

## Hilbert scales and approximation theory

### Lecture notes

#### The Eigenvalue problem for compact symmetric operators

In the following  $H$  denotes an (infinite dimensional) real Hilbert space with scalar product  $(\cdot, \cdot)$  and the norm  $\|\cdot\|$ . We will consider mappings  $K : H \rightarrow H$ . Unless otherwise noticed the standard assumptions on  $K$  are:

- i)  $K$  is symmetric, i.e. for all  $x, y \in H$  it holds  $(x, Ky) = (x, Ky)$
- ii)  $K$  is compact, i.e. for any (infinite) sequence  $\{x_n\}$  bounded in  $H$  contains a subsequence  $\{x_{n'}\}$  such that  $\{Kx_{n'}\}$  is convergent,
- iii)  $K$  is injective, i.e.  $Kx = 0$  implies  $x = 0$ .

A first consequence is

**Lemma:**  $K$  is bounded, i.e.

$$\|K\| := \sup_{x \neq 0} \frac{\|Kx\|}{\|x\|} .$$

**Lemma:** Let  $K$  be bounded, and fulfill condition i) from above, but not necessarily the two other condition ii) and iii). Then  $\|K\|$  equals

$$N(K) = \sup_{x \neq 0} \frac{|(x, Kx)|}{\|x\|} .$$

**Theorem:** There exists a countable sequence  $\{\lambda_i, \varphi_i\}$  of eigenelements and eigenvalues

$K\varphi_i = \lambda_i \varphi_i$  with the properties

- i) the eigenelements are pair-wise orthogonal, i.e.  $(\varphi_i, \varphi_k) = \delta_{i,k}$
- ii) the eigenvalues tend to zero, i.e.  $\lim_{i \rightarrow \infty} \lambda_i = 0$
- iii) the generalized Fourier sums  $S_n := \sum_{i=1}^n (x, \varphi_i) \varphi_i \rightarrow x$  with  $n \rightarrow \infty$  for all  $x \in H$
- iv) the Parseval equation

$$\|x\|^2 = \sum_i (x, \varphi_i)^2$$

holds for all  $x \in H$ .

## Hilbert Scales

Let  $H$  be a (infinite dimensional) Hilbert space with scalar product  $(\cdot, \cdot)$ , the norm  $\|\cdot\|$  and  $A$  be a linear operator with the properties

- i)  $A$  is self-adjoint, positive definite
- ii)  $A^{-1}$  is compact.

Without loss of generality, possible by multiplying  $A$  with a constant, we may assume

$$(x, Ax) \geq \|x\|^2 \quad \text{for all } x \in D(A)$$

The operator  $K = A^{-1}$  has the properties of the previous section. Any eigenvalue of  $K$  is also an eigenvalue of  $A$  to the eigenvalues being the inverse of the first. Now by replacing  $\lambda_i \rightarrow \lambda_i^{-1}$  we have from the previous section

- i) there is a countable sequence  $\{\lambda_i, \varphi_i\}$  with

$$A\varphi_i = \lambda_i \varphi_i, \quad (\varphi_i, \varphi_k) = \delta_{i,k} \quad \text{and} \quad \lim_{i \rightarrow \infty} \lambda_i = \infty$$

- ii) any  $x \in H$  is represented by

$$(*) \quad x = \sum_{i=1}^{\infty} (x, \varphi_i) \varphi_i \quad \text{and} \quad \|x\|^2 = \sum_{i=1}^{\infty} (x, \varphi_i)^2.$$

**Lemma:** Let  $x \in D(A)$ , then

$$(**) \quad Ax = \sum_{i=1}^{\infty} \lambda_i (x, \varphi_i) \varphi_i, \quad \|Ax\|^2 = \sum_{i=1}^{\infty} \lambda_i^2 (x, \varphi_i)^2,$$

$$(Ax, Ay) = \sum_{i=1}^{\infty} \lambda_i^2 (x, \varphi_i)(y, \varphi_i).$$

Because of (\*) there is a one-to-one mapping  $I$  of  $H$  to the space  $\hat{H}$  of infinite sequences of real numbers

$$\hat{H} := \{\hat{x} | \hat{x} = (x_1, x_2, \dots)\}$$

defined by

$$\hat{x} = Ix \quad \text{with} \quad x_i = (x, \varphi_i).$$

If we equip  $\hat{H}$  with the norm

$$\|\hat{x}\|^2 = \sum_{i=1}^{\infty} (x, \varphi_i)^2$$

then  $I$  is an isometry.

By looking at (\*\*) it is reasonable to introduce for non-negative  $\alpha$  the weighted inner products

$$(\hat{x}, \hat{y})_\alpha = \sum_i \lambda_i^\alpha (x, \varphi_i)(y, \varphi_i) = \sum_i \lambda_i^\alpha x_i y_i$$

and the norms

$$\|\hat{x}\|_\alpha^2 = (\hat{x}, \hat{x})_\alpha$$

Let  $\hat{H}_\alpha$  denote the set of all sequences with finite  $\alpha$ -norm. then  $\hat{H}_\alpha$  is a Hilbert space. The proof is the same as the standard one for the space  $l_2$ .

Similarly one can define the spaces  $H_\alpha$ : they consist of those elements  $x \in H$  such that  $Ix \in \hat{H}_\alpha$  with scalar product

$$(x, y)_\alpha = \sum_i \lambda_i^\alpha (x, \varphi_i)(y, \varphi_i) = \sum_i \lambda_i^\alpha x_i y_i$$

and norm

$$\|x\|_\alpha^2 = (x, x)_\alpha.$$

Because of the Parseval identity we have especially

$$(x, y)_0 = (x, y)$$

and because of (\*\*) it holds

$$\|x\|_2^2 = (Ax, Ax)_0, \quad H_2 = D(A).$$

The set  $\{H_\alpha | \alpha \geq 0\}$  is called a Hilbert scale. The condition  $\alpha \geq 0$  is in our context necessary for the following reasons:

Since the eigen-values  $\lambda_i$  tend to infinity we would have for  $\alpha < 0$ :  $\lim \lambda_i^\alpha \rightarrow 0$ . Then there exist sequences  $\hat{x} = (x_1, x_2, \dots)$  with

$$\|\hat{x}\|_2^2 < \infty, \quad \|\hat{x}\|_0^2 = \infty.$$

Because of Bessel's inequality there exists no  $x \in H$  with  $Ix = \hat{x}$ . This difficulty could be overcome by duality arguments which we omit here.

There are certain relations between the spaces  $\{H_\alpha | \alpha \geq 0\}$  for different indices:

**Lemma:** Let  $\alpha < \beta$ . Then

$$\|x\|_\alpha \leq \|x\|_\beta$$

and the embedding  $H_\beta \rightarrow H_\alpha$  is compact.

**Lemma:** Let  $\alpha < \beta < \gamma$ . Then

$$\|x\|_\beta \leq \|x\|_\alpha^\mu \|x\|_\gamma^\nu \text{ for } x \in H_\gamma$$

with  $\mu = \frac{\gamma - \beta}{\gamma - \alpha}$  and  $\nu = \frac{\beta - \alpha}{\gamma - \alpha}$ .

**Lemma:** Let  $\alpha < \beta < \gamma$ . To any  $x \in H_\beta$  and  $t > 0$  there is a  $y = y_t(x)$  according to

- i)  $\|x - y\|_\alpha \leq t^{\beta - \alpha} \|x\|_\beta$
- ii)  $\|x - y\|_\beta \leq \|x\|_\beta$ ,  $\|y\|_\beta \leq \|x\|_\beta$
- iii)  $\|y\|_\gamma \leq t^{-(\gamma - \beta)} \|x\|_\beta$ .

**Corollary:** Let  $\alpha < \beta < \gamma$ . To any  $x \in H_\beta$  and  $t > 0$  there is a  $y = y_t(x)$  according to

- i)  $\|x - y\|_\rho \leq t^{\beta - \rho} \|x\|_\beta$  for  $\alpha \leq \rho \leq \beta$
- ii)  $\|y\|_\sigma \leq t^{-(\sigma - \beta)} \|x\|_\beta$  for  $\beta \leq \sigma \leq \gamma$ .

**Remark:** Our construction of the Hilbert scale is based on the operator  $A$  with the two properties i) and ii). The domain  $D(A)$  of  $A$  equipped with the norm

$$\|Ax\|^2 = \sum_{i=1}^{\infty} \lambda_i^2(x, \varphi_i)^2$$

turned out to be the space  $H_2$  which is densely and compactly embedded in  $H = H_0$ . It can be shown that on the contrary to any such pair of Hilbert spaces there is an operator  $A$  with the properties i) and ii) such that

$$D(A) = H_2 \quad R(A) = H_0 \quad \text{and} \quad \|x\|_2 = \|Ax\|.$$

We give three examples of differential operator and singular integral operators, whereby the integral operators are related to each other by partial integration:

**Example 1:** Let  $H = L^2(0,1)$  and

$$Au := -u''$$

with

$$D(A) = \overset{\bullet}{W}_2^2(0,1) := \overset{\circ}{W}_2^1(0,1) \cap \overset{\circ}{W}_2^2(0,1).$$

Building on the orthogonal set of eigenpairs  $\{\lambda_i, \varphi_i\}$  of  $A_i$ , i.e.

$$-\varphi_i'' = \lambda_i \varphi_i \quad \varphi_i(0) = \varphi_i(1) = 0$$

it holds the inclusion

$$D(A) \subseteq H_A = H_1 = \overset{\circ}{W}_2^1(0,1) \subseteq L^2(0,1).$$

**Example 2:** Let  $H = L_{22}^*(\Gamma)$  with  $\Gamma := S^1(\mathbb{R}^2)$ , i.e.  $\Gamma$  is the boundary of the unit sphere. Then  $H$  is the space of integrable periodic function in  $\mathbb{R}$ . Let

$$(Au)(x) := -\oint \log 2 \sin \frac{x-y}{2} u(y) dy =: \oint k(x-y)u(y) dy$$

and

$$D(A) = H = L_{22}^*(\Gamma).$$

The Fourier coefficients of this convolution are

$$(Au)_\nu = k_\nu u_\nu = \frac{1}{2|\nu|} u_\nu$$

i.e. it holds  $D(A) \subseteq H_A = H_{-1/2}(\Gamma)$ .

A relation of this Fourier representation to the fractional function is given by

$$x - [x] - \frac{1}{2} = -\sum_1^\infty \frac{\sin 2\pi\nu x}{\pi\nu}$$

**Remark:** We give some further background and analysis of the even function

$$k(x) := -\ln \left| 2 \sin \frac{x}{2} \right| =: -\log \left| 2 \sin \frac{x}{2} \right|$$

Consider the model problem

$$\begin{aligned} -\Delta U &= 0 & \text{in } \Omega \\ U &= f & \text{on } \Gamma := \partial\Omega, \end{aligned}$$

whereby the area  $\Omega$  is simply connected with sufficiently smooth boundary. Let  $y = y(s) - s \in (0,1]$  be a parametrization of the boundary  $\partial\Omega$ . Then for fixed  $\bar{z}$  the functions

$$U(\bar{x}) = -\log|\bar{x} - \bar{z}|$$

Are solutions of the Laplace equation and for any  $L_1(\partial\Omega)$ - integrable function  $u = u(t)$  the function

$$(Au)(\bar{x}) := \oint_{\partial\Omega} \log|\bar{x} - u(t)| dt$$

is a solution of the model problem. In an appropriate Hilbert space  $H$  this defines an integral operator, which is coercive for certain areas  $\Omega$  and which fulfills the Garding inequality for general areas  $\Omega$ . We give the Fourier coefficient analysis in case of  $H = L_2^*(\Gamma)$  with  $\Gamma := S^1(R^2)$ , i.e.  $\Gamma$  is the boundary of the unit sphere. Let  $x(s) := (\cos(s), \sin(s))$  be a parametrization of  $\Gamma := S^1(R^2)$  then it holds

$$|x(s) - x(t)|^2 = \left( \begin{array}{c} \cos(s) - \cos(t) \\ \sin(s) - \sin(t) \end{array} \right)^2 = 2 - 2\cos(s-t) = 2(1 - \cos(2\frac{s-t}{2})) = 2 \left[ 2\sin^2 \frac{s-t}{2} \right] = 4\sin^2 \frac{s-t}{2}$$

and therefore

$$-\log|x(s) - x(t)| = -\log 2 \left| \sin \frac{s-t}{2} \right| = k(s-t)$$

The Fourier coefficients  $k_\nu$  of the kernel  $k(x)$  are calculated as follows

$$k_\nu := \frac{1}{2\pi} \oint k(x) e^{-i\nu x} dx = \frac{1}{2\pi} \int_0^{2\pi} \log \left| 2 \sin \frac{t}{2} \right| e^{-i\nu t} dt = \frac{2}{2\pi} \int_0^\pi \log \left| 2 \sin \frac{t}{2} \right| \cos(\nu t) dt = k_{-\nu}$$

As  $\varepsilon \log 2 \sin \frac{\varepsilon}{2} \rightarrow 0$  partial integration leads to

$$\begin{aligned} k_\nu &= \frac{1}{\nu\pi} \sin(\nu t) \Big|_0^\pi - \frac{1}{\nu\pi} \int_0^\pi \frac{2 \sin(\nu t) \cos \frac{t}{2}}{2 \sin \frac{t}{2}} dt = -\frac{1}{\nu\pi} \int_0^\pi \frac{\sin(\frac{2\nu+1}{2}t) - \sin(\frac{2\nu-1}{2}t)}{2 \sin \frac{t}{2}} dt \\ k_\nu &= -\frac{1}{\nu\pi} \int_0^\pi \left( \frac{1}{2} + \cos(t) \dots + \cos(\nu t) \right) - \left( \frac{1}{2} + \cos(t) \dots + \cos((\nu-1)t) \right) dt = -\frac{1}{\nu} \cdot \end{aligned}$$

## Extension and generalizations

For  $t > 0$  we introduce an additional inner product resp. norm by

$$(x, y)_{(t)}^2 = \sum_{i=1}^{\infty} e^{-\sqrt{\lambda_i} t} (x, \varphi_i)(y, \varphi_i)$$

$$\|x\|_{(t)}^2 = (x, x)_{(t)}^2 .$$

Now the factor have exponential decay  $e^{-\sqrt{\lambda_i} t}$  instead of a polynomial decay in case of  $\lambda_i^\alpha$ . Obviously we have

$$\|x\|_{(t)} \leq c(\alpha, t) \|x\|_\alpha \text{ for } x \in H_\alpha$$

with  $c(\alpha, t)$  depending only from  $\alpha$  and  $t > 0$ . Thus the  $(t)$ -norm is weaker than any  $\alpha$ -norm. On the other hand any negative norm, i.e.  $\|x\|_\alpha$  with  $\alpha < 0$ , is bounded by the  $0$ -norm and the newly introduced  $(t)$ -norm. It holds:

**Lemma:** Let  $\alpha > 0$  be fixed. The  $\alpha$ -norm of any  $x \in H_0$  is bounded by

$$\|x\|_{-\alpha}^2 \leq \delta^{2\alpha} \|x\|_0^2 + e^{t/\delta} \|x\|_{(t)}^2$$

with  $\delta > 0$  being arbitrary.

**Remark:** This inequality is in a certain sense the counterpart of the logarithmic convexity of the  $\alpha$ -norm, which can be reformulated in the form ( $\mu, \nu > 0, \mu + \nu > 1$ )

$$\|x\|_\beta^2 \leq \nu \varepsilon \|x\|_\gamma^2 + \mu e^{-\nu/\mu} \|x\|_\alpha^2$$

applying Young's inequality to

$$\|x\|_\beta^2 \leq (\|x\|_\alpha^2)^\mu (\|x\|_\gamma^2)^\nu .$$

The counterpart of lemma 4 above is

**Lemma:** Let  $t, \delta > 0$  be fixed. To any  $x \in H_0$  there is a  $y = y_t(x)$  according to

- i)  $\|x - y\| \leq \|x\|$
- ii)  $\|y\|_1 \leq \delta^{-1} \|x\|$
- iii)  $\|x - y\|_{(t)} \leq e^{-t/\delta} \|x\| .$

## Eigenfunctions and Eigendifferentials

Let  $H$  be a (infinite dimensional) Hilbert space with inner product  $(\cdot, \cdot)$ , the norm  $\|\cdot\|$  and  $A$  be a linear self-adjoint, positive definite operator, but we omit the additional assumption, that  $A^{-1}$  compact. Then the operator  $K = A^{-1}$  does not fulfill the properties leading to a discrete spectrum.

We define a set of projections operators onto closed subspaces of  $H$  in the following way:

$$R \rightarrow L(H, H)$$

$$\lambda \rightarrow E_\lambda := \int_{\lambda_0}^{\lambda} \varphi_\mu(\varphi_\mu, *) d\mu \quad , \quad \mu \in [\lambda_0, \infty) ,$$

i.e. 
$$dE_\lambda = \varphi_\lambda(\varphi_\lambda, *) d\lambda .$$

The spectrum  $\sigma(A) \subset C$  of the operator  $A$  is the support of the spectral measure  $dE_\lambda$ .

The set  $E_\lambda$  fulfills the following properties:

- i)  $E_\lambda$  is a projection operator for all  $\lambda \in R$
- ii) for  $\lambda \leq \mu$  it follows  $E_\lambda \leq E_\mu$  i.e.  $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$
- iii)  $\lim_{\lambda \rightarrow -\infty} E_\lambda = 0$  and  $\lim_{\lambda \rightarrow \infty} E_\lambda = Id$
- iv)  $\lim_{\substack{\mu \rightarrow \lambda \\ \mu > \lambda}} E_\mu = E_\lambda$  .

**Proposition:** Let  $E_\lambda$  be a set of projection operators with the properties i)-iv) having a compact support  $[a, b]$ . Let  $f : [a, b] \rightarrow R$  be a continuous function. Then there exists exactly one Hermitian operator  $A_f : H \rightarrow H$  with

$$(A_f x, x) = \int_{-\infty}^{\infty} f(\lambda) d(E_\lambda x, x) .$$

Symbolically one writes

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda .$$

Using the abbreviation

$$\mu_{x,y}(\lambda) := (E_\lambda x, y) \quad , \quad d\mu_{x,y}(\lambda) := d(E_\lambda x, y)$$

one gets

$$(Ax, y) = \int_{-\infty}^{\infty} \lambda d(E_\lambda x, y) = \int_{-\infty}^{\infty} \lambda d\mu_{x,x}(\lambda) \quad , \quad \|x\|_1^2 = \int_{-\infty}^{\infty} \lambda d\|E_\lambda x\|^2 = \int_{-\infty}^{\infty} \lambda d\mu_{x,x}(\lambda)$$

$$(A^2 x, y) = \int_{-\infty}^{\infty} \lambda^2 d(E_\lambda x, y) = \int_{-\infty}^{\infty} \lambda^2 d\mu_{x,x}(\lambda) \quad , \quad \|Ax\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\|E_\lambda x\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\mu_{x,x}(\lambda) .$$



The function  $\sigma(\lambda) := \|E_\lambda x\|^2$  is called the spectral function of  $A$  for the vector  $x$ . It has the properties of a distribution function.

It holds the following eigenpair relations

$$A\varphi_i = \lambda_i\varphi_i \quad A\varphi_\lambda = \lambda\varphi_\lambda \quad \|\varphi_\lambda\|^2 = \infty, \quad (\varphi_\lambda, \varphi_\mu) = \delta(\varphi_\lambda - \varphi_\mu).$$

The  $\varphi_\lambda$  are not elements of the Hilbert space. The so-called eigendifferentials, which play a key role in quantum mechanics, are built as superposition of such eigenfunctions.

Let  $I$  be the interval covering the continuous spectrum of  $A$ . We note the following representations:

$$x = \sum_1^\infty (x, \varphi_i) \varphi_i + \int_I \varphi_\mu (\varphi_\mu, x) d\mu, \quad Ax = \sum_1^\infty \lambda_i (x, \varphi_i) \varphi_i + \int_I \lambda \varphi_\mu (\varphi_\mu, x) d\mu$$

$$\|x\|^2 = \sum_1^\infty |(x, \varphi_i)|^2 + \int_I |(\varphi_\mu, x)|^2 d\mu,$$

$$\|x\|_1^2 = \sum_1^\infty \lambda_i |(x, \varphi_i)|^2 + \int_I \lambda |(\varphi_\mu, x)|^2 d\mu$$

$$\|x\|_2^2 = \|Ax\|^2 = \sum_1^\infty \lambda_i^2 |(x, \varphi_i)|^2 + \int_I \lambda^2 |(\varphi_\mu, x)|^2 d\mu.$$

**Example:** The location operator  $Q_x$  and the momentum operator  $P_x$  both have only a continuous spectrum. For positive energies  $\lambda \geq 0$  the Schrödinger equation

$$H\varphi_\lambda(x) = \lambda\varphi_\lambda(x)$$

delivers no element of the Hilbert space  $H$ , but linear, bounded functional with an underlying domain  $M \subset H$  which is dense in  $H$ . Only if one builds wave packages out of  $\varphi_\lambda(x)$  it results into elements of  $H$ . The practical way to find Eigen-differentials is looking for solutions of a distribution equation.