Mathematical issues concerning the Navier-Stokes equations and some of their generalizations

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This article primarily deals with internal, isothermal, unsteady flows of a class of incompressible fluids with both constant, and shear or pressure dependent viscosity that includes the Navier-Stokes fluid as a special subclass.

We begin with a description of such fluids within the framework of a continuum. We then discuss various ways in which the response of a fluid can depart from that of a Navier-Stokes fluid. Next, we introduce a general thermodynamic framework that has been successful in describing the disparate response of continua that includes those of inelasticity, solid-to-solid transformation, viscoelasticity, granular materials, blood and asphalt rheology etc. Here, it leads to a novel derivation of the constitutive equation for the Cauchy stress for fluids with constant, or shear or pressure, or density dependent viscosity within a full thermo-mechanical setting. One advantage of this approach consists in a transparent treatment of the constraint of incompressibility.

We then concentrate on mathematical analysis of three-dimensional unsteady flows of fluids with shear dependent viscosity that includes the Navier-Stokes model and Ladyshenskaya's model as special cases.

We are interested in the issues connected with mathematical self-consistency of the models, i.e., we are interested in knowing whether 1) flows exist for reasonable, but arbitrary initial data and for all instants of time, 2) flows are uniquely determined, 3) the velocity is bounded and 4) the large-time behavior of all possible

flows can be captured by a finite dimensional, small (compact) set attracting all flow trajectories exponentially.

For simplicity, we eliminate a choice of boundary conditions and their influence on flows assuming that all functions are spatially periodic with zero mean value over periodic cell. All results could be however extended to internal flows where the tangent component of the velocity satisfies Navier's slip at the boundary. Most of the results hold also for no-slip boundary conditions.

While the mathematical consistency understood in the above sense of the Navier-Stokes model in three dimension is not clear yet, we will show that Ladyzhenskaya's model and some of its generalization enjoy all above properties for certain range of parameters. Briefly, we also discuss further results related to further generalizations of the Navier-Stokes equations.

Keywords: incompressible fluid, Navier-Stokes fluid, non-Newtonian fluid, rheology, mathematical analysis

The contribution of J. Málek to this work is a part of the research project MSM 0021620839 financed by MSMT. K. R. Rajagopal thanks the National Science Foundation for its support. A part of this research was performed during the stay of J. Málek at Department of Mathematics, Texas A&M University.

The authors thank Miroslav Bulíček for his continuous help in the process of the preparation of this work, and Petr Kaplický for his critical comments to earlier drafts of this work.

In memory of

Olga Alexandrovna Ladyzhenskaya

March 7, 1922 - January 12, 2004

and

Jindřich Nečas

December 14, 1929 - December 6, 2002

Chapter A

Incompressible Fluids With Shear, Pressure and Density Dependent Viscosity from Point of View of Continuum Physics

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Mathematical Analysis of Flows of Fluids With Shear, Pressure and Density Dependent Viscosity

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Chapter A

Incompressible Fluids With Shear, Pressure and Density Dependent Viscosity from Point of View of Continuum Physics

1. Introduction

1.1. What is a fluid?

The meaning of words provided in even the most advanced of dictionaries, say the Oxford English Dictionary [1], will rarely serve the needs of a scientist or technologist adequately and this is never more evident than in the case of the meaning assigned to the word "fluid" in its substantive form: "A substance whose particles move freely among themselves, so as to give way before the slightest pressure." The inadequacy, in the present case, stems from the latter part of the sentence which states that fluids cannot resist pressure; more so as the above definition is immediately followed by the classification: "Fluids are divided into liquids which are incompletely elastic, and gases, which are completely so.". With regard to the first definition, as "Fluids" obviously include liquids such as water, which under normal ranges of pressure are essentially incompressible and can support a purely spherical state of stress without flowing the definition offered in the dictionary is, if not totally wrong†, at the very least confounding. Much, if not all of hydrostatics

† One could take the point of view that no body is perfectly incompressible and thus the body does deform, ever so slightly, due to the application of pressure. The definition however cannot be developed thusly as the intent is clearly that the body suffers significant deformation due to the slightest application of the pressure.

is based on the premise that most liquids are incompressible. Next, with regard to the classification of liquids being "incompletely elastic", we have to bear in mind that all real gases are not "completely elastic". The ideal gas model is of course purely elastic.

What then does one mean by a fluid? When we encounter the word “Fluid” for the first time in a physics course at school, we are told that a “fluid” is a body that takes the shape of a container. This meaning assigned to a fluid, can after due care, be used to conclude that a fluid is a body whose symmetry group is the unimodular group†. Such a definition is also not without difficulty. While a liquid takes the shape of the container partially if its volume is less than that of the container, a gas expands to always fill a container. The definition via symmetry groups can handle this difficulty in the sense that it requires densities to be constant while determining the symmetry group. However, this places an unnecessary restriction with regard to defining gases, as this is akin to defining a body on only a small subclass of processes that the body can undergo. We shall not get into a detailed discussion of these subtle issues here.

Another definition for a fluid that is quite common, specially with those conversant with the notion of stress, is that a fluid is a body that cannot support a shear stress, as opposed to pressure as required by the definition in [1]. A natural question that immediately arises is that of time scales. How long can a fluid body not support a shear stress? How does one measure this inability to support a shear stress? Is it with the naked eye or is it to be inferred with the aid of sophisticated instruments? Is the assessment to be made in one second, one day, one month or one year? These questions are not being raised merely from the philosophical standpoint. They have very practical pragmatic underpinnings. It is possible, say in the time scale of one hour, one might be unable to discern the flow or deformation that a body undergoes, with the naked eye. This is indeed the case with regard to the experiment on asphalt that has been going on for over seventy years (see Murali Krishnan and Rajagopal [62] for a description of the experiment). The earlier definition for the fluid cannot escape the issue of time scale either. One has to contend with how long it takes to attain the shape of the container.

† This statement is not strictly correct. A special subclass of fluids, those that are referred to as “Simple fluids” admit such an interpretation (see Noll [101], Truesdell and Noll [143]). However, it is possible that there exist anisotropic fluids whose symmetry group is not the unimodular group (see Rajagopal and Srinivasa [113]).

The importance of the notion of time scales was recognized by Maxwell. He observes [89]: "In the case of a viscous fluid it is time which is required, and if enough time is given, the very smallest force will produce a sensible effect, such as would require a very large force if suddenly applied. Thus a block of pitch may be so hard that you cannot make a dent in it by striking it with your knuckles; and yet it will in the course of time flaten itself by its weight, and glide downhill like a stream of water". The key words in the above remarks of Maxwell are "if enough time is given". Thus, what we can infer at best is whether a body is more or less fluid-like, i.e., within the time scales of the observation of our interest does a small shear stress produce a sensible deformation or does it not. Let us then accept to "understand" a "Fluid" as a body that, in the time scale of observation of interest, undergoes discernible deformation due to the application of a sufficiently small shear stress‡.

1.2. Navier-Stokes fluid model

The popular Navier-Stokes model traces its origin to the seminal work of Newton [99] followed by the penetrating studies by Navier [92], Poisson [103] and St-Venant [120], culminating in the definitive study of Stokes [135]†. In his immortal Principia, Newton [99] states: "The resistance arising from the want of lubricity in parts of the fluid is, other things being equal, proportional to the velocity with which the parts of the fluid are separated from one another." What is now popularly referred to as the Navier-Stokes model implies a linear relationship between the shear stress and the shear rate. However, it was recognized over a century ago that this want of lubricity need not be proportional to the shear stress. Trouton [141] observes "the rate of flow of the material under shearing stress cannot be in simple proportion to shear rate". However, the popular view persisted and was that

‡ We assume we can agree on what we mean by the time scale of observation of interest. It is also important to recognize that if the shear stress is too small, its effect, the flow, might not be discernible. Thus, we also have to contend with the notion of a spatial scale for a discerning movement and a force scale for discerning forces.

† It is interesting to observe what Stokes [135] has to say concerning the development of the fluid model that is referred to as the Navier-Stokes model. Stokes remarks: "I afterward found that Poisson had written a memoir on the same subject, and on referring to it found that he had arrived at the same equations. The method which he employed was however so different from mine that I feel justified in laying the later before this society .... The same equations have been obtained by Navier in the case of an incompressible fluid (Mém. de l'Académie, t. VI, p. 389), but his principles differ from mine still more than do Poisson's."

the rate of flow was proportional to the shear stress as evidenced by the following remarks of Bingham [10]: " When viscous substance, either a liquid or a gas, is subjected to a shearing stress, a continuous deformation results which is, within certain restrictions directly proportional to the shearing stress. This fundamental law of viscous flow ... ." Though Bingham offers a caveat "within certain restrictions", his immediate use of the terms "fundamental law of viscous flow" clearly indicates how well the notion of the proportional relations between a kinematical measure of flow and the shear stress was ingrained in the fluid dynamicist of those times.

We will record below, for the sake of discussion, the classical fluid models that bear the names of Euler, and Navier and Stokes.
Homogeneous Compressible Euler Fluid:

$$\mathbf{T} = -p(\varrho)\mathbf{I}\,. \tag{A.1.1}$$

Homogeneous Incompressible Euler Fluid:

$$\mathbf{T} = -p\mathbf{I}\,, \qquad \mathrm{tr}\mathbf{D} = 0\,. \tag{A.1.2}$$

Homogeneous Compressible Navier-Stokes Fluid:

$$\mathbf{T} = -p(\varrho)\mathbf{I} + \lambda(\varrho)\,(\mathrm{tr}\mathbf{D})\,\mathbf{I} + 2\mu(\varrho)\mathbf{D}\,. \tag{A.1.3}$$

Homogeneous Incompressible Navier-Stokes Fluid:

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}\,, \qquad \mathrm{tr}\mathbf{D} = 0\,. \tag{A.1.4}$$

In the above definitions, $\mathbf{T}$ denotes the Cauchy stress, ϱ is the density, λ and μ the bulk and shear moduli of viscosity and $\mathbf{D}$ the symmetric part of the velocity gradient. In equations (A.1.1) and (A.1.3) the pressure is defined through an equation of state, while in (A.1.2) and (A.1.4), it is the reaction force due to the constraint that the fluid be incompressible.

Within the course of this article we will confine our mathematical discussion mainly to the incompressible Navier-Stokes fluid model (A.1.4) and many of its generalizations.

A model that is not of the form (A.1.3) and (A.1.4) falls into the category of (compressible and incompressible) non-Newtonian fluids†. This exclusive definition

† Navier-Stokes fluids are usually referred to in the fluid mechanics literature as Newtonian fluids. The equations of motions for Newtonian fluids are referred to as the Navier-Stokes equations.

leads to innumerable fluid models and choices amongst them have to be based on observed response of real fluids that cannot be adequately captured by the above models. This leads us to a discussion of these observations.

1.3. Departures From Newtonian Behavior

We briefly list several typical non-Newtonian responses. In their description, detailed characterizations are given to those phenomena and corresponding models whose mathematical properties will be discussed in this paper. A reader interested in a more details on non–Newtonian fluids is referred for example to the monographs Truesdell and Noll [143], Schowalter [125] or Huilgol [54], or to the articles of J.M. Burgers in [14], or to the review article by Rajagopal [115].

Shear-Thinning/Shear-Thickening

Let us consider an unsteady simple shear flow in which the velocity field $\mathbf{v}$ is given by

$$\mathbf{v} = u(y,t)\mathbf{i}\,, \tag{A.1.5}$$

in a Cartesian coordinate system (x, y, z) with base vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, respectively, t denoting the time. We notice that (A.1.5) automatically meets

$$\operatorname{div}\mathbf{v} = \mathrm{tr}\mathbf{D} = 0, \tag{A.1.6}$$

and the only non-zero component for the shear stress corresponding to (A.1.3) or (A.1.4) is given by

$$T_{xy}(y,t) = \mu u_{,y}(y,t) \qquad \text{where } u_{,y} := \frac{du}{dy}\,, \tag{A.1.7}$$

i.e., the shear stress varies proportionally with respect to the gradient of the velocity, the constant of proportionality being the viscosity. Thus, the graph of the shear stress versus the velocity gradient (in this case the shear rate) is a straight line (see curve 3 in Fig. 1).

Let us consider a steady shearing flow, i.e., a flow wherein $u = u(y)$ and $\kappa := u_{,y} = \mathit{constant}$ at each point of the container occupied by the fluid. It is observed that in many fluids there is a considerable departure from the above relationship (A.1.7) between the shear stress and the shear rate. In some fluids it is observed that the relationship is as depicted by the curve 1 in Fig. 1, i.e. the generalized

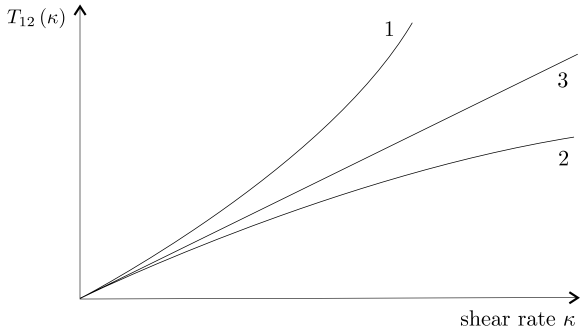

Figure 1. Shear Thinning/Shear Thickening

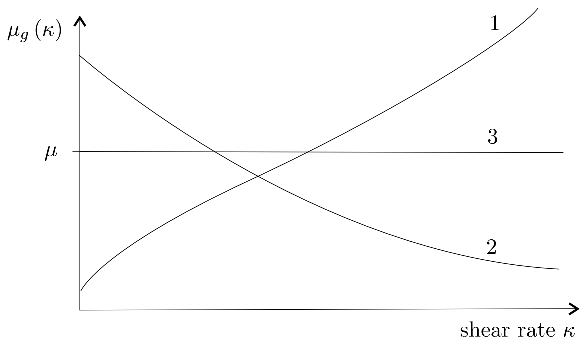

Figure 2. Generalized viscosity

viscosity which is defined through

$$\mu_g(\kappa) := \frac{T_{xy}}{\kappa}, \tag{A.1.8}$$

is monotonically increasing (cf. curve 1 in Fig. 2). Thus, in such fluids, the viscosity increases with the shear rate and they are referred to as shear-thickening fluids. On the other hand there are fluids for which the relationship between the shear stress and the shear rate is as depicted by curve 2 in Fig. 1. In such fluids, the generalized viscosity decreases with increasing shear rate and for this reason such fluids are called shear-thinning fluids. The Newtonian fluid is thus a very special fluid. It neither shear thins nor shear thickens.

The models with shear dependent viscosity are used in many areas of engineering science such as geophysics, glaciology, colloid mechanics, polymer mechanics, blood and food rheology, etc. An illustrative list of references for such models and their applications is given in [87].

Normal Stress Differences In Simple Shear Flows

Next, let us compute the normal stresses along the x, y and z direction for the simple shear flow (A.1.5). A trivial calculation leads to, in the case of models (A.1.3) and (A.1.4),

$$T_{xx} = T_{yy} = T_{zz} = -p\,,$$

and thus

$$T_{xx} - T_{yy} = T_{xx} - T_{zz} = T_{yy} - T_{zz} = 0\,.$$

That is the normal stress differences are zero in a Navier-Stokes fluid. However, it can be shown that some of the phenomena that are observed during the flows of fluids such as die-swell, rod-climbing, secondary flows in cylindrical pipes of non-circular cross-section, etc., have as their basis non-zero differences between these normal stresses.

Stress-Relaxation

When subject to a step change in strain ε (see Fig. 3 left), that results in a simple shear flow (A.1.5) to $\dot{\varepsilon}$ drawn at Fig. 3 right the stress $\sigma := T_{xy}$ in bodies modeled by (A.1.3) and (A.1.4) suffers an abrupt change that is undefined at the instant the strain has suffered a change and is zero at all other instants (see Fig. 4 right). On the other hand, there are many bodies that respond in the manner shown in Fig. 5. The graph at right depicts the fluid-like behavior as no stress is necessary to maintain a fixed strain, in the long run. The graph at left represents solid-like response. The Newtonian fluid model is incapable of describing stress-relaxation, a phenomenon exhibited by many real bodies. The important fact to recognize is that a Newtonian fluid stress relaxes instantaneously (see Fig. 4 right)†.

Creep

Next, let us consider a body that is subject to a step change in the stress (see Fig. 6). In the case of a Newtonian fluid the strain will increase linearly with time (see Fig. 7 at right). However, there are many bodies whose strain will vary as depicted in Fig. 8. The curve at left depicts solid-like behavior while the curve at right depicts fluid-like behavior. The response, which is referred to as “creep” as

† This does not mean that it has instantaneous elasticity.

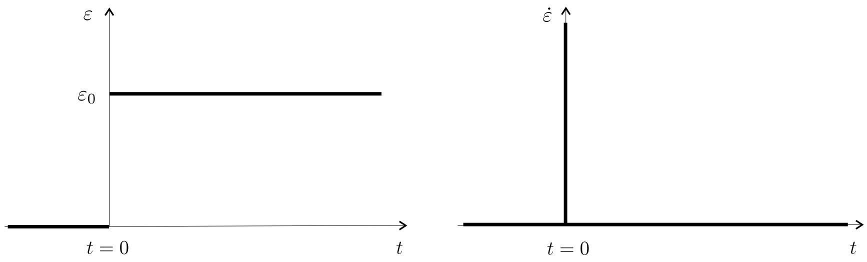

Figure 3. Stress-Relaxation test: response to a step change in strain (picture at left). The picture at right sketches its derivative.

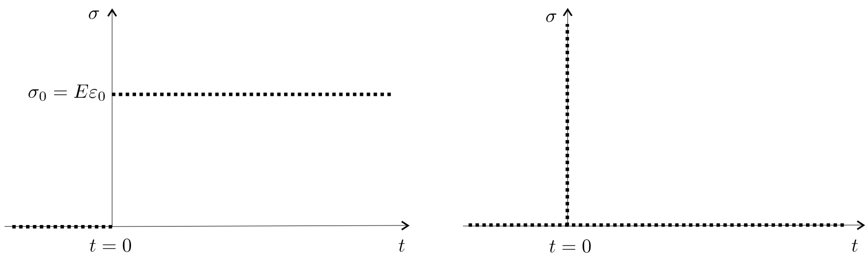

Figure 4. Shear Stress Response to a step change in strain for linear spring (at left) and Navier-Stokes fluid (at right)

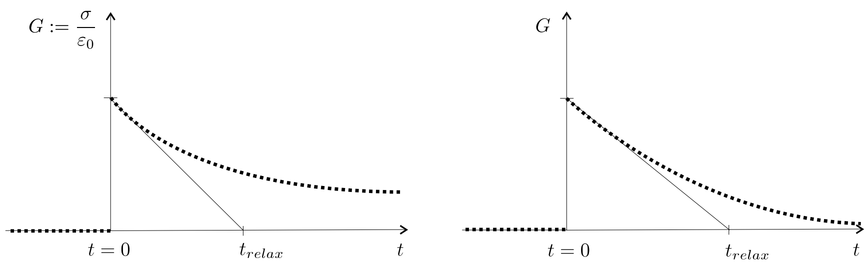

Figure 5. Stress-Relaxation for more realistic materials

the body flows while the stress is held constant. A Newtonian fluid creeps linearly with time. Many real fluids creep non-linearly with time.

Jump discontinuities in stresses

Yield stress

Bodies that have a threshold value for the stress before they can flow are supposed to exhibit the phenomenon of “yielding”, see Fig. 9. However, if one takes the point of view that a fluid is a body that cannot sustain shear, then by definition there can be no stress threshold to flow, which is the basic premise of the notion of

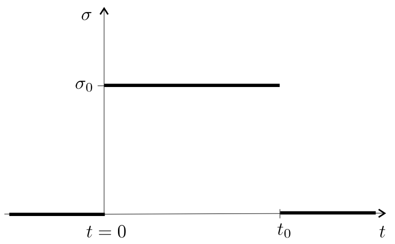

Figure 6. Creep test

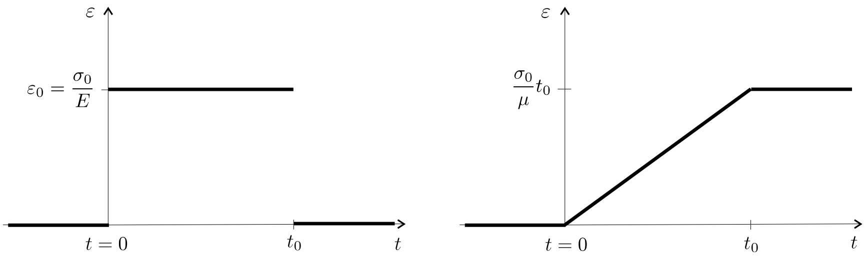

Figure 7. Deformation response to step change of shear stress for linear spring (left) and Newtonian fluid (right)

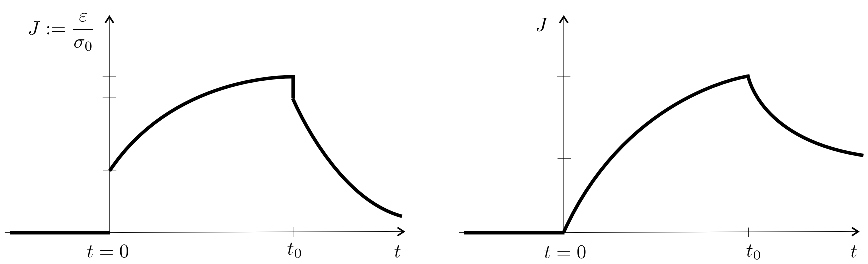

Figure 8. Creep of solid-like and fluid-like materials

a "yield stress". This is yet another example where the importance of time scales comes into play. It might seem, with respect to some time scale of observation, that the flow in a fluid is not discernible until a sufficient large stress is applied. This does not mean that the body in question can support small values of shear stresses, indefinitely. It merely means that the flow that is induced is not significant. A Newtonian fluid has no threshold before it can start flowing. A material responding as a Newtonian fluid once the yield stress is reached is called the Bingham fluid.

Activation criterion

It is possible that in some fluids, the response characteristics can change when a certain criterion, that could depend on the stress, strain rate or other kinematical quantitites, is met. An interesting example of the same is phenomena of coagulation

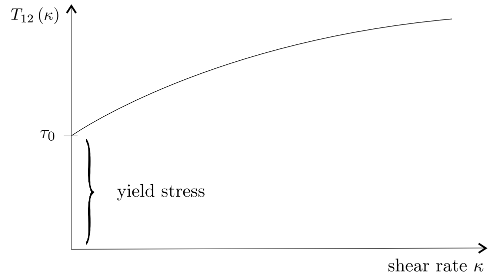

Figure 9. Yield Stress

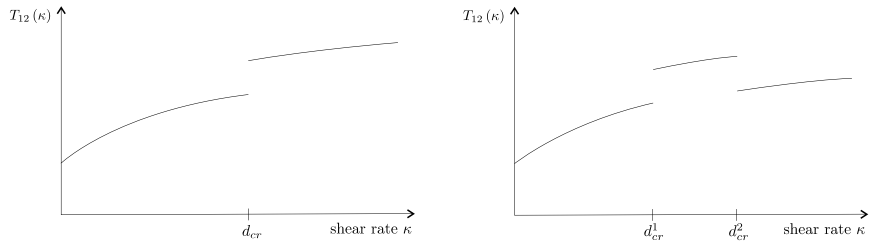

Figure 10. Activation and Deactivation of Fluids with Shear Dependent Viscosity modelled as jump discontinuities in stress

or dissolution of blood. Of course, here issues are more complicated as complex chemical reactions are taking place in the fluid of interest, blood.

Platelet activation is followed by their interactions with a variety of plasma proteins that leads to the aggregation of platelets which in turn leads to coagulation, i.e., the formation of clots. The activated platelets also serve as sites for enzyme complexes that play an important role in the formation of clots. These clots, as well as the original blood, are viscoelastic fluids, the clot being significantly more viscous that regular blood. In many situations the viscoelasticity is not consequential and can be ignored and the fluid can be approximated as a generalized Newtonian fluid. While the formation of the clot takes a finite length of time, we can neglect this with respect to a time scale of interest associated with the flowing blood. As the viscosity has increased considerably over a sufficiently short time, in the simple shear flow, the fluid could be regarded as suffering a jump discontinuity as depicted in Fig. 10 at left. On deforming the clot further, we notice a most interesing phenomenon. At a sufficiently high stress, dissolution of the clot takes place and the viscosity now undergoes a significant decrease close to its original value as depicted in Fig. 10 at right . Thus, in general "activation" can lead to either an increase of decrease in viscosity over a very short space of time whereby we can think of it as a jump. See [3] for more details.

Pressure-Thickening fluids - Fluids With Pressure Dependent Viscosities

Except for the activation criterion, the above departures from Newtonian response are at the heart of what is usually referred to as non-Newtonian fluid mechanics. We now turn to a somewhat different departure from the classical Newtonian model. Notice that the models (A.1.3) and (A.1.4) are explicit expressions for the stress, in terms of kinematical variable $\mathbf{D}$, and the density ϱ in the case of (A.1.3). If the equation of state relating the "thermodynamic pressure" p and the density ϱ is invertible, then we could express λ and μ as functions of the pressure. Thus, in the case of a compressible Navier-Stokes fluid the viscosity μ clearly depends on the pressure. The question to ask is if, in fluids that are usually considered as incompressible liquids such as water under normal operating conditions, the viscosity could be a function of the pressure? The answer to this question is an unequivocal yes by virtue of the fact that when the range of pressures to which the fluid is subject to is sufficiently large, while the density may vary by a few percent, the viscosity could vary by several orders of magnitude, in fact as much a factor of 10^8! Thus, it is reasonable to suppose a liquid to be incompressible while at the same time the viscosity is pressure dependent.

In the case of an incompressible fluid whose viscosity depends on both the pressure (mean normal stress) and the symmetric part of the velocity gradient, i.e., when the stress is given by the representation

$$\mathbf{T} = -p\mathbf{I} + \mu(p, \mathbf{D})\mathbf{D}\,. \tag{A.1.9}$$

As $p = -\frac{1}{3}\mathrm{tr}\mathbf{T}$, it becomes obvious that we have an implicit relationship between $\mathbf{T}$ and $\mathbf{D}$, and the constitutive relation is of the form

$$\mathbf{f}(\mathbf{T}, \mathbf{D}) = \mathbf{0}\,, \tag{A.1.10}$$

i.e., we have an implicit constitutive equation.

It immediately follows from (A.1.10) that

$$\frac{\partial \mathbf{f}}{\partial \mathbf{T}}\dot{\mathbf{T}} + \frac{\partial \mathbf{f}}{\partial \mathbf{D}}\dot{\mathbf{D}} = \mathbf{0}\,, \tag{A.1.11}$$

which can be expressed as

$$[\mathbf{A}(\mathbf{T}, \mathbf{D})]\dot{\mathbf{T}} + [\mathbf{B}(\mathbf{T}, \mathbf{D})]\dot{\mathbf{D}} = \mathbf{0}\,. \tag{A.1.12}$$

The constitutive relation (A.1.12) is more general that (A.1.10) as an implicit equation of the form (A.1.12) need not be integrable to yield an equation of the form (A.1.10).

A further generalization within the context of implicit constitutive relations for compressible bodies is the equation

$$\mathbf{g}(\varrho, \mathbf{T}, \mathbf{D}) = \mathbf{0}\,. \tag{A.1.13}$$

Before we get into a more detailed discussion of implicit models for fluids let us consider a brief history of fluids with pressure dependent viscosity. Stokes [135] recognized that in general the viscosity could depend upon the pressure. It is clear from his discussion that he is considering liquids such as water. Having recognized the dependence of the viscosity on the pressure, he makes the simplifying assumption "If we suppose μ to be independent of the pressure also, and substitute ... ". Having made the assumption that the viscosity is independent of the pressure, he feels the need to substantiate that such is indeed the case for a restricted class of flows, those in pipes and channels, according to the experiments of DuBuat [26]: "Let us now consider in what cases it is allowable to suppose μ to be independent of the pressure. It has been concluded by DuBuat from his experiments on the motion of water in pipes and canals, that the total retardation of the velocity due to friction is not increased by increasing the pressure .... I shall therefore suppose that for water, and by analogy for other incompressible fluids, μ is independent of the pressure."

While the range of pressures attained in DuBuat's experiment might justify the assumption made by Stokes for a certain class of problems, one cannot in general make such an assumption. There are many technologically significant problems such as elastohydrodynamics (see Szeri [136]) wherein the fluid is subject to such a range of pressure that the viscosity changes by several orders of magnitude. There is a considerable amount of literature concerning the variation of viscosity with pressure and an exhaustive discussion of the literature before 1931 can be found in the authoritative treatise on the physics of high pressure by Bridgman [12].

Andrade [5] suggested the viscosity depends on the pressure, density and temperature in the following manner

$$\mu(p, \varrho, \theta) = A\varrho^{1/2} \exp\left(\frac{B}{\theta}(p + D\varrho^2)\right)\,, \tag{A.1.14}$$

where A, B and D are constants. In the processes where the temperature is uniformly constant, in the case of many liquids, it would be reasonable to assume that the liquid is incompressible and the viscosity varies exponentially with the pressure. This is precisely the assumption that is made in studies in elastohydrodynamics.

One can carry out a formal analysis based on standard representation theorems for isotropic functions (see Spencer [133]) that requires that the (A.1.10) satisfying for all orthogonal tensors $\mathbf{Q}$

$$\mathbf{g}(\varrho, \mathbf{QTQ}^T, \mathbf{QDQ}^T) = \mathbf{Qg}(\varrho, \mathbf{T}, \mathbf{D})\mathbf{Q}^T$$

take the implicit constitutive relation

$$\begin{aligned} &\alpha_0\mathbf{I} + \alpha_1\mathbf{T} + \alpha_2\mathbf{D} + \alpha_3\mathbf{T}^2 + \alpha_4\mathbf{D}^2 + \alpha_5(\mathbf{TD} + \mathbf{DT}) \\ &\quad + \alpha_6(\mathbf{T}^2\mathbf{D} + \mathbf{DT}^2) + \alpha_7(\mathbf{TD}^2 + \mathbf{D}^2\mathbf{T}) + \alpha_8(\mathbf{T}^2\mathbf{D}^2 + \mathbf{D}^2\mathbf{T}^2) = 0\,, \end{aligned} \tag{A.1.15}$$

where the material moduli α_i $i = 0, \ldots, 8$ depend on

$$\varrho, \mathrm{tr}\mathbf{T}, \mathrm{tr}\mathbf{D}, \mathrm{tr}\mathbf{T}^2, \mathrm{tr}\mathbf{D}^2, \mathrm{tr}\mathbf{T}^3, \mathrm{tr}\mathbf{D}^3, \mathrm{tr}(\mathbf{TD}), \mathrm{tr}(\mathbf{T}^2\mathbf{D}), \mathrm{tr}(\mathbf{D}^2\mathbf{T}), \mathrm{tr}(\mathbf{T}^2\mathbf{D}^2)\,.$$

The model

$$\mathbf{T} = -p(\varrho)\mathbf{I} + \beta(\varrho, \mathrm{tr}\mathbf{T}, \mathrm{tr}\mathbf{D}^2)\mathbf{D}$$

is a special subclass of models of the form (A.1.15). The counterpart in the case of an incompressible fluid would be

$$\mathbf{T} = -p\mathbf{I} + \mu(p, \mathrm{tr}\mathbf{D}^2)\mathbf{D}\,, \qquad \mathrm{tr}\mathbf{D} = 0. \tag{A.1.16}$$

We shall later provide a thermodynamic basis for the development of the model (A.1.16).

2. Balance equations

2.1. Kinematics

We shall keep our discussion of kinematics to a bare minimum. Let $\mathcal{B}$ denote the abstract body and let $\kappa : \mathcal{B} \to \mathcal{E}$, where $\mathcal{E}$ is a three dimensional Euclidean space, be a placer and $\kappa(\mathcal{B})$ the configuration of the body. We shall assume that the placer is one to one. By a motion we mean a one parameter family of placers (see Noll [100]). It follows that if $\kappa_R(\mathcal{B})$ is some reference configuration, and $\kappa_t(\mathcal{B})$ a configuration at time t, then we can identify the motion with a mapping $\chi_{\kappa_R} : \kappa_R(\mathcal{B}) \times \mathbb{R} \to \kappa_t(\mathcal{B})$ such that†

$$x = \chi_{\kappa_R}(X, t)\,. \tag{A.2.1}$$

We shall suppose that χ_{κ_R} is sufficiently smooth to render the operations defined on it meaningful. Since χ_{κ_R} is one to one, we can define its inverse so that

$$X = \chi^{-1}_{\kappa_R}(x, t)\,. \tag{A.2.2}$$

Thus, any (scalar) property φ associated with an abstract body $\mathcal{B}$ can be expressed as (analogously we proceed for vectors or tensors)

$$\varphi = \varphi(P, t) = \hat{\varphi}(X, t) = \tilde{\varphi}(x, t)\,. \tag{A.2.3}$$

We define the following Lagrangean and Eulerian temporal and spatial derivatives:

$$\dot{\varphi} := \frac{\partial \hat{\varphi}}{\partial t}\,, \qquad \varphi_{,t} := \frac{\partial \tilde{\varphi}}{\partial t}\,, \qquad \nabla_X \varphi = \frac{\partial \hat{\varphi}}{\partial X}\,, \qquad \nabla_x \varphi := \frac{\partial \tilde{\varphi}}{\partial x}\,. \tag{A.2.4}$$

The Lagrangean and Eulerian divergence operators will be expressed as *Div* and *div*, respectively.

The velocity $\mathbf{v}$ and the acceleration $\mathbf{a}$ are defined through

$$\mathbf{v} = \frac{\partial \chi_{\kappa_R}}{\partial t} \qquad \mathbf{a} = \frac{\partial^2 \chi_{\kappa_R}}{\partial t^2}\,, \tag{A.2.5}$$

and the deformation gradient $\mathbf{F}_{\kappa_R}$ is defined through

$$\mathbf{F}_{\kappa_R} = \frac{\partial \chi_{\kappa_R}}{\partial X}\,. \tag{A.2.6}$$

The velocity gradient $\mathbf{L}$ and its symmetric part $\mathbf{D}$ are defined through

$$\mathbf{L} = \nabla_x \mathbf{v}\,, \qquad \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T)\,. \tag{A.2.7}$$

It immediately follows that

$$\mathbf{L} = \dot{\mathbf{F}}_{\kappa_R} \mathbf{F}^{-1}_{\kappa_R}\,. \tag{A.2.8}$$

It also follows from the notations and definitions given above, in particular from (A.2.4) and (A.2.5) that

$$\dot{\varphi} = \varphi_{,t} + \nabla_x \varphi \cdot \mathbf{v}\,. \tag{A.2.9}$$

† It is customary to denote x and X which are points in an Euclidean space as bold face quantities. We however choose not to do so. On the other hand, all vectors, and higher order tensors are indicated by bold face.

2.2. Balance of Mass - Incompressibility - Inhomogenity

The balance of mass in its Lagrangean form states that

$$\int_{\mathcal{P}_R} \varrho_R(X) dX = \int_{\mathcal{P}_t} \varrho(x,t) dx \qquad \text{for all } \mathcal{P}_R \subset \kappa_R(\mathcal{B}) \text{ with } \mathcal{P}_t := \chi_{\kappa_R}(\mathcal{P}_R, t)\,, \tag{A.2.10}$$

which immediately leads to, using the Substitution theorem,

$$\varrho(x,t)\, \det \mathbf{F}_{\kappa_R}(X,t) = \varrho_R(X)\,. \tag{A.2.11}$$

A body is *incompressible* if

$$\int_{\mathcal{P}_R} dX = \int_{\mathcal{P}_t} dx \qquad \text{for all } \mathcal{P}_R \subset \kappa_R(\mathcal{B})$$

which leads to

$$\det \mathbf{F}_{\kappa_R}(X,t) = 1 \qquad \text{for all } X \in \kappa_R(\mathcal{B})\,. \tag{A.2.12}$$

If $\det \mathbf{F}_{\kappa_R}$ is continuously differentiable with respect to time, then by virtue of the identity

$$\frac{d}{dt} \det \mathbf{F}_{\kappa_R} = \operatorname{div} \mathbf{v} \det \mathbf{F}_{\kappa_R}\,,$$

we conclude, since $\det \mathbf{F}_{\kappa_R} \neq 0$ that

$$\operatorname{div} \mathbf{v}(x,t) = 0 \quad \text{ for all } t \in \mathbb{R} \text{ and } x \in \kappa_t(\mathcal{B})\,. \tag{A.2.13}$$

It is usually in the above form that the constraint of incompressibility is enforced in fluid mechanics.

From the Eulerian perspective, the balance of mass takes the form

$$\frac{d}{dt} \int_{\mathcal{P}_t} \varrho \, dx = 0 \qquad \text{for all } \mathcal{P}_t \subset \kappa_t(\mathcal{B})\,. \tag{A.2.14}$$

It immediately follows that

$$\varrho_{,t} + (\nabla_x \varrho) \cdot \mathbf{v} + \varrho \operatorname{div} \mathbf{v} = 0 \iff \varrho_{,t} + \operatorname{div}(\varrho \mathbf{v}) = 0\,. \tag{A.2.15}$$

If the fluid is incompressible, it immediately follows from (A.2.15) that

$$\varrho_{,t} + (\nabla_x \varrho) \cdot \mathbf{v} = 0 \iff \dot{\varrho} = 0 \iff \varrho(t,x) = \varrho(0,X) = \varrho_R(X)\,. \tag{A.2.16}$$

That is, for a fixed particle, the density is constant, as a function of time. However, the density of a particle may vary from one particle to another. The fact that the density varies at certain location in space, does not imply that the fluid is not incompressible. This variation is due to the fact that the fluid is inhomogeneous, a concept that has not been grasped clearly in fluid mechanics (see Anand and Rajagopal [4] for a discussion).

2.3. Balance of Linear Momentum

The balance of linear momentum that originates from the second law of Newton in classical mechanics applied to each subset $\mathcal{P}_t = \chi_{\kappa_R}(\mathcal{P}_R, t)$ of the current configuration takes the form

$$\frac{d}{dt}\int_{\mathcal{P}_t} \rho\mathbf{v}\, dx = \int_{\mathcal{P}_t} \rho\mathbf{b}\, dx + \int_{\partial\mathcal{P}_t} \mathbf{T}^T\mathbf{n}\, dS\,, \tag{A.2.17}$$

where $\mathbf{T}$ denotes the Cauchy stress that is related to the surface traction $\mathbf{t}$ through $\mathbf{t} = \mathbf{T}^T\mathbf{n}$, and $\mathbf{b}$ denotes the specific body force. It then leads to the balance of linear momentum in its local Eulerian form:

$$\varrho\dot{\mathbf{v}} = \operatorname{div}\mathbf{T}^T + \varrho\mathbf{b}\,. \tag{A.2.18}$$

Two comments are in order.

First, considering the case when $\kappa_t(\mathcal{B}) = \kappa_R(\mathcal{B})$ for all $t \geq 0$ and setting $\Omega := \kappa_R(\mathcal{B})$, it is not difficult to conclude at least for incompressible fluids, that (A.2.17) and (A.2.14) imply that

$$\frac{d}{dt}\int_{O} \rho\mathbf{v}\, dx + \int_{\partial O} \Big[(\rho\mathbf{v})(\mathbf{v}\cdot\mathbf{n}) - \mathbf{T}^T\mathbf{n}\Big]\, dS = \int_{O} \rho\mathbf{b}\, dx\,, \tag{A.2.19}$$

$$\frac{d}{dt}\int_{O} \rho\, dx + \int_{\partial O} \rho(\mathbf{v}\cdot\mathbf{n}) dS = 0, \tag{A.2.20}$$

valid for all (fixed) subsets O of Ω.

When compared to (A.2.17), this formulation is more suitable for further consideration in those problems where the velocity field $\mathbf{v}$ is taken as a primitive field defined on $\Omega \times \langle 0, \infty)$ (i.e. it is not defined through (A.2.5)).

To illustrate this convenience, we give a simple analogy from classical mechanics: consider a motion of a mass-spring system described by the second order ordinary differential equations for the displacement from the equilibrium and compare it with a free fall of the mass captured by the first order ordinary differential equations for the velocity. In fluid mechanics, the velocity field is typically taken as primitive variable.

Second, the derivation of (A.2.18) from (A.2.17) and similarly (A.2.15) from (A.2.14) requires certain smoothness of particular terms. In analysis, the classical formulations of the balance equations (A.2.18) and (A.2.15) are usually starting points for definition of various kinds of solutions. Following Oseen [102] (see also [34], [35]), we want to emphasize that the notion of a weak solution (or suitable weak solution) is very natural for equations of continuum mechanics, since their weak formulation can be directly obtained from the original formulations of the balance laws (A.2.14) and (A.2.17) or better (A.2.19) and (A.2.20). This comment is equally applicable to the other balance equation of continuum physics as well.

2.4. Balance of Angular Momentum

In the absence of internal couples, the balance of angular momentum implies that the Cauchy stress is symmetric, i.e.,

$$\mathbf{T} = \mathbf{T}^T . \tag{A.2.21}$$

2.5. Balance of Energy

The local form of the balance of energy is

$$\varrho\dot{\epsilon} = \mathbf{T} \cdot \nabla\mathbf{v} - \operatorname{div}\mathbf{q} + \varrho r , \tag{A.2.22}$$

where ϵ denotes the internal energy, $\mathbf{q}$ denotes the heat flux vector and r the specific radiant heating.

2.6. Further Thermodynamic Considerations (The Second Law). Reduced dissipation equation

To know how a body is constituted and to distinguish one body from another, we need to know how bodies store energy. How, and how much of, this energy that is stored in a body can be recovered from the body. How much of the working on a body is converted to energy in thermal form (heat). What is the nature of the latent energy that is associated with the changes in phase that the body undergoes. What is the nature of the latent energy (which is different in general from latent heat). By what different means does a body produce the entropy? These are but few of the pieces of information that one needs to describe the response of the body. Merely knowing this information is insufficient to describe how the body will respond to external stimuli. A body's response has to meet the basic balance laws of mass, linear and angular momentum, energy and the second law of thermodynamics.

Various forms for the second law of thermodynamics have been proposed and are associated with the names of Kelvin, Plank, Clausius, Duhem, Caratheodory and others. Essentially, the second law states that the rate of entropy production has to be non-negative†. A special form of the second law, the Claussius-Duhem inequality, has been used, within the context of a continua, to obtain restrictions on allowable constitutive relations (see Coleman and Noll [20]). This is enforced by allowing the body to undergo arbitrary processes in which the second law is required to hold. The problem with such an approach is that the constitutive structure that we ascribe to a body is only meant to hold for a certain class of processes. The body might behave quite differently outside this class of processes. For instance, while rubber may behave like an elastic material in the sense that the stored energy depends only on the deformation gradient and this energy can be completely recovered in processes that are reasonably slow in some sense, the same rubber if deformed at exceedingly high strain rates crystallizes and not only does the energy that is stored not depend purely on the deformation gradient, all the energy that was supplied to the body cannot be recovered. Thus, the models for rubber depend on the process class one has in mind and this would not allow one to subject the body to arbitrary process. We thus find it more reasonable to assume the constitutive structures for the rate of entropy production, based on physical grounds, that are automatically non-negative.

Let us first introduce the second law of thermodynamics in the form

$$\varrho\theta\dot{\eta} \geq -\operatorname{div}\mathbf{q} + \frac{\mathbf{q}\cdot(\nabla_x\theta)}{\theta} + \varrho r\,, \tag{A.2.23}$$

where η denotes the specific entropy.

On introducing the specific Helmholtz potential ψ through

$$\psi := \epsilon - \theta\eta\,,$$

and using the balance of energy (A.2.22), we can express (A.2.23) as

$$\mathbf{T}\cdot\mathbf{L} - \varrho\dot{\psi} - \varrho\dot{\theta}\eta - \frac{\mathbf{q}\cdot(\nabla_x\theta)}{\theta} \geq 0\,. \tag{A.2.24}$$

The above inequality is usually referred to as the dissipation inequality. This inequality is commonly used in continuum mechanics to obtain restrictions on the constitutive relations. A serious difficulty with regard to such an approach becomes immediately apparent. No restrictions whatsoever can be placed on the radiant heating. More importantly, the radiant heating is treated as a quantity that adjusts itself to meet the balance of energy. But this is clearly unacceptable as the

† There is a disagreement as to whether this inequality ought to be enforced locally at every point in the body, or only globally, even from the point of view of statistical thermodynamics.

radiant heating has to be a constitutive specification. How a body responds to radiant heating is critical, especially in view of the fact that all the energy that our world receives is in the form of electromagnetic radiation which is converted to energy in its thermal form (see Rajagopal and Tao [114] for a discussion of these issues). As we shall be primarily interested in the mechanical response of fluids, we shall ignore the radiant heating altogether, but we should bear in mind the above observation when we consider more general processes.

We shall define the specific rate of entropy production ξ through

$$\xi := \mathbf{T} \cdot \mathbf{L} - \varrho\dot{\psi} - \varrho\dot{\theta}\eta - \frac{\mathbf{q} \cdot (\nabla_x \theta)}{\theta} \,. \tag{A.2.25}$$

We shall make constitutive assumptions for the rate of entropy production ξ and require that (A.2.25) hold in all admissible processes (see Green and Nagdhi [48]). Thus, the equation (A.2.25) will be used as a constraint that is to be met in all admissible processes. We shall choose ξ so that it is non-negative and thus the second law is automatically met.

We now come to a crucial step in our thermodynamic considerations. From amongst a class of admissible non-negative rate of entropy productions, we choose that which is maximal. This is asking a great deal more than the second law of thermodynamics. The rationale for the same is the following. Let us consider an isolated system. For such a system, it is well accepted that its entropy becomes a maximum and the system would reach equilibrium. The assumption that the rate of entropy production is a maximum ensures that the body attains its equilibrium as quickly as possible. Thus, this assumption can be viewed as an assumption of economy or an assumption of laziness, the system tries to get to the equilibrium state as quickly as possible, i.e., in the most economic manner. It is important to recognize that this is merely an assumption and not some deep principle of physics. The efficacy of the assumption has to be borne out by its predictions and to date the assumption has led to meaningful results in a wide variety of material behavior (see results pertinent to viscoelasticity [112], [113], classical plasticity ([109], [110]), twinning ([107], [108]), solid to solid phase transition [111]), crystallization in polymers ([117], [118]), single crystal supper alloys [104], etc.).

2.7. Isothermal flows at uniform temperature

Here, we shall restrict ourselves to flows that take place at constant temperature for the whole period of interest at all points of the body. Consequently, the equations governing such flows for a compressible fluid are

$$\dot{\varrho} = -\varrho \operatorname{div} \mathbf{v} \qquad \varrho\dot{\mathbf{v}} = \operatorname{div} \mathbf{T} + \varrho \mathbf{b}\,, \tag{A.2.26}$$

while for an incompressible fluid they take the form

$$\operatorname{div} \mathbf{v} = 0\,, \qquad \dot{\varrho} = 0\,, \qquad \varrho\dot{\mathbf{v}} = \operatorname{div} \mathbf{T} + \varrho \mathbf{b}\,. \tag{A.2.27}$$

Note that (A.2.24) and (A.2.25) reduce to

$$\mathbf{T} \cdot \mathbf{D} - \varrho\dot{\psi} = \xi \quad \text{ and } \quad \xi \geq 0\,. \tag{A.2.28}$$

In order to obtain a feel for the structure of the constitutive quantities appearing in (A.2.28), we consider first the Cauchy stress for the incompressible and compressible Euler fluid, and then for the incompressible and compressible Navier-Stokes fluid. Note that Euler fluids are ideal fluids in that there is no dissipation in any process undergone by the fluid, i.e., $\xi \equiv 0$ in all processes.

Compressible Euler fluid. Since $\xi \equiv 0$ and (A.1.1) implies

$$\mathbf{T} \cdot \mathbf{L} = -p(\varrho)\mathbf{I} \cdot \mathbf{L} = -p(\varrho)\mathrm{tr}\mathbf{L} = -p(\varrho)\mathrm{tr}\mathbf{D} = -p(\varrho) \operatorname{div} \mathbf{v}\,,$$

the reduced thermo-mechanical equation (A.2.28) simplifies to

$$\varrho\dot{\psi} = -p(\varrho) \operatorname{div} \mathbf{v}\,. \tag{A.2.29}$$

This suggests that it might be appropriate to consider ψ of the form

$$\psi = \Psi(\varrho)\,. \tag{A.2.30}$$

In fact, since an ideal fluid is an elastic fluid, it follows that its specific Helmoltz free energy ψ depends only on the deformation gradient $\mathbf{F}$. If we suppose that the symmetry group of a fluid is the unimodular group, then the balance of mass could lead one to conclude that ψ depends on the density ϱ.

Using $(\text{A.2.26})_1$, we then have from (A.2.30)

$$\dot{\psi} = \Psi_{,\varrho}(\varrho)\dot{\varrho} = -\varrho\Psi_{,\varrho}(\varrho) \operatorname{div} \mathbf{v}\,, \tag{A.2.31}$$

and we conclude from (A.2.29) and (A.2.31) that

$$p(\varrho) = \varrho^2 \Psi_{,\varrho}(\varrho) \,. \tag{A.2.32}$$

Incompressible Euler fluid. Since we are dealing with a homogeneous fluid we have $\varrho \equiv \varrho^*$, where ϱ^* is a positive constant. We also have

$$\dot{\Psi}(\varrho^*) = 0 \,, \quad \mathbf{T} \cdot \mathbf{L} = -p\mathbf{I} \cdot \mathbf{L} = -p(\varrho) \operatorname{div} \mathbf{v} = 0 \,, \quad \text{and } \xi \equiv 0 \,.$$

Thus, each term in (A.2.28) vanishes and (A.2.28) clearly holds.

Compressible Navier-Stokes fluid. Consider $\mathbf{T}$ of the form (A.1.3) and ψ of the form (A.2.30) fulfilling (A.2.32). Denoting $\mathbf{C}^\delta$ the deviatoric (traceless) part of any tensor $\mathbf{C}$, i.e., $\mathbf{C}^\delta = \mathbf{C} - \frac{1}{3}(\mathrm{tr}\mathbf{C})\mathbf{I}$, then we have

$$\begin{aligned}
\xi &= \mathbf{T} \cdot \mathbf{L} - \varrho \dot{\psi} \\
&= -p(\varrho) \operatorname{div} \mathbf{v} + 2\mu(\varrho)\mathbf{D} \cdot \mathbf{D} + \lambda(\varrho)(\mathrm{tr}\mathbf{D})^2 + \varrho^2 \Psi_{,\varrho}(\varrho) \\
&= 2\mu(\varrho)\mathbf{D} \cdot \mathbf{D} + \lambda(\varrho)(\mathrm{tr}\mathbf{D})^2 \\
&= 2\mu(\varrho)\mathbf{D}^\delta \cdot \mathbf{D}^\delta + \left(\lambda(\varrho) + \frac{2}{3}\mu(\varrho)\right)(\mathrm{tr}\mathbf{D})^2 \,.
\end{aligned}$$

Note that the second law of thermodynamics is met if $\mu(\varrho) \geq 0$ and $\lambda(\varrho) + \dfrac{2}{3}\mu(\varrho) \geq 0$.

Incompressible Navier-Stokes fluid. Similar considerations as those for the case of a compressible Navier-Stokes fluid imply

$$\xi = 2\mu\mathbf{D} \cdot \mathbf{D} = 2\mu|\mathbf{D}|^2 \,.$$

Note that for both the incompressible Euler and Navier-Stokes fluid we have

$$p = -\frac{1}{3}\mathrm{tr}\mathbf{T} \,.$$

2.8. Natural Configurations

Most bodies can exist stress free in more than one configuration and such configurations are referred to as "natural configurations" (see Eckart [28], Rajagopal [116]).Given a current configuration of a homogeneously deformed body, the stress-free configuration that the body takes on upon the removal of all external stimuli is the underlying "natural configuration" corresponding to the current configuration of the body. As a body undergoes a thermodynamic process, in general, the underlying natural configuration evolves. The evolution of this underlying natural configuration is determined by the maximization of the entropy production (see how this methodology is used in viscoelasticity ([112], [113]), classical plasticity ([109], [110]), twinning ([107], [108]), solid to solid phase transition [111]), crystallization in polymers ([117], [118]), single crystal super alloys [104]). In the case of the both incompressible and compressible Navier-Stokes fluids and the generalizations discussed here, the current configuration $\kappa_t(\mathcal{B})$ itself serves as the natural configuration.

3. The Constitutive Models For Compressible and Incompressible Navier-Stokes Fluids and Some of their Generalizations

3.1. Standard approach in continuum physics

The starting point for the development of the model for a homogeneous compressible Navier-Stokes fluid is the assumption that the Cauchy stress depends on the density and the velocity gradient, i.e.,

$$\mathbf{T} = \mathbf{f}(\varrho, \mathbf{L}) \,. \tag{A.3.1}$$

It follows from the assumption of frame-indifference that the stress can depend on the velocity gradient only through its symmetric part, i.e.,

$$\mathbf{T} = \mathbf{f}(\varrho, \mathbf{D}) \,. \tag{A.3.2}$$

The requirement the fluid be isotropic then implies that

$$\mathbf{f}(\varrho, \mathbf{D}) = \alpha_1 \mathbf{I} + \alpha_2 \mathbf{D} + \alpha_3 \mathbf{D}^2 \,, \tag{A.3.3}$$

where $\alpha_i = \alpha_i(\varrho, I_{\mathbf{D}}, II_{\mathbf{D}}, III_{\mathbf{D}})$, and

$$I_{\mathbf{D}} = \mathrm{tr}\mathbf{D}, \qquad II_{\mathbf{D}} = \frac{1}{2}[(\mathrm{tr}\mathbf{D})^2 - \mathrm{tr}\mathbf{D}^2] \,, \qquad III_{\mathbf{D}} = \det \mathbf{D} \,.$$

If we require that the stress be linear in $\mathbf{D}$, then we immediately obtain

$$\mathbf{T} = -p(\varrho)\mathbf{I} + \lambda(\varrho)(\mathrm{tr}\mathbf{D})\mathbf{I} + 2\mu(\varrho)\mathbf{D} \,, \tag{A.3.4}$$

which is the classical homogeneous compressible Navier-Stokes fluid.

Starting with the assumption that the fluid is incompressible and homogeneous, and

$$\mathbf{T} = \mathbf{g}(\mathbf{L}) \tag{A.3.5}$$

a similar procedure leads to (see Truesdell and Noll [143])

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}\,. \tag{A.3.6}$$

The standard procedure for dealing with constraints such as incompressibility, namely that the constraint reactions do no work (see Truesdell [142]) is fraught with several tacit assumptions (we shall not discuss them here) that restrict the class of models possible. For instance it will not allow the material modulus μ to depend on the Lagrange multiplier p. The alternate approach presented below attempts to avoid such drawbacks. Another general alternative procedure established in purely mechanical context has been recently developed in Rajagopal and Srinivasa [106].

3.2. Alternate approach

We provide below an alternate approach for deriving the constitutive relation for a homogeneous compressible and an incompressible Navier-Stokes fluid. Instead of assuming a constitutive equation for the stress as the starting point, we shall start assuming forms for the Helmholtz potential and the rate of dissipation, namely two scalars.

We first focus on the derivation of the constitutive equation for the Cauchy stress for the compressible Navier-Stokes fluid supposing that

$$\psi(x,t) = \Psi(\varrho(x,t))\,. \tag{A.3.7}$$

and

$$\begin{aligned}\xi = \Xi(\mathbf{D}) &= 2\,\mu(\varrho)\mathbf{D}\cdot\mathbf{D} + \lambda(\varrho)(\mathrm{tr}\mathbf{D})^2 \\ &= 2\,\mu(\varrho)|\mathbf{D}^\delta|^2 + (\lambda(\varrho) + \frac{2}{3}\mu(\varrho))(\mathrm{tr}\mathbf{D})^2\,, \\ &\text{where } \mu(\varrho) \geq 0\,, \quad \lambda(\varrho) + \frac{2}{3}\mu(\varrho) \geq 0\,.\end{aligned} \tag{A.3.8}$$

With such a choice of ξ the second law is automatically met, and (A.2.28) takes the form (cf. (A.2.31))

$$\xi = (\mathbf{T} + \varrho^2\Psi_{,\varrho}(\varrho)\mathbf{I})\cdot\mathbf{D}\,. \tag{A.3.9}$$

For a fixed $\mathbf{T}$ there are plenty of $\mathbf{D}$'s that satisfy (A.3.8) and (A.3.9). We pick a $\mathbf{D}$ such that $\mathbf{D}$ maximizes (A.3.8) and fulfils (A.3.9). This leads to a constrained maximization that gives the following necessary condition

$$\frac{\partial\Xi}{\partial\mathbf{D}} - \lambda_1(\mathbf{T} + \varrho^2\Psi_{,\varrho}(\varrho)\mathbf{I} - \frac{\partial\Xi}{\partial\mathbf{D}}) = 0\,,$$

or equivalently

$$\frac{1+\lambda_1}{\lambda_1}\frac{\partial\Xi}{\partial\mathbf{D}} = (\mathbf{T} + \varrho^2\Psi_{,\varrho}(\varrho)\mathbf{I})\,. \tag{A.3.10}$$

To eliminate the constraint we take scalar product of (A.3.10) with $\mathbf{D}$. Using (A.3.9), (A.3.10) and the fact that

$$\frac{\partial\Xi}{\partial\mathbf{D}} = 2(2\mu(\varrho)\mathbf{D} + \lambda(\varrho)(\mathrm{tr}\mathbf{D})\mathbf{I})\,, \tag{A.3.11}$$

we find that

$$\frac{1+\lambda_1}{\lambda_1} = \frac{\Xi}{\frac{\partial\Xi}{\partial\mathbf{D}}\cdot\mathbf{D}} = \frac{1}{2}\,. \tag{A.3.12}$$

Inserting (A.3.11) and (A.3.12) into (A.3.10) we obtain

$$\mathbf{T} = -\varrho^2\Psi_{,\varrho}(\varrho)\mathbf{I} + 2\mu(\varrho)\mathbf{D} + \lambda(\varrho)(\mathrm{tr}\mathbf{D})\mathbf{I}\,. \tag{A.3.13}$$

Finally, setting $p(\varrho) = \varrho^2\Psi_{,\varrho}(\varrho)$ we obtain the Cauchy stress for compressible Navier-Stokes fluid, cf. (A.1.3).

Next, we provide a derivation for an hierarchy of incompressible fluid models that generalize the incompressible Navier-Stokes fluid in the following sense: the viscosity can not only be a constant, but it can be a function that may depend on the density, the symmetric part of the velocity gradient $\mathbf{D}$ specifically through $\mathbf{D}\cdot\mathbf{D}$, or the mean normal stress, i.e. the pressure $p := -\frac{1}{3}\mathrm{tr}\mathbf{T}$, or it can depend on any or all of them. We shall consider the most general case within this setting by assuming that

$$\xi = \Xi(p,\varrho,\mathbf{D}) = 2\nu(p,\varrho,\mathbf{D}\cdot\mathbf{D})\,\mathbf{D}\cdot\mathbf{D}\,. \tag{A.3.14}$$

Clearly, if $\nu \geq 0$ then automatically $\xi \geq 0$, ensuring that the second law is complied with.

We assume that the specific Helmoltz potential ψ is of the form (A.3.7). By virtue of the fact that the fluid is incompressible, i.e.,

$$\mathrm{tr}\mathbf{D} = 0\,, \tag{A.3.15}$$

we obtain $\dot\varrho = 0$, $\dot\psi$ vanishes in (A.2.28) and we have from (A.2.28)

$$\mathbf{T}\cdot\mathbf{D} = \Xi\,. \tag{A.3.16}$$

Following the same procedure as that presented above, in case of a compressible fluid, we maximize Ξ with respect to $\mathbf{D}$ that is subject to the constraints (A.3.15) and (A.3.16). As the necessary condition for the extremum we obtain the equation

$$(1+\lambda_1)\Xi_{,\mathbf{D}} - \lambda_1\mathbf{T} - \lambda_0\mathbf{I} = 0\,, \tag{A.3.17}$$

where λ_0 and λ_1 are the Lagrange multipliers due to the constraints (A.3.15) and (A.3.16). We eliminate them as follows. Taking the scalar product of (A.3.17) with $\mathbf{D}$, and using (A.3.15) and (A.3.16) we obtain

$$\frac{1+\lambda_1}{\lambda_1} = \frac{\Xi}{\Xi_{,\mathbf{D}} \cdot \mathbf{D}} \,. \tag{A.3.18}$$

Note that

$$\Xi_{,\mathbf{D}} = 4\Big(\nu(p,\varrho,\mathbf{D}\cdot\mathbf{D}) + \nu_{,\mathbf{D}}(p,\varrho,\mathbf{D}\cdot\mathbf{D})\mathbf{D}\cdot\mathbf{D}\Big)\mathbf{D} \,. \tag{A.3.19}$$

Consequently, $\mathrm{tr}\Xi_{,\mathbf{D}} = 0$ by virtue of (A.3.15). Thus, taking the trace of (A.3.17) we have

$$-\frac{\lambda_0}{\lambda_1} = -p \quad \text{ with } p = -\frac{1}{3}\mathrm{tr}\mathbf{T} \,. \tag{A.3.20}$$

Using (A.3.17)–(A.3.20), we finally find that (A.3.17) takes the form

$$\mathbf{T} = -p\mathbf{I} + 2\,\nu(p,\varrho,\mathbf{D}\cdot\mathbf{D})\mathbf{D} \,. \tag{A.3.21}$$

Mathematical issues related to the system (A.2.27) with the constitutive equation (A.3.21) will be discussed in the second part of this treatise. The fluid given by (A.3.21) has the ability to shear thin, shear thicken and pressure thicken. After adding the yield stress or activation criterion, the model could capture phenomena connected with the development of discontinuous stresses. On the other hand, the model (A.2.27) together with (A.3.21) cannot stress relax or creep in a non-linear way, neither can it exhibit nonzero normal stress differences in a simple shear flow.

4. Boundary Conditions

No aspect of mathematical modeling has been neglected as that of determining appropriate boundary conditions. Mathematicians seem especially oblivious to the fact that boundary conditions are constitutive specifications. In fact, boundary conditions require an understanding of the nature of the bodies that are divided by the boundary. Boundaries are rarely sharp, with the constituents that abut either side of the boundary invariably exchanging molecules. In the case of the boundary between two liquids or a gas and a liquid this molecular exchange is quite obvious, it is not so in the case of a reasonably impervious solid boundary and a liquid. The ever popular "no-slip" (adherence) boundary condition is supposed to have had the imprimatur of Stokes behind it, but Stokes' opinions concerning the status of the "no-slip" condition are nowhere close to unequivocal as many investigators lead one to believe. A variety of suggestions were put forward by the pioneers of the field, Bernoulli, DuBuat, Navier, Poisson, Girard, Stokes and others, as to the condition that ought to be applied on the boundary between an impervious solid and a liquid. One fact that was obvious to all of them was that boundary conditions ought to be derived, just as constitutive relations are developed for the material in the bulk, even more so. This is made evident by Stokes [135] who makes this obvious with his remarks: "Besides the equations which must hold good at any point in the interior of the mass, it will be necessary to form also the equations which must be satisfied at the boundary." After emphasizing the need to derive the equations that ought to be applied at a boundary, Stokes [135] goes on to derive a variety of such boundary conditions.

That Stokes [135] was in two minds about the appropriateness of the "no-slip" boundary condition is evident from his following remarks: "DuBuat found by experiment that when the mean velocity of water flowing through a pipe is less than one inch in a second, the water near the inner surface of the pipe is at rest. If these experiments may be trusted, the conditions to be satisfied in the case of small velocities are those which first occurred to me ....", but he goes on to add: "I have said that when the velocity is not small the tangential force called into action by the sliding of water over the inner surface of the pipe varies nearly the square of the velocity ....". The key words that demand our attention are "the sliding of water over the inner surface". Sliding implies that Stokes believed that the fluid is slipping at the boundary. That he was far from convinced concerning the applicability of the "no-slip" condition is made crystal clear when he remarks: "The most interesting questions concerning the subject require for their solution a knowledge of the conditions which must be satisfied at the surface of solid in contact with the fluid, which, except in the case of very small motions, are unknown.". To Stokes the determination of appropriate boundary conditions was an open problem.

An excellent concise history concerning boundary conditions for fluids can be found in Goldstein [47]. We discuss briefly some of the boundary conditions that have been proposed for a fluid flowing past a solid impervious boundary.

Navier [92] derived a slip condition which can be duly generalized to the condition

$$\mathbf{v} \cdot \boldsymbol{\tau} = -K(\mathbf{T}\mathbf{n} \cdot \boldsymbol{\tau}), \qquad K \geq 0, \tag{A.4.1}$$

where $\mathbf{n}$ is the unit outward normal vector and $\boldsymbol{\tau}$ stands for a tangent vector at the boundary point; K is usually assumed to be a constant but it could however be assumed to be a function of the normal stresses and the shear rate, i.e.,

$$K = K(\mathbf{Tn} \cdot \mathbf{n}, |\mathbf{D}|^2)\,. \tag{A.4.2}$$

The above boundary conditions, when $K > 0$, is referred to as the slip boundary condition. If $K = 0$, we obtain the classical "no-slip" boundary condition.

Another boundary condition that is sometimes used, especially when dealing with non-Newtonian fluids, is the "threshold-slip" condition. This takes the form

$$\begin{aligned} &|\mathbf{Tn} \cdot \boldsymbol{\tau}| \le \alpha |\mathbf{Tn} \cdot \mathbf{n}| \implies \mathbf{v} \cdot \boldsymbol{\tau} = 0\,, \\ &|\mathbf{Tn} \cdot \boldsymbol{\tau}| > \alpha |\mathbf{Tn} \cdot \mathbf{n}| \implies \mathbf{v} \cdot \boldsymbol{\tau} \neq 0 \quad \text{and} \ -\gamma \frac{\mathbf{v} \cdot \mathbf{n}}{|\mathbf{v} \cdot \mathbf{n}|} = \mathbf{Tn} \cdot \boldsymbol{\tau}\,, \end{aligned} \tag{A.4.3}$$

where $\gamma = \gamma(\mathbf{Tn} \cdot \mathbf{n}, \mathbf{v} \cdot \boldsymbol{\tau})$.

The above condition implies that fluid will not slip until the ratio of the magnitude of shear stress and the magnitude of the normal stress exceeds a certain value. When it does exceed that value, it will slip and the slip velocity will depend on both the shear and normal stresses. It is also possible to require that γ depends on $|\mathbf{D}|^2$.

A much simpler condition that is commonly used is

$$\mathbf{v} \cdot \boldsymbol{\tau} = \begin{cases} v_{0\boldsymbol{\tau}} & \text{if } |\mathbf{Tn} \cdot \boldsymbol{\tau}| > \beta\,, \\ 0 & \text{if } |\mathbf{Tn} \cdot \boldsymbol{\tau}| \le \beta\,. \end{cases} \tag{A.4.4}$$

Thus the fluid will slip if the shear stress exceeds a certain value. Here, $v_{0\boldsymbol{\tau}}$ are given scalar functions for each $\boldsymbol{\tau}$ generating the tangent space at the boundary.

If the boundary is permeable, then in addition to the possibility of $\mathbf{v} \cdot \boldsymbol{\tau}$ not being equal to zero, we have to specify the normal component of the velocity $\mathbf{v} \cdot \mathbf{n}$. Several flows have been proposed for flows past porous media, however we shall not discuss them here.

In order to understand characteristic features of particular terms appearing in the system of PDEs, as well as their natural dependence, it is convenient to eliminate completely the presence of the boundary and boundary conditions on the flow, i.e., on the solution.

This can be realized in two way:

1) Assume that the fluid occupies the whole three-dimensional space with the velocity vanishing at $|x| \to +\infty$. Then starting with an initial-condition

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0 \in \mathbb{R}^3 \tag{A.4.5}$$

we are interested in knowing the properties of the velocity and the pressure of governing equations at any instant of the time $t > 0$ and any position $x \in \mathbb{R}^3$.
2) Assume that for $T, L \in (0,\infty)$

$$\begin{aligned} v_i, p :& [0,T] \times \mathbb{R}^3 \to \mathbb{R} \text{ are } L\text{-periodic at each direction } x_i, \\ \text{with } &\int_\Omega v_i \, dx = 0, \qquad \int_\Omega p \, dx = 0. \qquad i = 1,2,3 \end{aligned} \tag{A.4.6}$$

Here $\Omega = (0,L) \times (0,L) \times (0,L)$ is a periodic cell.

The advantage of the second case consists in the fact that we work on domain with a compact closure.

Chapter B

Mathematical Analysis of Flows of Fluids With Shear, Pressure and Density Dependent Viscosity

1. Introduction

1.1. A taxonomy of models

The objective of this chapter is to provide a survey of results regarding the mathematical analysis of the system of partial differential equations for the (unknown) density ρ, the velocity $\mathbf{v} = (v_1, v_2, v_3)$ and the pressure (mean normal stress) p, partial differential equations being

$$\begin{aligned} \rho_{,t} + \nabla\rho\cdot\mathbf{v} = 0, \qquad \operatorname{div}\mathbf{v} = 0, \\ \rho(\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v})) = -\nabla p + \operatorname{div}(2\nu(p,\rho,|\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}) + \rho\mathbf{b}, \end{aligned} \tag{B.1.1}$$

focusing however mostly on some of its simplifications specified below. The system (B.1.1) is exactly the system (A.2.27) with the constitutive equations (A.3.21) whose interpretation from the perspective of non-Newtonian fluid mechanics and the connection to compressible fluid models were discussed in Chapter A. Unlike (A.2.27) and (A.3.21) we use a different notation in order to express the equations in the form (B.1.1). First of all, owing to the incompressibility constraint, we have

$$\dot{\mathbf{v}} = \mathbf{v}_{,t} + [\nabla_x\mathbf{v}]\mathbf{v} = \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}),$$

where the tensor product $\mathbf{a}\otimes\mathbf{b}$ is the second order tensor with components

$$(\mathbf{a}\otimes\mathbf{b})_{ij} = a_i b_j \qquad \text{for any} \qquad \mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3).$$

Next note that in virtue of to $(\text{B.1.1})_2$, we can rewrite $(\text{B.1.1})_1$ as $\rho_{,t} + \operatorname{div}(\rho\mathbf{v}) = 0$. We also explicitly use the notation $\mathbf{D}(\mathbf{v})$ instead of $\mathbf{D}$ in order to clearly identify our interest concerning the velocity field. As discussed in Chapter A, the model (B.1.1) includes a lot of special important cases particularly for homogeneous fluids†. Note that for the case of a homogeneous fluid,(B.1.1) reduces to

$$\operatorname{div}\mathbf{v}=0,\qquad \mathbf{v}_{,t}+\operatorname{div}(\mathbf{v}\otimes\mathbf{v})=-\nabla p+\operatorname{div}(2\nu(p,|\mathbf{D}(\mathbf{v})|^2))\mathbf{D}(\mathbf{v}))+\mathbf{b},\quad (B.1.2)$$

obtained by multiplying $(B.1.1)_2$ by $\frac{1}{\rho_0}$, and relabelling the dynamic pressure $\frac{p}{\rho_0}$ and the dynamic viscosity $\frac{\nu(p,\rho_0,|\mathbf{D}(\mathbf{v})|^2)}{\rho_0}$ again as p and $\nu(p,|\mathbf{D}(\mathbf{v})|^2)$, respectively. For later reference, we give a list of several special models contained as a special subclasses of (B.1.2):

a) Fluids with pressure dependent viscosity where ν is independent of the shear rate, but depends on the pressure p:

$$\operatorname{div}\mathbf{v}=0,\qquad \mathbf{v}_{,t}+\operatorname{div}(\mathbf{v}\otimes\mathbf{v})-\operatorname{div}(\nu(p)[\nabla\mathbf{v}+(\nabla\mathbf{v})^T])=-\nabla p+\mathbf{b};\quad (B.1.3)$$

b) Fluids with shear dependent viscosity with the viscosity independent of the pressure:

$$\operatorname{div}\mathbf{v}=0,\qquad \mathbf{v}_{,t}+\operatorname{div}(\mathbf{v}\otimes\mathbf{v})-\operatorname{div}\mathbf{S}(\mathbf{D}(\mathbf{v}))=-\nabla p+\mathbf{b};\quad (B.1.4)$$

here we introduce the notation

$$\mathbf{S}(\mathbf{D}(\mathbf{v}))=2\nu(|\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}).\quad (B.1.5)$$

This class of fluids includes:

c) Ladyzhenskaya's fluids‡ with $\nu(|\mathbf{D}(\mathbf{v})|^2)=\nu_0+\nu_1|\mathbf{D}(\mathbf{v})|^{r-2}$, where $r>2$ is fixed, ν_0 and ν_1 are positive numbers:

$$\operatorname{div}\mathbf{v}=0,\qquad \mathbf{v}_{,t}+\operatorname{div}(\mathbf{v}\otimes\mathbf{v})-\nu_0\triangle\mathbf{v}-2\nu_1\operatorname{div}(|\mathbf{D}(\mathbf{v})|^{r-2}\mathbf{D}(\mathbf{v}))=-\nabla p+\mathbf{b}\quad (B.1.6)$$

d) Power-law fluids with $\nu(|\mathbf{D}(\mathbf{v})|^2)=\nu_1|\mathbf{D}(\mathbf{v})|^{r-2}$ where $r\in(1,\infty)$ is fixed and ν_1 is a positive number:

$$\operatorname{div}\mathbf{v}=0,\qquad \mathbf{v}_{,t}+\operatorname{div}(\mathbf{v}\otimes\mathbf{v})-2\nu_1\operatorname{div}(|\mathbf{D}(\mathbf{v})|^{r-2}\mathbf{D}(\mathbf{v}))=-\nabla p+\mathbf{b}\quad (B.1.7)$$

† Recall that in our setting a fluid is homogeneous if for some positive number ρ_0 the density fulfils $\rho(x,t)=\rho_0$ for all time instants $t\ge 0$ and all $x\in\kappa_t(\mathcal{B})$.

‡ For $r=3$ this system of PDEs is frequently called Smagorinski's model of turbulence, see [131]. Then ν_0 is molecular viscosity and ν_1 is the turbulent viscosity.

e) Navier-Stokes fluids with $\nu(p, |\mathbf{D}(\mathbf{v})|^2) = \nu_0$ (ν_0 being a positive number):

$$\operatorname{div} \mathbf{v} = 0, \qquad \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \nu_0 \triangle \mathbf{v} = -\nabla p + \mathbf{b}. \tag{B.1.8}$$

The equations of motions (B.1.8) for a Navier-Stokes fluid are referred to as the Navier-Stokes equations (NSEs), the equations (B.1.6) for a Ladyzhenskaya's fluid will be referred to as the Ladyzhenskaya's equations. A fluid captured by Ladyzhenskaya's equations reduces to NSEs (B.1.8) by taking $\nu_1 = 0$ in (B.1.6) and to power-law fluids by setting $\nu_0 = 0$. Note also that setting $r = 2$ in the Ladyzhenskaya's equations we again obtain NSEs with the constant viscosity $2(\nu_0 + \nu_1)$.

1.2. Mathematical self-consistency of the models

Irrespective of how accurately a model of our choice approximates the real behavior of a fluid, mathematical analysis is interested in questions concerning its *mathematical self-consistency*‡.

We say that a model is mathematically self-consistent if it exhibits at least the following properties:

(I) *large-time and large-data existence:* Completing the model by a reasonable set of boundary conditions and considering a smooth, but arbitrary initial data, the model should admit a solution for all positive time instants.

(II) *large-time and large-data uniqueness:* The motion is fully determined by its initial, boundary and other data and depends on them continuously; particularly, such a motion is unique for a given set of data.

(III) *large-time and large-data regularity:* The physical quantities, such as the velocity in the case of fluids, are bounded.

These three requirements thus form a minimal set of mathematical properties that one would like any *evolutionary* model of (classical) mechanics to exhibit, particularly any of the models (B.1.3) up to (B.1.8).

A discussion of the current state of results with regard the properties **(I)**, **(II)** and **(III)** for the above models forms the backbone of the remaining part of this article. Towards purpose, we eliminate the influence of the boundary by considering spatially periodic problem only, cf.(A.4.6). On the other hand, we do not apply tools that are just suitable for periodic functions (such as Fourier series) but rather use

‡ See a video record of Caffarelli's presentation of the 3^{rd} millenium problem "Navier-Stokes and smoothness"[15].

tools and approaches that can be used under more general conditions for other boundary-value problem, as well.

1.3. Weak solution: a natural notion of solution for PDEs of the continuum physics

The tasks **(I)-(III)** require to know what is meant by solution. We obtain a hint from the balance of linear momentum for each (measurable) subset of the body (A.2.17), as recognized already by Oseen [102]. Note that (A.2.19) requires some integrability of the first derivatives of the velocity and the integrability of the pressure, while the classical formulation† (B.1.8) is based on the knowledge of the second derivatives of $\mathbf{v}$ and the gradient of p. Oseen [102] not only observed this discrepancy between (A.2.19) and (B.1.8), but he also proposed and derived a notion of *weak solution* directly from the original formulation‡ of the balance of linear momentum (A.2.19).

To be more specific, following the procedure outlined by Oseen [102] (for other approaches see also [34] p. 55, and [35]) it is possible to conclude directly from (A.2.19) and (A.2.20) that ρ, $\mathbf{v}$ and $\mathbf{T}$ fulfil for all $t > 0$

$$\begin{aligned} -\int_0^t \int_\Omega (\rho\mathbf{v})(\tau, x) \cdot \boldsymbol{\varphi}_{,\tau}(\tau, x)\, dx\, d\tau \ &+ \int_\Omega (\rho\mathbf{v})(t, x) \cdot \boldsymbol{\varphi}(t, x)\, dx \\ - \int_\Omega (\rho\mathbf{v})(0, x) \cdot \boldsymbol{\varphi}(0, x)\, dx &- \int_0^t \int_\Omega (\rho\mathbf{v} \otimes \mathbf{v}) \cdot \nabla\boldsymbol{\varphi}\, dx\, d\tau \\ + \int_0^t \int_\Omega \mathbf{T} \cdot \nabla\boldsymbol{\varphi}\, dx\, d\tau &= \int_0^t \int_\Omega \rho\mathbf{b} \cdot \boldsymbol{\varphi}\, dx\, d\tau \end{aligned} \tag{B.1.9}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}\left(-\infty, +\infty; \left(\mathcal{C}^\infty_{per}\right)^3\right)$ and

$$\begin{aligned} -\int_0^t \int_\Omega \rho(\tau, x)\xi_{,t}(\tau, x)\, dx\, d\tau \ &+ \int_\Omega \rho(t, x)\xi(t, x)\, dx \\ - \int_\Omega \rho(0, x)\xi(0, x)\, dx &- \int_0^t \int_\Omega \rho\mathbf{v} \cdot \nabla\xi\, dx\, d\tau = 0 \end{aligned} \tag{B.1.10}$$

for all $\xi \in \mathcal{D}\left(-\infty, +\infty; \mathcal{C}^\infty_{per}\right)$.

Identities (B.1.9) and (B.1.10) are exactly weak forms of the equations (B.1.1). Neither Oseen nor later on Leray [74] used the word "weak" in their understanding of solution, but both of them work with it. While Oseen established the results

† Oseen in his monograph [102] treats the Navier-Stokes fluids and their linearizations only.
‡ See also [34] and [35].

concerning local-in-time existence, uniqueness and regularity for large data, Leray [74] proved large-time and large-data existence for weak solutions of the Navier-Stokes equations, verifying thus **(I)**, leaving as open the tasks **(II)** and **(III)**. These tasks are still unresolved to our knowledge. The tasks **(II)** and **(III)** for the Navier-Stokes equation (B.1.8) represent the third millenium problem of the Clay Mathematical Institute [33].

The next issue concerns the function spaces where the solution satisfying (B.1.9) and (B.1.10) are to be found.

There is an interesting link between the constitutive theory via the maximization of the entropy production presented in Chapter A and the choice of function spaces where weak solutions are constructed. We showed earlier how the form of the constitutive equation for the Cauchy stress can be determined knowing the constitutive equations for the specific Helmoltz free energy ψ and for the rate of dissipation ξ by maximizing w.r.t $\mathbf{D}$'s fulfilling the reduced thermomechanical equation and the divergenceless condition as the constraint. Here, we show that the form of ψ determines function spaces for ρ, while the form of ξ determines the function space for $\mathbf{v}$. This link would become even more transparent for more complex problems ([88] for example).

Consider ψ and ξ of the form

$$\psi = \Psi(\rho) \qquad \text{and} \qquad \xi = 2\nu(p, \rho, \mathbf{D}\cdot\mathbf{D})\mathbf{D}\cdot\mathbf{D}. \tag{B.1.11}$$

Assume that ρ fulfills

$$0 \le \sup_{0\le t\le T} \int_\Omega \rho\Psi(\rho(t,x))\, dx < \infty, \tag{B.1.12}$$

and

$$0 \le \int_0^T \int_\Omega \nu(p, \rho, \mathbf{D}(\mathbf{v})\cdot\mathbf{D}(\mathbf{v}))\mathbf{D}(\mathbf{v})\cdot\mathbf{D}(\mathbf{v})\, dxdt < \infty. \tag{B.1.13}$$

If for example $\Psi(\rho) = \rho^\gamma$ with $\gamma > 1$ and $\nu(p, \rho, \mathbf{D}(\mathbf{v})\cdot\mathbf{D}(\mathbf{v})) = \nu_0$, then (B.1.12) and (B.1.13) imply that

$$\rho \in L^\infty(0, T; L^{\gamma+1}_{per}) \qquad \text{and} \qquad \mathbf{D}(\mathbf{v}) \in L^2(0, T; L^2_{per}). \tag{B.1.14}$$

In general, depending on specific structure of Ψ, (B.1.12) implies that

$$\rho \in L^\infty(0, T; X_\Psi) \ \text{ for some space } X_\Psi. \tag{B.1.15}$$

If $\Psi(\rho) = \rho^\gamma$, then $X_\Psi = L^{\gamma+1}_{per}$. Similarly, depending on the form of ν, one can conclude that

$$\mathbf{D}(\mathbf{v}) \in Y_{dis} \text{ or } \mathbf{v} \in X_{dis} \text{ for certain function spaces } Y_{dis} \text{ and } X_{dis}, \text{ respectively.}$$

In case of the constant viscosity $Y_{dis} = L^2(0,T;L^2_{per})$ and $X_{dis} = L^2(0,T;W^{1,2}_{per}(\Omega))$.

Note that the reduced thermomechanical equation (A.2.28) requires that

$$\mathbf{T}\cdot\mathbf{D}(\mathbf{v}) = \xi + \rho\dot{\psi} = 2\nu(p,\rho,\mathbf{D}(\mathbf{v})\cdot\mathbf{D}(\mathbf{v}))\mathbf{D}(\mathbf{v})\cdot\mathbf{D}(\mathbf{v}) + \rho\dot{\Psi}(\rho). \tag{B.1.16}$$

Now, if we formally set $\boldsymbol{\varphi} = \mathbf{v}$ in (B.1.9) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \varrho|\mathbf{v}|^2\,dx + \int_\Omega \mathbf{T}\cdot\mathbf{D}(\mathbf{v})\,dx = \int_\Omega \varrho\mathbf{b}\cdot\mathbf{v}\,dx \tag{B.1.17}$$

and using Eq. (B.1.16) we see that the second term in (B.1.17) can be expressed as

$$\begin{aligned}\int_\Omega \mathbf{T}(p,\varrho,\mathbf{D}(\mathbf{v}))\cdot\mathbf{D}(\mathbf{v}))\,dx &= \int_\Omega \Xi(p,\varrho,\mathbf{D}(\mathbf{v}))\,dx + \int_\Omega \varrho\dot{\psi}\,dx\\ &= \int_\Omega \Xi(p,\varrho,\mathbf{D}(\mathbf{v}))\,dx + \int_\Omega \frac{d}{dt}(\varrho\psi)\,dx\\ &= \int_\Omega \nu(p,\varrho,|\mathbf{D}(\mathbf{v})|^2)\,|\mathbf{D}(\mathbf{v})|^2\,dx + \frac{d}{dt}\int_\Omega \varrho\psi\,dx\,,\end{aligned} \tag{B.1.18}$$

where we used the fact that $\dot{\varrho} = 0$ (see (B.1.1)).

Assume that ρ_0 and $\mathbf{v}_0$ are Ω-periodic functions satisfying (α_1, α_2 being positive constants)

$$\varrho_0 \in X_\Psi \quad \text{and } \alpha_1 \le \varrho_0 \le \alpha_2\,, \tag{B.1.19}$$

$$\mathbf{v}_0 \in L^2(\Omega) \qquad \text{and} \qquad \operatorname{div}\mathbf{v}_0 = 0. \tag{B.1.20}$$

Then the fact that ρ fulfills the transport equation implies that

$$\alpha_1 \le \rho(x,t) \le \alpha_2 \qquad \text{for all } (x,t) \in \Omega\times(0,+\infty). \tag{B.1.21}$$

Consequently, it follows from (B.1.17)-(B.1.18) and (B.1.19)-(B.1.20) that (for all $T > 0$)

$$\mathbf{v} \in L^\infty(0,T;L^2_{per}) \cap X_{dis} \text{ and } \rho \in L^\infty(0,T;X_\Psi). \tag{B.1.22}$$

Specific description depends on the behavior of the viscosity with respect to $\mathbf{D}$, p and ρ respectively. See Subsection 7.3 for further details.

1.4. Models and their invariance with respect to scaling

Solutions of the equations for power-law fluids (B.1.7) considered for $r \in (1,3)$ are invariant with respect to the scaling

$$\begin{aligned} \mathbf{v}^\lambda(t,x) &= \lambda^{\frac{r-1}{3-r}}\mathbf{v}(\lambda^{\frac{2}{3-r}}t, \lambda x), \\ p^\lambda(t,x) &= \lambda^{2\frac{r-1}{3-r}}p(\lambda^{\frac{2}{3-r}}t, \lambda x). \end{aligned} \tag{B.1.23}$$

It means that if $(\mathbf{v}, p)$ solves (B.1.7) with $\mathbf{b} = \mathbf{0}$, then $(\mathbf{v}^\lambda, p^\lambda)$ solves (B.1.7) as well. Note that NSEs are also included by setting $r = 2$ in (B.1.7).

Applying this scaling we may magnify the flow near the point of interest located inside the fluid domain. Studying the behavior of the averaged rate of dissipation $d(\mathbf{v})$ defined through

$$d(\mathbf{v}) := \int_{-1}^{0}\int_{\mathcal{B}_1(0)} \xi(\mathbf{D}(\mathbf{v}))\, dx\, dt = 2\nu_1 \int_{-1}^{0}\int_{\mathcal{B}_1(0)} |\mathbf{D}(\mathbf{v})|^r\, dx\, dt \tag{B.1.24}$$

for $d(\mathbf{v}^\lambda)$ as $\lambda \to \infty$, we can give the following classification of the problem:

$$\text{if } d(\mathbf{v}^\lambda) \begin{cases} \to 0 \\ \to A \in (0,\infty) \text{ as } \lambda \to \infty \\ \to \infty \end{cases} \quad \text{then the problem is} \begin{cases} \text{supercritical,} \\ \text{critical,} \\ \text{subcritical.} \end{cases}$$

Roughly speaking, we may say that for a subcritical problem the zooming (near possible singularity) is penalized by $d(\mathbf{v}^\lambda)$ as $\lambda \to \infty$, while for supercritical case the energy dissipated out the system is insensitive measure of this magnification.

Because of this, standard regularity techniques based on difference quotient methods should in principle works for subcritical case, while supercritical problems are difficult to handle without any additional information and they are even difficult to treat since weak formulation are not suitable for the application of finer regularity techniques. The Navier-Stokes equations in three spatial dimension represent a supercritical problem.

In order to overcome a drawback of "supercritical" problems to fully exploit fine regularity techniques, Caffareli, Kohn and Nirenberg [16] introduced the notion of *suitable weak solution*, and established its existence. A key new property of thus suitable form of weak solution is the *local energy inequality.*

For (B.1.1) this is formally achieved by taking a sum of two identities: the first one is obtained by setting $\boldsymbol{\varphi} = \mathbf{v}\phi$ in (B.1.9) and the second one by setting $\xi = \frac{|\mathbf{v}|^2}{2}\phi$

in (B.1.10), with $\phi \in \mathcal{D}(-\infty,+\infty;\mathcal{C}^\infty_{per})$ satisfying $\phi(x,t)\geq 0$ for all t, x. The local energy inequality thus reads

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (\rho|\mathbf{v}|^2\phi)(x,t)\,dx \;+\int_0^t\int_\Omega (\mathbf{T}\cdot\mathbf{D}(\mathbf{v})\phi)\,(x,\tau)\,dx\,d\tau \;\leq\\
&\frac{1}{2}\int_\Omega \rho_0(x)|\mathbf{v}_0(x)|^2\phi(0,x)\,dx \;+\frac{1}{2}\int_0^t\int_\Omega \rho|\mathbf{v}|^2\,(\mathbf{v}\cdot\nabla\phi+\phi_{,t})\,(x,\tau)\,dx\,d\tau\\
&-\int_0^t\int_\Omega (\mathbf{T}\mathbf{v}\cdot\nabla\phi-\rho\mathbf{b}\cdot\mathbf{v}\phi)\,(x,\tau)\,dx\,d\tau.
\end{aligned} \tag{B.1.25}$$

Using this tool, Caffarelli, Kohn and Nirenberg [16] were able to give a significant improvement in characterizing the structure of possible singularities for the Navier-Stokes equations in three dimensions. (Section 6 addresses this issue.)

Since in three spatial dimension $d(\mathbf{v}^\lambda)$ defined through (B.1.24) fulfils

$$d(\mathbf{v}^\lambda) = 2\nu_1\lambda^{\frac{5r-11}{3-r}}\int_{-\frac{1}{\lambda^{\frac{2}{3-r}}}}^0\int_{\mathcal{B}_{1/\lambda}(0)}|\mathbf{D}(\mathbf{v}^1)|^r\,dy\,d\tau, \qquad r\in(1,3) \tag{B.1.26}$$

we see that the evolutionary equations for power-law fluids represent a subcritical problem as $r>\frac{11}{5}$. Thus, the power-law fluids model should be mathematically treatable for $r\geq\frac{11}{5}$. The same is true for the Ladyzhenskaya's equations (B.1.6) which has in comparison with the power-law fluid model one better property: the viscosity $\nu(\mathbf{D})=\nu_0+\nu_1|\mathbf{D}|^{r-2}$ and consequently the corresponding nonlinear operator $-\operatorname{div}((\nu_0+\nu_1|\mathbf{D}(\mathbf{v})|^{r-2})\mathbf{D}(\mathbf{v}))$ are **not** degenerate (while for power-law fluid, $\nu(\mathbf{D})=\nu_1|\mathbf{D}|^{r-2}$ as $|\mathbf{D}|\to 0$ degenerates for $r>2$, and becomes singular for $r<2$).

The Ladyzhenskaya's equations with $r\geq\frac{11}{5}$ **are** mathematically self-consistent; large-time existence of weak solution (task **(I)**) has been proved by Ladyzhenskaya (see [72], [65]), she also established large-time uniqueness (task **(II)**) for $r\geq\frac{5}{2}$. Task **(II)** and **(III)** for $r\geq\frac{11}{5}$ were completed by Bellout, Bloom and Nečas in [8], Málek, Nečas and Růžička in [83] and by Málek, Nečas, Rokyta and Růžička in [81], although the boundedness of the velocity is perhaps explicitly stated in this contribution for the first time. The results in [83] give however the most difficult steps in this direction. Sections 3,4 and 5 focus on this topic.

Mathematical self-consistency of the Ladyzhenskaya's equations (and some of its generalization) is the central issue of this contribution. After introducing the notion of weak solution and suitable weak solution to equations for fluids with shear dependent viscosity (B.1.4) that includes the NSEs, Ladyzhenskaya's equations and power-law fluids as special cases, we deal with large-time existence of these models in Section 3 and using two methods we establish the existence of suitable weak solution for $r > \frac{9}{5}$, and the existence of weak solution satisfying only global energy inequality for $r > \frac{8}{5}$.

Regularity of such solution is studied in Section 4 and established for $r \geq \frac{11}{5}$. Particularly, if $r \geq \frac{11}{5}$ we conclude that the velocity is bounded. We also outline how the higher differentiability technique can be used as a tool in the existence theory. Uniqueness and large-time behavior is addressed in Section 5.

The short Section 6 gives a survey of the results dealing with structure of possible singularities of flows for the Navier-Stokes fluid.

The final Section 7 states briefly results on large-time and large-data existence for further models namely for homogeneous fluid with pressure dependent viscosity and for inhomogeneous fluids with density or shear dependent viscosity.

2. Definitions of (suitable) weak solutions

Before giving a precise formulation of (suitable) weak solution to the system of PDEs (B.1.4)-(B.1.5) describing unsteady flows of fluids with shear dependent viscosity, we need to specify the assumptions characterizing the structure of the tensor $\mathbf{S} = 2\nu(|\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v})$, and to define function spaces we work with.

2.1. Assumptions concerning the stress tensor

Let us compute the expression

$$\frac{\partial \mathbf{S}(\mathbf{A})}{\partial \mathbf{A}} \cdot (\mathbf{B} \otimes \mathbf{B}) := \sum_{i,j,k=1}^{3} \frac{\partial \mathbf{S}_{ij}(\mathbf{A})}{\partial \mathbf{A}_{kl}} \mathbf{B}_{ij} \mathbf{B}_{kl}$$

for the Cauchy stress of Ladyzhenskaya's fluids and for power-law fluids. In the case of Ladyzhenskaya's fluid, i.e., when

$$\mathbf{S}(\mathbf{A}) = 2\left(\nu_0 + \nu_1 |\mathbf{A}|^{r-2}\right)\mathbf{A} \qquad (r > 2) \tag{B.2.1}$$

we have

$$\frac{\partial \mathbf{S}(\mathbf{A})}{\partial \mathbf{A}} \cdot (\mathbf{B} \otimes \mathbf{B}) = 2\left(\nu_0 + \nu_1 |\mathbf{A}|^{r-2}\right)|\mathbf{B}|^2 + 2\nu_1(r-2)|\mathbf{A}|^{r-4}(\mathbf{A} \cdot \mathbf{B})^2, \tag{B.2.2}$$

which implies

$$\begin{aligned}\frac{\partial \mathbf{S}(\mathbf{A})}{\partial \mathbf{A}} \cdot (\mathbf{B} \otimes \mathbf{B}) \geq 2\left(\nu_0 + \nu_1 |\mathbf{A}|^{r-2}\right)|\mathbf{B}|^2 \geq C_1\left(1 + |\mathbf{A}|^{r-2}\right)|\mathbf{B}|^2 \\ \text{with } C_1 = 2\min(\nu_0, \nu_1)\end{aligned} \tag{B.2.3}$$

and

$$\frac{\partial \mathbf{S}(\mathbf{A})}{\partial \mathbf{A}} \cdot (\mathbf{B} \otimes \mathbf{B}) \leq C_2 \left(1 + |\mathbf{A}|^{r-2}\right) |\mathbf{B}|^2 \text{ with } C_2 = 2 \max(\nu_0, \nu_1(r-1)), \quad \text{(B.2.4)}$$

while for power-law fluids (set $\nu_0 = 0$ in (B.2.2))

$$\frac{\partial \mathbf{S}(\mathbf{A})}{\partial \mathbf{A}} \cdot (\mathbf{B} \otimes \mathbf{B}) \geq \begin{cases} 2\nu_1 |\mathbf{A}|^{r-2} |\mathbf{B}|^2 & \text{if } r \geq 2 \\ 2\nu_1 (r-1) |\mathbf{A}|^{r-2} |\mathbf{B}|^2 & \text{if } r < 2 \end{cases} \quad \text{(B.2.5)}$$

and

$$\frac{\partial \mathbf{S}(\mathbf{A})}{\partial \mathbf{A}} \cdot (\mathbf{B} \otimes \mathbf{B}) \leq \begin{cases} 2\nu_1 (r-1) |\mathbf{A}|^{r-2} |\mathbf{B}|^2 & \text{if } r \geq 2 \\ 2\nu_1 |\mathbf{A}|^{r-2} |\mathbf{B}|^2 & \text{if } r < 2. \end{cases} \quad \text{(B.2.6)}$$

Motivated by the inequalities (B.2.3)-(B.2.4) for Ladyzhenskaya's fluid and (B.2.5)-(B.2.6) for the power-law fluid we put the following assumption on $\mathbf{S}$.

Let κ be either 0 or 1. We assume that

$$\mathbf{S} : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3} \;\text{ with }\; \mathbf{S}(\mathbf{0}) = \mathbf{0} \;\text{ fulfils }\; \mathbf{S} \in \left[\mathcal{C}^1\left(\mathbb{R}^{3\times 3}\right)\right]^{3\times 3}, \quad \text{(B.2.7)}$$

and there are two positive constants C_1 and C_2 such that for a certain $r \in (1, \infty)$ and for all $\mathbf{0} \neq \mathbf{A}, \mathbf{B} \in \mathbb{R}^{3\times 3}_{sym}$

$$C_1 \left(\kappa + |\mathbf{A}|^{r-2}\right) |\mathbf{B}|^2 \leq \frac{\partial \mathbf{S}(\mathbf{A})}{\partial \mathbf{A}} \cdot (\mathbf{B} \otimes \mathbf{B}) \leq C_2 \left(\kappa + |\mathbf{A}|^{r-2}\right) |\mathbf{B}|^2. \quad \text{(B.2.8)}$$

We also use convention that $\kappa = 0$ if $r < 2$.

2.2. Function spaces

We primarily deal with functions defined on $\mathbb{R}^3$ that are periodic with the periodic cell $\Omega = (0, L)^3$. The space $\mathcal{C}^\infty_{per}$ consists of smooth Ω-periodic functions.

Let r be such that $1 \leq r < \infty$. The Lebesgue space L^r_{per} is introduced as the closure of $\mathcal{C}^\infty_{per}$-functions with $\int_\Omega f(x)\, dx = 0$ where the closure is made w.r.t. the norm $\|\cdot\|_r$, where $\|f\|_r^r = \int_\Omega |f(x)|^r\, dx$. The Sobolev space $W^{1,r}_{per}$ is the space of Ω-periodic Lebesgue-measurable functions $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\partial_{x_i} f$ exists in a weak sense and f and $\partial_{x_i} f$ belong to L^r_{per}. Both L^r_{per} and $W^{1,r}_{per}$ are Banach spaces with the norms $\|f\|_r := \left(\int_\Omega |f(x)|^r\, dx\right)^{\frac{1}{r}}$ and $\|f\|_{1,r} := \left(\int_\Omega |\nabla_x f(x)|^r\, dx\right)^{\frac{1}{r}}$, respectively.

Let $(X, \|\cdot\|_X)$ be a Banach space of scalar functions defined on Ω. Then X^3 represents the space of vector-valued functions whose all components belong to X.

Also, X^* denotes the dual space to X. For $r' = \frac{r}{r-1}$, we usually write $W^{-1,r'}_{per}$ instead of $\left(W^{1,r}_{per}\right)^*$.

We also introduce the space $W^{1,r}_{per,\mathrm{div}}$ being a closed subspace of $\left(W^{1,r}_{per}\right)^3$ defined as the closure (w.r.t the norm $\|\cdot\|_{1,r}$) of all smooth Ω-periodic functions $\mathbf{v}$ with the zero mean value such that $\operatorname{div}\mathbf{v} = 0$. Note that $W^{1,r}_{per,\mathrm{div}} = \{\mathbf{v} \in W^{1,r}_{per}, \operatorname{div}\mathbf{v} = 0\}$.

If Y is any Banach space, $T \in (0,\infty)$ and $1 \le q \le \infty$, then $L^q(0,T;Y)$ denotes the Bochner space formed by functions $g : (0,T) \to Y$ such that, for $1 \le q < \infty$, $\|g\|_{L^q(0,T;Y)} := \left(\int_0^T \|g(t)\|_Y^q \, dt\right)^{\frac{1}{q}}$ is finite. The norm in $L^\infty(0,T;Y)$ is defined as infimum of $\sup_{t\in[0,T]\setminus E} \|g(t)\|_Y$, where infimum is taken over all subsets E of $[0,T]$ having zero Lebesgue measure.

Also, if X is a reflexive Banach space, then X_{weak} denotes the space equipped with the weak topology. Thus, for example

$$\mathcal{C}(0,T;X_{weak}) \equiv \Big\{\boldsymbol{\varphi} \in L^\infty(0,T;X); \langle\boldsymbol{\varphi}(\cdot),h\rangle \in \mathcal{C}(\langle 0,T\rangle) \text{ for all } h \in X^*\Big\}.$$

Let $1 < \alpha, \beta < \infty$. Let X be a Banach space, and let X_0, X_1 be separable and reflexive Banach spaces satisfying $X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$. Then the *Aubin-Lions lemma* [76] says that the space

$$W := \big\{v \in L^\alpha(0,T;X_0); v_{,t} \in L^\beta(0,T;X_1)\big\}$$

is compactly embedded into $L^\alpha(0,T;X)$, i.e., $W \hookrightarrow\hookrightarrow L^\alpha(0,T;X)$.

2.3. Definition of Problem ($\mathcal{P}$) and its (suitable) weak solutions

Our main task is to study the mathematical properties of *Problem* ($\mathcal{P}$) consisting of:

- four partial differential equations (B.1.4) with $\mathbf{S}$ satisfying (B.2.7)-(B.2.8),
- the spatially periodic requirement (A.4.6),
- the initial condition

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0(\cdot) \qquad \text{in } \mathbb{R}^3. \tag{B.2.9}$$

Let $T > 0$ be fixed, but arbitrary number. We assume that given functions $\mathbf{b}$ and $\mathbf{v}_0$ fulfil

$$\mathbf{b} \in \left(L^r(0,T;W^{1,r}_{per})\right)^* = L^{r'}(0,T;W^{-1,r'}_{per}), \tag{B.2.10}$$

and

$$\operatorname{div}\mathbf{v}_0 = 0 \text{ in } (\mathcal{C}^\infty_{per})^* \qquad \text{and} \qquad \mathbf{v}_0 \in L^2_{per}. \tag{B.2.11}$$

Let (B.2.7) and (B.2.8) hold.

We say that $(\mathbf{v}, p) = (v_1, v_2, v_3, p)$ is a *suitable weak solution* to Problem $(\mathcal{P})$ provided that

$$\mathbf{v} \in \mathcal{C}(0,T; L^2_{weak}(\Omega)) \cap L^r(0,T; W^{1,r}_{per,\mathrm{div}}) \cap L^{\frac{5r}{3}}(0,T; L^{\frac{5r}{3}}); \tag{B.2.12}$$

$$\mathbf{v}_{,t} \in \begin{cases} L^{r'}(0,T; W^{-1,r'}_{per}) \\ L^{\frac{5r}{6}}(0,T; W^{-1,\frac{5r}{6}}_{per}) \end{cases} \text{and } p \in \begin{cases} L^{r'}(0,T; L^{r'}_{per}) & \text{for } r \geq \dfrac{11}{5}, \\ L^{\frac{5r}{6}}(0,T; L^{\frac{5r}{6}}_{per}) & \text{for } r < \dfrac{11}{5}; \end{cases} \tag{B.2.13}$$

$$\lim_{t\to 0_+} \|\mathbf{v}(t) - \mathbf{v}_0\|_2^2 = 0; \tag{B.2.14}$$

$$\begin{aligned} &\int_0^T \langle \mathbf{v}_{,t}(t), \boldsymbol{\varphi}(t)\rangle - ((\mathbf{v}\otimes\mathbf{v})(t), \nabla\boldsymbol{\varphi}(t)) + (\mathbf{S}(\mathbf{D}(\mathbf{v}(t))), \mathbf{D}(\boldsymbol{\varphi}(t))) \\ &-(p(t), \operatorname{div}\boldsymbol{\varphi}(t))\, dt = \int_0^T \langle \mathbf{b}(t), \boldsymbol{\varphi}(t)\rangle\, dt \quad \text{for all } \boldsymbol{\varphi} \in L^s(0,T; W^{1,s}_{per}) \\ &\quad \text{with } \ s = r \ \text{ if } \ r \geq \frac{11}{5} \quad \text{ and } \quad s = \frac{5r}{5r-6} \ \text{ if } \ \frac{6}{5} < r < \frac{11}{5}; \end{aligned} \tag{B.2.15}$$

$$\begin{aligned} &\frac{1}{2}\int_\Omega \left(|\mathbf{v}|^2\phi\right)(t,x)\, dx + \int_0^t\int_\Omega \mathbf{S}(\mathbf{D}(\mathbf{v}))\cdot\mathbf{D}(\mathbf{v})\phi\, dx\, d\tau \\ &\leq \frac{1}{2}\int_\Omega |\mathbf{v}_0(x)|^2\phi(0,x)\, dx + \int_0^t\int_\Omega \frac{|\mathbf{v}|^2}{2}\phi_{,t}\, dx\, d\tau + \int_0^t \langle \mathbf{b}, \mathbf{v}\phi\rangle\, d\tau \\ &\quad + \int_0^t\int_\Omega \left(\frac{|\mathbf{v}|^2}{2}\mathbf{v} + p\mathbf{v} - \mathbf{S}(\mathbf{D}(\mathbf{v}))\mathbf{v}\right)\cdot\nabla\phi\, dx\, d\tau \end{aligned} \tag{B.2.16}$$

valid for all $\phi \in \mathcal{D}(-\infty, +\infty; \mathcal{C}^\infty_{per})$, $\phi \geq 0$ and for almost all $t \in (0, T\rangle$.

- If $r \geq 3$, the assertion $\mathbf{v} \in L^{\frac{5r}{3}}(0,T; L^{\frac{5r}{3}})$ in (B.2.12) can be improved. For example, if $r > 3$, $\mathbf{v}$ being in $L^r(0,T; W^{1,r})$ implies $\mathbf{v} \in L^r(0,T; \mathcal{C}^{0,\frac{r-3}{3r}})$. Since our interest is focused on $r \in (1,3)$ we do not discuss this alternative in what follows.
- Note that (B.2.12) and (B.2.13) ensure that all terms in (B.2.15) have sense for $r > \frac{6}{5}$, while all terms in (B.2.16) are finite if $r > \frac{9}{5}$.
- Note that taking $\phi \equiv 1$, one can conclude from (B.2.16) the standard energy inequality

$$\frac{1}{2}\|\mathbf{v}(t)\|_2^2 + \int_0^t (\mathbf{S}(\mathbf{D}(\mathbf{v})), \mathbf{D}(\mathbf{v}))\, d\tau \leq \frac{1}{2}\|\mathbf{v}_0\|_2^2 + \int_0^t \langle \mathbf{b}, \mathbf{v}\rangle\, d\tau \tag{B.2.17}$$

has sense provided that the right hand side is finite.

We say that $(\mathbf{v}, p)$ is *weak solution* to Problem $(\mathcal{P})$ if (B.2.12)–(B.2.15) and (B.2.17) hold.

- For the NSEs, local energy inequality takes slightly different form due to the linearity of $\mathbf{S}$ that allows to perform the integration per parts once more. This gives

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \left(|\mathbf{v}|^2\phi\right)(t,x)\,dx + \nu_0\int_0^t\int_\Omega |\nabla\mathbf{v}|^2\phi\,dx\,d\tau \\
&\le \frac{1}{2}\int_\Omega |\mathbf{v}_0(x)|^2\phi(0,x)\,dx + \int_0^t\int_\Omega p\mathbf{v}\cdot\nabla\phi\,dx\,d\tau + \int_0^t \langle \mathbf{b},\mathbf{v}\phi\rangle\,d\tau \\
&\qquad + \int_0^t\int_\Omega \frac{|\mathbf{v}|^2}{2}\left(\phi_{,t} + \nu_0\triangle\phi + \mathbf{v}\cdot\nabla\phi\right)\,d\tau
\end{aligned} \tag{B.2.18}$$

valid for all $\phi \in \mathcal{D}(-\infty,+\infty;\mathcal{C}^\infty_{per})$, $\phi \ge 0$ and for almost all $t \in (0,T\rangle$.

• Note that for $r \ge \frac{11}{5}$ we can set $\boldsymbol{\varphi} = \mathbf{v}$ or $\boldsymbol{\varphi} = \mathbf{v}\phi$ in (B.2.15) and derive (B.2.16) and (B.2.17) in the form of *equality*. Also, for $r \ge \frac{11}{5}$, we have $\mathbf{v} \in \mathcal{C}(0,T;L^2_{per})$ that follows from the fact that $\mathbf{v} \in L^r(0,T;W^{1,r}_{per})$ and $\mathbf{v}_{,t} \in \left(L^r(0,T;W^{1,r}_{per})\right)^*$.

• Notice that the assumption (B.2.8) holds for the Ladyzhenskaya's equations with $\kappa = 1$, and for power-law fluids with $\kappa = 0$.

2.4. Useful inequalities

We first obtain several inequalities that are consequences of (B.2.7) and (B.2.8). Since

$$\begin{aligned}
(\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B}))\cdot(\mathbf{A}-\mathbf{B}) &= \int_0^1 \frac{d}{ds}\mathbf{S}_{ij}(\mathbf{B}+s(\mathbf{A}-\mathbf{B}))ds\,(\mathbf{A}-\mathbf{B})_{ij} \\
&= \int_0^1 \frac{\partial \mathbf{S}_{ij}}{\partial \mathbf{A}_{kl}}(\mathbf{B}+s(\mathbf{A}-\mathbf{B}))(\mathbf{A}-\mathbf{B})_{kl}(\mathbf{A}-\mathbf{B})_{ij}\,ds,
\end{aligned}$$

it follows from the first inequality in (B.2.8) that

$$(\mathbf{S}(\mathbf{A})-\mathbf{S}(\mathbf{B}))\cdot(\mathbf{A}-\mathbf{B}) \ge C_1\int_0^1 (\kappa + |\mathbf{B}+s(\mathbf{A}-\mathbf{B})|^{r-2})ds|\mathbf{A}-\mathbf{B}|^2 \ge 0. \tag{B.2.19}$$

If $r \ge 2$, (B.2.19) then implies (see Lemma 5.1.19 in [81] or [23])

$$(\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B}))\cdot(\mathbf{A}-\mathbf{B}) \ge C_3 \begin{cases} |\mathbf{A}-\mathbf{B}|^r & \text{if } \kappa = 0, \\ |\mathbf{A}-\mathbf{B}|^2 + |\mathbf{A}-\mathbf{B}|^r & \text{if } \kappa = 1. \end{cases} \tag{B.2.20}$$

Consequently, for $\mathbf{A} = \mathbf{D}(\mathbf{u})$ and $\mathbf{B} = \mathbf{D}(\mathbf{v})$ we have

$$((\mathbf{S}(\mathbf{D}(\mathbf{u})) - \mathbf{S}(\mathbf{D}(\mathbf{v})), \mathbf{D}(\mathbf{u}-\mathbf{v})) \ge C_3 \begin{cases} \|\mathbf{D}(\mathbf{u}-\mathbf{v})\|_r^r & \text{if } \kappa = 0, \\ \|\mathbf{D}(\mathbf{u}-\mathbf{v})\|_2^2 + \|\mathbf{D}(\mathbf{u}-\mathbf{v})\|_r^r & \text{if } \kappa = 1. \end{cases} \tag{B.2.21}$$

If $r < 2$ (and $\kappa = 0$), we show below that (B.2.19) implies

$$(\mathbf{S}(\mathbf{D}(\mathbf{u})) - \mathbf{S}(\mathbf{D}(\mathbf{v})), \mathbf{D}(\mathbf{u}-\mathbf{v}))\,\|\mathbf{D}(\mathbf{u}) + \mathbf{D}(\mathbf{v})\|_r^{2-r} \ge C_4\|\mathbf{D}(\mathbf{u}-\mathbf{v})\|_r^2. \tag{B.2.22}$$

Consequently, setting $\mathbf{v} \equiv \mathbf{0}$ in (B.2.21) and (B.2.22) we conclude

$$(\mathbf{S}(\mathbf{D}(\mathbf{u})), \mathbf{D}(\mathbf{u})) \geq C_5 \|\mathbf{D}(\mathbf{u})\|_r^r. \tag{B.2.23}$$

In fact it directly follows from (B.2.19) that $\mathbf{S}$ is strictly monotone, i.e.

$$(\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B})) \cdot (\mathbf{A} - \mathbf{B}) > 0 \text{ if } \mathbf{A} \neq \mathbf{B}.$$

Also, if $\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v}) \in L^r(0,T; L^r(\Omega)^{3\times 3})$, (B.2.22) and Hölder's inequality lead to

$$\begin{aligned} &\int_0^T \|\mathbf{D}(\mathbf{u}-\mathbf{v})\|_r^r \, dt \\ &\qquad \leq C_6 \left(\int_0^T (\mathbf{S}(\mathbf{D}(\mathbf{u})) - \mathbf{S}(\mathbf{D}(\mathbf{v})), \mathbf{D}(\mathbf{u}-\mathbf{v}))\, dt \right)^{\frac{r}{2}} \end{aligned} \tag{B.2.24}$$

To see (B.2.22) we use Hölder inequality and inequality $|\mathbf{B} + s(\mathbf{A}-\mathbf{B})| \leq |\mathbf{A}| + |\mathbf{B}|$ valid for all $s \in (0,1)$ in the following calculation, where $\mathbf{D}(s)$ abbreviates $\mathbf{D}(\mathbf{v}) + s\mathbf{D}(\mathbf{u}-\mathbf{v})$:

$$\begin{aligned} &\|\mathbf{D}(\mathbf{u}-\mathbf{v})\|_r^r = \int_\Omega |\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{v})|^r \, dx \\ &= \int_\Omega \left[\Big(\int_0^1 |\mathbf{D}(s)|^{r-2} ds \Big) |\mathbf{D}(\mathbf{u}-\mathbf{v})|^2 \right]^{\frac{r}{2}} \left(\int_0^1 |\mathbf{D}(s)|^{r-2} ds \right)^{\frac{-r}{2}} dx \\ &\leq c \left(\int_\Omega \Big[(\mathbf{S}(\mathbf{D}(\mathbf{v})) - \mathbf{S}(\mathbf{D}(\mathbf{u}))) \cdot (\mathbf{D}(\mathbf{u}-\mathbf{v}))\, dx \Big] \right)^{\frac{r}{2}} \left(\int_\Omega \Big(\int_0^1 |\mathbf{D}(s)|^{r-2} ds \Big)^{\frac{-r}{2-r}} dx \right)^{\frac{2-r}{2}} \\ &\leq \left(\int_\Omega \Big[(\mathbf{S}(\mathbf{D}(\mathbf{v})) - \mathbf{S}(\mathbf{D}(\mathbf{u}))) \cdot (\mathbf{D}(\mathbf{u}-\mathbf{v}))\, dx \Big] \right)^{\frac{r}{2}} \|\mathbf{D}(\mathbf{u}) + \mathbf{D}(\mathbf{v})\|_r^{\frac{2-r}{2} r} \end{aligned}$$

which implies (B.2.22).

Analogously, we could check that for $r \in (1,2)$ and $\theta \in (\frac{1}{r}, 1)$ it holds for $\mathbf{D}(\mathbf{u})$, $\mathbf{D}(\mathbf{v}) \in L^r(0,T; L^r(\Omega)^{3\times 3})$

$$\int_0^T \|\mathbf{D}(\mathbf{u}-\mathbf{v})\|_{r\theta}^{r\theta} \, dt \leq \tilde{C}_6 \left(\int_0^T \int_\Omega [\mathbf{S}(\mathbf{D}(\mathbf{u})) - \mathbf{S}(\mathbf{D}(\mathbf{v})) \cdot \mathbf{D}(\mathbf{u}-\mathbf{v}))]^\theta \, dx\, dt \right)^{\frac{r}{2}} \tag{B.2.25}$$

where $\tilde{C}_6$ depends on $|\Omega|$ and T, and the L^r-norms of $\mathbf{D}(\mathbf{u})$ and $\mathbf{D}(\mathbf{v})$.

It also follows from (B.2.7) and (B.2.8):

$$\begin{aligned} \mathbf{S}(\mathbf{A}) = \mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{0}) &= \int_0^1 \frac{d}{ds} \mathbf{S}(s\mathbf{A})\, ds = \int_0^1 \frac{\partial \mathbf{S}(s\mathbf{A})}{\partial \mathbf{A}} \cdot \mathbf{A}\, ds \\ &\leq C_2 \int_0^1 (\kappa + s^{r-2} |\mathbf{A}|^{r-2})\, ds\, |\mathbf{A}| \leq C_2 \kappa |\mathbf{A}| + C_2 \frac{1}{r-1} |\mathbf{A}|^{r-1}. \end{aligned}$$

Consequently, using the convection that $\kappa = 0$ if $r < 2$, we have

$$|\mathbf{S}(\mathbf{A})| \leq C_2\kappa|\mathbf{A}| + C_2\frac{1}{r-1}|\mathbf{A}|^{r-1} \leq C_0(\kappa + |\mathbf{A}|)^{r-1}. \tag{B.2.26}$$

If $r \in (1, +\infty)$, Korn's inequality states, see [98] or [91] that there is $C_N > 0$ such that

$$\|\nabla\mathbf{u}\|_r \leq C_N\|\mathbf{D}(\mathbf{u})\|_r \text{ for all } \mathbf{u} \in W^{1,r}_{per} \text{ or } W^{1,r}_0. \tag{B.2.27}$$

3. Existence of a (suitable) weak solution

3.1. Formulation of the results and bibliographical notes

The aim of this section is to establish the following result on large-time and large-data existence of suitable weak solution to unsteady flows of fluids with shear dependent viscosity.

Theorem 3.1. *Assume that* $\mathbf{S} : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}$ *satisfies* (B.2.7) *and* (B.2.8) *with fixed parameter* r. *Let also* $\mathbf{b}$ *and* $\mathbf{v}_0$ *fulfil* (B.2.10) *and* (B.2.11), *respectively. If*

$$r > \frac{8}{5}, \tag{B.3.1}$$

then there is a weak solution to Problem $(\mathcal{P})$.
If in addition

$$r > \frac{9}{5}, \tag{B.3.2}$$

then there is a suitable weak solution to Problem $(\mathcal{P})$.
Finally, if

$$r \geq \frac{11}{5}, \tag{B.3.3}$$

then the local energy equality hold and $\mathbf{v} \in \mathcal{C}(0, T; L^2_{per})$.

Mathematical analysis of (B.1.4)-(B.1.5) was initiated by Ladyzhenskaya (see [72], [65]) who prove the existence of weak solution for $r \geq \frac{11}{5}$ treating homogeneous Dirichlet (i.e. no-slip) boundary condition. Her approach based on a combination of monotone operator theory together with the compactness arguments works for easier boundary-value problems, as spatial periodic problem or Navier's slip as well. Provided that the viscosity depends on the full velocity gradient, i.e. $\nu = \nu(|\nabla\mathbf{v}|^2)$, the same results are presented in the book of Lions [76]. For a complementary reading, see Kaniel [55].

The following table gives an overview of the results and methods available in two and general dimensions. We discuss below the results in three dimensions in detail.

$\boxed{\text{Spatially-periodic problem}}$	$d = 2$	$d \geq 3$	Refs.
monotone operators + compactness $\Rightarrow$	$r \geq 2$	$r \geq 1 + \dfrac{2d}{d+2}$	[65],[76]
monotone operators $+L^{\infty}$-function $\Rightarrow$	$r > \dfrac{3}{2}$	$r \geq \dfrac{2(d+1)}{d+2}$	[43]
regularity technique (higher-differentiability) $\Rightarrow$	$r > 1$	$r > \dfrac{3d}{d+2}$	[81]
$\mathcal{C}^{1,\alpha}$ − regularity $\Rightarrow$	$r \geq \dfrac{4}{3}$	?	[58]
$\boxed{\text{Dirichlet problem (No-slip boundary)}}$			
monotone operators + compactness $\Rightarrow$	$r \geq 2$	$r \geq 1 + \dfrac{2d}{d+2}$	[72], [65], [76]
regularity technique (higher-differentiability) $\Rightarrow$	$r \geq \dfrac{3}{2}$	$r \geq 2$ $(d = 3)$	[57], [82]
$\mathcal{C}^{1,\alpha}$ − regularity $\Rightarrow$	$r \geq 2$	?	[56].

Note that Theorem 3.1 covers also the large-time and large-data existence for the NSEs, the results obtained for the Cauchy problem in the fundamental article by Leray [74], and extended to bounded domains with no-slip boundary conditions by Hopf [52] and to notion of suitable weak solution by Caffarelli, Kohn and Nierenberg [16]. The technique of monotone operators [65] or [76] however does not cover these results (as (B.3.3) does not include $r = 2$).

This gap in the existence theory was filled by the result presented by Málek, Nečas, Rokyta and Růžička in [81], see [8] and [83] for the first appearance. The method based on the regularity technique to obtain fractional higher differentiability gives, among other results, the existence of weak solution for r fulfilling (B.3.2). This concerns the spatially periodic problem (A.4.6). For no-slip boundary conditions, the existence for $r \geq 2$ is established in [82]. The idea of this method will be explained in Section 4.

Later on in [43], using the facts that the nonlinear operator is strictly monotone and only L^∞-test function are permitted, Frehse, Málek and Steinhauer extended in some sense the existence theory for non-linear parabolic systems with L^1-right hand side performed for example in [11] and proved the existence of weak solution for $r > \frac{8}{5}$. In the following subsection the proof of Theorem 3.1 is established using the approach from [65] for $r \geq \frac{11}{5}$ and from [43] for $r \in (\frac{8}{5}, \frac{11}{5})$. Note that the last result for $r \in (\frac{8}{5}, 2)$ for no-slip boundary conditions is not completely solved yet.

Frehse and Málek conjecture in [42] that one can exploit the restriction that only Lipschitz test function are admissible and establish the existence of weak solution for $r > \frac{6}{5}$. See [44] for details on this technique for time independent problems.

3.2. Definition of an approximate Problem ($\mathcal{P}^{\varepsilon,\eta}$) and apriori estimates

Let $\eta > 0$ and $\varepsilon > 0$ be fixed. If $u \in L^1_{loc}(\mathbb{R}^3)$, then $u{*}\omega^\eta := \frac{1}{\eta^3} \int_{\mathbb{R}^3} \omega \left(\frac{x-y}{\eta}\right) u(y)\, dy$ with $\omega(\cdot) \in \mathcal{D}(B_1(0))$, $\omega \geq 0$, ω being radially symmetric, $\int_{B_1(0)} \omega = 1$, is the standard regularization of a function u.

We consider *Problem* ($\mathcal{P}^{\varepsilon,\eta}$) to find $(\mathbf{v},p) := (\mathbf{v}^{\varepsilon,\eta}, p^{\varepsilon,\eta})$ such that†

$$\operatorname{div}\mathbf{v} + \varepsilon|p|^\alpha p = 0 \qquad \text{with} \begin{cases} \alpha = \dfrac{2-r}{r-1} & \text{for} \quad r \geq \dfrac{11}{5}, \\ \alpha = \dfrac{5r-12}{6} & \text{for} \quad r < \dfrac{11}{5}, \end{cases} \tag{B.3.10}$$

$$\mathbf{v}_{,t} + \operatorname{div}\left((\mathbf{v} * \omega^\eta) \otimes \mathbf{v}\right) - \operatorname{div}\mathbf{S}(\mathbf{D}(\mathbf{v})) = -\nabla p + \mathbf{b}, \tag{B.3.11}$$

and $(\mathbf{v}^{\varepsilon,\eta}, p^{\varepsilon,\eta})$ are Ω-periodic functions fulfilling (A.4.6) and each $\mathbf{v}^{\varepsilon,\eta}$ starts with the initial value specified in (B.2.9).

Note that (B.3.10)-(B.3.11) is tantamount to

$$\mathbf{v}_{,t} + \operatorname{div}\left((\mathbf{v} * \omega^\eta) \otimes \mathbf{v}\right) - \operatorname{div}\mathbf{S}(\mathbf{D}(\mathbf{v})) - \left(\frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1}} \nabla\Big(\frac{\operatorname{div}\mathbf{v}}{|\operatorname{div}\mathbf{v}|^{\frac{\alpha}{\alpha+1}}}\Big) = \mathbf{b}, \tag{B.3.12}$$

with $p := -\left(\frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1}} |\operatorname{div}\mathbf{v}|^{-\frac{\alpha}{\alpha+1}} \operatorname{div}\mathbf{v}$ defined after solving for $\mathbf{v} = \mathbf{v}^{\varepsilon,\eta}$ (B.3.12) together with (B.2.9) and (A.4.6). This kind of approximation is in the literature called quasi-compressible approximation or the problem with penalized divergenceless constraint. Although in (B.3.12) three non-linear operators appear, the solvability of (A.3.10) is not difficult due to the fact that the first operator $\operatorname{div}\left((\mathbf{v} * \omega^\eta) \otimes \mathbf{v}\right)$ is compact, the second and third one (for $\varepsilon > 0$ fixed) are monotone and the fol-

† There is a clear hint regarding the choice of α. Applying formally div to $(\text{B.1.4})_2$ with $\mathbf{b} = \mathbf{0}$, we obtain

$$p = (-\triangle)^{-1} \operatorname{div}\operatorname{div}\left(\mathbf{v}\otimes\mathbf{v} - \mathbf{S}(\mathbf{D}(\mathbf{v}))\right). \tag{B.3.4}$$

Since the energy inequality implies that

$$\mathbf{v} \in L^\infty(0,T; L^2_{per}) \cap L^r(0,T; W^{1,r}_{per}), \tag{B.3.5}$$

the interpolation inequality $\|u\|_q \leq \|u\|_2^{\frac{2(3r-q)}{q(5r-6)}} \|u\|_{\frac{3r}{3-r}}^{\frac{3r}{q}\frac{q-2}{5r-6}}$ (for $r < 3$) leads to

$$\mathbf{v} \in L^{\frac{5r}{3}}(0,T; L^{\frac{5r}{3}}_{per}). \tag{B.3.6}$$

Consequently

$$\mathbf{v}\otimes\mathbf{v} \in L^{\frac{5r}{6}}(0,T; L^{\frac{5r}{6}}_{per}). \tag{B.3.7}$$

Due to (B.2.26), $\mathbf{S}(\mathbf{D}(\mathbf{v}))$ behaves as $|\nabla\mathbf{v}|^{r-1}$ and thus

$$\mathbf{S}(\mathbf{D}(\mathbf{v})) \in L^{r'}(0,T; L^{r'}_{per}). \tag{B.3.8}$$

It thus follows from (B.3.4), (B.3.7) and (B.3.8) that

$$p \in L^q(0,T; L^q_{per}) \qquad \text{where } q = \min\left\{\frac{5r}{6}, \frac{r}{r-1}\right\}. \tag{B.3.9}$$

If $r \geq \frac{11}{5}$, then $q = \frac{r}{r-1}$ while for $r < \frac{11}{5}$, $q = \frac{5r}{6}$. In both cases, we choose α in such way that $\alpha + 2 = q$.

lowing estimates are available:

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|_2^2 + \int_\Omega \mathbf{S}(\mathbf{D}(\mathbf{v}))\cdot\mathbf{D}(\mathbf{v})\,dx + \left\{\begin{matrix} \varepsilon\|p\|_{\alpha+2}^{\alpha+2} \\ \left(\dfrac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1}}\|\operatorname{div}\mathbf{v}\|_{\frac{\alpha+2}{\alpha+1}}^{\frac{\alpha+2}{\alpha+1}} \end{matrix}\right\} = \langle \mathbf{b},\mathbf{v}\rangle \le \|\mathbf{b}\|_{-1,r'}\|\nabla\mathbf{v}\|_r. \tag{B.3.13}$$

Note that

$$\alpha+2 = \begin{cases} r' \\ \dfrac{5r}{6} \end{cases}, \quad \alpha+1 = \begin{cases} \dfrac{1}{r-1} \\ \dfrac{5r-6}{6} \end{cases} \text{ and } \quad \frac{\alpha+2}{\alpha+1} = \begin{cases} r \\ \dfrac{5r}{5r-6} \end{cases} \quad \text{for} \quad \begin{matrix} r \ge \dfrac{11}{5} \\ r < \dfrac{11}{5} \end{matrix}. \tag{B.3.14}$$

Inequality (B.2.23) and Korn's inequality (B.2.27) then allow us to conclude from (B.3.13) that

$$\begin{aligned}\sup_{t\in[0,T]}\|\mathbf{v}^{\varepsilon,\eta}(t)\|_2^2 + \int_0^T \|\nabla\mathbf{v}^{\varepsilon,\eta}(t)\|_r^r\,dt + \varepsilon\int_0^T \|p^{\varepsilon,\eta}(t)\|_{\alpha+2}^{\alpha+2}\,dt + \\ + \left(\frac{1}{\varepsilon}\right)^{\frac{1}{1+\alpha}}\int_0^T \|\operatorname{div}\mathbf{v}^{\varepsilon,\eta}(t)\|_{\frac{\alpha+2}{\alpha+1}}^{\frac{\alpha+2}{\alpha+1}}\,dt \le K,\end{aligned} \tag{B.3.15}$$

where K is an absolute constant depending on $\|\mathbf{b}\|_{L^{r'}(0,T;W^{-1,r'}_{per})}$, $\|\mathbf{v}_0\|_2$ and the constant C_0^{-1} from (B.2.26).

It follows from the first two terms (implying that $\mathbf{v}^{\varepsilon,\eta}$ belongs to $L^\infty(0,T;L^2_{per})\cap L^r(0,T;W^{1,r}_{per})$ uniformly w.r.t. ε and η) that for $r\in(1,3)$

$$\int_0^T \|\mathbf{v}^{\varepsilon,\eta}\|_{\frac{5r}{3}}^{\frac{5r}{3}}\,dt \le K. \tag{B.3.16}$$

Due to (B.3.15) and (B.2.26) we also have

$$\int_0^T \|\mathbf{S}(\mathbf{D}(\mathbf{v}))\|_{r'}^{r'}\,dt \le K. \tag{B.3.17}$$

Looking at the equation (B.3.12) and using the estimates (B.3.15)–(B.3.17) we see that the third and fifth term belong to $L^{r'}(0,T;W^{-1,r'}_{per}(\Omega))$ (even uniformly w.r.t. ε and η), the second term $\operatorname{div}\Big((\mathbf{v}*\omega^\eta)\otimes\mathbf{v})\Big)$ belongs to $L^{r'}(0,T;W^{-1,r'}_{per}(\Omega))$ uniformly w.r.t. ε and η for $r\ge\frac{11}{5}$, for $r<\frac{11}{5}$ it belongs to $L^{\frac{5r}{6}}(0,T;W^{-1,\frac{5r}{6}}_{per}(\Omega))$ (again uniformly w.r.t. ε and η). The term $(\varepsilon)^{-\frac{1}{\alpha+1}}\operatorname{div}\Big(|\operatorname{div}\mathbf{v}|^{\frac{-\alpha}{\alpha+1}}(\operatorname{div}\mathbf{v})\,\mathbf{I}\Big)$ also belongs to $L^{r'}(0,T;W^{-1,r'}_{per}(\Omega))$ for $r\ge\frac{11}{5}$, and to $L^{\frac{5r}{6}}(0,T;W^{-1,\frac{5r}{6}}_{per}(\Omega))$ otherwise, however *not* uniformly w.r.t. $\varepsilon>0$.

As a consequence of this consideration, we have

$$\mathbf{v}^{\varepsilon,\eta}_{,t} \in \begin{cases} L^{r'}(0,T;W^{-1,r'}_{per}) & \text{for } r \geq \dfrac{11}{5}, \\ L^{\frac{5r}{6}}(0,T;W^{-1,\frac{5r}{6}}_{per}) & \text{for } r < \dfrac{11}{5}. \end{cases} \tag{B.3.18}$$

Obviously, if we eliminate the term $(\varepsilon)^{-\frac{1}{\alpha+1}} \operatorname{div}\Big(|\operatorname{div}\mathbf{v}|^{\frac{-\alpha}{\alpha+1}}(\operatorname{div}\mathbf{v})\,\mathbf{I}\Big)$ using divergenceless test functions we obtain the estimates that are uniform w.r.t. ε. Doing so, we obtain

$$\int_0^T \|\mathbf{v}^{\varepsilon,\eta}_{,t}\|^{\alpha+2}_{W^{-1,\alpha+2}_{per,div}}\,dt \leq K. \tag{B.3.19}$$

3.3. Solvability of an approximative problem

In this subsection ε and η are fixed, and thus we write $(\mathbf{v},p)$ instead $(\mathbf{v}^{\varepsilon,\eta},p^{\varepsilon,\eta})$. Based on (B.3.15) we set

$$X := \begin{cases} L^r(0,T;W^{1,r}_{per}) & \text{if } r \geq \dfrac{11}{5}, \\ \{\mathbf{u} \in L^r(0,T;W^{1,r}_{per}); \operatorname{div}\mathbf{u} \in L^{\frac{5r}{5r-6}}(0,T;L^{\frac{5r}{5r-6}})\} & \text{if } r < \dfrac{11}{5}. \end{cases}$$

Let $\{\boldsymbol{\omega}^s\}_{s=1}^{\infty}$ be a basis of X. We construct a solution to (B.3.10)-(B.3.11), more precisely to (B.3.12) via Galerkin approximations $\{\mathbf{v}^N\}_{N=1}^{\infty}$ being of the form

$$\mathbf{v}^N(t,x) = \sum_{s=1}^{N} c_s^N(t)\boldsymbol{\omega}^s(x),$$

where $\mathbf{c}^N := \{c_s^N(t)\}_{s=1}^{\infty}$ solve the system of ordinary differential equations:

$$\begin{aligned} &\frac{d}{dt}(\mathbf{v}^N,\boldsymbol{\omega}^s) - ((\mathbf{v}^N * \omega^\eta)\otimes\mathbf{v}^N,\boldsymbol{\omega}^s) + (\mathbf{S}(\mathbf{D}(\mathbf{v}^N)),\mathbf{D}(\boldsymbol{\omega}^s)) \\ &\quad + \frac{1}{\varepsilon^{\frac{1}{\alpha+1}}}(|\operatorname{div}\mathbf{v}^N|^{\frac{-\alpha}{\alpha+1}}\operatorname{div}\mathbf{v}^N,\operatorname{div}\boldsymbol{\omega}^s) = \langle\mathbf{b},\boldsymbol{\omega}^s\rangle \quad \text{for } s=1,2,\ldots,N. \end{aligned} \tag{B.3.20}$$

Due to linearity of the second component in all expressions, we obtain (B.3.13) for $\mathbf{v}^N$ that leads to (B.3.15), (B.3.16) and (B.3.17) for $\mathbf{v}^N$. Local-in-time existence of solution to (B.3.20) follows from Caratheodory theory, global-in-time existence is then consequence of (B.3.15), or its variant for $\mathbf{v}^N$. It also follows from (B.3.15), (B.3.17) and (B.3.18) that there is a subsequence $\{\mathbf{v}^n\}_{n=1}^{\infty} \subset \{\mathbf{v}^N\}_{N=1}^{\infty}$ and $\mathbf{v} \in$

$X \cap L^\infty(0,T;L^2_{per})$, $\overline{\mathbf{S}} \in L^{r'}(0,T;L^{r'}(\Omega)^{3\times 3})$ and $\overline{P} \in L^{\alpha+2}(0,T;L^{\alpha+2})$ such that

$$\mathbf{v}^n \rightharpoonup \mathbf{v} \qquad \begin{array}{l}\text{weakly in } X\\ \text{*-weakly in } L^\infty(0,T;L^2_{per}),\end{array} \tag{B.3.21}$$

$$\mathbf{v}^n_{,t} \rightharpoonup \mathbf{v}_{,t} \qquad \text{weakly in } L^{\alpha+2}(0,T;W^{-1,\alpha+2}_{per}), \tag{B.3.22}$$

$$\mathbf{S}(\mathbf{D}(\mathbf{v}^n)) \rightharpoonup \overline{\mathbf{S}} \qquad \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)^{3\times 3}), \tag{B.3.23}$$

$$P(\mathbf{v}^n) := |\operatorname{div}\mathbf{v}^n|^{\frac{-\alpha}{\alpha+1}}\operatorname{div}\mathbf{v}^n \rightharpoonup \overline{P} \qquad \text{weakly in } L^{\alpha+2}(0,T;L^{\alpha+2}_{per}), \tag{B.3.24}$$

and thanks to Aubin-Lions compactness lemma (cf. [81], Lemma 1.2.48 or [76], Section 1.5)

$$\mathbf{v}^n \to \mathbf{v} \text{ strongly in } L^r(0,T;L^q_{per}) \text{ for all } q \in \langle 1, \frac{3r}{3-r}). \tag{B.3.25}$$

Simple arguments then lead to the conclusion that $\mathbf{v}$, $\overline{\mathbf{S}}$ and $\overline{P}$ fulfil (for almost all $t \in (0,T\rangle$)

$$0 = \int_0^t \left\{ \langle \mathbf{v}_{,t}, \boldsymbol{\varphi}\rangle - ((\mathbf{v} * \omega^\eta)\otimes \mathbf{v}, \nabla\boldsymbol{\varphi}) + (\overline{\mathbf{S}}, \mathbf{D}(\boldsymbol{\varphi})) + \left(\frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1}} (\overline{P}, \operatorname{div}\boldsymbol{\varphi}) - \langle \mathbf{b}, \boldsymbol{\varphi}\rangle \right\} d\tau \tag{B.3.26}$$

for all $\boldsymbol{\varphi} \in X$. Particularly, for $\boldsymbol{\varphi} = \mathbf{v}$ we have

$$0 = \frac{1}{2}(\|\mathbf{v}(t)\|_2^2 - \|\mathbf{v}_0\|_2^2) + \int_0^t \left[(\overline{\mathbf{S}}, \mathbf{D}(\mathbf{v})) + \left(\frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1}} (\overline{P}, \operatorname{div}\mathbf{v}) - \langle \mathbf{b}, \mathbf{v}\rangle \right] d\tau. \tag{B.3.27}$$

Since for $\boldsymbol{\psi} \in X$:

$$0 \le \int_0^t (\mathbf{S}(\mathbf{D}(\mathbf{v}^n)) - \mathbf{S}(\mathbf{D}(\boldsymbol{\psi})), \mathbf{D}(\mathbf{v}^n) - \mathbf{D}(\boldsymbol{\psi})) + \left(\frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1}} (P(\mathbf{v}^n) - P(\boldsymbol{\psi}), \operatorname{div}(\mathbf{v}^n - \boldsymbol{\psi}))$$

we use the equation (B.3.13) with $\mathbf{v}^n$ instead of $\mathbf{v}$ to replace the term

$$\int_0^t (\mathbf{S}(\mathbf{D}(\mathbf{v}^n)), \mathbf{D}(\mathbf{v}^n)) + \left(\frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1}} (P(\mathbf{v}^n), \operatorname{div}\mathbf{v}^n) d\tau$$

and pass to the limit as $n \to \infty$. Using (B.3.21)-(B.3.25) we conclude

$$0 \le \int_0^t \left[(\overline{\mathbf{S}} - \mathbf{S}(\mathbf{D}(\boldsymbol{\psi})), \mathbf{D}(\mathbf{v}) - \mathbf{D}(\boldsymbol{\psi})) + \left(\frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1}} (\overline{P} - P(\boldsymbol{\psi}), \operatorname{div}(\mathbf{v} - \boldsymbol{\psi})) \right] d\tau \tag{B.3.28}$$

for all $\boldsymbol{\psi} \in X$.

A possible choice $\boldsymbol{\psi} = \mathbf{v} \pm \lambda\boldsymbol{\varphi}$, $\lambda > 0$, and continuity of the operators in (B.3.28) (for $\lambda \to 0_+$) then imply

$$0 = \int_0^t (\overline{\mathbf{S}} - \mathbf{S}(\mathbf{D}(\mathbf{v})), \mathbf{D}(\boldsymbol{\varphi})) + \left(\frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1}} (\overline{P} - P(\mathbf{v}), \operatorname{div} \boldsymbol{\varphi})\, d\tau, \tag{B.3.29}$$

that says

$$\int_0^t \left\{ (\overline{\mathbf{S}}, \mathbf{D}(\boldsymbol{\varphi})) + \left(\frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1}} (\overline{P}, \operatorname{div} \boldsymbol{\varphi}) \right\} d\tau$$
$$= \int_0^t \left\{ (\mathbf{S}(\mathbf{D}(\mathbf{v})), \mathbf{D}(\boldsymbol{\varphi})) + \left(\frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1}} (P(\mathbf{v}), \operatorname{div} \boldsymbol{\varphi}) \right\} d\tau.$$

and (B.3.26) leads to the equation for $(\mathbf{v}, p) = (\mathbf{v}^{\varepsilon,\eta}, p^{\varepsilon,\eta})$

$$\int_0^t \langle \mathbf{v}_{,t}, \boldsymbol{\varphi}\rangle - ((\mathbf{v} * \omega^\eta) \otimes \mathbf{v}, \nabla \boldsymbol{\varphi}) + (\mathbf{S}(\mathbf{D}(\mathbf{v})), \mathbf{D}(\boldsymbol{\varphi}))\, d\tau$$
$$+ \int_0^t \Big(\underbrace{\left(\frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1}} |\operatorname{div} \mathbf{v}|^{\frac{-\alpha}{1+\alpha}} \operatorname{div} \mathbf{v}}_{=:-p}, \operatorname{div} \boldsymbol{\varphi} \Big)\, d\tau = \int_0^t \langle \mathbf{b}, \boldsymbol{\varphi}\rangle\, d\tau \tag{B.3.30}$$

valid for all $\boldsymbol{\varphi} \in X$.

3.4. Further uniform estimates w.r.t ε and η

Recall first that $\{\mathbf{v}^{\varepsilon,\eta}\}$ fulfil (B.3.15), (B.3.17) and (B.3.19) where K is independent of ε and η. Next, we focus on the uniform estimates of the pressure $p^{\varepsilon,\eta} := -\frac{1}{\varepsilon^{\frac{1}{\alpha+1}}} |\operatorname{div} \mathbf{v}^{\varepsilon,\eta}|^{\frac{-\alpha}{1+\alpha}} \operatorname{div} \mathbf{v}^{\varepsilon,\eta}$. We start with taking in (B.3.30) $\boldsymbol{\varphi} = \nabla h^{\varepsilon,\eta}$, where $h^{\varepsilon,\eta}$ solves

$$\triangle h^{\varepsilon,\eta} = |p^{\varepsilon,\eta}|^\alpha p^{\varepsilon,\eta} \quad \text{in } \mathbb{R}^3,\ h^{\varepsilon,\eta} \text{ being } \Omega\text{-periodic}, \int_\Omega h^{\varepsilon,\eta}(x)\, dx = 0 \tag{B.3.31}$$

with the estimate

$$\|h^{\varepsilon,\eta}\|_{2,q} \le C \|p^{\varepsilon,\eta}\|_{(\alpha+1)q}^{\alpha+1} \qquad q \in (1, \infty), \tag{B.3.32}$$

where C is independent of ε and η, but it may depend on q and L. As the result we obtain

$$\int_0^t \|p^{\varepsilon,\eta}\|_{\alpha+2}^{\alpha+2} d\tau \le \int_0^t \langle \mathbf{v}_{,t}^{\varepsilon,\eta}, \nabla h^{\varepsilon,\eta}\rangle d\tau$$
$$+ \int_0^t \int_\Omega |\mathbf{v}^{\varepsilon,\eta}||\mathbf{v}^{\varepsilon,\eta} * \omega^\eta||\nabla^{(2)} h^{\varepsilon,\eta}| dx\, d\tau \tag{B.3.33}$$
$$+ \int_0^t \int_\Omega |\mathbf{S}(\mathbf{D}(\mathbf{v}^{\varepsilon,\eta}))||\mathbf{D}(\nabla h^{\varepsilon,\eta})| dx\, d\tau := I_1 + I_2 + I_3.$$

Terms I_2 and I_3 and (B.3.32) suggest to set q such that $q(\alpha+1)$-norm for p on right hand side of (B.3.32) equals to $(\alpha+2)$-norm of p on the left hand side of (B.3.33). This gives $q := \frac{\alpha+2}{\alpha+1}$ and it is easy to check that using (B.3.16) and (B.3.17), I_2 and I_3 are then controlled†

Focusing on I_1, we observe first that $\mathbf{v}^{\varepsilon,\eta}$ can be decomposed into the sum

$$\mathbf{v}^{\varepsilon,\eta} = \mathbf{v}^{\varepsilon,\eta}_{div} + \nabla g^{\mathbf{v}^{\varepsilon,\eta}} \qquad \text{(Helmholtz decomposition)} \tag{B.3.34}$$

where

$$\operatorname{div} \mathbf{v}^{\varepsilon,\eta}_{div} = 0, \qquad \mathbf{v}^{\varepsilon,\eta}_{div} \text{ and } g^{\mathbf{v}^{\varepsilon,\eta}} \text{ are } \Omega - \text{periodic} \tag{B.3.35}$$

and

$$-\triangle g^{\mathbf{v}^{\varepsilon,\eta}} = -\operatorname{div} \mathbf{v}^{\varepsilon,\eta} = \varepsilon |p^{\varepsilon,\eta}|^\alpha p^{\varepsilon,\eta}, \qquad \int_\Omega g^{\mathbf{v}^{\varepsilon,\eta}}\, dx = 0. \tag{B.3.36}$$

Thus, $\frac{g^{\mathbf{v}^{\varepsilon,\eta}}}{\varepsilon} = -h^{\varepsilon,\eta}$ as follows from (B.3.31) and (B.3.36) and unique solvability of the Laplace equation in the considered class of functions. Consequently

$$\begin{aligned}\int_0^t \langle \mathbf{v}^{\varepsilon,\eta}_{,t}, \nabla h^{\varepsilon,\eta}\rangle\, d\tau &= \int_0^t \langle \nabla g^{\mathbf{v}^{\varepsilon,\eta}}_{,t}, \nabla h^{\varepsilon,\eta}\rangle d\tau = -\varepsilon \int_0^t \langle \nabla \frac{g^{\mathbf{v}^{\varepsilon,\eta}}_{,t}}{\varepsilon}, \nabla(-h^{\varepsilon,\eta})\rangle\, d\tau \\ &= -\varepsilon \int_0^t \frac{1}{2}\frac{d}{dt} \|\nabla \frac{g^{\mathbf{v}^{\varepsilon,\eta}}_{,t}}{\varepsilon}\|_2^2 d\tau = -\frac{1}{2\varepsilon}(\|g^{\mathbf{v}^{\varepsilon,\eta}}(t)\|_2^2 - \|g^{\mathbf{v}^{\varepsilon,\eta}}(0)\|_2^2) \le 0\end{aligned} \tag{B.3.37}$$

as $\triangle g^{\mathbf{v}^{\varepsilon,\eta}}(0) = \operatorname{div} \mathbf{v}^{\varepsilon,\eta}(0) = 0 \Rightarrow g^{\mathbf{v}^{\varepsilon,\eta}}(0) = 0$. (The reader may wish to perform this argument how to treat the term I_1 first for smooth approximations $(\mathbf{v}^{\varepsilon,\eta}_m, p^{\varepsilon,\eta}_m)$ and then pass to the limit as $m \to \infty$, whereas $\mathbf{v}^{\varepsilon,\eta}_m$ follows from the density of smooth functions in $L^r(0,T;W^{1,r}_{per})$ and $p^{\varepsilon,\eta}_m := -(\varepsilon)^{\frac{1}{\alpha+1}} |\operatorname{div} \mathbf{v}^{\varepsilon,\eta}_m|^{-\frac{\alpha}{\alpha+1}} \operatorname{div} \mathbf{v}^{\varepsilon,\eta}_m$.) Thus, it follows from (B.3.33)-(B.3.37) that†

$$\int_0^T \|p^{\varepsilon,\eta}\|^{\alpha+2}_{\alpha+2}\, d\tau \le K. \tag{B.3.38}$$

Consequently, we can strengthen (B.3.19) to conclude from (B.3.30) that

$$\int_0^T \|\mathbf{v}^{\varepsilon,\eta}_{,t}\|^{\alpha+2}_{W^{-1,\alpha+2}_{per}}\, d\tau \le K. \tag{B.3.39}$$

† To be more explicit, considering for example the term I_3 we have

$$|I_3| \le \left(\int_0^t \|\mathbf{S}(\mathbf{D}(\mathbf{v}^{\varepsilon,\eta}))\|^{\alpha+2}_{\alpha+2} d\tau\right)^{\frac{1}{\alpha+2}} \left(\int_0^t \|\nabla^2 h^{\varepsilon,\eta}\|^{\frac{\alpha+2}{\alpha+1}}_{\frac{\alpha+2}{\alpha+1}} d\tau\right)^{\frac{\alpha+1}{\alpha+2}} \le KC \left(\int_0^t \|p^{\varepsilon,\eta}\|^{\alpha+2}_{\alpha+2} d\tau\right)^{\frac{\alpha+1}{\alpha+2}},$$

where we used the fact that $\alpha+2 \ge r'$ for arbitrary $r > 1$.

† Note that this step can be repeated without any change for the Navier's boundary conditions, it is however open in general for no-slip boundary conditions due to the fact that ∇h is not an admissible function.

The estimates (B.3.38) and (B.3.39) are uniform w.r.t. η. If we however relax this requirement and use the fact that $\mathbf{v} * \omega^\eta$ is a smooth function for $\eta > 0$ fixed we obtain, proceeding as above,

$$\int_0^T \|p^{\varepsilon,\eta}\|_{r'}^{r'}\, d\tau \le C(\eta^{-1}) \tag{B.3.40}$$

and

$$\int_0^T \|\mathbf{v}_{,t}^{\varepsilon,\eta}\|_{W_{per}^{-1,r'}}^{r'}\, d\tau \le C(\eta^{-1}). \tag{B.3.41}$$

3.5. Limit $\varepsilon \to 0$

For fixed $\eta > 0$, we establish in this section the existence of (suitable) weak solution to the problem

$$(\mathcal{P}^\eta) \qquad \begin{cases} \operatorname{div}\mathbf{v} = 0,\ \mathbf{v}_{,t} + \operatorname{div}((\mathbf{v} * \omega^\eta) \otimes \mathbf{v}) - \operatorname{div}(\mathbf{S}(\mathbf{D}(\mathbf{v}))) = -\nabla p + \mathbf{b} \\ v_i, p \quad \Omega-\text{periodic with} \quad \displaystyle\int_\Omega v_i\, dx = \int_\Omega p\, dx = 0 \quad \text{for } i = 1,2,3 \\ \mathbf{v}(0,\cdot) = \mathbf{v}_0 \quad \text{in } \Omega, \end{cases}$$

if the parameter r appearing in (B.2.8) fulfils

$$r > \frac{8}{5}. \tag{B.3.42}$$

Using the estimates (B.3.15), (B.3.17), (B.3.40) and (B.3.41) uniform w.r.t. $\varepsilon > 0$, an letting $\varepsilon \to 0$ we can find a sequence $\{\mathbf{v}^n, p^n\}$ chosen from $\{\mathbf{v}^{\varepsilon,\eta}, p^{\varepsilon,\eta}\}$, and a limit element $\{\mathbf{v}, p\} := \{\mathbf{v}^\eta, p^\eta\}$ such that

$$\mathbf{v}^n \rightharpoonup \mathbf{v} \quad \text{weakly in } L^r(0,T; W^{1,r}_{per}) \text{ and } {}^*\text{-weakly in } L^\infty(0,T; L^2_{per}) \tag{B.3.43}$$

$$\mathbf{v}^n_{,t} \rightharpoonup \mathbf{v}_{,t} \quad \text{weakly in } L^{r'}(0,T; W^{-1,r'}_{per}) \tag{B.3.44}$$

$$\mathbf{v}^n \to \mathbf{v} \quad \text{strongly in } \begin{cases} L^r(0,T; L^q_{per}) \text{ for all } q \in \langle 1, \frac{3r}{3-r}) \\ L^s(0,T; L^s_{per}) \text{ for all } s \in \langle 1, \frac{5r}{3}) \end{cases} \tag{B.3.45}$$

$$\mathbf{S}(\mathbf{D}(\mathbf{v}^n)) \rightharpoonup \overline{\mathbf{S}} \quad \text{weakly in } L^{r'}(0,T; L^{r'}_{per}) \tag{B.3.46}$$

and

$$p^n \rightharpoonup p \quad \text{weakly in } L^{r'}(0,T; L^{r'}_{per}). \tag{B.3.47}$$

It also follows from the fourth term in (B.3.15) (as $\varepsilon \to 0$) that

$$\operatorname{div}\mathbf{v} = 0 \qquad \text{a.e. in } (0,T) \times \Omega. \tag{B.3.48}$$

Consider (B.3.30) for $\{\mathbf{v}^n, p^n\}$ instead of $\{\mathbf{v}, p\} = \{\mathbf{v}^{\varepsilon,\eta}, p^{\varepsilon,\eta}\}$, we can pass to the limit as $n \to \infty$ and obtain with help of (B.3.43)-(B.3.47)

$$\begin{aligned}\int_0^t \Big\{ \langle \mathbf{v}_{,t}, \boldsymbol{\varphi}\rangle - ((\mathbf{v} * \omega^\eta) \otimes \mathbf{v}, \nabla\boldsymbol{\varphi}) + (\mathbf{S}(\mathbf{D}(\mathbf{v})), \mathbf{D}(\boldsymbol{\varphi})) - (p, \operatorname{div}\boldsymbol{\varphi}) \Big\} d\tau \\ = \int_0^t \langle \mathbf{b}, \boldsymbol{\varphi}\rangle d\tau \qquad \text{for all } \boldsymbol{\varphi} \in L^r(0,T,W^{1,r}_{per})\,.\end{aligned} \tag{B.3.49}$$

provided that we show that

$$\overline{\mathbf{S}} = \mathbf{S}(\mathbf{D}(\mathbf{v})) \qquad \text{a.e. in } (0,T) \times \Omega. \tag{B.3.50}$$

For this purpose, we consider again (B.3.30) for $(\mathbf{v}^n, p^n)$ and set $\boldsymbol{\varphi} = \mathbf{v}^n - \mathbf{v}$ therein. Then

$$\begin{aligned}\int_0^T \langle \mathbf{v}^n_{,t} - \mathbf{v}_{,t}, \mathbf{v}^n - \mathbf{v}\rangle dt + \int_0^T (\mathbf{S}(\mathbf{D}(\mathbf{v}^n)) - \mathbf{S}(\mathbf{D}(\mathbf{v})), \mathbf{D}(\mathbf{v}^n - \mathbf{v}))dt \\ = -\int_0^T \langle \mathbf{v}_{,t}, \mathbf{v}^n - \mathbf{v}\rangle dt + (\mathbf{S}(\mathbf{D}(\mathbf{v})), \mathbf{D}(\mathbf{v}^n - \mathbf{v}))dt + \langle \mathbf{b}, \mathbf{v}^n - \mathbf{v}\rangle dt \\ - \int_0^T ((\mathbf{v}^n * \omega^\eta) \otimes \mathbf{v}^n, \nabla(\mathbf{v}^n - \mathbf{v}))dt.\end{aligned} \tag{B.3.51}$$

Let $n \to \infty$. The first term on the right hand side of (B.3.51) vanishes due to weak convergence (B.3.43), the last integral that equals to $\int_0^T (\operatorname{div}(\mathbf{v}^n * \omega^\eta)\mathbf{v}^n, \mathbf{v}^n - \mathbf{v})dt + \int_0^T ((\mathbf{v}^n * \omega^\eta) \otimes (\mathbf{v}^n - \mathbf{v}), \nabla\mathbf{v}^n)$ also vanishes since (B.3.45) and $|\nabla\mathbf{v}^n|\,|\mathbf{v}^n|$ is uniformly integrable if $r > \frac{8}{5}$.

Consequently, using (B.3.51) and (B.2.20) resp. (B.2.21), we have†

$$\lim_{n\to\infty} \int_0^T \|\mathbf{D}(\mathbf{v}^n) - \mathbf{D}(\mathbf{v})\|_r^r dt = 0, \tag{B.3.52}$$

Thus $\mathbf{D}(\mathbf{v}^n) \to \mathbf{D}(\mathbf{v})$ a.e. in $(0,T)\times\Omega$ (at least for subsequence) and Vitali's Lemma (see [81] Lemma 2.1 or Dunford and Schvartz [27]) and (B.3.17) give $\mathbf{S}(\mathbf{D}(\mathbf{v}^n)) \to \mathbf{S}(\mathbf{D}(\mathbf{v}))$ a.e. in $(0,T) \times \Omega$ that implies (B.3.50).

† We also used the fact that

$$\int_0^T \langle \mathbf{v}^n_{,t} - \mathbf{v}_{,t}, \mathbf{v}^n - \mathbf{v}\rangle dt = \frac{1}{2}\|\mathbf{v}^n(T) - \mathbf{v}(T)\|_2^2 - \frac{1}{2}\|\mathbf{v}^n(0) - \mathbf{v}(0)\|_2^2 = \frac{1}{2}\|\mathbf{v}^n(T) - \mathbf{v}(T)\|_2^2.$$

This requires to check that $\mathbf{v}^n(0) = \mathbf{v}(0) = \mathbf{v}_0$. We skip it however here and show it later for more difficult case.

Taking $\boldsymbol{\varphi} = \mathbf{v}\phi, \phi \in \mathcal{D}(-\infty,\infty;\mathcal{C}^\infty_{per})$ in (B.3.49) we conclude the *local energy equality*

$$\frac{1}{2}\int_\Omega \left(|\mathbf{v}|^2\phi\right)(t,x)\,dx + \int_0^t\int_\Omega \mathbf{S}(\mathbf{D}(\mathbf{v}))\cdot\mathbf{D}(\mathbf{v})\phi\,dx\,d\tau$$
$$= \frac{1}{2}\int_\Omega |\mathbf{v}_0(x)|^2\phi(0,x)\,dx + \frac{1}{2}\int_0^t\int_\Omega |\mathbf{v}|^2\phi_{,t}\,dx\,d\tau + \int_0^t \langle \mathbf{b},\mathbf{v}\phi\rangle\,d\tau \tag{B.3.53}$$
$$+\int_0^t\int_\Omega\left(\frac{|\mathbf{v}|^2}{2}(\mathbf{v}*\omega^\eta) + p\mathbf{v} - \mathbf{S}(\mathbf{D}(\mathbf{v}))\mathbf{v}\right)\cdot\nabla\phi\,dx\,d\tau$$

Also taking $\boldsymbol{\varphi} = \mathbf{v}$ in (B.3.49) we have global energy equality

$$\frac{1}{2}\|\mathbf{v}(t)\|_2^2 + \int_0^t (\mathbf{S}(\mathbf{D}(\mathbf{v})),\mathbf{D}(\mathbf{v}))\,d\tau = \frac{1}{2}\|\mathbf{v}_0\|_2^2 + \int_0^t \langle \mathbf{b},\mathbf{v}\rangle d\tau \tag{B.3.54}$$

and thanks to lower-semicontinuity of the norms w.r.t. weak convergence it follows from (B.3.15),(B.3.16), (B.3.17), (B.3.38) and (B.3.39) that $(\mathbf{v},p) = (\mathbf{v}^\eta,p^\eta)$ fulfils the following estimates that are uniform w.r.t. $\eta > 0$:

$$\sup_{t\in[0,T]}\|\mathbf{v}^\eta(t)\|_2^2 + \int_0^T \|\nabla\mathbf{v}^\eta\|_r^r\,dt + \int_0^T \|\mathbf{v}^\eta\|_{\frac{5r}{3}}^{\frac{5r}{3}}\,dt \le K, \tag{B.3.55}$$

$$\int_0^T \|\mathbf{S}(\mathbf{D}(\mathbf{v}^\eta))\|_{r'}^{r'}\,dt \le K, \tag{B.3.56}$$

$$\int_0^T \|p^\eta\|_{\alpha+2}^{\alpha+2}\,dt \le K \qquad \text{with } \alpha+2 = \begin{cases} r' & \text{if } r \ge \frac{11}{5} \\ \frac{5r}{6} & \text{if } r < \frac{11}{5}\end{cases} \tag{B.3.57}$$

$$\int_0^T \|\mathbf{v}^\eta_{,t}\|_{W^{-1,\alpha+2}_{per}}^{\alpha+2}\,dt \le K. \tag{B.3.58}$$

3.6. Limit $\eta\to 0$, the case $r \ge \frac{11}{5}$

If $r \ge \frac{11}{5}$, the available uniform estimates coincides with those needed to pass to the limit in Subsection 3.5. Thus, we proceed as above. The quadratic convective term requires

$$\mathbf{v}^n \to \mathbf{v} \qquad \text{strongly in } L^2(0,T;L^2(\Omega)). \tag{B.3.59}$$

This follows from the Aubin-Lions lemma provided that

$$r > \frac{6}{5},$$

which is of course trivially fulfilled here (and also in the next Subsection). The other argument coincides with those used in Subsection 3.5. For $r \ge \frac{11}{5}$, we have thus the existence of weak solution fulfilling energy equality, local energy equality, etc. The proof of Theorem 3.1 in the case $r \ge \frac{11}{5}$ is complete.

3.7. Limit $\eta \to 0$, the case $\frac{8}{5} < r < \frac{11}{5}$

We start observing that if $\mathbf{u} \in L^{\frac{5r}{3}}(0,T;L^{\frac{5r}{3}}_{per})$ and $\nabla\mathbf{u} \in L^r(0,T;L^r_{per})$ then

$$[\nabla\mathbf{u}](\mathbf{u} * \omega^\eta) \in L^1(0,T;L^1_{per}) \tag{B.3.60}$$

uniformly w.r.t. $\eta > 0$ provided that

$$r \geq \frac{8}{5}. \tag{B.3.61}$$

Thus, introducing for $r > \frac{8}{5}$ and $\delta \in (0, \frac{5}{8}(r - \frac{8}{5}))$ the space of divergenceless functions

$$X_\delta := \{\boldsymbol{\varphi} \in L^r(0,T;W^{1,r}_{per,\mathrm{div}}) \cap L^{\frac{1+\delta}{\delta}}(0,T;L^{\frac{1+\delta}{\delta}}_{per})\}\,, \tag{B.3.62}$$

and using the fact that

$$-((\mathbf{v} * \omega^\eta) \otimes \mathbf{v}, \nabla\boldsymbol{\varphi}) = ([\nabla\mathbf{v}](\mathbf{v} * \omega^\eta), \boldsymbol{\varphi}), \tag{B.3.63}$$

it follows from (B.3.49) that

$$\|\mathbf{v}^\eta_{,t}\|_{X^*_\delta} \leq K \qquad \text{uniformly w.r.t. } \eta > 0. \tag{B.3.64}$$

Letting η tend to zero, and using (B.3.55)-(B.3.58), (B.3.64) and the Aubin-Lions compactness lemma, we find a subsequence $\{(\mathbf{v}^k, p^k)\}_{k\in\mathbb{N}}$ and "its weak limit" $\{(\mathbf{v}, p)\}$ such that ($r < \frac{11}{5}$)

$$\mathbf{v}^k \rightharpoonup \mathbf{v} \qquad \text{weakly in } L^r(0,T;W^{1,r}_{per}) \text{ and *-weakly in } L^\infty(0,T;L^2_{per}), \tag{B.3.65}$$

$$\mathbf{v}^k_{,t} \rightharpoonup \mathbf{v}_{,t} \qquad \text{weakly in } L^{\frac{5r}{6}}(0,T;W^{-1,\frac{5r}{6}}_{per})\text{and in } X^*_\delta, \tag{B.3.66}$$

$$\mathbf{v}^k \to \mathbf{v} \qquad \text{strongly in } L^r(0,T;L^q_{per}) \text{ for all } q \in \langle 1, \frac{3r}{3-r}), \tag{B.3.67}$$

$$\mathbf{v}^k \to \mathbf{v} \qquad \text{strongly in } L^s(0,T;L^2_{per}) \text{ for all } s > 1, \text{ if } r > \frac{6}{5}, \tag{B.3.68}$$

$$\mathbf{v}^k \to \mathbf{v} \qquad \text{a.e. in } (0,T) \times \Omega, \tag{B.3.69}$$

$$p^k \rightharpoonup p \qquad \text{weakly in } L^{\frac{5r}{6}}(0,T;L^{\frac{5r}{6}}_{per}), \tag{B.3.70}$$

and there is $\overline{\mathbf{S}} \in L^{r'}(0,T;L^{r'}_{per})$ so that

$$\mathbf{S}(\mathbf{D}(\mathbf{v}^k)) \rightharpoonup \overline{\mathbf{S}} \quad \text{weakly in } L^{r'}(0,T;L^{r'}_{per}).. \tag{B.3.71}$$

Also, it follows from (B.3.68) that

$$\mathbf{v}^k(t) \to \mathbf{v}(t) \qquad \text{strongly in } L^2_{per} \text{ for all } t \in [0,T] \setminus N, \tag{B.3.72}$$

where N has zero one-dimensional Lebesgue measure.

In order to identify $\overline{\mathbf{S}}$ with $\mathbf{S}(\mathbf{D}(\mathbf{v}))$ we showed in previous sections that this follows from

$$\lim_{n\to\infty} \int_0^T \int_\Omega (\mathbf{S}(\mathbf{D}(\mathbf{v}^n)) - \mathbf{S}(\mathbf{D}(\mathbf{v}))) \cdot (\mathbf{D}(\mathbf{v}^n) - \mathbf{D}(\mathbf{v}))\, dx\, dt = 0\,, \tag{B.3.73}$$

using the fact that this integral operator is uniformly monotone (note that it would suffice to know that this operator is strictly monotone).

Here, we will show a condition weaker than (B.3.73), namely:

(*) for every $\varepsilon^* > 0$ and for some $\theta \in (\frac{1}{r}, 1)$ there is a subsequence $\{\mathbf{v}^n\}_{n=1}^\infty$ of $\{\mathbf{v}^k\}_{k=1}^\infty$ such that

$$\lim_{n\to\infty} \int_0^T \int_\Omega \Big[(\mathbf{S}(\mathbf{D}(\mathbf{v}^n)) - \mathbf{S}(\mathbf{D}(\mathbf{v}))) \cdot (\mathbf{D}(\mathbf{v}^n) - \mathbf{D}(\mathbf{v}))\Big]^\theta dx\, dt \le \varepsilon^*.$$

Once **(*)** is proved, we take $\varepsilon^*_m \to 0$ and for each $m \in \mathbb{N}$ select gradually (not relabelled) subsequences so that the Cantor diagonal sequence (again not relabelled) fulfils

$$\mathbf{D}(\mathbf{v}^n) \to \mathbf{D}(\mathbf{v}) \quad \text{a.e. in } (0,T) \times \mathbb{R}^3. \tag{B.3.74}$$

Vitali's theorem and (B.3.74) then imply $\overline{\mathbf{S}} = \mathbf{S}(\mathbf{D}(\mathbf{v}))$ a.e. in $(0,T) \times \mathbb{R}^3$. The convergences (B.3.65)-(B.3.71) and (B.3.74) clearly suffice to pass to the limit from the weak formulation of Problem $(\mathcal{P}^\eta)$ to the weak form of Problem $(\mathcal{P})$.

It remains to verify **(*)**. For this purpose we set

$$g^k := \big(|\nabla \mathbf{v}^k|^r + |\nabla \mathbf{v}|^r + (|\mathbf{S}(\mathbf{D}(\mathbf{v}^k))| + |\mathbf{S}(\mathbf{D}(\mathbf{v}))|)(|\mathbf{D}(\mathbf{v}^k)| + |\mathbf{D}(\mathbf{v})|)\big)\,. \tag{B.3.75}$$

Clearly $g^k \ge 0$ and

$$0 \le \int_0^T \int_\Omega g^k\, dx\, dt \le K. \qquad (K > 1) \tag{B.3.76}$$

We prove the following property (K is referred to (B.3.76))

(**) for every $\varepsilon^{**} > 0$ there is $L \le \dfrac{\varepsilon^{**}}{K}$, $\{\mathbf{v}^n\}_{n=1}^\infty \subset \{\mathbf{v}^k\}_{k=1}^\infty$ and sets $E^n := \{(x,t) \in (0,T) \times \Omega; L^2 \le |\mathbf{v}^n(t,x) - \mathbf{v}(t,x)| < L\}$ such that

$$\int_E g^n\, dx\, dt \le \varepsilon^{**}.$$

To see it, we fix $\varepsilon^{**} \in (0,1)$, set $L_1 = \frac{\varepsilon^*}{K}$ and take $N \in \mathbb{N}$ such that for $N\varepsilon^{**} > K$ (K refers to (B.3.76)). Defining iteratively $L_i = L_{i-1}^2$ for $i = 2, 3, \dots N$, we set

$$E^{k,i} = \{(t,x) \in (0,T) \times \Omega;\ L_i^2 \le |\mathbf{v}^k(t,x) - \mathbf{v}(t,x)| < L_i\}\ (i = 1, 2, \dots N.)$$

For $k \in \mathbb{N}$ fixed, $E^{k,i}$ are mutually disjoint. Consequently,

$$\sum_{i=1}^{N} \int_{E^{k,i}} g^k \, dx \, dt \le K.$$

As $N\varepsilon^{**} > K$, for each $k \in \mathbb{N}$ there is $i_0(k) \in \{1, \dots N\}$ such that

$$\int_{E^{k,i_0(k)}} g^k \, dx \, dt \le \varepsilon^{**}.$$

However, $i_0(k)$ are taken from finite set of indices. Then, there has to be a sequence $\{\mathbf{v}^n\} \subset \{\mathbf{v}^k\}$ such that $i_0(n) = i_0^*$ for each n ($i_0^* \in \{1, 2, \dots N\}$ fixed). The property **(**)** is then proved setting $L = L_{i_0^*}$ and $E^n = E^{n,i_0^*}$.

Returning to our aim to verify **(*)**, we consider $(\mathbf{v}^n, p^n)$ satisfying (B.3.49), (B.3.53), (B.3.54) and having all convergence properties stated in (B.3.65)-(B.3.71) and **(**)**, and we set $\boldsymbol{\varphi}$ in (B.3.49) of the form

$$\boldsymbol{\varphi}^n := \mathbf{h}^n - \nabla z := (\mathbf{v}^n - \mathbf{v}) \left(1 - \min\left(\frac{|\mathbf{v}^n - \mathbf{v}|}{L}, 1\right)\right) - \nabla z^n, \tag{B.3.77}$$

where L comes from **(**)**, and z^n solves for

$$f^n := \operatorname{div}\left((\mathbf{v}^n - \mathbf{v}) \left(1 - \min\left(\frac{|\mathbf{v}^n - \mathbf{v}|}{L}, 1\right)\right)\right) = \operatorname{div} \mathbf{h}^n$$

the problem

$$-\triangle z^n = f^n \quad z^n \text{ being } \Omega\text{-periodic}\,, \int_\Omega z^n \, dx = 0. \tag{B.3.78}$$

We summarize the properties of $\mathbf{h}^n$, z^n and $\boldsymbol{\varphi}^n$. Introducing Q^n through $Q^n := \{(t,x) \in (0,T) \times \Omega; |\mathbf{v}^n(t,x) - \mathbf{v}(t,x)| < L\}$ we note first that

$$\mathbf{h}^n = \mathbf{0} \quad \text{ on } (0,T) \times \Omega \setminus Q^n, \tag{B.3.79}$$

and

$$|\mathbf{h}(t,x)| \le L \quad \text{ for all } (t,x) \in (0,T) \times \Omega. \tag{B.3.80}$$

Consequently, owing to (B.3.69) and Lebesgue's theorem we have for all $s \in \langle 1, \infty)$

$$\int_0^T \|\mathbf{h}^n\|_s^s \, dt \to 0 \qquad \text{as } n \to \infty, \tag{B.3.81}$$

and due to L^s-theory for the Laplace-operator, it follows from (B.3.78) that

$$\int_0^T \|\nabla z^n\|_s^s \to 0 \qquad \text{as } n \to \infty. \tag{B.3.82}$$

From (B.3.81) and (B.3.82) it follows ($\boldsymbol{\varphi}^n$ defined in (B.3.77))

$$\int_0^T \|\boldsymbol{\varphi}^n\|_s^s \, dt \to 0 \qquad \text{as } n \to \infty. \tag{B.3.83}$$

Next, (χ_Z denotes the characteristic function of a set Z)

$$f^n = \operatorname{div} \mathbf{h}^n = (\mathbf{v}^n - \mathbf{v}) \cdot \frac{(\mathbf{v}^n - \mathbf{v})_j}{L} \frac{\nabla(\mathbf{v}^n - \mathbf{v})_j}{|\mathbf{v}^n - \mathbf{v}|} \chi_{Q^n}.$$

Splitting Q^n into E^n (introduced in **(**)**) and its complement, and using the fact that $|f^n|^r \le |\nabla(\mathbf{v}^n - \mathbf{v})|^r \le |\nabla \mathbf{v}^n|^r + |\nabla \mathbf{v}|^r$ on E^n and $|f^n|^r \le L(|\nabla \mathbf{v}^n|^r + |\nabla \mathbf{v}|^r \le \frac{\varepsilon^*}{K}(|\nabla \mathbf{v}^n|^r + |\nabla \mathbf{v}|^r)$ on $Q^n \setminus E^n$ we conclude from **(**)** that

$$\int_0^T \|f^n\|_{r,Q^n}^r \, dt \le 2\varepsilon^*, \tag{B.3.84}$$

and using L^r-regularity for the Laplace operator and (B.3.78)

$$\int_0^T \|\nabla^{(2)} z^n\|_{r,Q^n}^r \, dt \le 2C_{reg}\varepsilon^*. \tag{B.3.85}$$

Note also that

$$\boldsymbol{\varphi}^n \rightharpoonup 0 \qquad \text{weakly in } L^r(0,T; W^{1,r}_{per}) \text{ and also in } X_\delta, \tag{B.3.86}$$

where X_δ is defined in (B.3.62).

Inserting $\boldsymbol{\varphi}^n$ of the form (B.3.77) into (B.3.49) we obtain (note that the term with pressure vanishes as $\operatorname{div} \boldsymbol{\varphi}^n = 0$)

$$\begin{aligned} \int_0^T \langle \mathbf{v}^n_{,t} - \mathbf{v}_{,t}, \boldsymbol{\varphi}^n \rangle \, dt &+ \int_0^T (\mathbf{S}(\mathbf{D}(\mathbf{v}^n)) - \mathbf{S}(\mathbf{D}(\mathbf{v})), \mathbf{D}(\boldsymbol{\varphi}^n)) \, dt \\ &= -\int_0^T ((\mathbf{v}^n * \omega^{\frac{1}{n}})[\nabla \mathbf{v}^n], \boldsymbol{\varphi}^n) \, dt + \int_0^T \langle \mathbf{b}, \boldsymbol{\varphi}^n \rangle \, dt \\ &- \int_0^T \langle \mathbf{v}_{,t}, \boldsymbol{\varphi}^n \rangle \, dt - \int_0 (\mathbf{S}(\mathbf{D}(\mathbf{v})), \mathbf{D}(\boldsymbol{\varphi}^n)) \, dt. \end{aligned} \tag{B.3.87}$$

It is not difficult to see that all terms on the right hand side of (B.3.87) tend to 0: the first one due to (B.3.60) and (B.3.83), the third one due to the fact that $\mathbf{v}_{,t} \in X_\delta^*$ and (B.3.86) holds, and the second and fourth terms also due to (B.3.86).

Let $H : \langle 0, \infty) \to \mathbb{R}$ satisfies $H(0) = 0$ and $H'(s) = (1 - \min(\frac{\sqrt{s}}{L}, 1))$. Then the first term on the left hand side is non-negative as

$$\int_0^T \langle \mathbf{v}^n_{,t} - \mathbf{v}_{,t}, \boldsymbol{\varphi}^n \rangle \, dt = \int_0^T \langle \mathbf{v}^n_{,t} - \mathbf{v}_{,t}, \mathbf{h}^n \rangle \, dt = H(|\mathbf{v}^n - \mathbf{v}|^2(T)) \ge 0.$$

We thus conclude from (B.3.87) that

$$\begin{aligned} &\lim_{n\to\infty} \int_{Q^n} (\mathbf{S}(\mathbf{D}(\mathbf{v}^n)) - \mathbf{S}(\mathbf{D}(\mathbf{v})))(\mathbf{D}(\mathbf{v}^n) - \mathbf{D}(\mathbf{v}))) \, dx \, dt \\ &\le \lim_{n\to\infty} \int_{Q^n} (|\mathbf{S}(\mathbf{D}(\mathbf{v}^n))| + |\mathbf{S}(\mathbf{D}(\mathbf{v}))|) \Big(|\nabla(\mathbf{v}^n - \mathbf{v})| + |\nabla^{(2)} z^n| \Big) \, dx \, dt \,. \end{aligned} \tag{B.3.88}$$

Arguing analogously asin the derivation of

$$\lim_{n\to\infty} \int_{Q^n} (\mathbf{S}(\mathbf{D}(\mathbf{v}^n)) - \mathbf{S}(\mathbf{D}(\mathbf{v})))(\mathbf{D}(\mathbf{v}^n) - \mathbf{D}(\mathbf{v}))) \, dx \, dt \le C\varepsilon^* \tag{B.3.89}$$

Since

$$\begin{aligned} &\int_0^T \int_\Omega \Big[(\mathbf{S}(\mathbf{D}(\mathbf{v}^n)) - \mathbf{S}(\mathbf{D}(\mathbf{v})))(\mathbf{D}(\mathbf{v}^n - \mathbf{v})) \Big]^\theta dx \, dt \\ &\qquad = \int_{Q^n} [\ldots]^\theta \, dx \, dt + \int_{(0,T)\times\Omega\setminus Q^n} [\ldots]^\theta \, dx \, dt \\ &\qquad \overset{\text{Hölder}}{\le} \left(\int_{Q^n} [\ldots] \, dx \, dt \right)^\theta |Q^n|^{1-\theta} \\ &\qquad\quad + \left(\int_{(0,T)\times\Omega\setminus Q^n} [\ldots] \, dx \, dt \right)^\theta |\{(t,x); |\mathbf{v}^n - \mathbf{v}| > L\}|^{1-\theta} \le C^* \varepsilon^*, \end{aligned}$$

where we apply (B.3.89) to handle the first term and the convergence in measure to treat the second term, we finally conclude that **(*)** holds.

3.8. Continuity w.r.t. time in weak topology of L^2_{per}

With all convergences established in previous sections, particularly with (B.3.65)-(B.3.70) and (B.3.74), it is straightforward to conclude that $(\mathbf{v}, p)$ fulfils weak identity (B.2.15). Taking here $\boldsymbol{\varphi}$ of the form

$$\boldsymbol{\varphi}(\tau, x) = \chi_{\langle t_0, t\rangle}(\tau) \tilde{\boldsymbol{\varphi}}(x), \text{ where } \tilde{\boldsymbol{\varphi}} \in W^{1,s}_{per} \text{ and } t_0, t \in \langle 0, T\rangle,$$

we obtain

$$\begin{aligned} (\mathbf{v}(t), \tilde{\boldsymbol{\varphi}}) - (\mathbf{v}(t_0), \tilde{\boldsymbol{\varphi}}) = \int_{t_0}^t & (\mathbf{v}(\tau) \otimes \mathbf{v}(\tau), \nabla \tilde{\boldsymbol{\varphi}}) - (\mathbf{S}(\mathbf{D}(\mathbf{v})), \mathbf{D}(\tilde{\boldsymbol{\varphi}})) \\ & + \langle \mathbf{b}(\tau), \tilde{\boldsymbol{\varphi}} \rangle + (p(\tau), \operatorname{div} \tilde{\boldsymbol{\varphi}}) \, d\tau. \end{aligned}$$

This implies (for $r > \frac{6}{5}$)

$$|(\mathbf{v}(t), \tilde{\boldsymbol{\varphi}}) - (\mathbf{v}(t_0), \tilde{\boldsymbol{\varphi}})| \le c \int_{t_0}^{t} \Big(\|\mathbf{v}(\tau)\|^2_{\frac{5r}{3}} + \|\nabla \mathbf{v}(\tau)\|_r^{r-1} + \|\mathbf{b}(\tau)\|_{-1,r'} + \|p(\tau)\|_{\alpha+2} \Big)\, d\tau \, \|\tilde{\boldsymbol{\varphi}}\|_{1,s}. \tag{B.3.90}$$

Using also Hölder's inequalities over time, we have

$$\begin{aligned} |(\mathbf{v}(t), \tilde{\boldsymbol{\varphi}}) - (\mathbf{v}(t_0), \tilde{\boldsymbol{\varphi}})| \le c \Big(& |t-t_0|^{\frac{5r-6}{5r}} \left(\int_{t_0}^{t} \|\mathbf{v}(\tau)\|^{\frac{5r}{3}}_{\frac{5r}{3}} \, d\tau \right)^{\frac{6}{5r}} \\ & + |t-t_0|^{\frac{1}{r}} \left(\int_{t_0}^{t} \|\nabla \mathbf{v}(\tau)\|_r^r \, d\tau \right)^{\frac{1}{r'}} \\ & + |t-t_0|^{\frac{1}{r}} \left(\int_{t_0}^{t} \|\mathbf{b}(\tau)\|_{-1,r'}^{r'} \, d\tau \right)^{\frac{1}{r'}} \\ & + |t-t_0|^{\frac{\alpha+1}{\alpha+2}} \left(\int_{t_0}^{t} \|p(\tau)\|_{\alpha+2}^{\alpha+2} \, d\tau \right)^{\frac{1}{\alpha+2}} \Big) \|\tilde{\boldsymbol{\varphi}}\|_{1,s}. \end{aligned} \tag{B.3.91}$$

Since all integrals are finite, (B.3.91) leads to the conclusion that $(\mathbf{v}(\cdot), \tilde{\boldsymbol{\varphi}})$ is continuous at t_0 for all $\tilde{\boldsymbol{\varphi}} \in W^{1,s}_{per}(\Omega)$. In other words,

$$\mathbf{v} \in \mathcal{C}(0, T; (W^{1,s}_{per})^*_{weak}) \tag{B.3.92}$$

or

$$\lim_{t \to t_0} (\mathbf{v}(t) - \mathbf{v}(t_0), \tilde{\boldsymbol{\varphi}}) = 0 \quad \text{for all } \tilde{\boldsymbol{\varphi}} \in W^{1,s}_{per} \quad \text{and for all } t_0 \in \langle 0, T\rangle \tag{B.3.93}$$

Since $\mathbf{v} \in L^\infty(0, T; L^2_{per})$ and $W^{1,s}_{per}$ is dense in L^2_{per}, we see that $\mathbf{v} \in \mathcal{C}(\langle 0, T\rangle; L^2_{weak})$, which is $(B.2.12)_1$.

3.9. (Local) Energy equality and inequality

If $r \ge \frac{11}{5}$, (B.2.15) permits to take $\boldsymbol{\varphi} = \mathbf{v}$ or $\boldsymbol{\varphi} = \mathbf{v}\phi$ which implies both energy equality and its local version. If $\frac{11}{5} > r > \frac{8}{5}$, we take $\limsup_{n\to\infty}$ of (B.3.54) where $\mathbf{v}$ means $\mathbf{v}^n$, and $t \in \langle 0, T\rangle \setminus N$ with N introduced in (B.3.72). Since

- $\limsup_{n\to\infty} \{a_n + b_n\} \ge \limsup_{n\to\infty} \{a_n\} + \liminf_{n\to\infty} \{b_n\}$,
- $\int_0^t \int_\Omega \mathbf{S}(\mathbf{D}(\mathbf{v})) \cdot \mathbf{D}(\mathbf{v}) \, dx \, d\tau \le \liminf_{n\to\infty} \int_0^t \int_\Omega \mathbf{S}(\mathbf{D}(\mathbf{v}^n)) \cdot \mathbf{D}(\mathbf{v}^n) \, dx \, d\tau$,
- $\mathbf{v}^n_0 = \mathbf{v}^n(0)$ for all $n \in \mathbb{N}$,
- (B.3.72) and (B.3.65) hold,

we see that energy inequality (B.2.17) directly follows.

Similarly we argue letting $n \to \infty$ in (B.3.53). Here we in addition need to pass to the limit in terms

$$\int_0^t \int_\Omega \left(\frac{|\mathbf{v}^n|^2}{2} (\mathbf{v}^n * \omega^{\frac{1}{n}}) + p^n \mathbf{v}^n \right) \cdot \nabla\phi \, dx \, d\tau,$$

that follows from (B.3.68), (B.3.69) and $(\text{B.3.55})_3$ provided that

$$r > \frac{9}{5}.$$

3.10. Attainment of the initial condition

The property (B.2.14) is an easy consequence of energy inequality (B.2.17) and the following operations

$$\begin{aligned} \|\mathbf{v}(t) - \mathbf{v}_0\|_2^2 &= \|\mathbf{v}(t)\|_2^2 + \|\mathbf{v}_0\|_2^2 - 2(\mathbf{v}(t), \mathbf{v}_0) \\ &= \|\mathbf{v}(t)\|_2^2 - \|\mathbf{v}_0\|_2^2 - 2(\mathbf{v}(t) - \mathbf{v}_0, \mathbf{v}_0) \\ &\overset{(\text{B.2.17})}{\leq} -2\int_0^t \left[(\mathbf{S}(\mathbf{D}(\mathbf{v})), \mathbf{D}(\mathbf{v})) - \langle \mathbf{b}, \mathbf{v} \rangle \right] \, d\tau - 2(\mathbf{v}(t) - \mathbf{v}_0, \mathbf{v}_0). \end{aligned} \tag{B.3.94}$$

Letting $t \to 0_+$ in (B.3.94) we conclude (B.2.14) from $(\text{B.2.12})_1$ and the fact that $(\mathbf{S}(\mathbf{D}(\mathbf{v})), \mathbf{D}(\mathbf{v})) - \langle \mathbf{b}, \mathbf{v} \rangle \in L^1(0,T)$.

4. On smoothness of flows

4.1. A survey of regularity results

The alternate topic of this section can be *higher differentiability* of weak solution $(\mathbf{v}, p)$ of Problem $(\mathcal{P})$.

For simplicity, we set

$$\mathbf{b} = \mathbf{0}.$$

Since we deal with spatially periodic problem we are free of technical difficulties due to localization.

For $j = 1, 2, 3$, $\mathbf{e}^j$ denotes the basis vector in $\mathbb{R}^3 (\mathbf{e}^j = (\delta_{1j}, \delta_{2j}, \delta_{31})$, δ_{ij} being the Kronecker delta). Let $\delta_0 > 0$ be fixed. Introducing for $h \in (0, \delta_0)$ the notation

$$\triangle_j^h z(t,x) = z^{[+h\mathbf{e}^j]}(t,x) - z(t,x) := z(t, x + h\mathbf{e}^j) - z(t,x),$$

it is not difficult to observe that (B.2.15) implies

$$\begin{aligned}\langle[\triangle_j^h \mathbf{v}]_{,t},\boldsymbol{\varphi}\rangle + \int_\Omega \Big[\mathbf{S}(\mathbf{D}(\mathbf{v}))^{[+h\mathbf{e}^j]} - \mathbf{S}(\mathbf{D}(\mathbf{v}))\Big]\cdot \mathbf{D}(\boldsymbol{\varphi})\,dx \\ = \Big((\triangle_j^h \mathbf{v})\otimes \mathbf{v}^{[+h\mathbf{e}^j]},\nabla\boldsymbol{\varphi}\Big) + \big(\mathbf{v}\otimes\triangle_j^h\mathbf{v},\nabla\boldsymbol{\varphi}\big) + \big(\triangle_j^h p,\operatorname{div}\boldsymbol{\varphi}\big)\,,\end{aligned} \tag{B.4.1}$$

that holds for all $\boldsymbol{\varphi}\in L^s(0,T;W^{1,s}_{per})$ with $s=r$ if $r\geq\frac{11}{5}$ and $s=\frac{5r}{5r-6}$ if $\frac{6}{5}\leq r<\frac{11}{5}$ almost everywhere in $(0,T)$. It is a direct consequence of these requirements on $\boldsymbol{\varphi}$ in (B.4.1) and (B.2.15) that we can put $\boldsymbol{\varphi}=\triangle_j^h\mathbf{v}$ in (B.4.1) only if $r\geq\frac{11}{5}$. In order to relax such an apriori bound on r, we can use instead of (B.4.1) the weak formulation of Problem $(\mathcal{P}^\eta)$. Then for $(\mathbf{v},p)=(\mathbf{v}^\eta,p^\eta)$ we have

$$\begin{aligned}\langle[\triangle_j^h \mathbf{v}]_{,t},\boldsymbol{\varphi}\rangle + \int_\Omega \Big[\mathbf{S}(\mathbf{D}(\mathbf{v}))^{[+h\mathbf{e}^j]} - \mathbf{S}(\mathbf{D}(\mathbf{v}))\Big]\cdot \mathbf{D}(\boldsymbol{\varphi})\,dx = \big(\triangle_j^h p,\operatorname{div}\boldsymbol{\varphi}\big) \\ + \Big(((\triangle_j^h \mathbf{v})*\omega^\eta)\otimes \mathbf{v}^{[+h\mathbf{e}^j]},\nabla\boldsymbol{\varphi}\Big) + \big((\mathbf{v}*\omega^\eta)\otimes\triangle_j^h\mathbf{v},\nabla\boldsymbol{\varphi}\big)\end{aligned} \tag{B.4.2}$$

valid for a.a. $t\in(0,T)$ and for all $\boldsymbol{\varphi}\in L^r(0,T;W^{1,r}_{per})$.

In (B.4.2), we are allowed to take $\boldsymbol{\varphi}=\triangle_j^h\mathbf{v}$ in (B.4.2), having the same restriction on r needed for the solvability of (B.4.2), i.e., $r>\frac{8}{5}$. We aim to obtain higher differentiability estimates uniformly w.r.t. $\eta>0$. Inserting $\boldsymbol{\varphi}=\triangle_j^h\mathbf{v}$ into (B.4.2), noting that $\operatorname{div}\triangle_j^k\mathbf{v}=0$ implying that the term involving the pressure as well as the last term appearing in (B.4.2) vanish, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\triangle_j^h\mathbf{v}\|_2^2 + \Big(\mathbf{S}(\mathbf{D}(\mathbf{v}))^{[+h\mathbf{e}^j]} - \mathbf{S}(\mathbf{D}(\mathbf{v})),[\mathbf{D}(\mathbf{v})]^{[+h\mathbf{e}^j]}-\mathbf{D}(\mathbf{v})\Big) \\ = -\Big((\triangle_j^h\mathbf{v}*\omega^\eta)\otimes\triangle_j^h\mathbf{v},\nabla\mathbf{v}^{[+h\mathbf{e}^j]}\Big) \leq \|\nabla\mathbf{v}\|_r\|\triangle_j^h\mathbf{v}\|^2_{\frac{2r}{r-1}}\,.\end{aligned} \tag{B.4.3}$$

Using (B.2.21) for $r\geq 2$ (or (B.2.22) for $r<2$), it follows from (B.4.3) that

$$\frac{1}{2}\frac{d}{dt}\|\triangle_j^h\mathbf{v}\|_2^2 + \|\triangle_j^h\mathbf{D}(\mathbf{v})\|_2^2 + \|\triangle_j^h\mathbf{D}(\mathbf{v})\|_r^r \leq \|\nabla\mathbf{v}\|_r\|\triangle_j^h\mathbf{v}\|^2_{\frac{2r}{r-1}}\,. \tag{B.4.4}$$

If one concludes from (B.4.4) some higher differentiability estimates (even fractional ones suffice), the compact embedding theorem (the Aubin-Lions lemma) then lead to almost everywhere convergence for the velocity gradients. We can then pass to limit, as $\eta\to 0$, from (B.3.49) to (B.2.15). The higher differentiability estimates thus represent another method to establish the large-time and large-data existence of weak solution to Problem $(\mathcal{P})$.

Carrying on the original contribution by Málek, Nečas and Růžička [83] and Bellout, Bloom and Nečas [8], see also [81], where however smoother approximations of Problem $(\mathcal{P})$ are considered†, it seems very likely that the outlined procedure is workable and one can thus find (fractional) higher differentability estimates for $r > \frac{9}{5}$. It is of interest to mention that this range for r's coincides with that required for the existence of suitable weak solution.

To be more precise, the following results are in place (see [83], [8], [81], [84]).

Theorem 4.1. *(i) If $r \geq \frac{11}{5}$ then there is a weak solution $(\mathbf{v}, p)$ to Problem $(\mathcal{P})$ fulfilling*

$$\left.\begin{array}{l} \sup\limits_{t\in\langle 0,T\rangle} \|\nabla\mathbf{v}(t)\|_2^2 \\ \displaystyle\int_0^T (\kappa\|\nabla^2\mathbf{v}\|_2^2 + \|\mathbf{D}(\mathbf{v})\|^r_{\mathcal{N}^{\frac{2}{r},r}(\Omega)} + \|\nabla\mathbf{v}\|_{3r}^r)\, dt \end{array}\right\} \leq C(\|\nabla\mathbf{v}_0\|_2) \tag{B.4.5}$$

$$\int_0^T \|\mathbf{v}_{,t}\|_2^2 dt + \sup_{t\in\langle 0,T\rangle} \|\nabla\mathbf{v}(t)\|_r^r \leq C(\|\nabla\mathbf{v}_0\|_r) \tag{B.4.6}$$

$$\left.\begin{array}{l} \sup\limits_{t\in\langle 0,T\rangle} \|\mathbf{v}_{,t}\|_2^2 \\ \displaystyle\kappa\int_0^T \|\nabla\mathbf{v}_{,t}\|_2^2 dt + \int_0^T\int_\Omega |\mathbf{D}(\mathbf{v})|^{r-2}|\mathbf{D}(\mathbf{v}_{,t})|^2 dx\, dt \end{array}\right\} \leq C(\|\mathbf{v}_0\|_{2,q}), \tag{B.4.7}$$

where $\kappa = 0$ or 1 according to (B.2.8)*, $q > 3$ and*

$$\|z\|_{\mathcal{N}^{\alpha,r}} := \left(\sup_{0<h\leq\delta_0}\int_\Omega \frac{|z(x+h)-z(x)|^r}{h^{\alpha r}}\, dx\right)^{\frac{1}{r}}.$$

(ii) if $r \in \langle 2, \frac{11}{5})$ then there is a weak solution $(\mathbf{v}, p)$ to Problem $(\mathcal{P})$ such that $\mathbf{v}$ fulfills

$$\kappa\int_0^T \|\nabla^2\mathbf{v}\|_2^{2\frac{3r-5}{r+1}} dt + \int_0^T \|\nabla\mathbf{v}\|_{W^{s,r}}^{\frac{r^2(3r-5)}{3(r^2-3r+4)}} dt < \infty \quad s\in(0,\frac{2}{r}), \tag{B.4.8}$$

(iii) if $r \in (\frac{9}{5}, 2)$ then there is $(\mathbf{v}, p)$ to Problem $(\mathcal{P})$ such that $\mathbf{v}$ fulfills

$$\int_0^T \|\nabla^2\mathbf{v}\|_r^{\frac{r(5r-9)}{(-r^2+8r-9)}} dt \leq \infty. \tag{B.4.9}$$

In particular, for the spatially-periodic problem described by the Navier-Stokes equations in three dimensions, it follows from Theorem 4.1 that (set $r = 2$ in

† In [83], the estimates are derived directly for Galerkin approximations using smooth basis of functions. In [8] a multipolar fluid model is used as a smooth approximation. Both approximations thus allow us to differentiate the equations of the approximative problems.

(B.4.8)) there is a weak solution $(\mathbf{v}, p)$ such that

$$\int_0^T \|\nabla^2 \mathbf{v}\|_2^{2/3} dt < \infty, \tag{B.4.10}$$

the result established by Foias, Guillopé and Temam [37].

Regarding a no-slip boundary condition, Málek, Nečas and Růžička [82] considered the case $r \geq 2$ with $\kappa = 1$ and showed that:

- if $r \geq \frac{9}{4}$ (and $r < 3$) then there is a weak solution that fulfils

$$\begin{aligned} \mathbf{v} &\in L^{\frac{2}{2-r}}(0,T; W^{2,\frac{6}{r+1}}(\Omega))^3 \cup L^2(0,T; W^{2,2}_{loc}), \\ \mathbf{v}_{,t} &\in L^2(0,T; L^2(\Omega)^3), \\ \int_0^T \int_{\Omega_0} &(1 + |\mathbf{D}(\mathbf{v})|)^{r-2} |\mathbf{D}(\nabla \mathbf{v})|^2 dx\, dt \leq K \quad \text{for all } \Omega_0 \subset\subset \Omega, \end{aligned} \tag{B.4.11}$$

- if $r \in \langle 2, \frac{9}{4})$ then

$$\int_0^T \|\nabla^2 \mathbf{v}\|_{\frac{6}{r+1}}^{\frac{2}{3}\frac{2r-3}{r-1}} dt \leq K < \infty. \tag{B.4.12}$$

Note that (B.4.12) implies (B.4.10) even for Dirichlet (no-slip) boundary value problem.

Instead of proving (B.4.5)-(B.4.10) rigorously, we rather provide a cascade of formal inequalities that form however essence of correct arguments. Details and many extensions can be found† in [81], [82] and [24]. This cascade consists of three levels of inequalities, considering the energy inequality as level zero.

Level 1: Differentiate $(B.1.4)_2$ w.r.t. x_s and scalarly multiply the result with $\frac{\partial \mathbf{v}}{\partial x_s}$.

Level 2: Multiply $(B.1.4)_2$ with $\mathbf{v}_{,t}$.

Level 3: Differentiate $(B.1.4)_2$ w.r.t. time t and use $\mathbf{v}_{,t}$ as the multiplier.

For $r \geq \frac{11}{5}$ the procedure leads to (B.4.5)-(B.4.7). For the Navier-Stokes equations inequality (B.4.5) *is not* available and there is a plenty of results in literature asking the question what are the conditions implying (B.4.5). The well-known are so-called Prodi-Serrin conditions‡ saying that (B.4.5) holds provided that

$$\mathbf{v} \in L^q(0,T; L^s) \quad \text{with } \frac{2}{q} + \frac{3}{s} \leq 1, \quad s \geq 3. \tag{B.4.13}$$

If $s > 3$, the result is established by Serrin in [130]. The most interesting limiting case $L^\infty(0,T; L^3)$ has been covered recently by Escauriaza, Seregin and Šverák

† Dealing with approximations different than those used in Section 3.

‡ Prodi asks for conditions implying uniqueness of weak solution, see [105]. It revealed that the criterion coincides with that for regularity.

([30], [31], [128]). Other regularity criteria are expressed in terms of the velocity gradient (see [7] for example), the vorticity (see [18]), the pressure ([9], [93], [129]), or just one component of the velocity ([95], [96]) or the velocity gradient (see [18], [30]). The result in [61] and [134] extends (B.4.13) to the class $L^2(0,T;BMO)$. The regularity criteria expressed in terms of eigenvalues and eigenfunctions of the symmetric part of the velocity gradient were established in [96], [97].

While fractional higher differentiability result, as that mentioned in (B.4.10), gives compactness of velocity gradient, say in all $L^q(0,T;L^q_{per})$, $q<2$, they do give any improvement on the regularity of the velocity or its gradient alone. In terms of our "level" inequalities (Doering and Gibbon [25] talk about the ladder where each split bar corresponds to a level above) for the Navier-Stokes equations in three dimensions, it is not known how to make the first step from ground (level zero) to the first rail (level 1). It is however proved that once the level 1 is achieved (in fact (B.4.13) or other criteria suffice), $L^\infty(0,T;L^2_{per})$ integrability of any spatial or time derivatives of any order is available, provided that data ($\mathbf{v}_0$ and $\mathbf{b}$) are smooth enough.

For Ladyzhenskaya's equations, or for Problem $(\mathcal{P})$ with $\kappa=1$, Theorem 4.1 states that if $r\geq\frac{11}{5}$, the first three levels (B.4.5)-(B.4.9) (of the ladder) are accessible. It is however open how to proceed to high levels. More precisely, using (B.4.7) and using (B.1.4) we rewrite Problem $(\mathcal{P})$ as

$$\operatorname{div}(\mathbf{v}\otimes\mathbf{v})-\operatorname{div}\mathbf{S}(\mathbf{D}(\mathbf{v}))+\nabla p=-\mathbf{v}_{,t}\in L^\infty(0,T;L^2_{per}), \tag{B.4.14}$$

and we observe that we can apply the higher differentiability technique to almost all time instants. Doing so we conclude that

Theorem 4.2. *If $r\geq\frac{11}{5}$ then there is $(\mathbf{v},p)$ solution to Problem $(\mathcal{P})$ (with $\kappa=1$) such that*

$$\sup_t\|\nabla^2\mathbf{v}(t)\|_2^2+\sup_t\|\nabla\mathbf{v}(t)\|^r_{\mathcal{N}^{\frac{2}{r},r}(\Omega)}+\sup_t\|\nabla\mathbf{v}(t)\|^r_{3r}\leq K<\infty. \tag{B.4.15}$$

In particular, $\mathbf{v}$ is bounded in $(0,T)\times\mathbb{R}^3$.

Thus, the task **(III)** large-time and large-data regularity is in sense given in Subsection B.1.2 fulfilled. The question if then also $\nabla\mathbf{v}$ is bounded or Hölder continuous has been however unanswered yet.

In next subsections we formally establish (B.4.5)-(B.4.7), and also (B.4.15). We also discuss related results on local-in-time existence of solutions with integrable second derivatives.

4.2. A cascade of inequalities for Ladyzhenskaya's equations

In this part, we consider the Ladyzhenskaya's equations (B.1.6). It means we deal with the system (B.1.4) where $\mathbf{S}(\mathbf{D}(\mathbf{v})) = \nu_0 + \nu_1|\mathbf{D}(\mathbf{v})|^{r-2}\mathbf{D}(\mathbf{v})$. Note also that (B.2.8) holds with $\kappa = 1$. In the sequel we sometimes use the specific structure of $\mathbf{S}$, sometimes we refer to (B.2.8).

• *Derivation of* (B.4.5). We formally differentiate (B.1.4) with respect to spatial variable x_s and take scalar product of the result with $\frac{\partial \mathbf{v}}{\partial x_s}$. After summing over $s = 1, 2, 3$ and integrating by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{v}\|_2^2 + \int_\Omega \frac{\partial\mathbf{S}(\mathbf{D}(\mathbf{v}))}{\partial\mathbf{D}} \cdot \mathbf{D}(\nabla\mathbf{v}) \otimes \mathbf{D}(\nabla\mathbf{v})\, dx = -\int_\Omega \frac{\partial v_k}{\partial x_s}\frac{\partial v_i}{\partial x_k}\frac{\partial v_i}{\partial x_s}. \quad \text{(B.4.16)}$$

Using (B.2.8) we conclude that

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{v}\|_2^2 + C_1\|\nabla^2\mathbf{v}\|_2^2 + C_1 J_r(\mathbf{v}) \le \|\nabla\mathbf{v}\|_3^3, \quad \text{(B.4.17)}$$

where

$$J_r(\mathbf{v}) = \int_\Omega |\mathbf{D}(\mathbf{v})|^{r-2}|\mathbf{D}(\nabla\mathbf{v})|^2\, dx.$$

Since

$$J_r(\mathbf{v}) \ge c^*\|\nabla\mathbf{v}\|_{3r}^r, \quad \text{(B.4.18)}$$

see [81], p. 227, and

$$J_r(\mathbf{v}) \ge c^{**}\|\mathbf{D}(\mathbf{v})\|^r_{\mathcal{N}^{\frac{2}{r},r}}\,, \quad \text{(B.4.19)}$$

see [84] for the proof, then (B.4.17) and the energy inequality (B.2.17) implies (B.4.5) if $r \ge 3$. If $r < 3$, then we incorporate the interpolation inequalities,

$$\begin{aligned} \|z\|_3 &\le \|z\|_r^{\frac{r-1}{2}}\|z\|_{3r}^{\frac{3-r}{2}} \\ \|z\|_3 &\le \|z\|_2^{2\frac{r-1}{3r-2}}\|z\|_{3r}^{\frac{r}{3r-2}}, \end{aligned} \quad \text{(B.4.20)}$$

and use the splitting $\|\nabla\mathbf{v}\|_3^3 = \|\nabla\mathbf{v}\|_3^{3\alpha}\|\nabla\mathbf{v}\|_3^{3(1-\alpha)}$ for $\alpha \in \langle 0, 1\rangle$. Then, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{v}\|_2^2 + C_1\|\nabla^2\|_2^2 + \frac{C_1}{2}J_r(\mathbf{v}) + C_1^*\|\nabla\mathbf{v}\|_{3r}^r \\ &\qquad \le \|\nabla\mathbf{v}\|_r^{3\alpha\frac{r-1}{2}}\|\nabla\mathbf{v}\|_2^{6(1-\alpha)\frac{r-1}{3r-2}}\|\nabla\mathbf{v}\|_{3r}^{\frac{3\alpha}{2}(3-r)+\frac{3r(1-\alpha)}{3r-2}}. \end{aligned} \quad \text{(B.4.21)}$$

Setting $Q_1 := \frac{3\alpha}{2}\frac{(3-r)}{r} + \frac{3(1-\alpha)}{3r-2}$, $Q_2 := \frac{3\alpha}{2}\frac{r-1}{r}$ and $Q_3 := 3(1-\alpha)\frac{r-1}{3r-2}$, we apply Young's inequality with $\delta = \frac{1}{Q_1}$. Requiring also that $Q_2\delta' = 1$, i.e. $Q_2 + Q_1 = 1$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{v}\|_2^2 + 2C_1\|\nabla^2\mathbf{v}\|_2^2 + C_1 J_r(\mathbf{v}) + C_1^*\|\nabla\mathbf{v}\|_r^r \le c\|\nabla\mathbf{v}\|_r^r\|\nabla\mathbf{v}\|_2^{2\lambda}, \tag{B.4.22}$$

where

$$\lambda := 2\frac{3-r}{3r-5}. \tag{B.4.23}$$

Since

$$\lambda \le 1 \Leftrightarrow r \ge \frac{11}{5},$$

we obtain (B.4.5) applying Gronwall lemma.

• *Derivation of* (B.4.6). The scalar multiplication of (B.1.4) with $\mathbf{v}_{,t}$ and the integration over Ω leads to

$$\|\mathbf{v}_{,t}\|_2^2 - (\operatorname{div}\mathbf{S}(\mathbf{D}(\mathbf{v})), \mathbf{v}_{,t}) + (\mathbf{v}_{,t}\otimes\mathbf{v}, \nabla\mathbf{v}) = 0\,. \tag{B.4.24}$$

Using the specific form of $\mathbf{S}$ and the integration by parts we obtain

$$\begin{aligned}\|\mathbf{v}_{,t}\|_2^2 + \frac{\nu_0}{2}\frac{d}{dt}\|\nabla\mathbf{v}\|_2^2 + \frac{\nu_1}{r}\frac{d}{dt}\|\mathbf{D}(\mathbf{v})\|_r^r &= (\mathbf{v}_{,t}\otimes\mathbf{v}, \nabla\mathbf{v}) \\ &\le \frac{1}{2}\|\mathbf{v}_{,t}\|_2^2 + \|\,|\mathbf{v}|\,|\nabla\mathbf{v}|\,\|_2^2.\end{aligned} \tag{B.4.25}$$

Since the estimate of the last term can be established with help of $W^{1,3r} \hookrightarrow L^\infty$

$$\int_\Omega |\mathbf{v}|^2|\nabla\mathbf{v}|^2\,dx \le \|\mathbf{v}\|_\infty^2\|\nabla\mathbf{v}\|_2^2 \le C(\sup_t\|\nabla\mathbf{v}(t)\|_2^2)\|\nabla\mathbf{v}\|_{3r}^2, \tag{B.4.26}$$

we see that (B.4.6) follows after integrating (B.4.25) over time and applying (B.4.5).

It is also possible to conclude from (B.4.24) that

$$\|\mathbf{v}_{,t}(0)\|_2^2 \le C(\|\mathbf{v}_0\|_{2,q})\,. \tag{B.4.27}$$

• *Derivation of* (B.4.7). A formal differentiation of (B.1.4) with respect to time, and a multiplication of the result with $\mathbf{v}_{,t}$ leads to

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}_{,t}\|_2^2 + \int_\Omega \frac{\partial\mathbf{S}(\mathbf{D}(\mathbf{v}))}{\partial\mathbf{D}}\cdot\mathbf{D}(\mathbf{v}_{,t})\otimes\mathbf{D}(\mathbf{v}_{,t})\,dx = (\mathbf{v}_{,t}\otimes\mathbf{v}, \nabla\mathbf{v}_{,t}), \tag{B.4.28}$$

since $(p_{,t}, \operatorname{div}\mathbf{v}_{,t}) = 0$ and $(\mathbf{v}\otimes\mathbf{v}_{,t}, \nabla\mathbf{v}_{,t}) = (\mathbf{v}, \nabla\frac{|\mathbf{v}_{,t}|^2}{2}) = -(\operatorname{div}\mathbf{v}, \frac{|\mathbf{v}_{,t}|^2}{2}) = 0$. Using (B.2.8), and applying Hölder's inequality to the right-hand side of (B.4.28) gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{v}_{,t}\|_2^2 + \kappa\|\nabla\mathbf{v}_{,t}\|_2^2 + \int_\Omega |\mathbf{D}(\mathbf{v})|^{r-2}|\mathbf{D}(\mathbf{v}_{,t})|^2\,dx& \\ \le \frac{\kappa}{2}\|\nabla\mathbf{v}_{,t}\|_2^2 + \|\mathbf{v}\|_\infty^2\|\mathbf{v}_{,t}\|_2^2& \\ \le \frac{\kappa}{2}\|\nabla\mathbf{v}_{,t}\|_2^2 + C\|\nabla\mathbf{v}\|_{3r}^2\|\mathbf{v}_{,t}\|_2^2.&\end{aligned} \tag{B.4.29}$$

The Gronwall lemma and (B.4.27) completes the formal proof of (B.4.7).

4.3. Boundedness of the velocity

Derivation of (B.4.15). Since $r \geq \frac{11}{5}$, (B.4.5)-(B.4.7) hold. We proceed similarly as in obtaining (B.4.16), the term with the time derivative $\mathbf{v}_{,t}$ is however treated differently:

$$(\nabla \mathbf{v}_{,t}, \nabla \mathbf{v}) = -(\mathbf{v}_{,t}, \triangle \mathbf{v}) \leq \|\mathbf{v}_{,t}\|_2 \|\nabla^2 \mathbf{v}\|_2 \,.$$

Using then the Hölder inequality and (B.4.18) we have instead of (B.4.17)

$$C_1 \|\nabla^2 \mathbf{v}\|_2^2 + C_1 J_r(\mathbf{v}) + \|\nabla \mathbf{v}\|_{3r}^r \leq \|\mathbf{v}_{,t}\|_2^2 + \|\nabla \mathbf{v}\|_3^3. \tag{B.4.30}$$

The interpolation inequality $\|z\|_3 \leq \|z\|_r^{\frac{r-1}{2}} \|z\|_{3r}^{\frac{3-r}{2}}$ then gives

$$C_1 \|\nabla^2 \mathbf{v}(t)\|_2^2 + C_1 J_r(\mathbf{v}(t)) + C_1 \|\nabla \mathbf{v}\|_{3r}^r \leq \|\mathbf{v}_{,t}(t)\|_2^2 + (\|\nabla \mathbf{v}(t)\|_r^r)^{3\frac{r-1}{2r}} (\|\nabla \mathbf{v}(t)\|_{3r}^r)^{\frac{3-r}{2r}} \,. \tag{B.4.31}$$

As $\sup_t \|\mathbf{v}_{,t}(t)\|_2^2 < \infty$ due to (B.4.7) and $\sup_t \|\nabla \mathbf{v}(t)\|_r^r < K \leq \infty$ owing to (B.4.6), and $\frac{3-r}{2r} < 1$, we obtain (B.4.15). Since $W^{2,2}(\Omega) \hookrightarrow \mathcal{C}^{0,\frac{1}{6}}(\Omega)$, we conclude from (B.4.15) that

$$\mathbf{v} \in L^\infty(0, T; \mathcal{C}^{0,\frac{1}{6}}(\Omega)). \tag{B.4.32}$$

In particular, $\mathbf{v}$ is bounded in $(0, T) \times \Omega$.

4.4. Fractional higher differentiability

Let $\frac{5}{3} < r < \frac{11}{5} (\Leftrightarrow \lambda > 1)$. Since

$$J_r(\mathbf{v}) \geq C_{11} \begin{cases} \kappa \|\nabla^2 \mathbf{v}\|_2^2 + \|\mathbf{D}(\mathbf{v})\|_{\mathcal{N}^{\frac{2}{r},r}(\Omega)}^r & \text{if } r \geq 2 \\ C \dfrac{\|\nabla^2 \mathbf{v}\|_r^2}{(1 + \|\nabla \mathbf{v}\|_r)^{2-r}} & \text{if } r < 2, \end{cases}$$

it follows from (B.4.22), see [81] for details, that

$$\kappa \int_0^T \frac{\|\nabla^2 \mathbf{v}\|_2^2}{(1 + \|\nabla \mathbf{v}\|_2^2)^\lambda} + \frac{\|\mathbf{D}(\mathbf{v})\|_{\mathcal{N}^{\frac{2}{r},r}(\Omega)}^r}{(1 + \|\nabla \mathbf{v}\|_2^2)^\lambda} \, dt \leq \infty \qquad \text{if } r \geq 2 \tag{B.4.33}$$

and

$$\int_0^T \frac{\|\nabla^2 \mathbf{v}\|_r^2}{(1 + \|\nabla \mathbf{v}\|_r)^{2-r}} \frac{1}{(1 + \|\nabla \mathbf{v}\|_2^2)^\lambda} \, dt \leq \infty \qquad \text{if } r < 2. \tag{B.4.34}$$

Hölder's inequality and the energy inequality then leads to (B.4.8) and (B.4.10). Using such estimates, we can then apply interpolation inequalities to obtain fractional higher differentiability with the exponent greater than one. For example, for the Navier-Stokes equations we know from (B.4.8) and (B.2.12) that

$$\int_0^T \|\mathbf{v}\|_{1,2}^2 \, dt < \infty \qquad \text{and} \qquad \int_0^T \|\mathbf{v}\|_{2,2}^{\frac{2}{3}} \, dt < \infty.$$

This then implies

$$\int_0^T \|\mathbf{v}\|_{1+s,2}^{\frac{2}{2s+1}} \, dt < \infty \qquad \text{and} \qquad \frac{2}{2s+1} \geq 1 \Leftrightarrow s \leq \frac{1}{2}, s \in \langle 0, 1 \rangle.$$

4.5. Short-time or small-data existence of "smooth" solution

Inequalities of the type (B.4.22) that can be rewritten in a simplified form

$$y'(t) \leq g(t) y(t)^{\lambda}, \quad \text{where} \quad y(t) \geq 0 \ \text{ and } g \in L^1(0,T), \tag{B.4.35}$$

serve, if $\lambda > 1$, as the key in proving either short-time and large-data or large-time and small-data existence of "smooth" solution.

Note that (B.4.22) takes the form of (B.4.35) with $y(t) = \|\nabla \mathbf{v}\|_2^2$, $g(t) = \|\nabla \mathbf{v}\|_r^r$ and $\lambda = 2\frac{3-r}{3r-5}$, and the energy inequality (B.2.17) implies that for all $T > 0$

$$\int_0^T \|\nabla \mathbf{v}\|_r^r \leq c \|\mathbf{v}_0\|_2^2 \tag{B.4.36}$$

If $\lambda > 1$, (B.4.35) is tantamount to

$$y(t) \leq \frac{y(0)}{(1 - I(t)(\lambda - 1)[y(0)]^{\lambda - 1})^{\frac{1}{\lambda - 1}}} \text{ with } I(t) := \int_0^t g(\tau) \, d\tau. \tag{B.4.37}$$

and we observe that

$$\sup_t y(t) \leq K < \infty$$

provided that

$$1 - I(t)(\lambda - 1)[y(0)]^{\lambda - 1} \geq \frac{1}{2}. \tag{B.4.38}$$

In the case of (B.4.22), the condition (B.4.38) reads

$$2(\lambda - 1)\Big(\int_0^t \|\nabla \mathbf{v}\|_r^r \, d\tau\Big) \|\nabla \mathbf{v}_0\|_2^{2(\lambda - 1)} \leq 1. \tag{B.4.39}$$

It follows from (B.4.36) that (B.4.39) holds for all $t > 0$ provided that

$$2(\lambda - 1) c \|\mathbf{v}_0\|_2^2 \|\nabla \mathbf{v}_0\|_2^{2(\lambda - 1)} \leq 1. \tag{B.4.40}$$

Thus, if $\mathbf{v}_0 \in W^{1,2}_{per}$ fulfils (B.4.40), there is a solution $\mathbf{v}$ such that for all $T > 0$

$$\sup_{t\in(0,T)} \|\nabla \mathbf{v}(t)\|_2^2 \le 2\|\nabla \mathbf{v}_0\|_2^2.$$

Since $\int_0^t \|\nabla \mathbf{v}\|_r^r \, d\tau \to 0$ as $t \to 0_+$, it also follows from (B.4.37) and (B.4.39) that for any $\mathbf{v}_0 \in W^{1,2}_{per}$ there is $t^* > 0$ such that weak solution $\mathbf{v}$ fulfils

$$\sup_{t\in(0,t^*)} \|\nabla \mathbf{v}(t)\|_2^2 \le 2\|\nabla \mathbf{v}_0\|_2^2. \tag{B.4.41}$$

In order to have an explicite bound on the length t^*, one can proceed slightly differently starting again from the inequality (B.4.17). If we apply only the second interpolation inequality from the (B.4.20) we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{v}\|_2^2 + C_1\|\nabla^2 \mathbf{v}\|_2^2 + \frac{C_1}{2}J_r(\mathbf{v}) + \|\nabla \mathbf{v}\|_{3r}^r \le c\|\nabla \mathbf{v}\|_2^{6\frac{r-1}{3r-2}} \|\nabla \mathbf{v}\|_{3r}^{\frac{3r}{3r-2}}. \tag{B.4.42}$$

Since $\frac{3}{3r-2} < 1$ if and only if $r > \frac{5}{3}$, Young's inequality leads to

$$\frac{d}{dt}\|\nabla \mathbf{v}\|_2^2 + 2C_1\|\nabla^2 \mathbf{v}\|_2^2 + C_1 J_r(\mathbf{v}) + \|\nabla \mathbf{v}\|_{3r}^r \le c\|\nabla \mathbf{v}\|_2^{6\frac{r-1}{3r-5}}. \tag{B.4.43}$$

This is inequality of the type

$$y' \le c y^\mu \qquad \text{with } \mu = \frac{3(r-1)}{3r-5} > 1.$$

Proceeding as above we observe that (B.4.41) holds provided that

$$0 < t^* \le \frac{1}{2(\mu-1)\|\nabla \mathbf{v}_0\|_2^{2(\mu-1)}}.$$

To summarize, the following results follow from (B.4.22) (and a discussion above on level inequalities) for the Navier-Stokes equations and the Ladyzhenskaya's equations.

• three-dimensional flows driven by Navier-Stokes equations starting with smooth initial flow $\mathbf{v}_0$ are smooth on certain time interval $(0, t^*)$. Also, if smooth initial condition $\mathbf{v}_0$ fulfils (B.4.40), large-time and small-data existence of smooth solution takes place. See [59] [76], [102], [22], [66], [77], [140] or [138].

• three-dimensional flows of power-law fluid or driven by Ladyzhenskaya's equations with $r \in (\frac{5}{3}, \frac{11}{5})$ fulfil† (B.4.5)-(B.4.7) on certain $(0, t^*)$ for any smooth initial data. In particular, $\mathbf{v}$ is bounded on $(0, t^*)$, Also, if $\mathbf{v}_0$ fulfils (B.4.40), large-time (and

† Strictly speaking the inequalities (B.4.5)-(B.4.7) hold only for $r \ge 2$. If $r < 2$, different norms appear in (B.4.5)-(B.4.7) (see [81], [87]).

small-data) existence of flows $\mathbf{v}$ fulfilling (B.4.5)-(B.4.7) and (B.4.15) is valid. Again $\mathbf{v}$ remains bounded provided $\mathbf{v}_0 \in W^{2,q}$, $q > 3$.

Large-time existence of $\mathcal{C}(0,T;W^{2,q})$-solutions for small data $\mathbf{v}_0 \in W^{2,q}$, $q > 3$ is also established in [2]. An improvement in the short-time and lasrge-data existence from the range $r > \frac{5}{3}$ up to $r > \frac{7}{5}$ is presented in [24].

5. Uniqueness and large-data behavior

The aim, to show internal mathematical consistency for Ladyzhenskaya's equations if $r \geq \frac{11}{5}$, will be completed by establishing two results on continuous dependence of flows on data, implying uniqueness. As a consequence, the asymptotic structure of all possible flows as $t \to \infty$ can be studied. We present results on existence of *exponential attractor*. This is a *compact* set in the function space of initial conditions, *invariant* with respect to solution semigroup, having *finite dimensional fractal dimension* and attracting all trajectories *exponentially*.

5.1. Uniquely determined flows described by Ladyzhenskaya's equations

Theorem 5.1. *Let $(\mathbf{v}^1, p^1)$ and $(\mathbf{v}^2, p^2)$ be two weak solutions to Problem $(\mathcal{P})$ corresponding to data $(\mathbf{v}_0^1, \mathbf{b}^1)$ and $(\mathbf{v}_0^2, \mathbf{b}^2)$, respectively. If*

$$r \geq \frac{5}{2} \tag{B.5.1}$$

and

$$\mathbf{v}_0^i \in L^2_{per} \text{ and } \mathbf{b}^i \in (L^{r'}(0,T;W^{-1,r'}_{per})) \quad (i = 1, 2), \tag{B.5.2}$$

then

$$\sup_t \|\mathbf{v}^1(t) - \mathbf{v}^2(t)\|_2^2 \leq h(\mathbf{v}_0^1 - \mathbf{v}_0^2, \mathbf{b}^1 - \mathbf{b}^2) \tag{B.5.3}$$

where

$$h(\boldsymbol{\omega}_0, \mathbf{g}, \mathbf{v}_0^2, \mathbf{b}^2) := c_1 \left(\|\boldsymbol{\omega}_0\|_2^2 + \int_0^T \|\mathbf{g}\|^{r'}_{(W^{1,r}_{per})^*} \right) \exp c_2 \left(\|\mathbf{v}_0^2\|_2^2 + \int_0^T \|\mathbf{b}^2\|^{r'}_{(W^{1,r}_{per})^*} \right).$$

Also,

$$\int_0^T \|\nabla(\mathbf{v}^1 - \mathbf{v}^2)\|_2^2 + \|\nabla(\mathbf{v}^1 - \mathbf{v}^2)\|_r^r \, dt \leq c h(\mathbf{v}_0^1 - \mathbf{v}_0^2, \mathbf{b}^1 - \mathbf{b}^2), \tag{B.5.4}$$

and

$$\int_0^T \|\nabla(p^1 - p^2)\|_{r'}^{r'} \, dt \leq c h(\mathbf{v}_0^1 - \mathbf{v}_0^2, \mathbf{b}^1 - \mathbf{b}^2). \tag{B.5.5}$$

In particular, Problem $\mathcal{P}$ is uniquely solvable in the class of weak solution.

Proof: Taking the difference of (B.2.15) considered for $(\mathbf{v}^1, p^1)$ from (B.2.15) for $(\mathbf{v}^2, p^2)$ we come, for $r \geq \frac{11}{5}$, to the identity for $\boldsymbol{\omega} = \mathbf{v}^2 - \mathbf{v}^1$ and $q = p^2 - p^1$

$$\begin{aligned} &\langle \boldsymbol{\omega}_{,t}, \boldsymbol{\varphi}\rangle + (\mathbf{S}(\mathbf{D}(\mathbf{v}^2)) - \mathbf{S}(\mathbf{D}(\mathbf{v}^1)), \mathbf{D}(\boldsymbol{\varphi})) \\ &\qquad = (q, \operatorname{div} \boldsymbol{\varphi}) + \langle \mathbf{b}^2 - \mathbf{b}^2, \boldsymbol{\varphi}\rangle + (\boldsymbol{\omega} \otimes \mathbf{v}^2, \nabla \boldsymbol{\varphi}) + (\mathbf{v}^1 \otimes \boldsymbol{\omega}, \nabla \boldsymbol{\varphi}) \end{aligned} \tag{B.5.6}$$

valid for all $\boldsymbol{\varphi} \in W^{1,r}_{per}$ and a.a. $t \in \langle 0, T\rangle$. Taking $\boldsymbol{\varphi} = \boldsymbol{\omega}$ and observing $(q, \operatorname{div} \boldsymbol{\omega}) = 0$ and $(\mathbf{v}^1 \otimes \boldsymbol{\omega}, \nabla \boldsymbol{\omega}) = 0$, (B.5.6) implies

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\omega}\|_2^2 + (\mathbf{S}(\mathbf{D}(\mathbf{v}^2)) - \mathbf{S}(\mathbf{D}(\mathbf{v}^1)), \mathbf{D}(\mathbf{v}^2 - \mathbf{v}^1)) \\ &\qquad = \langle \mathbf{b}^1 - \mathbf{b}^2, \boldsymbol{\omega}\rangle - (\boldsymbol{\omega} \otimes \boldsymbol{\omega}, \nabla \mathbf{v}^2). \end{aligned} \tag{B.5.7}$$

Monotone properties of $\mathbf{S}$, i.e. (B.2.21) and (B.2.22), Korn's inequality and duality estimates allow us to treat the term with $\mathbf{b}^1 - \mathbf{b}^2$, then yield

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\omega}\|_2^2 + \nu_0\|\nabla\boldsymbol{\omega}\|_2^2 + \frac{\nu_1}{2}\|\nabla\boldsymbol{\omega}\|_r^r \leq c\|\mathbf{b}^2 - \mathbf{b}^1\|^{r'}_{(W^{1,r}_{per})^*} + \int_\Omega |\boldsymbol{\omega}|^2|\nabla\mathbf{v}^2|\, dx. \tag{B.5.8}$$

Using also

$$\begin{aligned} \int_\Omega |\boldsymbol{\omega}|^2|\nabla\mathbf{v}^2|\, dx &\leq \|\nabla\mathbf{v}^2\|_r\|\boldsymbol{\omega}\|^2_{\frac{2r}{r-1}} \leq \|\nabla\mathbf{v}^2\|_r\|\boldsymbol{\omega}\|_2^{\frac{2r-3}{r}}\|\boldsymbol{\omega}\|_6^{\frac{3}{r}} \\ &\leq c\|\nabla\mathbf{v}^2\|_r\|\boldsymbol{\omega}\|_2^{\frac{2r-3}{r}}\|\nabla\boldsymbol{\omega}\|_2^{\frac{3}{r}} \\ &\leq \frac{\nu_0}{2}\|\nabla\boldsymbol{\omega}\|_2^2 + c\|\nabla\mathbf{v}^2\|_r^{\frac{2r}{2r-3}}\|\boldsymbol{\omega}\|_2^2, \end{aligned}$$

it follows from (B.5.8) that

$$\begin{aligned} &\frac{d}{dt}\|\boldsymbol{\omega}\|_2^2 + \left[\nu_0\|\nabla\boldsymbol{\omega}\|_2^2 + \nu_1\|\nabla\boldsymbol{\omega}\|_r^r\right] \\ &\qquad \leq c\left(\|\mathbf{b}^1 - \mathbf{b}^2\|^{r'}_{(W^{1,r}_{per})^*} + \|\nabla\mathbf{v}^2\|_r^{\frac{2r}{2r-3}}\|\boldsymbol{\omega}\|_2^2\right). \end{aligned} \tag{B.5.9}$$

Neglecting the terms standing in square brackets, the Gronwall lemma completes then the proof of (B.5.3) provided that $\frac{2r}{2r-3} \leq r$, which is exactly the condition (B.5.1). The energy inequality (B.2.17) and (B.2.23) to estimate $\int_0^T \|\nabla\mathbf{v}^2\|_r^{\frac{2r}{2r-3}}\, dt$ is also used.

Integrating (B.5.9) over time between 0 and T, using (B.5.3) to control $\sup_t \|\boldsymbol{\omega}\|_2^2$ lead then to (B.5.4).

To conclude (B.5.5), we set $\boldsymbol{\varphi} := \nabla h$ in (B.5.6), where h solves

$$\begin{aligned} &\triangle h = |q|^{\frac{2-r}{r-1}}q - \frac{1}{|\Omega|}\int_\Omega |q|^{\frac{2-r}{r-1}}q \\ &h \text{ is } \Omega-\text{periodic}, \int_\Omega h\, dx = 0. \end{aligned} \tag{B.5.10}$$

Then,

$$\|\boldsymbol{\varphi}\|_{1,r} \le \|h\|_{2,r} \le c\| \, |q|^{\frac{1}{r-1}} \|_r \le c\|q\|_{r'}^{\frac{1}{r-1}}. \tag{B.5.11}$$

Since $\langle \boldsymbol{\omega}_{,t}, \boldsymbol{\varphi}\rangle = \langle \boldsymbol{\omega}_{,t}, \nabla h\rangle = 0$, (B.5.6) with $\boldsymbol{\varphi} = \nabla h$ then leads to

$$\begin{aligned} \|q\|_{r'}^{r'} &\le c\int_\Omega |\nabla\boldsymbol{\omega}||\nabla\boldsymbol{\varphi}| + (|\mathbf{D}(\mathbf{v}^1)| + |\mathbf{D}(\mathbf{v}^2)|)^{r-2}|\mathbf{D}(\boldsymbol{\omega})||\mathbf{D}(\boldsymbol{\varphi})| \, dx \\ &+ \|\mathbf{b}^1 - \mathbf{b}^2\|_{W_{per}^{-1,r'}} \|\nabla\boldsymbol{\varphi}\|_r + \|\mathbf{v}^1 + \mathbf{v}^2\|_{\frac{3r}{2(2r-3)}} \|\boldsymbol{\omega}\|_{\frac{3r}{3-r}} \|\nabla\boldsymbol{\varphi}\|_r \\ &\le \|\nabla\boldsymbol{\omega}\|_2\|\nabla\boldsymbol{\varphi}\|_2 + \|\nabla\boldsymbol{\omega}\|_r\|\nabla\mathbf{v}^1 + \nabla\mathbf{v}^2\|_r^{r-2}\|\nabla\boldsymbol{\varphi}\|_r \\ &+ \|\mathbf{b}^1 - \mathbf{b}^2\|_{W_{per}^{-1,r'}} \|\nabla\boldsymbol{\varphi}\|_r + \|\mathbf{v}^1 + \mathbf{v}^2\|_{\frac{3r}{2(2r-3)}} \|\nabla\boldsymbol{\omega}\|_r\|\nabla\boldsymbol{\varphi}\|_r. \end{aligned} \tag{B.5.12}$$

Using (B.5.11), Young's inequality, the fact that $\mathbf{v}^1+\mathbf{v}^2 \in L^{\frac{5r}{3}}(0,T; L_{per}^{\frac{5r}{3}}) \cap L^r(0,T; W_{per}^{1,r})$, and finally (B.5.4), we obtain (B.5.5). $\square$

Theorem 5.2. *Let* $(\mathbf{v}^1, p^1)$ *and* $(\mathbf{v}^2, p^2)$ *be two weak solution to Problem* $(\mathcal{P})$ *corresponding to data* $(\mathbf{v}_0^1, \mathbf{b}^1)$ *and* $(\mathbf{v}_0^2, \mathbf{b}^2)$ *respectively. If*

$$r \ge \frac{11}{5}, \tag{B.5.13}$$

$(\mathbf{v}_0^1, \mathbf{b}^1)$ *fulfills* (B.5.2) *and*

$$\mathbf{v}_0^2 \in W_{per}^{1,2}(\Omega) \text{ and } \mathbf{b}^2 \in L^2(0,2; L_{per}^2), \tag{B.5.14}$$

then for all $t \in (0,T\rangle$ *the inequalities* (B.5.3)-(B.5.5) *hold with*

$$h(\boldsymbol{\omega}_0, \mathbf{g}, \mathbf{v}_0^2, \mathbf{b}^2) := c_1 \left(\|\boldsymbol{\omega}_0\|_2^2 + \int_0^T \|\mathbf{g}\|_{(W_{per}^{1,r})^*}^{r'} \right) \exp c_2 \left(\|\mathbf{v}_0^2\|_{1,2}^2 + \int_0^T \|\mathbf{b}^2\|_2^2 \right).$$

Consequently, a weak solution fulfilling in addition (B.4.5) *is unique in the class of weak solutions. In another words, if data fulfils* (B.5.14) *Problem* $(\mathcal{P})$ *is uniquely solvable.*

Proof: Since $(\mathbf{v}_0^2, \mathbf{b}^2)$ fulfils (B.5.14), Theorem 4.1 implies

$$\int_0^T \|\nabla\mathbf{v}^2\|_{3r}^r \, dt \le c(\|\nabla\mathbf{v}_0^2\|_2^2 + \int_0^T \|\mathbf{b}^2\|_2^2 \, dt).$$

Proceeding step by steps as in the proof of Theorem 5.1, we estimate the right hand side of (B.5.8) as follows:

$$\begin{aligned} \int_\Omega |\boldsymbol{\omega}|^2|\nabla\mathbf{v}^2| \, dx &\le \|\nabla\mathbf{v}^2\|_{3r}\|\boldsymbol{\omega}\|_{\frac{6r}{3r-1}}^2 \le \|\nabla\mathbf{v}^2\|_{3r}\|\boldsymbol{\omega}\|_2^{2\frac{2r-1}{2r}}\|\nabla\boldsymbol{\omega}\|_2^{\frac{2}{r}} \\ &\le \frac{\nu_0}{2}\|\nabla\boldsymbol{\omega}\|_2^2 + c\|\nabla\mathbf{v}^2\|_{3r}^{\frac{2r}{2r-1}}\|\boldsymbol{\omega}\|_2^2. \end{aligned}$$

As $\frac{2r}{2r-1} \le r$ for $r \ge \frac{11}{5}$, the remaining part of the proof coincides with that of Theorem 5.1. $\square$

Uniqueness of weak solution of Problem $(\mathcal{P})$ for $r \ge \frac{5}{2}$ is stated in [65], see also [76], uniqueness for $r \ge \frac{11}{5}$ is mentioned in [78] and [81].

5.2. Large-time behavior - the method of trajectories

Not only are the Navier-Stokes equations the first system of nonlinear partial differential equations for which the methods of functional analysis were applied and developed†, the Navier-Stokes equations, at least in two dimension, serve also as the first system of equations of mathematical physics to which the theory of dynamical systems was addressed and further extended‡. The restriction to two dimensional flows is due to missing uniqueness and lack of regularity in three spatial dimensions.

Owing to uniquely determined flows $(\mathbf{v}, p)$ of the Navier-Stokes fluid in two spatial dimensions, the mapping

$$S_t : L^2_{per} \to L^2_{per} \text{ such that } S_t\mathbf{v}_0 = \mathbf{v}(t)$$

posses the semigroup property, i.e.,

$$S_0 = Id \text{ and } S_{t+s} = S_t S_s \text{ for all } t, s \ge 0. \tag{B.5.15}$$

We recall definitions of several basic notions. For later use, let $(X, \|\cdot\|_X)$ be a normed spaces and $S_t : X \to X$ having the properties (B.5.15). A bounded set $B \subset X$ is said to be *uniformly absorbing* if for all $B_0 \subset X$ bounded there is $t_0 = t(B_0)$ such that $S_t B_0 \subset B$ for all $t \ge t_0$. A set $\tilde{B} \subset X$ is *positively invariant* w.r.t S_t if $S_t\tilde{B} \subset \tilde{B}$ for all $t \ge 0$. If there is a bounded set $B^* \subset Y \hookrightarrow\hookrightarrow X$ that is uniformly absorbing all bounded sets in X and that is positively invariant, then

$$\mathcal{A} := \bigcap_{s>0} \overline{\bigcup_{t \ge s} S_t B^*}$$

is called *global attractor* as it shares the following properties: (i) $\mathcal{A}$ is compact in X, (ii) $S_t\mathcal{A} = \mathcal{A}$ for all $t \ge 0$, i.e., $\mathcal{A}$ is invariant w.r.t. S_t and (iii) $\mathcal{A}$ attracts all

† We can refer to [74], [52], [76], [65], [137], [22], [73], etc.
‡ As general reference, we can give [63], [39], [68], [21], [139], [50], [29], [46], [69].

bounded sets of X, which means that¶ for all $B \subset X$ bounded

$$\mathrm{dist}_X(S_t B, \mathcal{A}) \to 0 \text{ as } t \to \infty.$$

Compactness of the global attractor recalls for the question of the finite dimension of large-time dynamics. For a compact set $C \subset X$, the fractal dimension $d_f^X(C)$ is defined as

$$d_f^X(C) := \limsup_{\varepsilon \to 0+} \frac{\log N_\varepsilon^X(C)}{\log \frac{1}{\varepsilon}},$$

where $N_\varepsilon^X(C)$ is the minimal number of ε-balls needed to cover C. According to Foias and Olson [38] if $d_f^X(C) < \frac{m}{2}, m \in \mathbb{N}$, then C can be placed into the graph of a Hölder continuous mapping from $\mathbb{R}^m$ onto C. This mapping is a projector if X is a Hilbert space. Thus the finiteness of the fractal dimension $d_f^X(\mathcal{A})$ and its estimates from above (and even more importantly from below) give a worth characterization of large-time dynamics. The following elementary criterium holds (see [85], Lemma 1.3):

$$\text{(***)} \qquad \begin{aligned} &\text{Let } (Y, \|\cdot\|_Y) \hookrightarrow\hookrightarrow (X, \|\cdot\|_X) \text{ and } C \subset X \text{ be bounded.} \\ &\text{If there is } L : X \to Y, \text{ being Lipschitz continuous on } C, \\ &\text{and } LC \subset C, \text{ then } d_f^X(C) < \infty. \end{aligned}$$

To ensure an exponential rate of attraction, Eden, Foias, Nikolaenko and Temam [29] enlarge the global attractor and introduce the notion called *exponential attractor*. This is a subset of B^* having the following properties: (i) $\mathcal{E}$ is compact in X, (ii) $\mathcal{E}$ is positively invariant w.r.t. S_t, (iii) $d_f^X(\mathcal{E}) < \infty$ and (iv) there are $\alpha_1, \alpha_2 > 0$ such that $\mathrm{dist}_X(S_t B^*, \mathcal{E}) \leq \alpha_1 e^{-\alpha_2 t}$ for all $t \geq 0$.

For two dimensional flows of the Navier-Stokes fluids, the existence of global (minimal B) attractor $\mathcal{A} \subset L^2_{per}$ was established by Ladyzhenskaya in [63]. Estimates on its fractal dimension were first studied by Foias and Temam [40], see also [68] for similar criterion. The up-to-date best estimates, based on the method of Lyapunov exponents, are due to Constantin and Foias (see [22] for example). A proof of the existence of exponential attractor is presented in [29].

It is natural to ask if the large-time dynamics of three-dimensional flows driven by the Ladyzhenskaya's equations share the same properties as two-dimensional

¶ The Hausdorff distance $\mathrm{dist}_X(A, B)$ of two sets $A, B \subset X$ is defined as

$$\mathrm{dist}_X(A, B) = \sup_{x \in A} \inf_{y \in B} \|x - y\|_X.$$

NSEs. The first result in this direction is due to Ladyzhenskaya [67] and [70] who proved the existence of global attractor for $r \geq \frac{5}{2}$, leaving however open the quaestion of its dimension. Need to say that neither Ladyzhenskaya criterion requiring that orthonormal projectors commute with the nonlinear operator $\operatorname{div}(\mathbf{S}(\mathbf{D}(\mathbf{v})))$ nor the method of Lyapunov exponents requiring the (not available) regularity results for the linearized problem, can hardly be applicable to show finiteness of the fractal dimension of attractors. The criterium **(***)** cannot be also used for $X = L^2_{per}$ and $Y = W^{1,2}_{per}$. It is however elementary to verify **(***)** for $X = L^2(0,\ell; L^2_{per})$ and $Y := \{\mathbf{u} \in L^2(0,\ell; W^{1,2}_{per}), \mathbf{u}_{,t} \in L^1(0,\ell; (W^{3,2}_{per})^*)\}$ where, $\ell > 0$ is fixed.

This is the first motivation to work with the set of ℓ-trajectories rather than with single values $\mathbf{v}(t) \in L^2_{per}$. The second motivation comes from uniqueness result formulated in Theorem 5.2 for $r \in \langle \frac{11}{5}, \frac{5}{2})$. We are not sure if just one trajectory starts from any $\mathbf{v}_0 \in L^2_{per}$ (Theorem 5.2 says it is true if $\mathbf{v}_0$ is smoother, namely $\mathbf{v}_0 \in W^{1,2}_{per}$). However, once we fix any ℓ-trajectory starting at $\mathbf{v}_0 \in L^2_{per}$, we know that it has uniquely defined continuation, as almost all values of the ℓ-trajectory belong to $W^{1,2}_{per}$. Thus the operators

$$L_t : L^2(0,\ell; L^2_{per}) \to L^2(0,T; L^2_{per}), \tag{B.5.16}$$

that appends to any ℓ-trajectory χ its uniquely defined shift at time t, have the semigroup property (B.5.15).

Following Málek and Pražák [85], using the semigroup (B.5.16) it is not only possible to find $\mathcal{A}_\ell \subset L^2(0,\ell; L^2_{per})$, the global attractor with respect to the semigroup L_t and with help of **(***)** to show that its fractal dimension is finite, but introducing $\mathcal{A} \subset L^2_{per}$ as set of all end-points of ℓ-trajectories belonging to $\mathcal{A}_\ell$, it easily follows from Lipschitz (or at least Hölder) continuity of the mapping $e : \chi \in \mathcal{A}_\ell \to \chi(\ell) \in \mathcal{A}$ that $\mathcal{A}$ is attractor with respect to original dynamics, with finite fractal dimension. The same approach gives also the existence of exponential attractor.

Theorem 5.3. *Let* $\mathbf{b} \in L^2_{per}$ *be time independent. Consider Problem* ($\mathcal{P}$) *with* $r \geq \frac{11}{5}$ *and* $\kappa = 1$ *in* (B.2.8)*, and with* $\mathbf{v}_0 \in L^2_{per}$*. Then this dynamical system possesses*

- *a global attractor* $\mathcal{A} \subset L^2_{per}$ *with finite fractal dimension,*
- *an exponential attractor* $\mathcal{E}$*.*

In both cases, explicit upper bounds on $d_f^X(\mathcal{A})$ and $d_f^X(\mathcal{E})$ with $X = L^2_{per}$ are available.

We refer to Málek and Pražák [85] for explanation of the method of trajectories, that originates in [78], and for the proof of Theorem 5.3. Explicit upper bounds on $d_f^{L^2_{per}}(\mathcal{A})$ are given in [86]. See also [13] for a comparison of the estimates for two-dimensional flows obtained by the method of Lyapunov exponents on one hand and the method **(***)** on the other hand.

As the extreme case of the method of trajectories one can consider Sell's study of ∞-trajectories of three-dimensional Navier-Stokes equations, see [126], suitable to treat ill-posed problem.

It is worth of mentioning a Ladyzhenskaya's counterexample to uniqueness of weak solutions to three-dimensional NSEs in a domain varying with time, see [64].

6. On structure of possible singularities for flows of Navier-Stokes fluid

It is hardly possible to cover all aspects related to the mathematical analysis of the Navier-Stokes equations. For other important aspects, different viewpoints and further references we refer the reader to the monographs by Constantin and Foias [22], Temam [140], von Wahl [145], Lions [77], Sohr [132], Ladhyzenskaya [65], Lemarié-Rieusset [73] and Cannone [17], as well as to the survey (or key) articles by Leray [74], Serrin [130], Heywood [51], Galdi [45], Wiegner [146], Kozono [60], among others.

Consider a (suitable) weak solution of the Navier-Stokes equations with $\mathbf{b} = \mathbf{0}$ and with an initial condition $\mathbf{v}_0 \in W^{k,2}(\Omega)$ for all $k \in \mathbb{N}$. Then, following also discussion in Section 5, there is certainly $T^* > 0$ such that $\mathbf{v}$ is a smooth flow on $[0, T^*]$. Even more, such $\mathbf{v}$ is uniquely determined in class of weak solutions. Since $\mathbf{v} \in L^2(0, \infty; W^{1,2}_{per})$ there is T^{**} such that $\mathbf{v}_0 := \mathbf{v}(T^{**})$ fulfils (B.4.40) implying that $\mathbf{v}$ is smooth on $[T^{**}, \infty)$. Thus possible singularities lie somewhere between T^* and T^{**}. Set

$$\sigma = \{t \in \langle 0, \infty), \limsup_{\tau \to t} \|\nabla \mathbf{v}(\tau)\|_2 = +\infty\}.$$

Since $\mathbf{v} \in L^2(0, \infty; W^{1,2}_{per})$, the Lebesgue measure of σ is zero.

The program to study the structure of possible singularities was initiated by J. Leray [74], who showed that even $\frac{1}{2}$-Hausdorff dimension of σ is zero, $\langle 0, T\rangle \setminus \sigma$

can be written as $\bigcup_{j=1}^{\infty}(a_j, b_j)$ and if $t^* \in \sigma$ then $\|\mathbf{v}(t)\|_{1,2} \geq \frac{C}{\sqrt{t^*-t}}$ as $t \to t^*_-$. Leray proposed to construct a weak solution exhibiting the singularity at t^* in the form†

$$\mathbf{v}(t,x) = \lambda(t)\mathbf{U}(\lambda(t)x), p(t,x) = \lambda^2(t)P(\lambda(t)x) \text{ with } \lambda(t) = \sqrt{2a(t^*-t)}, \quad \text{(B.6.1)}$$

where $a > 0$, and showed that if there is a nontrivial solution $(\mathbf{U}, P)$ of the system

$$\operatorname{div}\mathbf{U} = 0, \quad -2a\mathbf{U} + \operatorname{div}(\mathbf{y}\otimes\mathbf{U}) + \operatorname{div}(\mathbf{U}\otimes\mathbf{U}) - \nu_0\triangle\mathbf{U} + \nabla P = 0, \qquad \text{(B.6.2)}$$

($\mathbf{y}$ is a generic point of $\mathbb{R}^3$), and if $\mathbf{U} \in L^\infty(\mathbb{R}^3) \cap L^2(\mathbb{R}^2)$, then $(\mathbf{v}, p)$ of the form (B.6.1) is a weak solution of the Navier-Stokes equations, being singular at $t = t^*$.

Based on an observation that $\frac{|\mathbf{U}|^2}{2} + P + a\mathbf{y}\cdot\mathbf{U}$ satisfies the maximum principle, Nečas, Růžička and Šverák [94] show that in the class of weak solutions satisfying $\mathbf{U} \in L^3(\mathbb{R}^3)$, the system (B.6.2) admits only trivial solution, $\mathbf{U} \equiv \mathbf{0}$. Tsai [144] prove the same under more general assumptions namely if $\mathbf{U} \in L^q(\mathbb{R}^3)$ for $q > 3$ or if $\mathbf{v}$ fulfils energy inequality considered on any ball $B \subset \mathbb{R}^3$. Clearly, the implication $\mathbf{U} \in W^{1,2}(\mathbb{R}^3) \implies \mathbf{U} \equiv \mathbf{0}$ follows from the result established in [94]. An elementary proof of this implication is given in [79], where also so-called pseudo-selfsimilar solutions are introduced. Their nonexistence is established in [90].

Note that $\mathbf{v}$ of the form (B.6.1) is not only in $L^\infty(0,T;L^2)$ but also in $L^\infty(0,T;L^3)$ provided $\mathbf{U} \in L^3$. Note also that the self-similar transformation (B.1.23) is meaningful in any conical domain. This recalls for the possibility to construct singular solution of the form (B.6.1) in cones. Escauriaza, Seregin and Šverák [30], [32], [31] show, using an approach different from that used in [94], that such solution does not exist, at least at the half-space.

Consider all points (t, x) such that $\mathbf{v}$ is bounded (or Hölder continuous) at certain parabolic neighborhood of (t, x). Let S be the complement of such set in $\langle 0, +\infty) \times \mathbb{R}^3$. Scheffer [121], [122] and [123] started to study the Hausdorff dimension of the set of singularities S. Caffareli, Kohn and Nierenberg [16], introducing the notion of suitable weak solution and proving its existence, finalized these studies by showing that one-dimensional parabolic Hausdorf measure of S is zero. Simplification of the proof and certain improvements of the technique called *partial regularity* can be found in [75], [71] and [127], or [19].

To give a better description of the result by Caffareli, Kohn and Nierenberg, we recall the definition of (parabolic) Hausdorf measures and related statements.

† The form of $(\mathbf{v}, p)$ can be also motivated by the self-similar scaling (B.1.23).

For a countable collection $\mathcal{Q} = \bigcup_{i\in\mathbb{N}} B_{\rho_i}(\mathbf{y}_i)$ in $\mathbb{R}^s$, set $S(\alpha) = \sum_{i=1}^{\infty} \rho_i^{\alpha}$. Then α-dimensional Hausdorf measure $H^{\alpha}(F)$ of a Borel set $F \subset \mathbb{R}^s$ is defined as

$$H^{\alpha}(F) = \lim_{\delta\to 0+} \inf_{\mathcal{Q}} \{S(\alpha); F \subset \bigcup_{i=1}^{\infty} B_{r_i}(x_i), \sup_{i\in\mathbb{N}} r_i < \delta\}.$$

Similarly, for a countable collection $\mathcal{Q}^{par} = \bigcup_{i\in\mathbb{N}} Q_{r_i}(t_i, x_i)$ of parabolic balls $Q_{r_i}(t_i, x_i) = \{(\tau, \mathbf{y}); |\tau - t_i| < r_i^2, |\mathbf{y} - x_i| < r_i\}$, set $S^{par}(\alpha) = \sum_{i=1}^{\infty} r_i^{\alpha}$. Then α-dimensional parabolic Hausdorf measure $P^{\alpha}(E)$ of a Borel set $E \subset \mathbb{R} \times \mathbb{R}^3$ is defined as

$$P^{\alpha}(E) = \lim_{\delta\to 0+} \inf_{\mathcal{Q}^{par}} \{S^{par}(\alpha); E \subset \bigcup_{i=1}^{\infty} Q_{r_i}(t_i, x_i), \sup_{i\in\mathbb{N}} r_i < \delta\}.$$

Clearly,

$$P^{\alpha}(E) = 0 \Leftrightarrow \forall\, \varepsilon > 0\ \exists \mathcal{Q}^{par} = \bigcup_{i=1}^{\infty} Q_{r_i}(t_i, x_i) \ \text{ such that } \sum_{i\in\mathbb{N}} r_i^{\alpha} < \varepsilon. \tag{B.6.3}$$

If $P^{\alpha}(E) < \infty$, then $P^{\alpha'} = 0$ for all $\alpha' > \alpha$ and $P^{\alpha''}(E) = +\infty$ for all $\alpha'' < \alpha$. If $\alpha \in \mathbb{N}$ and $P^{\alpha}(E) < \infty$, then E is homeomorphic to a subset in $\mathbb{R}^{\alpha}$.

The following characterization of smooth points is known due to Caffarelli, Kohn and Nirenberg [16]:

Theorem 6.1. *Let $(\mathbf{v}, p)$ be a suitable weak solution to the Navier-Stokes equations. There is a universal constant $\varepsilon^* > 0$ such that if*

$$\frac{1}{R} \int_{Q_R(t_0,x_0)} |\nabla \mathbf{v}|^2\, dx\, dt < \varepsilon^*, \tag{B.6.4}$$

then for any $k \in \mathbb{N} \cup \{0\}$, the functions $(t, x) \mapsto \nabla^k \mathbf{v}(t, x)$ are Hölder continuous in $Q_{\frac{R}{2}}(t_0, x_0)$ and

$$\sup_{(\tau,\mathbf{y})\in Q_{\frac{R}{2}}(t_0,x_0)} |\nabla^k \mathbf{v}| \le C_k R^{-(k+1)}, \tag{B.6.5}$$

C_k being a universal constant.

Thus, if (t^*, x^*) is a singular point, there is $Q_{R^*}(t^*, x^*)$ such that

$$\int_{Q_{R^*}(t^*,x^*)} |\nabla \mathbf{v}|^2\, d\tau\, dx \ge \varepsilon^* R^*. \tag{B.6.6}$$

Clearly, $\bigcup_{(t^*,x^*)\in S} Q_{R^*}(t^*, x^*)$ is a collection of (parabolic) balls that cover S. Since the four-dimensional Lebesgue measure of S is zero, the four-dimensional Lebesgue measure of discovering collection can be made arbitrarily small. Vitali's covering lemma then provides the existence of a countable subcollection of mutually disjoint balls such that

$$S \subset \bigcup_{i=1}^{\infty} Q_{5R_i}(t^i, x^i) \quad (t^i, x^i) \in S),$$

and the four-dimensional Lebesgue measure of $\bigcup_{i=1}^{\infty} Q_{R_i}(t^i, x^i)$ is small as needed, say less than $\varepsilon^*\varepsilon/5$, $\varepsilon > 0$ arbitrary. Then

$$\sum_{i=1}^{\infty} 5R_i \leq \frac{5}{\varepsilon^*} \sum_{i=1}^{\infty} \int_{Q_{R_i}(t^i,x^i)} |\nabla \mathbf{v}|^2 = \frac{5}{\varepsilon^*} \int_{\left\{Q_{R_i}(t^i,x^i), R_i<\delta\right\}} |\nabla \mathbf{v}|^2 < \varepsilon.$$

According to (B.6.3), $P^1(S) = 0$ and S cannot be a curve in $\mathbb{R}^+ \times \mathbb{R}^3$. Consequently,

- weak solutions of the Navier-Stokes equations in two dimensions are smooth,
- axially symmetric flows cannot have the singularity outside the set $r = 0$,
- the result on zero $\frac{1}{2}$-dimensional Hausdorff measure of singular times σ follows, due to inequality $H^{\frac{1}{2}}(\sigma) \leq cP^1(S)$.

Schaffer in [124] constructs an irregular (non-physical) $\mathbf{b}$ satisfying $\mathbf{b} \cdot \mathbf{v} \leq 0$ so that for any $\delta > 0$ the Hausdorff dimension of singular points is above $1-\delta$ showing thus optimality of the Caffarelli, Kohn, Nirenberg result.

We refer to the above mentioned literature for further details.

7. Other incompressible fluid models

As we were invited to address both physical and analytical aspects to fluids with pressure dependent viscosities in Volume of Handbook of Mathematical Fluid Dynamics (edited by S. Friedlander and D. Serre), we only briefly comment available results here.

7.1. Fluids with pressure-dependent viscosity

To our knowledge, there is no large-time and large-data existence result to system of partial differential equations of the form (B.1.3). Even more, no results on large-time existence for small data or short-time existence for large-data seems to be in place. Renardy [119] obtained local existence and uniqueness result in higher Sobolev spaces not only assuming the viscosity fulfils

$$\lim_{p \to +\infty} \frac{\nu(p)}{p} = 0, \tag{B.7.1}$$

that clearly contradicts to experiments† but also requiring an additional conditions on eigenvalues of $\mathbf{D}(\mathbf{v})$ in terms of $\frac{\partial \nu}{\partial p}$.

Gazzola does not assume (B.7.1). He however establishes only short time existence of smooth solution for small data under very restrictive conditions, both on the almost conservative specific body forces $\mathbf{b}$ and initial data.

7.2. Fluids with pressure and shear dependent viscosities

Considering apparently a more complicated model for fluids, namely (B.1.2), where the viscosity is not only a function of p, but depends also on the shear rate, it has been observed by Málek, Nečas and Rajagopal [80] that for certain specific forms of viscosities, large-time and large-data existence takes place. More precisely, assuming that for a $\mathcal{C}^1$-function $\mathbf{S}$ of the form $\mathbf{S}(p, \mathbf{D}(\mathbf{v})) = \nu(p, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v})$ there are two positive constants C_1, C_2 such that for all $\mathbf{0} \neq \mathbf{A}, \mathbf{B} \in \mathbb{R}^{3\times 3}_{sym}$ and for all $q \in \mathbb{R}$

$$C_1(1 + |\mathbf{A}|^2)^{\frac{r-2}{2}} |\mathbf{B}|^2 \leq \frac{\partial \mathbf{S}(q, \mathbf{A})}{\partial \mathbf{A}} \cdot (\mathbf{B} \otimes \mathbf{B}) \leq C_2(1 + |\mathbf{A}|^2)^{\frac{r-2}{2}} |\mathbf{B}|^2 \qquad \text{(B.7.3)}$$

and

$$\left| \frac{\partial \mathbf{S}(q, \mathbf{A})}{\partial q} \right| \leq \gamma_0 (1 + |\mathbf{A}|^2)^{\frac{r-2}{4}}, \quad \text{with} \ \ \gamma_0 = \min\left(\frac{1}{2}, \frac{C_1}{4C_2}\right), \qquad \text{(B.7.4)}$$

Málek, Nečas and Rajagopal [80] established the following result.

Theorem 7.1. *Let* $\mathbf{S}$ *satisfy* (B.7.3) *and* (B.7.4) *with* $r \in (\frac{9}{5}, 2)$. *Let* $\mathbf{v}_0 \in W^{1,2}_{per}$ *and* $g \in L^2(0, T)$. *Then there is a (suitable) weak solution* $(\mathbf{v}, p)$ *to* (B.1.2) *subjected to spatially periodic conditions* (B.2.9) *and the requirement* $\int_\Omega p(t, x)\, dx = g(t)$ *for* $t \in (0, T)$ *such that*

$$\mathbf{v} \in \mathcal{C}(0, T; L^2_{weak}) \cap L^r(0, T; W^{1,r}_{per}) \cap L^{\frac{5r}{3}}(0, T; L^{\frac{5r}{3}}) \qquad \text{(B.7.5)}$$

$$p \in L^{\frac{5r}{6}}(0, T; L^{\frac{5r}{6}}) \qquad \text{(B.7.6)}$$

Moreover, if $r \in \langle \frac{5}{3}, 2)$ *there is a solution* $(\mathbf{v}, p)$ *such that*

$$\mathbf{v} \in L^\infty(0, T^*; W^{1,2}_{per,\mathrm{div}}) \cap L^r(0, T^*; W^{2,r}_{div}) \qquad \text{(B.7.7)}$$

$$p \in L^2(0, T^*; W^{1,2}). \qquad \text{(B.7.8)}$$

† In fact, in most popular engineering models the relationship between ν and p is exponential, i.e.,

$$\nu(p) = \exp(\alpha_0 p), \quad \alpha_0 > 0. \qquad \text{(B.7.2)}$$

Here, $T^* > 0$ is arbitrary if $\mathbf{v}_0$ is sufficiently small or T^* is small enough if $\mathbf{v}_0$ is arbitrary.

It may be of interest to mention that considering instead of (B.7.2) the viscosity of the form

$$\nu(p, |\mathbf{D}(\mathbf{v})|^2) = \begin{cases} (1 + A + |\mathbf{D}|^2)^{\frac{r-2}{2}} & \text{if} \quad p < 0, \\ (A + \exp(-\alpha q p) + |\mathbf{D}|^2)^{\frac{r-2}{2}} & \text{if} \quad p \geq 0, \end{cases}$$

the assumptions (B.7.3) and (B.7.4) are fulfilled provided that

$$2\alpha q(2 - r) \leq (r - 1)A^{\frac{2-r}{2}}.$$

This can be achieved by taking one of the parameters α or q small enough, or A large enough or r close enough to 2.

Another examples and the proof of Theorem 7.1 can be found in [80]. Two-dimensional flows are studied in [53] and [13]. In the latter, large-time behavior, based on uniqueness result, is also studied via the method of trajectories. A step towards the treatment of other boundary conditions is made in [41].

7.3. Inhomogeneous incompressible fluids

Here, we give references to results relevant to analysis of the partial differential equations (B.1.1). The first results deal with $\mathbf{T}$ of the form

$$\mathbf{T} = -p\mathbf{I} + 2\mu(\rho)\mathbf{D}(\mathbf{v}).$$

Large-time and large-data existence of weak solution established by Novosibirsk school prior 1990 is presented by Antontsev, Kazhikov and Monakhov in [6]. A profound exposition is given in the first chapter of the monograph by P. L. Lions [77].

The fluids with μ depending on $|\mathbf{D}(\mathbf{v})|^2$ were analysed in in [36], were Fernadéz-Cara, Guillén and Ortega proved existence of weak solution to (B.1.1) with

$$\mathbf{T} = -\mathbf{I} + \left(\mu_0 + \mu_1 |\mathbf{D}(\mathbf{v})|^{r-2}\right) \mathbf{D}(\mathbf{v})$$

for $r \geq \frac{12}{5}$. This result, treating homogeneous Dirichlet, i.e., (no-slip) boundary conditions, was recently improved by Guillén-Gonzáles [49], in the case of spatially periodic problem to $r \geq 2$.

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