp M_2} \ \text{ if } \ \|x_0\| < \delta_1(\varepsilon).$$

If we consider the solution $x(\cdot, t_0, x_0)$ such that $\|x_0\| < \delta(\varepsilon) = \min\left\{\frac{\varepsilon}{2}, \delta_1(\varepsilon)\right\}$, we have

$$\begin{aligned}
\|x(t, t_0, x_0)\| &\leq \|x_0\| + \|x(t, t_0, x_0) - x_0\| \\
&\leq \|x_0\| + \int_{t_0}^{t} D(s, \|x_0\|) \exp\left(\int_{s}^{t} P(u, \|x_0\|)\, du\right) ds \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2 \exp M_2} \exp M_2 = \varepsilon
\end{aligned}$$

for all $t \geq t_0$. This mean that the trivial solution of (4.1) is uniformly stable.
■

Corollary 212 *If the function f verifies he relation (4.11), $I(t,0) \equiv 0$ for all $t \in [\alpha, \infty)$ and the following condition*

$$\text{there exists a } \delta_0 > 0 \text{ such that } \int_\alpha^\infty I(s, \delta_0)\, ds < M_1 \tag{4.80}$$
$$\text{and } \int_\alpha^\infty \frac{\partial I}{\partial x}(s, \delta)\, ds \leq M_2 \text{ for all } 0 < \delta \leq \delta_0$$

holds, then the trivial solution $x \equiv 0$ of the system (4.1) is uniformly stable.

Corollary 213 *If the function f satisfies the relation (4.13), $K(0) \equiv 0$ and the following condition*

$$\int_\alpha^\infty C(s)\, ds < \infty \text{ and there exists a } \delta_0 > 0 \text{ such that } \frac{dK}{dx} \tag{4.81}$$
$$\text{is bounded in } (0, \delta_0),$$

holds, then the trivial solution of (4.1) is uniformly stable.

Finally, we have

Theorem 214 *Let us suppose that the function f verifies the relation (4.35) and the condition*

$$\text{there exists a } \delta_0 > 0 \text{ such that } \int_\alpha^\infty S(u, \delta)\, du \leq \tilde{M} \text{ if} \tag{4.82}$$
$$0 \leq \delta \leq \delta_0,$$

holds. Then the trivial solution $x \equiv 0$ of (4.1) is uniformly stable.

Proof. Since

$$\|f(t,x) - f(t, x+y)\| \leq S(t, \|x\|)\, \|y\|$$

for every $x, y \in \mathbb{R}^n$, putting $y = -x$ we obtain

$$\|f(t,x)\| \leq S(t, \|x\|)\, \|x\|$$

for all $t \in [\alpha, \infty)$ and $x \in \mathbb{R}^n$.

Let $\varepsilon > 0$. If

$$\|x_0\| < \delta(\varepsilon) = \min\left(\frac{\varepsilon}{2}, \delta_0, \frac{\varepsilon}{2\tilde{M}\exp\tilde{M}}\right)$$

we obtain

$$\begin{aligned}
\|x(t,t_0,x_0)\| &\le \|x_0\| + \|x(t,t_0,x_0) - x_0\| \\
&\le \|x_0\| + \int_{t_0}^{t} \|f(s,x_0)\| \exp\left(\int_s^t S(u,\|x_0\|)\,du\right) ds \\
&\le \|x_0\| + \|x_0\| \int_\alpha^\infty S(u,\|x_0\|)\,du \exp\left(\int_\alpha^\infty S(u,\|x_0\|)\,du\right) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2\tilde{M}\exp\tilde{M}}\tilde{M}\exp\tilde{M} = \varepsilon
\end{aligned}$$

for all $t \ge t_0$. And the theorem is thus proved. ■

Corollary 215 *If the function f satisfies the relation (4.37) and the following condition:*

$$\text{there exists a } \delta_0 > 0 \text{ such that } R \text{ is bounded in } (0,\delta_0) \tag{4.83}$$
$$\text{and } \int_\alpha^\infty C(s)\,ds < \infty,$$

holds, then the trivial solution $x \equiv 0$ of (4.1) is uniformly stable.

4.6 Theorems of Uniform Asymptotic Stability

Let us consider the non-homogeneous system

$$\frac{dx}{dt} = A(t)\,x + f(t,x), \; t \in [\alpha,\infty), \tag{4.84}$$

where $A : [\alpha,\infty) \to B(\mathbb{R}^n)$, $f : [\alpha,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous and $f(t,0) \equiv 0$ for all $t \in [\alpha,\infty)$.

The main purpose of this section is to give some theorems of uniform asymptotic stability for the trivial solution $x \equiv 0$ of the system (4.84).

Theorem 216 *If the trivial solution of the system of first approximation is uniformly asymptotically stable, the mapping f satisfies the relation (4.3), $L(t,0) \equiv 0$ for all $t \in [\alpha, \infty)$ and*

$$\text{there exists a } \delta_0 > 0 \text{ such that } \int_\alpha^\infty M(s,\delta)\,ds \le \tilde{M} < \infty \tag{4.85}$$

for all $0 \le \delta \le \delta_0$,

then the trivial solution $x \equiv 0$ of the system (4.84) is uniformly asymptotically stable.

Proof. Let $x(\cdot, t_0, x_0)$ be a solution of the Cauchy problem $(A; f, t_0, x_0)$. Then $x(\cdot, t_0, x_0)$ verifies the integral equation

$$x(t,t_0,x_0) = C(t,t_0)\,x_0 + C(t,t_0)\int_{t_0}^t C(t_0,s)\,f(s, x(s,t_0,x_0))\,ds \tag{4.86}$$

for all $t \ge t_0$.

Passing at norms, we obtain

$$\|x(t,t_0,x_0)\| \le \|C(t,t_0)\|\,x_0 + \|C(t,t_0)\|\int_{t_0}^t \|C(t_0,s)\|\,L(s, \|x(s,t_0,x_0)\|)\,ds$$

for all $t \ge t_0$.

Since the trivial solution of the linear system is uniformly asymptotically stable, then there exists $\beta > 0$, $m > 0$ such that

$$\|C(t,t_0)\| \le \beta e^{-m(t-t_0)},\ t \ge t_0.$$

It follows that

$$\|x(t,t_0,x_0)\| \le \beta e^{-m(t-t_0)}\,\|x_0\| + \beta e^{-m(t-t_0)}\int_{t_0}^t \beta e^{-m(t_0-s)} L(s, \|x(s,t_0,x_0)\|)\,ds$$

for all $t \ge t_0$.

Applying Lemma 74, we obtain

$$\begin{aligned}
&\|x(t,t_0,x_0)\| \\
&\quad\leq \beta e^{-m(t-t_0)}\|x_0\| + \beta e^{-m(t-t_0)}\int_{t_0}^{t}\beta e^{-m(t_0-s)}L\left(s,\beta e^{-m(s-t_0)}\|x_0\|\right) \\
&\qquad\times \exp\left(\int_s^t M\left(u,\beta e^{-m(u-t_0)}\right)\|x_0\|\,du\right)ds
\end{aligned}$$

for all $t \geq t_0$.

By the condition (2.1), we have

$$\begin{aligned}
\beta e^{-m(t_0-s)}L\left(s,\beta e^{-m(s-t_0)}\right) &\leq \beta e^{-m(t_0-s)}M(s,0)\,\beta e^{-m(s-t_0)}\|x_0\| \\
&= \beta^2 M(s,0)\|x_0\| \quad \text{for all} \quad s \geq t_0.
\end{aligned}$$

We obtain

$$\begin{aligned}
\|x(t,t_0,x_0)\| \leq \beta e^{-m(t-t_0)}\|x_0\| &+ \beta^3 e^{-m(t-t_0)}\|x_0\|\int_\alpha^t M(u,0) \\
&\times \exp\left(\int_\alpha^t M\left(s,\beta e^{-m(s-t)}\|x_0\|\right)ds\right)du
\end{aligned}$$

If $\|x_0\| < \frac{\delta_0}{\beta}$, we have

$$\|x(t,t_0,x_0)\| \leq \left(\beta + \beta^3\tilde{M}\exp\tilde{M}\right)e^{-m(t-t_0)}\|x_0\|$$

for all $t \geq t_0$, which means that the trivial solution of (4.84) is uniformly asymptotically stable. ■

Another result is embodied in the following theorem.

Theorem 217 *If the trivial solution of the system by the first approximation is uniformly asymptotically stable, f verifies the relation (4.3), $L(t,0) \equiv 0$ for all $t \in [\alpha,\infty)$ and*

$$\int_\alpha^\infty M(s,\delta)\,ds \leq \tilde{M} < \infty \quad \textit{for all} \quad \delta \geq 0, \tag{4.87}$$

then the trivial solution $x \equiv 0$ of the system (4.84) is globally exponentially stable, i.e.,

$$\|x(t,t_0,x_0)\| \leq \beta e^{-m(t-t_0)}\|x_0\|$$

for all $x_0 \in \mathbb{R}^n$ and $t \geq t_0$.

Proof. Let $x(\cdot, t_0, x_0)$ be the solution of the Cauchy problem $(A; f, t_0, x_0)$. By similar computation we have the estimate

$$\begin{aligned}\|x(t,t_0,x_0)\| &\le \beta e^{-m(t-t_0)}\|x_0\| + \beta e^{-m(t-t_0)}\|x_0\|\int_\alpha^t M(u,0) \\ &\quad\times \exp\left(\int_u^t M\left(s, \beta e^{-m(s-t_0)}\|x_0\|\right) ds\right) du \\ &\le \beta e^{-m(t-t_0)}\|x_0\| + \beta^3 e^{-m(t-t_0)}\tilde{M}\exp\tilde{M} \\ &= \left(\beta + \beta^3\tilde{M}\exp\tilde{M}\right) e^{-m(t-t_0)}\|x_0\|\end{aligned}$$

for all $t \ge t_0$ and $x_0 \in \mathbb{R}^n$.

The theorem is thus proved. ■

Corollary 218 *If the trivial solution $x \equiv 0$ of the linear system by the first approximation is uniformly asymptotically stable, f satisfies the relation (4.5), $G(t,0) \equiv 0$ for all $t \in [\alpha, \infty)$ and*

$$\int_\alpha^\infty N(s)\, ds < \infty, \tag{4.88}$$

then the trivial solution $x \equiv 0$ of the system (4.84) is global exponentially stable.

Corollary 219 *If the trivial solution $x \equiv 0$ of the linear system by the first approximation is uniformly asymptotically stable, f verifies the relation (4.7), $H(0) \equiv 0$, and*

$$\int_\alpha^\infty C(s)\, ds < \infty \tag{4.89}$$

then the trivial solution of (4.84) is global exponentially stable.

We can now give another result.

Theorem 220 *If the fundamental matrix of the solution of the linear system of first approximation satisfies the condition:*

$$\|C(t,t_0)\| \le \beta e^{-m(t-t_0)},\ \beta, m > 0,\ t \ge t_0. \tag{4.90}$$

f verifies the relation (4.9), $D(t,0) \equiv 0$ *for all* $t \in [\alpha, \infty)$ *and*

$$\text{there exists a } \delta_0 > 0 \text{ such that } \int_\alpha^\infty e^{ms} D(s, \delta_0)\, ds \le M_1 \tag{4.91}$$
$$\text{and } \int_\alpha^\infty P(s,\delta)\, ds \le M_2 \text{ for all } 0 < \delta \le \delta_0,$$

then the trivial solution (4.84) is uniformly asymptotically stable.

Proof. Let $x(\cdot, t_0, x_0)$, $x_0 \neq 0$, be the solution of the Cauchy problem $(A; f, t_0, x_0)$. By similar computation we have the estimate

$$\begin{aligned} &\|x(t,t_0,x_0)\| \\ &\le \beta e^{-m(t-t_0)} \|x_0\| + \beta e^{-m(t-t_0)} \int_{t_0}^t \beta e^{-m(t_0-s)} D\left(s, \beta e^{-m(s-t_0)} \|x_0\|\right) \\ &\qquad \times \exp\left(\int_s^t M\left(u, \beta e^{-m(u-t_0)} \|x_0\|\right) du\right) ds,\ t \ge t_0. \end{aligned}$$

Since $D(t,0) \equiv 0$ for all $t \in [\alpha, \infty)$ and D is continuous, then there exists a $\delta_1(\varepsilon) > \alpha$ such that

$$\int_\alpha^\infty e^{ms} D(s, \beta \|x_0\|)\, ds < \frac{\varepsilon}{2\beta^2 \exp M_2}$$

for all x_0 with $\|x_0\| < \frac{\delta_1(\varepsilon)}{\beta}$.

Putting

$$\delta(\varepsilon) = \min\left\{\frac{\delta_0}{\beta}, \frac{\varepsilon}{2\beta}, \frac{\delta_1(\varepsilon)}{\beta}\right\},$$

then, for every x_0 with $\|x_0\| < \delta(\varepsilon)$, we have

$$\|x(t,t_0,x_0)\| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for all } t \ge t_0.$$

On the other hand, if $\|x_0\| < \frac{\delta_0}{\beta}$, we have

$$\|x(t,t_0,x_0)\| \le e^{-m(t-t_0)}\delta_0 + \beta^2 e^{-mt} M_1 \exp M_2$$

However,

$$e^{-m(t-t_0)} \le \frac{\varepsilon}{2} \ \text{ iff } \ t \ge -\frac{1}{m} \ln\left(\frac{\varepsilon}{2\beta^2 M_1 \exp M_2}\right).$$

Putting

$$T(\varepsilon) := \min\left\{-\frac{1}{m}\ln\left(\frac{\varepsilon}{2\delta_0}\right), -\frac{1}{m}\ln\left(\frac{\varepsilon}{2\beta^2 M_1 \exp M_2}\right)\right\}$$

with $\varepsilon > 0$ sufficiently small, we obtain

$$t - t_0 \geq T(\varepsilon) \quad \text{implies} \quad \|x(t, t_0, x_0)\| \leq \varepsilon,$$

which means that the trivial solution of (4.84) is uniformly asymptotically stable. ■

Corollary 221 *If the fundamental matrix of the solutions of the linear system by the first approximation satisfies the condition (4.90), f verifies the relation (4.11), $I(t,0) \equiv 0$ for all $t \in [\alpha, \infty)$ and*

$$\text{there exists a } \delta_0 > 0 \text{ such that } \int_\alpha^\infty e^{ms} I(s, \delta_0)\, ds \leq M_1 \tag{4.92}$$
$$\text{and } \int_\alpha^\infty \frac{\partial I}{\partial x}(s, \delta)\, ds \leq M_2 \text{ for all } 0 < \delta \leq \delta_0,$$

then the trivial solution of (4.84) is uniformly asymptotically stable.

Corollary 222 *If the fundamental matrix of the solution of the linear system by the first approximation satisfies the condition (4.90), f verifies the relation (4.13), $K(0) \equiv 0$ for all $t \in [\alpha, \infty)$ and*

$$\text{there exists a } \delta_0 > 0 \text{ such that } \frac{dK}{dx} \text{ is bounded in} \tag{4.93}$$
$$(0, \delta_0) \text{ and } \int_\alpha^\infty e^{ms} C(s)\, ds < \infty,$$

then the trivial solution $x \equiv 0$ of the system (4.84) is uniformly asymptotically stable.

Another result is embodied in the following theorem.

Theorem 223 *If the trivial solution of the linear system of the first approximation is uniformly asymptotically stable, f satisfies the relation (4.35) and the following condition*

$$\text{there exists a } \delta_0 > 0 \text{ such that } \int_\alpha^\infty S(s, \delta)\, ds \leq \tilde{M} \tag{4.94}$$
$$\text{for all } \leq \delta \leq \delta_0 \text{ holds}$$

then the trivial solution of (4.84) is uniformly asymptotically stable.

Proof. Let $x(\cdot, t_0, x_0)$ be the solution of the Cauchy problem $(A; f, t_0, x_0)$. Applying Lemma 199, we have the estimation

$$\begin{aligned}&\|x(t,t_0,x_0) - C(t,t_0)x_0\| \\ &\qquad \leq \|C(t,t_0)\| \int_{t_0}^{t} \|C(t_0,s) f(s, C(s,t_0)x_0)\| \\ &\qquad \times \exp\left(\int_s^t S(u, \|C(u,t_0)x_0\|) \|C(u,t_0)\| \|C(t_0,u)\| \, du\right) ds.\end{aligned}$$

By simple computation and since the trivial solution of the linear system by the first approximation is uniformly asymptotically stable, we have:

$$\begin{aligned}\|x(t,t_0,x_0)\| \leq \beta e^{-m(t-t_0)} \|x_0\| + \beta e^{-m(t-t_0)} \|x_0\| \int_{t_0}^{t} S\left(s, \beta e^{-m(s-t_0)} \|x_0\|\right) \\ \times \exp\left(\int_s^t S\left(u, \beta e^{-m(u-t_0)} \|x_0\|\right) du\right) ds,\end{aligned}$$

for all $t \geq t_0$.

If $\|x_0\| < \frac{\delta_0}{\beta}$, we obtain

$$\|x(t,t_0,x_0)\| \leq \beta\left(1 + \tilde{M} \exp \tilde{M}\right) e^{-m(t-t_0)} \|x_0\|$$

for all $t \geq t_0$.

The theorem is thus proved. ■

Similarly we can prove

Theorem 224 *If the trivial solution of the linear system is asymptotically stable, f satisfies the relation (4.35) and the following condition*

$$\int_{\alpha}^{\infty} S(u,\delta)\, du \leq \tilde{M} \quad \text{for all } \delta \geq 0, \text{ is valid,} \tag{4.95}$$

then the trivial solution of (4.84) is globally exponentially stable.

Corollary 225 *If the trivial solution of the linear system by the first approximation is uniformly asymptotically stable, f satisfies the relation (4.37) and the following condition is true:*

$$\begin{aligned}&\int_{\alpha}^{\infty} C(s)\, ds < \infty \text{ and there exists a } \delta_0 > 0 \text{ such that} \\ &R \text{ is bounded in } (0, \delta_0),\end{aligned} \tag{4.96}$$

then the trivial solution of (4.84) is globally exponentially stable.

4.7 Theorems of Global Asymptotic Stability

Let us consider the non-homogeneous system

$$\frac{dx}{dt} = A(t)\,x + f(t,x)\,,\ t \in [\alpha,\infty)\,, \tag{4.97}$$

where $A : [\alpha,\infty) \to B(\mathbb{R}^n)$, $f : [\alpha,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$ are continuous $f(t,0) \equiv 0$, for all $t \in [\alpha,\infty)$ and f is local Lipschitzian in x on $\mathbb{R}^n$.

Further, we shall assume that the fundamental matrix of the solution of the corresponding homogeneous system satisfies the condition

$$\|C(t,t_0)\| \le \beta e^{-m(t-t_0)},\ t \ge t_0, \tag{4.98}$$

i.e., the trivial solution of the linear system by the first approximation is uniformly asymptotically stable.

Definition 226 *The trivial solution $x \equiv 0$ of the system (4.97) is said to be globally asymptotic stable iff for every $t_0 \in [\alpha,\infty)$ and $x_0 \in \mathbb{R}^n$, the corresponding solution $x(\cdot,t_0,x_0)$ is defined in $[t_0,\infty)$ and $\lim\limits_{t\to\infty} \|x(t,t_0,x_0)\| = 0$.*

The following theorem is valid.

Theorem 227 *Let us suppose that the function f satisfies the relation (4.3) and the following conditions*

$$\int_\alpha^\infty e^{ms} L\left(s,\frac{\delta}{e^{ms}}\right) ds < \infty, \tag{4.99}$$

$$\int_\alpha^\infty M\left(s,\frac{\delta}{e^{ms}}\right) ds < \infty \quad \text{for all} \quad \delta > 0, \tag{4.100}$$

hold. Then the trivial solution of (4.97) is globally asymptotically stable.

Proof. Let $t \in [\alpha,\beta)$, $x_0 \in \mathbb{R}^n$ and $x(\cdot,t_0,x_0)$ be the solution of $(A;f,t_0,x_0)$ defined in the maximal interval of existence $[t_0,T)$. We have

$$x(t,t_0,x_0) = C(t,t_0)\,x_0 + C(t,t_0)\int_{t_0}^t C(t_0,s)\,f(s,x(s,t_0,x_0))\,ds$$

for all $t \in [t_0,T)$.

Passing at norms and applying Lemma 74, we obtain the estimation

$$\|x(t,t_0,x_0)\| \leq \beta e^{-m(t-t_0)}\|x_0\| + \beta e^{-mt}\int_{t_0}^{t} e^{ms} L\left(s,\beta e^{-m(s-t_0)}\|x_0\|\right) \times \exp\left(\int_s^t M\left(u,\beta e^{-m(u-t_0)}\|x_0\|\right)du\right)ds$$

for all $t\in[t_0,T)$.

By the relations (4.99) and (4.100), we have

$$\|x(t,t_0,x_0)\| \leq \beta e^{-m(t-t_0)}\|x_0\| + \beta e^{-mt}\int_{\alpha}^{\infty} e^{ms} L\left(s,\frac{\beta e^{mt_0}\|x_0\|}{e^{ms}}\right)ds \times \exp\left(\int_{\alpha}^{\infty} M\left(s,\frac{\beta e^{mt_0}\|x_0\|}{e^{ms}}\right)ds\right) \quad (4.101)$$

for all $t\in[t_0,T)$, which implies that $\lim_{t\to T}\|x(t,t_0,x_0)\|<\infty$ and by Theorem 10 of [10, pp. 49], it results that $\lim_{t\to\infty}\|x(t,t_0,x_0)\|=0$ and the theorem is proved. ■

Theorem 228 *Let us suppose that the function f satisfies the relation (4.3) and the following conditions*

$$\lim_{t\to\infty} L\left(t,\frac{\delta}{e^{mt}}\right)=0, \quad (4.102)$$

$$\int_{\alpha}^{\infty} M\left(s,\frac{\delta}{e^{ms}}\right)ds<\infty \quad \text{for all } \delta\geq 0, \quad (4.103)$$

hold. Then the trivial solution of (4.97) is globally asymptotically stable.

Proof. Let $t_0\in[\alpha,\infty)$, $x_0\in\mathbb{R}^n$ and $x(\cdot,t_0,x_0)$ be the solution of $(A;f,t_0,x_0)$ defined in the maximal interval of existence $[t_0,T)$. We have

$$\|x(t,t_0,x_0)\| \leq \beta e^{-m(t-t_0)}\|x_0\| + \beta e^{-mt}\int_{t_0}^{t} e^{ms} L\left(s,\beta e^{-m(s-t_0)}\|x_0\|\right) \times \exp\left(\int_s^t M\left(u,\beta e^{-m(u-t_0)}\|x_0\|\right)du\right)ds$$

for all $t \in [t_0, T)$.

Let us consider the function $g : [t_0, \infty) \to \mathbb{R}^+$ given by

$$g(t) = \frac{\beta}{e^{mt}} \int_{t_0}^{t} e^{ms} L\left(s, \frac{\beta e^{mt_0} \|x_0\|}{e^{ms}}\right) ds.$$

Then g is continuous in $[t_0, \infty)$ and

$$\lim_{t\to\infty} g(t) = \lim_{t\to\infty} \frac{e^{mt} L\left(t, \frac{\beta e^{mt_0}\|x_0\|}{e^{mt}}\right)}{me^{mt}} = 0$$

for all $x \in \mathbb{R}^n$. It results that

$$g(t) \leq \tilde{M}(t_0, x_0) \text{ for all } t \in [t_0, \infty),$$

which implies that

$$\|x(t, t_0, x_0)\| \leq \beta e^{-m(t-t_0)} \|x_0\| + \tilde{M}(t_0, x_0) \int_{\alpha}^{t} M\left(u, \frac{\beta e^{mt_0} \|x_0\|}{e^{mu}}\right) du$$

for all $t \in [t_0, T)$.

Since

$$\lim_{t\to T} \|x(t, t_0, x_0)\| < \infty,$$

it results that $x(\cdot, t_0, x_0)$ is defined in $[t_0, \infty)$ and by (4.7) we obtain $\lim_{t\to\infty} \|x(t, t_0, x_0)\| = 0$ and the theorem is proved. ■

Further, we shall prove another theorem of global asymptotic stability.

Theorem 229 *Let us assume that the mapping f verifies the relation (4.3) and the following conditions*

$$\lim_{t\to\infty} \left(\int_{\alpha}^{t} M\left(s, \frac{\delta}{e^{ms}}\right) ds - mt\right) = -\infty, \tag{4.104}$$

$$\int_{\alpha}^{\infty} \frac{e^{ms} L\left(s, \frac{\delta}{e^{ms}}\right)}{\exp\left(\int_{\alpha}^{s} M\left(u, \frac{\delta}{e^{mu}}\right) du\right)} ds < \infty \quad \textit{for all} \quad \delta \geq 0 \tag{4.105}$$

hold. Then the trivial solutions of (4.97) are globally asymptotically stable.

Proof. Let $t_0 \in [\alpha, \infty)$, $x_0 \in \mathbb{R}^n$ and $x(\cdot, t_0, x_0)$ be the solution of the Cauchy problem $(A; f, t_0, x_0)$ defined in the maximal interval of existence $[t_0, T)$. By similar computation, we have

$$\|x(t, t_0, x_0)\| \leq \beta e^{-m(t-t_0)} \|x_0\| + \exp\left(\int_\alpha^t M\left(s, \frac{\beta e^{mt_0} \|x_0\|}{e^{ms}}\right) ds - mt\right)$$
$$\times \int_\alpha^\infty \frac{e^{ms} L\left(s, \frac{\beta e^{mt_0}\|x_0\|}{e^{ms}}\right)}{\exp\left(\int_\alpha^s M\left(u, \frac{\beta e^{mt_0}\|x_0\|}{e^{mu}}\right) du\right)} ds$$

for all $t \in [t_0, T)$, which means that $\lim\limits_{\substack{t \to T \\ t < T}} \|x(t, t_0, x_0)\| < \infty$ from where it results that $T = \infty$.

On the other hand, we have that $\lim\limits_{t \to \infty} \|x(t, t_0, x_0)\| = 0$ and the theorem is proved. ■

Finally, we shall prove

Theorem 230 *If the function f verifies the relation (4.3) and the following conditions*

$$M\left(t, \frac{\delta}{e^{mt}}\right) \leq \frac{k(\delta)}{t}, \ k \text{ is nonnegative continuous in } [0, \infty), \tag{4.106}$$

$$\int_\alpha^\infty \frac{e^{ms} L\left(s, \frac{\delta}{e^{ms}}\right)}{s^{k(\delta)}} ds < \infty \quad \text{for all} \quad \delta \geq 0 \tag{4.107}$$

hold. Then the trivial solution of the system (4.97) is globally asymptotically stable.

Proof. If $x(\cdot, t_0, x_0)$ is the solution of $(A; f, t_0, x_0)$ defined in the maximal interval of existence $[t_0, T)$, we have, by (4.106) and (4.107) that

$$\lim_{t \to T} \|x(t, t_0, x_0)\| < \infty;$$

from where results $T = \infty$.

On the other hand, we have that $\lim\limits_{t \to \infty} \|x(t, t_0, x_0)\| = 0$ and the theorem is proved. ■

Remark 231 *If we assume that the function f satisfies the relations (4.5), (4.7), (4.9), (4.11), and (4.13) or the particular conditions (4.15) and (4.17), we can obtain a large number of corollaries and consequences for the above theorems. We omit the details.*

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