Lecture Notes on Superconductivity: Condensed Matter and QCD

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I. INTRODUCTION

Superconductivity is one of the most fascinating chapters of modern physics. It has been a continuous source of inspiration for different realms of physics and has shown a tremendous capacity of cross-fertilization, to say nothing of its numerous technological applications. Before giving a more accurate definition of this phenomenon let us however briefly sketch the historical path leading to it. Two were the main steps in the discovery of superconductivity. The former was due to Kamerlingh Onnes (Kamerlingh Onnes, 1911) who discovered that the electrical resistance of various metals, e. g. mercury, lead, tin and many others, disappeared when the temperature was lowered below some critical value T_c. The actual values of T_c varied with the metal, but they were all of the order of a few K, or at most of the order of tenths of a K. Subsequently perfect diamagnetism in superconductors was discovered (Meissner and Ochsenfeld, 1933). This property not only implies that magnetic fields are excluded from superconductors, but also that any field originally present in the metal is expelled from it when lowering the temperature below its critical value. These two features were captured in the equations proposed by the brothers F. and H. London (London and London, 1935) who first realized the quantum character of the phenomenon. The decade starting in 1950 was the stage of two major theoretical breakthroughs. First, Ginzburg and Landau (GL) created a theory describing the transition between the superconducting and the normal phases (Ginzburg and Landau, 1950). It can be noted that, when it appeared, the GL theory looked rather phenomenological and was not really appreciated in the western literature. Seven years later Bardeen, Cooper and Schrieffer (BCS) created the microscopic theory that bears their name (Bardeen *et al.*, 1957). Their theory was based on the fundamental theorem (Cooper, 1956), which states that, for a system of many electrons at small T, any weak attraction, no matter how small it is, can bind two electrons together, forming the so called Cooper pair. Subsequently in (Gor'kov, 1959) it was realized that the GL theory was equivalent to the BCS theory around the critical point, and this result vindicated the GL theory as a masterpiece in physics. Furthermore Gor'kov proved that the fundamental quantities of the two theories, i.e. the BCS parameter gap Δ and the GL wavefunction ψ, were related by a proportionality constant and ψ can be thought of as the Cooper pair wavefunction in the center-of-mass frame. In a sense, the GL theory was the prototype of the modern effective theories; in spite of its limitation to the phase transition it has a larger field of application, as shown for example by its use in the inhomogeneous cases, when the gap is not uniform in space. Another remarkable advance in these years was the Abrikosov's theory of the type II superconductors (Abrikosov, 1957), a class of superconductors allowing a penetration of the magnetic field, within certain critical values.

The inspiring power of superconductivity became soon evident in the field of elementary particle physics. Two pioneering papers (Nambu and Jona-Lasinio, 1961a,b) introduced the idea of generating elementary particle masses through the mechanism of dynamical symmetry breaking suggested by superconductivity. This idea was so fruitful that it eventually was a crucial ingredient of the Standard Model (SM) of the elementary particles, where the masses are generated by the formation of the Higgs condensate much in the same way as superconductivity originates from the presence of a gap. Furthermore, the Meissner effect, which is characterized by a penetration length, is the origin, in the elementary particle physics language, of the masses of the gauge vector bosons. These masses are nothing but the inverse of the penetration length.

With the advent of QCD it was early realized that at high density, due to the asymptotic freedom property (Gross and Wilczek, 1973; Politzer, 1973) and to the existence of an attractive channel in the color interaction, diquark condensates might be formed (Bailin and Love, 1984; Barrois, 1977; Collins and Perry, 1975; Frautschi, 1978). Since these condensates break the color gauge symmetry, the subject took the name of color superconductivity. However, only in the last few years this has become a very active field of research; these developments are reviewed in (Alford, 2001; Hong, 2001; Hsu, 2000; Nardulli, 2002; Rajagopal and Wilczek, 2001). It should also be noted that color superconductivity might have implications in astrophysics because for some compact stars, e.g. pulsars, the baryon densities necessary for color superconductivity can probably be reached.

Superconductivity in metals was the stage of another breakthrough in the 1980s with the discovery of high T_c superconductors.

Finally we want to mention another development which took place in 1964 and which is of interest also in QCD. It originates in high-field superconductors where a strong magnetic field, coupled to the spins of the conduction electrons, gives rise to a separation of the Fermi surfaces corresponding to electrons with opposite spins. If the separation is too high the pairing is destroyed and there is a transition (first-order at small temperature) from the superconducting state to the normal one. In two separate and contemporary papers, (Larkin and Ovchinnikov, 1964) and (Fulde and Ferrell, 1964), it was shown that a new state could be formed, close to the transition line. This state that hereafter will be called LOFF[1] has the feature of exhibiting an order parameter, or a gap, which is not a constant, but has a space variation whose typical wavelength is of the order of the inverse of the difference in the Fermi energies of the pairing electrons. The space modulation of the gap arises because the electron pair has non zero total momentum and it is a rather peculiar phenomenon that leads to the possibility of a non uniform or anisotropic ground state, breaking translational and rotational symmetries. It has been also conjectured that the typical inhomogeneous ground state might have a periodic or, in other words, a crystalline structure. For this reason other names of this phenomenon are inhomogeneous or anisotropic or crystalline superconductivity.

In these lectures notes I used in particular the review papers by (Polchinski, 1993), (Rajagopal and Wilczek, 2001), (Nardulli, 2002), (Schafer, 2003) and (Casalbuoni and Nardulli, 2003). I found also the following books quite useful (Schrieffer, 1964), (Tinkham, 1995), (Ginzburg and Andryushin, 1994), (Landau *et al.*, 1980) and (Abrikosov *et al.*, 1963).

A. Basic experimental facts

As already said, superconductivity was discovered in 1911 by Kamerlingh Onnes in Leiden (Kamerlingh Onnes, 1911). The basic observation was the disappearance of electrical resistance of various metals (mercury, lead and thin) in a very small range of temperatures around a critical temperature T_c characteristic of the material (see Fig. 1). This is particularly clear in experiments with persistent currents in superconducting rings. These currents have been observed to flow without measurable decreasing up to one year allowing to put a lower bound of 10^5 years on their decay time. Notice also that good conductors have resistivity at a temperature of several degrees K, of the order of 10^{-6} ohm cm, whereas the resistivity of a superconductor is lower that 10^{-23} ohm cm. Critical temperatures for typical superconductors range from 4.15 K for mercury, to 3.69 K for tin, and to 7.26 K and 9.2 K for lead and niobium respectively.

In 1933 Meissner and Ochsenfeld (Meissner and Ochsenfeld, 1933) discovered the **perfect diamagnetism**, that is the magnetic field **B** penetrates only a depth $\lambda \simeq 500$ Å and is excluded from the body of the material.

One could think that due to the vanishing of the electric resistance the electric field is zero within the material and

[1] In the literature the LOFF state is also known as the FFLO state.

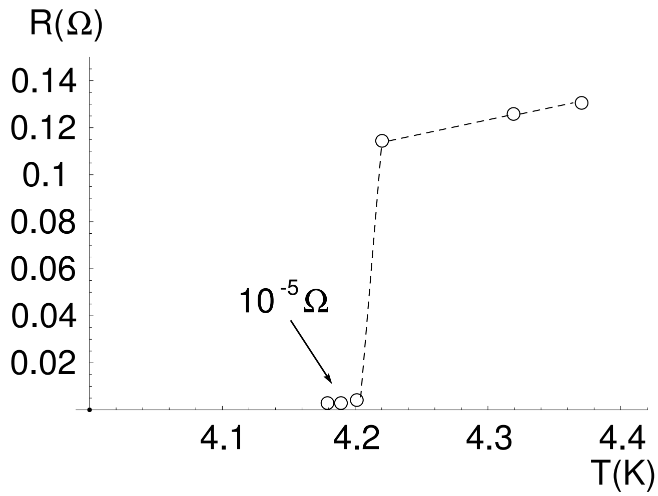

FIG. 1 *Data from Onnes' pioneering works. The plot shows the electric resistance of the mercury vs. temperature.*

therefore, due to the Maxwell equation

$$\nabla \wedge \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\,, \tag{1.1}$$

the magnetic field is frozen, whereas it is expelled. This implies that superconductivity will be destroyed by a critical magnetic field H_c such that

$$f_s(T) + \frac{H_c^2(T)}{8\pi} = f_n(T)\,, \tag{1.2}$$

where $f_{s,n}(T)$ are the densities of free energy in the the superconducting phase at zero magnetic field and the density of free energy in the normal phase. The behavior of the critical magnetic field with temperature was found empirically to be parabolic (see Fig. 2)

$$H_c(T) \approx H_c(0)\left[1 - \left(\frac{T}{T_c}\right)^2\right]\,. \tag{1.3}$$

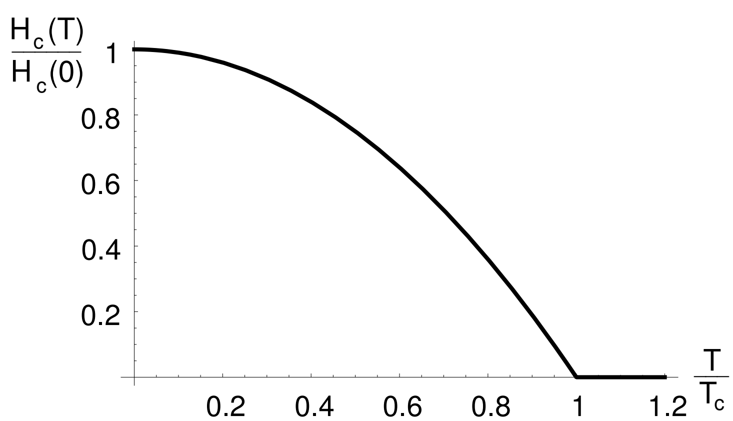

FIG. 2 *The critical field vs. temperature.*

The critical field at zero temperature is of the order of few hundred gauss for superconductors as Al, Sn, In, Pb, etc. These superconductors are said to be "soft". For "hard" superconductors as Nb_3Sn superconductivity stays up to values of 10^5 gauss. What happens is that up to a "lower" critical value H_{c1} we have the complete Meissner effect. Above H_{c1} the magnetic flux penetrates into the bulk of the material in the form of vortices (Abrikosov vortices) and the penetration is complete at $H = H_{c2} > H_{c1}$. H_{c2} is called the "upper" critical field.

At zero magnetic field a second order transition at $T = T_c$ is observed. The jump in the specific heat is about three times the the electronic specific heat of the normal state. In the zero temperature limit the specific heat decreases exponentially (due to the energy gap of the elementary excitations or quasiparticles, see later).

An interesting observation leading eventually to appreciate the role of the phonons in superconductivity (Frolich, 1950), was the **isotope effect**. It was found (Maxwell, 1950; Reynolds *et al.*, 1950) that the critical field at zero temperature and the transition temperature T_c vary as

$$T_c \approx H_c(0) \approx \frac{1}{M^\alpha}\,, \tag{1.4}$$

with the isotopic mass of the material. This makes the critical temperature and field larger for lighter isotopes. This shows the role of the lattice vibrations, or of the phonons. It has been found that

$$\alpha \approx 0.45 \div 0.5 \tag{1.5}$$

for many superconductors, although there are several exceptions as Ru, Mo, etc.

The presence of an energy gap in the spectrum of the elementary excitations has been observed directly in various ways. For instance, through the threshold for the absorption of e.m. radiation, or through the measure of the electron tunnelling current between two films of superconducting material separated by a thin (≈ 20 Å) oxide layer. In the case of Al the experimental result is plotted in Fig. 3. The presence of an energy gap of order T_c was suggested by Daunt and Mendelssohn (Daunt and Mendelssohn, 1946) to explain the absence of thermoelectric effects, but it was also postulated theoretically by Ginzburg (Ginzburg, 1953) and Bardeen (Bardeen, 1956). The first experimental evidence is due to Corak et al. (Corak *et al.*, 1954, 1956) who measured the specific heat of a superconductor. Below T_c the specific heat has an exponential behavior

$$c_s \approx a\,\gamma\,T_c e^{-bT_c/T}\,, \tag{1.6}$$

whereas in the normal state

$$c_n \approx \gamma T\,, \tag{1.7}$$

with $b \approx 1.5$. This implies a minimum excitation energy per particle of about $1.5T_c$. This result was confirmed experimentally by measurements of e.m. absorption (Glover and Tinkham, 1956).

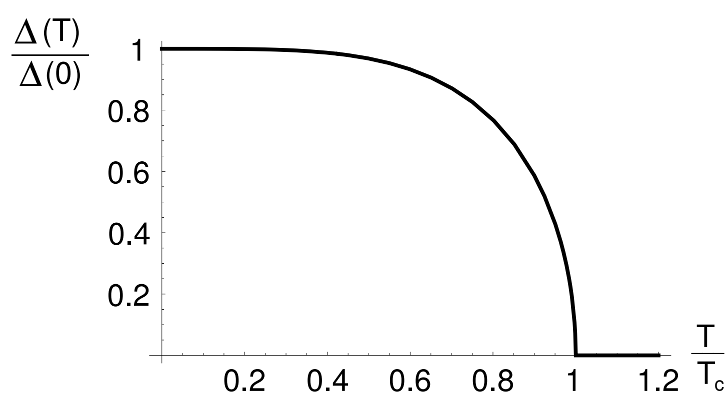

FIG. 3 *The gap vs. temperature in Al as determined by electron tunneling.*

B. Phenomenological models

In this Section we will describe some early phenomenological models trying to explain superconductivity phenomena. From the very beginning it was clear that in a superconductor a finite fraction of electrons forms a sort of condensate or "macromolecule" (superfluid) capable of motion as a whole. At zero temperature the condensation is complete over all the volume, but when increasing the temperature part of the condensate evaporates and goes to form a weakly interacting normal Fermi liquid. At the critical temperature all the condensate disappears. We will start to review the first two-fluid model as formulated by Gorter and Casimir.

1. Gorter-Casimir model

This model was first formulated in 1934 (Gorter and Casimir, 1934a,b) and it consists in a simple ansatz for the free energy of the superconductor. Let x represents the fraction of electrons in the normal fluid and $1-x$ the ones in the superfluid. Gorter and Casimir assumed the following expression for the free energy of the electrons

$$F(x,T) = \sqrt{x}\, f_n(T) + (1-x)\, f_s(T), \tag{1.8}$$

with

$$f_n(T) = -\frac{\gamma}{2}T^2, \quad f_s(T) = -\beta = \text{constant}, \tag{1.9}$$

The free-energy for the electrons in a normal metal is just $f_n(T)$, whereas $f_s(T)$ gives the condensation energy associated to the superfluid. Minimizing the free energy with respect to x, one finds the fraction of normal electrons at a temperature T

$$x = \frac{1}{16}\frac{\gamma^2}{\beta^2}T^4. \tag{1.10}$$

We see that $x = 1$ at the critical temperature T_c given by

$$T_c^2 = \frac{4\beta}{\gamma}. \tag{1.11}$$

Therefore

$$x = \left(\frac{T}{T_c}\right)^4. \tag{1.12}$$

The corresponding value of the free energy is

$$F_s(T) = -\beta\left(1 + \left(\frac{T}{T_c}\right)^4\right). \tag{1.13}$$

Recalling the definition (1.2) of the critical magnetic field, and using

$$F_n(T) = -\frac{\gamma}{2}T^2 = -2\beta\left(\frac{T}{T_c}\right)^2 \tag{1.14}$$

we find easily

$$\frac{H_c^2(T)}{8\pi} = F_n(T) - F_s(T) = \beta\left(1 - \left(\frac{T}{T_c}\right)^2\right)^2, \tag{1.15}$$

from which

$$H_c(T) = H_0\left(1 - \left(\frac{T}{T_c}\right)^2\right), \tag{1.16}$$

with

$$H_0 = \sqrt{8\pi\beta}. \tag{1.17}$$

The specific heat in the normal phase is

$$c_n = -T\frac{\partial^2 F_n(T)}{\partial T^2} = \gamma T, \tag{1.18}$$

whereas in the superconducting phase

$$c_s = 3\gamma T_c \left(\frac{T}{T_c}\right)^3. \tag{1.19}$$

This shows that there is a jump in the specific heat and that, in general agreement with experiments, the ratio of the two specific heats at the transition point is 3. Of course, this is an "ad hoc" model, without any theoretical justification but it is interesting because it leads to nontrivial predictions and in reasonable account with the experiments. However the postulated expression for the free energy has almost nothing to do with the one derived from the microscopical theory.

2. The London theory

The brothers H. and F. London (London and London, 1935) gave a phenomenological description of the basic facts of superconductivity by proposing a scheme based on a two-fluid type concept with superfluid and normal fluid densities n_s and n_n associated with velocities $\mathbf{v}_s$ and $\mathbf{v}_n$. The densities satisfy

$$n_s + n_n = n, \tag{1.20}$$

where n is the average electron number per unit volume. The two current densities satisfy

$$\frac{\partial \mathbf{J}_s}{\partial t} = \frac{n_s e^2}{m}\mathbf{E} \qquad (\mathbf{J}_s = -en_s\mathbf{v}_s), \tag{1.21}$$

$$\mathbf{J}_n = \sigma_n \mathbf{E} \qquad (\mathbf{J}_n = -en_n\mathbf{v}_n). \tag{1.22}$$

The first equation is nothing but the Newton equation for particles of charge $-e$ and density n_s. The other London equation is

$$\boldsymbol{\nabla} \wedge \mathbf{J}_s = -\frac{n_s e^2}{mc}\mathbf{B}. \tag{1.23}$$

From this equation the Meissner effect follows. In fact consider the following Maxwell equation

$$\boldsymbol{\nabla} \wedge \mathbf{B} = \frac{4\pi}{c}\mathbf{J_s}, \tag{1.24}$$

where we have neglected displacement currents and the normal fluid current. By taking the curl of this expression and using

$$\boldsymbol{\nabla} \wedge \boldsymbol{\nabla} \wedge \mathbf{B} = -\boldsymbol{\nabla}^2\mathbf{B}, \tag{1.25}$$

in conjunction with Eq. (1.23) we get

$$\boldsymbol{\nabla}^2\mathbf{B} = \frac{4\pi n_s e^2}{mc^2}\mathbf{B} = \frac{1}{\lambda_L^2}\mathbf{B}, \tag{1.26}$$

with the *penetration depth* defined by

$$\lambda_L(T) = \left(\frac{mc^2}{4\pi n_s e^2}\right)^{1/2}. \tag{1.27}$$

Applying Eq. (1.26) to a plane boundary located at $x = 0$ we get

$$B(x) = B(0)e^{-x/\lambda_L}, \tag{1.28}$$

showing that the magnetic field vanishes in the bulk of the material. Notice that for $T \to T_c$ one expects $n_s \to 0$ and therefore $\lambda_L(T)$ should go to ∞ in the limit. On the other hand for $T \to 0$, $n_s \to n$ and we get

$$\lambda_L(0) = \left(\frac{mc^2}{4\pi ne^2}\right)^{1/2}. \tag{1.29}$$

In the two-fluid theory of Gorter and Casimir (Gorter and Casimir, 1934a,b) one has

$$\frac{n_s}{n} = 1 - \left(\frac{T}{T_c}\right)^4, \tag{1.30}$$

and

$$\lambda_L(T) = \frac{\lambda_L(0)}{\left[1 - \left(\frac{T}{T_c}\right)^4\right]^{1/2}}. \tag{1.31}$$

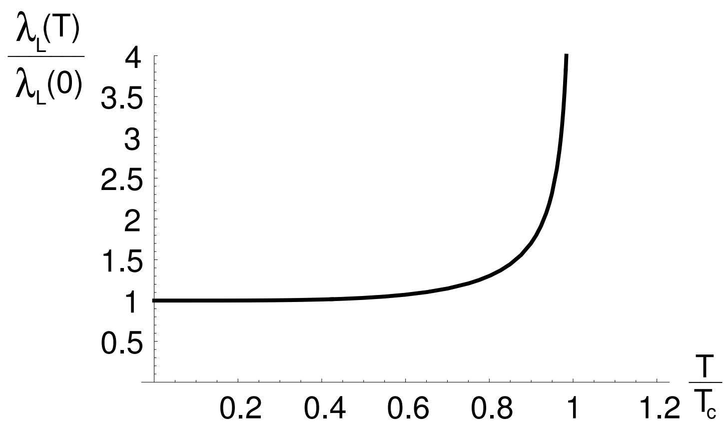

FIG. 4 *The penetration depth vs. temperature.*

This agrees very well with the experiments. Notice that at T_c the magnetic field penetrates all the material since λ_L diverges. However, as shown in Fig. 4, as soon as the temperature is lower that T_c the penetration depth goes very close to its value at $T = 0$ establishing the Meissner effect in the bulk of the superconductor.

The London equations can be justified as follows: let us assume that the wave function describing the superfluid is not changed, at first order, by the presence of an e.m. field. The canonical momentum of a particle is

$$\mathbf{p} = m\mathbf{v} + \frac{e}{c}\mathbf{A}. \tag{1.32}$$

Then, in stationary conditions, we expect

$$\langle \mathbf{p} \rangle = 0, \tag{1.33}$$

or

$$\langle \mathbf{v}_s \rangle = -\frac{e}{mc}\mathbf{A}, \tag{1.34}$$

implying

$$\mathbf{J}_s = en_s\langle \mathbf{v}_s \rangle = -\frac{n_s e^2}{mc}\mathbf{A}. \tag{1.35}$$

By taking the time derivative and the curl of this expression we get the two London equations.

3. Pippard non-local electrodynamics

Pippard (Pippard, 1953) had the idea that the local relation between $\mathbf{J}_s$ and $\mathbf{A}$ of Eq. (1.35) should be substituted by a non-local relation. In fact the wave function of the superconducting state is not localized. This can be seen as follows: only electrons within T_c from the Fermi surface can play a role at the transition. The corresponding momentum will be of order

$$\Delta p \approx \frac{T_c}{v_F} \tag{1.36}$$

and

$$\Delta x \gtrsim \frac{1}{\Delta p} \approx \frac{v_F}{T_c}. \tag{1.37}$$

This define a characteristic length (Pippard's coherence length)

$$\xi_0 = a\frac{v_F}{T_c}, \tag{1.38}$$

with $a \approx 1$. For typical superconductors $\xi_0 \gg \lambda_L(0)$. The importance of this length arises from the fact that impurities increase the penetration depth $\lambda_L(0)$. This happens because the response of the supercurrent to the vector potential is smeared out in a volume of order ξ_0. Therefore the supercurrent is weakened. Pippard was guided by a work of Chamber[2] studying the relation between the electric field and the current density in normal metals. The relation found by Chamber is a solution of Boltzmann equation in the case of a scattering mechanism characterized by a mean free path l. The result of Chamber generalizes the Ohm's law $\mathbf{J}(\mathbf{r}) = \sigma\mathbf{E}(\mathbf{r})$

$$\mathbf{J}(\mathbf{r}) = \frac{3\sigma}{4\pi l}\int \frac{\mathbf{R}(\mathbf{R}\cdot\mathbf{E}(\mathbf{r}'))e^{-R/l}}{R^4}\,d^3\mathbf{r}', \quad \mathbf{R} = \mathbf{r} - \mathbf{r}'. \tag{1.39}$$

If $\mathbf{E}(\mathbf{r})$ is nearly constant within a volume of radius l we get

$$\mathbf{E}(\mathbf{r})\cdot\mathbf{J}(\mathbf{r}) = \frac{3\sigma}{4\pi l}|\mathbf{E}(\mathbf{r})|^2\int \frac{\cos^2\theta\, e^{-R/l}}{R^2}\,d^3\mathbf{r}' = \sigma|\mathbf{E}(\mathbf{r})|^2, \tag{1.40}$$

implying the Ohm's law. Then Pippard's generalization of

$$\mathbf{J}_s(\mathbf{r}) = -\frac{1}{c\Lambda(T)}\mathbf{A}(\mathbf{r}), \quad \Lambda(T) = \frac{e^2 n_s(T)}{m}, \tag{1.41}$$

is

$$\mathbf{J}(\mathbf{r}) = -\frac{3\sigma}{4\pi\xi_0\Lambda(T)c}\int \frac{\mathbf{R}(\mathbf{R}\cdot\mathbf{A}(\mathbf{r}'))e^{-R/\xi}}{R^4}\,d^3\mathbf{r}', \tag{1.42}$$

with an effective coherence length defined as

$$\frac{1}{\xi} = \frac{1}{\xi_0} + \frac{1}{l}, \tag{1.43}$$

and l the mean free path for the scattering of the electrons over the impurities. For almost constant field one finds as before

$$\mathbf{J}_s(\mathbf{r}) = -\frac{1}{c\Lambda(T)}\frac{\xi}{\xi_0}\mathbf{A}(\mathbf{r}). \tag{1.44}$$

Therefore for pure materials ($l \to \infty$) one recover the local result, whereas for an impure material the penetration depth increases by a factor $\xi_0/\xi > 1$. Pippard has also shown that a good fit to the experimental values of the parameter a appearing in Eq. (1.38) is 0.15, whereas from the microscopic theory one has $a \approx 0.18$, corresponding to

$$\xi_0 = \frac{v_F}{\pi\Delta}. \tag{1.45}$$

This is obtained using $T_c \approx .56\,\Delta$, with Δ the energy gap (see later).

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[2] Chamber's work is discussed in (Ziman, 1964)

4. The Ginzburg-Landau theory

In 1950 Ginzburg and Landau (Ginzburg and Landau, 1950) formulated their theory of superconductivity introducing a complex wave function as an order parameter. This was done in the context of Landau theory of second order phase transitions and as such this treatment is strictly valid only around the second order critical point. The wave function is related to the superfluid density by

$$n_s = |\psi(\mathbf{r})|^2. \tag{1.46}$$

Furthermore it was postulated a difference of free energy between the normal and the superconducting phase of the form

$$F_s(T) - F_n(T) = \int d^3\mathbf{r} \left(-\frac{1}{2m^*}\psi^*(\mathbf{r})|(\boldsymbol{\nabla} + ie^*\mathbf{A})|^2\psi(\mathbf{r}) + \alpha(T)|\psi(\mathbf{r})|^2 + \frac{1}{2}\beta(T)|\psi(\mathbf{r})|^4 \right), \tag{1.47}$$

where m^* and e^* were the effective mass and charge that in the microscopic theory turned out to be $2m$ and $2e$ respectively. One can look for a constant wave function minimizing the free energy. We find

$$\alpha(T)\psi + \beta(T)\psi|\psi|^2 = 0, \tag{1.48}$$

giving

$$|\psi|^2 = -\frac{\alpha(T)}{\beta(T)}, \tag{1.49}$$

and for the free energy density

$$f_s(T) - f_n(T) = -\frac{1}{2}\frac{\alpha^2(T)}{\beta(T)} = -\frac{H_c^2(T)}{8\pi}, \tag{1.50}$$

where the last equality follows from Eq. (1.2). Recalling that in the London theory (see Eq. (1.27))

$$n_s = |\psi|^2 \approx \frac{1}{\lambda_L^2(T)}, \tag{1.51}$$

we find

$$\frac{\lambda_L^2(0)}{\lambda_L^2(T)} = \frac{|\psi(T)|^2}{|\psi(0)|^2} = \frac{1}{n}|\psi(T)|^2 = -\frac{1}{n}\frac{\alpha(T)}{\beta(T)}. \tag{1.52}$$

From Eqs. (1.50) and (1.52) we get

$$n\alpha(T) = -\frac{H_c^2(T)}{4\pi}\frac{\lambda_L^2(T)}{\lambda_L^2(0)} \tag{1.53}$$

and

$$n^2\beta(T) = \frac{H_c^2(T)}{4\pi}\frac{\lambda_L^4(T)}{\lambda_L^4(0)}. \tag{1.54}$$

The equation of motion at zero em field is

$$-\frac{1}{2m^*}\boldsymbol{\nabla}^2\psi + \alpha(T)\psi + \beta(T)|\psi|^2\psi = 0. \tag{1.55}$$

We can look at solutions close to the constant one by defining $\psi = \psi_e + f$ where

$$|\psi_e|^2 = -\frac{\alpha(T)}{\beta(T)}. \tag{1.56}$$

We find, at the lowest order in f

$$\frac{1}{4m^*|\alpha(T)|}\boldsymbol{\nabla}^2 f - f = 0. \tag{1.57}$$

This shows an exponential decrease which we will write as

$$f \approx e^{-\sqrt{2}r/\xi(T)}, \tag{1.58}$$

where we have introduced the Ginzburg-Landau (GL) coherence length

$$\xi(T) = \frac{1}{\sqrt{2m^*|\alpha(T)|}}. \tag{1.59}$$

Using the expression (1.50) for $\alpha(T)$ we have also

$$\xi(T) = \sqrt{\frac{2\pi n}{m^* H_c^2(T)} \frac{\lambda_L(0)}{\lambda_L(T)}}. \tag{1.60}$$

Recalling that ($t = T/T_c$)

$$H_c(T) \approx \left(1 - t^2\right), \qquad \lambda_L(T) \approx \frac{1}{\left(1 - t^4\right)^{1/2}}, \tag{1.61}$$

we see that also the GL coherence length goes to infinity for $T \to T_c$

$$\xi(T) \approx \frac{1}{H_c(T)\lambda_L(T)} \approx \frac{1}{(1-t^2)^{1/2}}. \tag{1.62}$$

It is possible to show that

$$\xi(T) \approx \frac{\xi_0}{(1-t^2)^{1/2}}. \tag{1.63}$$

Therefore the GL coherence length is related but not the same as the Pippard's coherence length. A useful quantity is

$$\kappa = \frac{\lambda_L(T)}{\xi(T)}, \tag{1.64}$$

which is finite for $T \to T_c$ and approximately independent on the temperature. For typical pure superconductors $\lambda \approx 500$ Å, $\xi \approx 3000$ Å, and $\kappa \ll 1$.

C. Cooper pairs

One of the pillars of the microscopic theory of superconductivity is that electrons close to the FErmi surface can be bound in pairs by an attractive arbitrary weak interaction (Cooper, 1956). First of all let us remember that the Fermi distribution function for $T \to 0$ is nothing but a θ-function

$$f(E,T) = \frac{1}{e^{(E-\mu)/T} + 1}, \qquad \lim_{T \to 0} f(E,T) = \theta(\mu - E), \tag{1.65}$$

meaning that all the states are occupied up to the Fermi energy

$$E_F = \mu, \tag{1.66}$$

where μ is the chemical potential, as shown in Fig. 5.

The key point is that the problem has an enormous degeneracy at the Fermi surface since there is no cost in free energy for adding or subtracting a fermion at the Fermi surface (here and in the following we will be quite liberal in speaking about thermodynamic potentials; in the present case the relevant quantity is the grand potential)

$$\Omega = E - \mu N \to (E \pm E_F) - (N \pm 1) = \Omega. \tag{1.67}$$

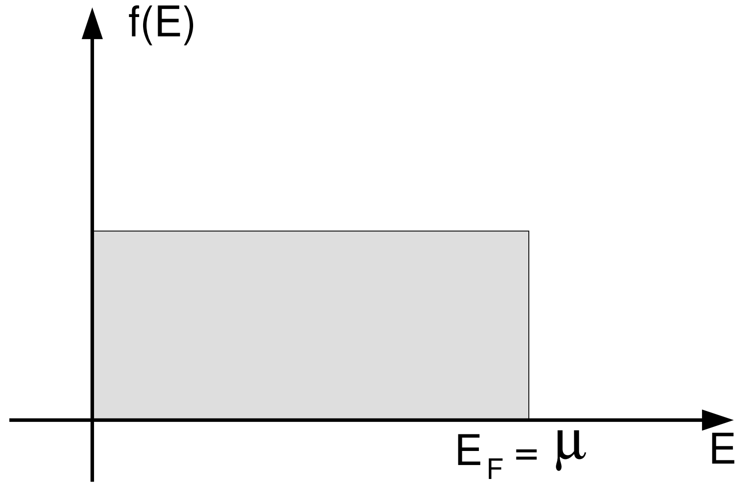

FIG. 5 *The Fermi distribution at zero temperature.*

This observation suggests that a condensation phenomenon can take place if two fermions are bounded. In fact, suppose that the binding energy is E_B, then adding a bounded pair to the Fermi surface we get

$$\Omega \to (E + 2E_F - E_B) - \mu(N+2) = -E_B. \tag{1.68}$$

Therefore we get more stability adding more bounded pairs to the Fermi surface. Cooper proved that two fermions can give rise to a bound state for an arbitrary attractive interaction by considering the following simple model. Let us add two fermions at the Fermi surface at $T = 0$ and suppose that the two fermions interact through an attractive potential. Interactions among this pair and the fermion sea in the Fermi sphere are neglected except for what follows from Fermi statistics. The next step is to look for a convenient two-particle wave function. Assuming that the pair has zero total momentum one starts with

$$\psi_0(\mathbf{r}_1 - \mathbf{r}_2) = \sum_{\mathbf{k}} g_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)}. \tag{1.69}$$

Here and in the following we will switch often back and forth from discretized momenta to continuous ones. We remember that the rule to go from one notation to the other is simply

$$\sum_{\mathbf{k}} \to \frac{L^3}{(2\pi)^3} \int d^3\mathbf{k}, \tag{1.70}$$

where L^3 is the quantization volume. Also often we will omit the volume factor. This means that in this case we are considering densities. We hope that from the context it will be clear what we are doing. One has also to introduce the spin wave function and properly antisymmetrize. We write

$$\psi_0(\mathbf{r}_1 - \mathbf{r}_2) = (\alpha_1\beta_2 - \alpha_2\beta_1) \sum_{\mathbf{k}} g_{\mathbf{k}} \cos(\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)), \tag{1.71}$$

where α_i and β_i are the spin functions. This wave function is expected to be preferred with respect to the triplet state, since the "cos" structure gives a bigger probability for the fermions to stay together. Inserting this wave function inside the Schrödinger equation

$$\left[-\frac{1}{2m}\left(\boldsymbol{\nabla}_1^2 + \boldsymbol{\nabla}_2^2\right) + V(\mathbf{r}_1 - \mathbf{r}_2)\right]\psi_0(\mathbf{r}_1 - \mathbf{r}_2) = E\psi_0(\mathbf{r}_1 - \mathbf{r}_2), \tag{1.72}$$

we find

$$(E - 2\epsilon_{\mathbf{k}})g_{\mathbf{k}} = \sum_{\mathbf{k}' > \mathbf{k}_F} V_{\mathbf{k},\mathbf{k}'}\, g_{\mathbf{k}'}, \tag{1.73}$$

where $\epsilon_{\mathbf{k}} = |\mathbf{k}|^2/2m$ and

$$V_{\mathbf{k},\mathbf{k}'} = \frac{1}{L^3}\int V(\mathbf{r})\, e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r})}\, d^3\mathbf{r}. \tag{1.74}$$

Since one looks for solutions with $E < 2\epsilon_{\mathbf{k}}$, Cooper made the following assumption on the potential:

$$V_{\mathbf{k},\mathbf{k}'} = \begin{cases} -G & k_F \le |\mathbf{k}| \le k_c \\ 0 & \text{otherwise} \end{cases} \tag{1.75}$$

with $G > 0$ and $\epsilon_{\mathbf{k}_F} = E_F$. Here a cutoff k_c has been introduced such that

$$\epsilon_{k_c} = E_F + \delta \tag{1.76}$$

and $\delta \ll E_F$. This means that one is restricting the physics to the one corresponding to degrees of freedom close to the Fermi surface. The Schrödinger equation reduces to

$$(E - 2\epsilon_{\mathbf{k}})g_{\mathbf{k}} = -G \sum_{\mathbf{k}'>\mathbf{k}_F} g_{\mathbf{k}'}. \tag{1.77}$$

Summing over $\mathbf{k}$ we get

$$\frac{1}{G} = \sum_{\mathbf{k}>\mathbf{k}_F} \frac{1}{2\epsilon_{\mathbf{k}} - E}. \tag{1.78}$$

Replacing the sum with an integral we obtain

$$\frac{1}{G} = \int_{k_F}^{k_c} \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2\epsilon_{\mathbf{k}} - E} = \int_{E_F}^{E_F+\delta} \frac{d\Omega}{(2\pi)^3} k^2 \frac{dk}{d\epsilon_{\mathbf{k}}}\frac{d\epsilon}{2\epsilon - E}. \tag{1.79}$$

Introducing the density of states at the Fermi surface for two electrons with spin up and down

$$\rho = 2\int \frac{d\Omega}{(2\pi)^3} k^2 \frac{dk}{d\epsilon_{\mathbf{k}}}, \tag{1.80}$$

we obtain

$$\frac{1}{G} = \frac{1}{4}\rho \log \frac{2E_F - E + 2\delta}{2E_F - E}. \tag{1.81}$$

Close to the Fermi surface we may assume $k \approx k_F$ and

$$\epsilon_{\mathbf{k}} = \mu + (\epsilon_{\mathbf{k}} - \mu) \approx \mu + \left.\frac{\partial \epsilon_{\mathbf{k}}}{\partial \mathbf{k}}\right|_{\mathbf{k}=\mathbf{k}_F} \cdot (\mathbf{k} - \mathbf{k}_F) = \mu + \mathbf{v}_F(\mathbf{k}) \cdot \boldsymbol{\ell}, \tag{1.82}$$

where

$$\boldsymbol{\ell} = \mathbf{k} - \mathbf{k}_F \tag{1.83}$$

is the "residual momentum". Therefore

$$\rho = \frac{k_F^2}{\pi^2 v_F}. \tag{1.84}$$

Solving Eq. (1.81) we find

$$E = 2E_F - 2\delta \frac{e^{-4/\rho G}}{1 - e^{-4/\rho G}}. \tag{1.85}$$

For most classic superconductor

$$\rho G < 0.3, \tag{1.86}$$

In this case (weak coupling approximation. $\rho G \ll 1$) we get

$$E \approx 2E_F - 2\delta e^{-4/\rho G}. \tag{1.87}$$

We see that a bound state is formed with a binding energy

$$E_B = 2\delta e^{-4/\rho G}. \tag{1.88}$$

The result is not analytic in G and cannot be obtained by a perturbative expansion in G. Notice also that the bound state exists regardless of the strength of G. Defining

$$N = \sum_{\mathbf{k}>\mathbf{k}_F} g_{\mathbf{k}}, \tag{1.89}$$

we get the wave function

$$\psi_0(\mathbf{r}) = N \sum_{\mathbf{k}>\mathbf{k}_F} \frac{\cos(\mathbf{k}\cdot\mathbf{r})}{2\epsilon_{\mathbf{k}} - E}. \tag{1.90}$$

Measuring energies from E_F we introduce

$$\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - E_F. \tag{1.91}$$

from which

$$\psi_0(\mathbf{r}) = N \sum_{\mathbf{k}>\mathbf{k}_F} \frac{\cos(\mathbf{k}\cdot\mathbf{r})}{2\xi_{\mathbf{k}} + E_B}. \tag{1.92}$$

We see that the wave function in momentum space has a maximum for $\xi_{\mathbf{k}} = 0$, that is for the pair being at the Fermi surface, and falls off with $\xi_{\mathbf{k}}$. Therefore the electrons involved in the pairing are the ones within a range E_B above E_F. Since for $\rho G \ll 1$ we have $E_B \ll \delta$, it follows that the behavior of $V_{\mathbf{k},\mathbf{k}'}$ far from the Fermi surface is irrelevant. Only the degrees of freedom close to the Fermi surface are important. Also using the uncertainty principle as in the discussion of the Pippard non-local theory we have that the size of the bound pair is larger than v_F/E_B. However the critical temperature turns out to be of the same order as E_B, therefore the size of the Cooper pair is of the order of the Pippard's coherence length $\xi_0 = av_F/T_c$.

1. The size of a Cooper pair

It is an interesting exercise to evaluate the size of a Cooper pair defined in terms of the mean square radius of the pair wave function

$$\bar{R}^2 = \frac{\int |\psi_0(\mathbf{r})|^2 |\mathbf{r}|^2 d^3\mathbf{r}}{\int |\psi_0(\mathbf{r})|^2 d^3\mathbf{r}}. \tag{1.93}$$

Using the expression (1.69) for ψ_0 we have

$$|\psi_0(\mathbf{r})|^2 = \sum_{\mathbf{k},\mathbf{k}'} g_{\mathbf{k}} g^*_{\mathbf{k}'} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} \tag{1.94}$$

and

$$\int |\psi_0(\mathbf{r})|^2 d^3\mathbf{r} = L^3 \sum_{\mathbf{k}} |g_{\mathbf{k}}|^2. \tag{1.95}$$

Also

$$\int |\psi_0(\mathbf{r})|^2 |\mathbf{r}|^2 d^3\mathbf{r} = \int \sum_{\mathbf{k}\mathbf{k}'} \left[-i\nabla_{\mathbf{k}'} g^*_{\mathbf{k}'}\right] \left[i\nabla_{\mathbf{k}} g_{\mathbf{k}}\right] e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} d^3\mathbf{r} = L^3 \sum_{\mathbf{k}} |\nabla_{\mathbf{k}} g_{\mathbf{k}}|^2. \tag{1.96}$$

Therefore

$$\bar{R}^2 = \frac{\sum_{\mathbf{k}} |\nabla_{\mathbf{k}} g_{\mathbf{k}}|^2}{\sum_{\mathbf{k}} |g_{\mathbf{k}}|^2}. \tag{1.97}$$

Recalling that

$$g_{\mathbf{k}} \approx \frac{1}{2\epsilon_{\mathbf{k}} - E} = \frac{1}{2\xi_{\mathbf{k}} + E_B}, \tag{1.98}$$

we obtain

$$\sum_{\mathbf{k}} |\nabla_{\mathbf{k}} g_{\mathbf{k}}|^2 \approx \sum_{\mathbf{k}} \frac{1}{(2\xi_{\mathbf{k}} + E_B)^4} \left| 2\frac{\partial \epsilon_{\mathbf{k}}}{\partial \mathbf{k}} \right|^2 = 4v_F^2 \sum_{\mathbf{k}} \frac{1}{(2\xi_{\mathbf{k}} + E_B)^4}. \tag{1.99}$$

Going to continuous variables and noticing that the density of states cancel in the ratio we find

$$\bar{R}^2 = 4v_F^2 \frac{\displaystyle\int_0^\infty \frac{d\xi}{(2\xi + E_B)^4}}{\displaystyle\int_0^\infty \frac{d\epsilon}{(2\xi + E_B)^2}} = 4v_F^2 \frac{\displaystyle -\frac{1}{3}\frac{1}{(2\xi + E_B)^3}\Big|_0^\infty}{\displaystyle -\frac{1}{2\xi + E_B}\Big|_0^\infty} = \frac{4}{3}\frac{v_F^2}{E_B^2}, \tag{1.100}$$

where, due to the convergence we have extended the integrals up to infinity. Assuming E_B of the order of the critical temperature T_c, with $T_c \approx 10$ K and $v_F \approx 10^8$ cm/s, we get

$$\bar{R} \approx 10^{-4} \text{ cm} \approx 10^4 \text{ Å}. \tag{1.101}$$

The order of magnitude of $\bar{R}$ is the same as the coherence length ξ_0. Since one electron occupies a typical size of about $(2 \text{ Å})^3$, this means that in a coherence volume there are about 10^{11} electrons. Therefore it is not reasonable to construct a pair wavefunction, but we need a wave function taking into account all the electrons. This is made in the BCS theory.

D. Origin of the attractive interaction

The problem of getting an attractive interaction among electrons is not an easy one. In fact the Coulomb interaction is repulsive, although it gets screened in the medium by a screening length of order of $1/k_s \approx 1$ Å. The screened Coulomb potential is given by

$$V(\mathbf{q}) = \frac{4\pi e^2}{q^2 + k_s^2}. \tag{1.102}$$

To get attraction is necessary to consider the effect of the motion of the ions. The rough idea is that one electron polarizes the medium attracting positive ions. In turn these attract a second electron giving rise to a net attraction between the two electrons. To quantify this idea is necessary to take into account the interaction among the electrons and the lattice or, in other terms, the interactions among the electrons and the phonons as suggested by (Frolich, 1952). This idea was confirmed by the discovery of the **isotope effect**, that is the dependence of T_c or of the gap from the isotope mass (see Section I.A). Several calculations were made by (Pines, 1958) using the "jellium model". The potential in this model is (de Gennes, 1989)

$$V(\mathbf{q}, \omega) = \frac{4\pi e^2}{q^2 + k_s^2}\left(1 + \frac{\omega_{\mathbf{q}}^2}{\omega^2 - \omega_{\mathbf{q}}^2}\right). \tag{1.103}$$

Here $\omega_{\mathbf{q}}$ is the phonon energy that, for a simple linear chain, is given by

$$\omega_{\mathbf{q}} = 2\sqrt{\frac{k}{M}}\sin(qa/2), \tag{1.104}$$

where a is the lattice distance, k the elastic constant of the harmonic force among the ions and M their mass. For $\omega < \omega_{\mathbf{q}}$ the phonon interaction is attractive at it may overcome the Coulomb force. Also, since the cutoff to be used in the determination of the binding energy, or for the gap, is essentially the Debye frequency which is proportional to $\omega_{\mathbf{q}}$ one gets naturally the isotope effect.

II. EFFECTIVE THEORY AT THE FERMI SURFACE

A. Introduction

It turns out that the BCS theory can be derived within the Landau theory of Fermi liquids, where a conductor is treated as a gas of nearly free electrons. This is because one can make use of the idea of quasiparticles, that is electrons dressed by the interaction. A justification of this statement has been given in (Benfatto and Gallavotti, 1990; Polchinski, 1993; Shankar, 1994). Here we will follow the treatment given by (Polchinski, 1993). In order to define an effective field theory one has to start identifying a scale which, for ordinary superconductivity (let us talk about this subject to start with) is of the order of tens of eV. For instance,

$$E_0 = m\alpha^2 \approx 27\ eV \tag{2.1}$$

is the typical energy in solids. Other possible scales as the ion masses M and velocity of light can be safely considered to be infinite. In a conductor a current can be excited with an arbitrary small field, meaning that the spectrum of the charged excitations goes to zero energy. If we are interested to study these excitations we can try to construct our effective theory at energies much smaller than E_0 (the superconducting gap turns out to be of the order of $10^{-3}\ eV$). Our first problem is then to identify the quasiparticles. The natural guess is that they are spin $1/2$ particles as the electrons in the metal. If we measure the energy with respect to the Fermi surface the most general free action can be written as

$$S_{\rm free} = \int dt\, d^3\mathbf{p}\left[i\psi_\sigma^\dagger(\mathbf{p})i\partial_t\psi_\sigma(\mathbf{p}) - (\epsilon(\mathbf{p}) - \epsilon_F)\psi_\sigma^\dagger(\mathbf{p})\psi_\sigma(\mathbf{p})\right]\ . \tag{2.2}$$

Here σ is a spin index and ϵ_F is the Fermi energy. The ground state of the theory is given by the Fermi sea with all the states $\epsilon(\mathbf{p}) < \epsilon_F$ filled and all the states $\epsilon(\mathbf{p}) > \epsilon_F$ empty. The Fermi surface is defined by $\epsilon(\mathbf{p}) = \epsilon_F$. A simple example is shown in Fig. 6.

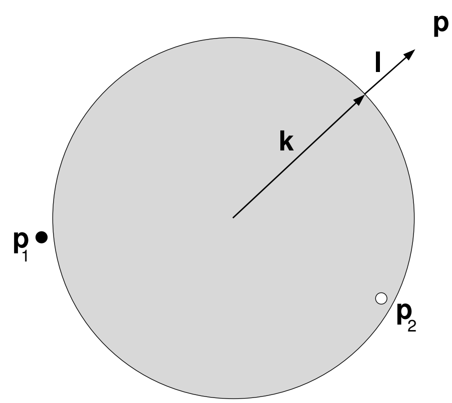

FIG. 6 *A spherical Fermi surface. Low lying excitations are shown: a particle at* $\mathbf{p}_1$ *and a hole at* $\mathbf{p}_2$*. The decomposition of a momentum as the Fermi momentum* $\mathbf{k}$*, and the residual momentum* $\mathbf{l}$ *is also shown.*

The free action defines the scaling properties of the fields. In this particular instance we are interested at the physics very close to the Fermi surface and therefore we are after the scaling properties for $\epsilon \to \epsilon_F$. Measuring energies with respect to the Fermi energy we introduce a scaling factor $s < 1$. Then, as the energy scales to zero the momenta must scale toward the Fermi surface. It is convenient to decompose the momenta as follows (see also Fig. 6)

$$\mathbf{p} = \mathbf{k} + \mathbf{l}\ . \tag{2.3}$$

Therefore we get

$$E \to s\,E,\quad \mathbf{k} \to \mathbf{k},\quad \mathbf{l} \to s\mathbf{l}\ . \tag{2.4}$$

We can expand the second term in Eq. (2.2) obtaining

$$\epsilon(\mathbf{p}) - \epsilon_F = \frac{\partial\epsilon(\mathbf{p})}{\partial\mathbf{p}}\Big|_{\mathbf{p}=\mathbf{k}} \cdot (\mathbf{p} - \mathbf{k}) = l v_F(\mathbf{k})\ , \tag{2.5}$$

where

$$\mathbf{v}_F(\mathbf{k}) = \frac{\partial \epsilon(\mathbf{p})}{\partial \mathbf{p}}\Big|_{\mathbf{p}=\mathbf{k}} . \tag{2.6}$$

Notice that $\mathbf{v}_F(\mathbf{k})$ is a vector orthogonal to the Fermi surface. We get

$$S_{\text{free}} = \int dt\, d^3\mathbf{p} \left[\psi_\sigma^\dagger(\mathbf{p})\,(i\partial_t - l v_F(\mathbf{k}))\,\psi_\sigma(\mathbf{p})\right] . \tag{2.7}$$

The various scaling laws are

$$\begin{array}{ll} dt \to s^{-1} dt, & d^3\mathbf{p} = d^2\mathbf{k} dl \to s d^2\mathbf{k} dl \\ \partial_t \to s\partial_t, & l \to sl . \end{array} \tag{2.8}$$

Therefore, in order to leave the free action invariant the fields must scale as

$$\psi_\sigma(\mathbf{p}) \to s^{-1/2}\psi_\sigma(\mathbf{p}) . \tag{2.9}$$

Our analysis goes on considering all the possible interaction terms compatible with the symmetries of the theory and looking for the relevant ones. The symmetries of the theory are the electron number and the spin $SU(2)$, since we are considering the non-relativistic limit. We ignore also possible complications coming from the real situation where one has to do with crystals. The possible terms are:

1. Quadratic terms:

$$\int dt\, d^2\mathbf{k}\, dl\, \mu(\mathbf{k})\psi_\sigma^\dagger(\mathbf{p})\psi_\sigma(\mathbf{p}) . \tag{2.10}$$

   This is a relevant term since it scales as s^{-1} but it can be absorbed into the definition of the Fermi surface (that is by $\epsilon(\mathbf{p})$. Further terms with time derivatives or powers of l are already present or they are irrelevant.

2. Quartic terms:

$$\int \prod_{i=1}^{4}\left(d^2\mathbf{k}_i\, dl_i\right)\left(\psi^\dagger(\mathbf{p}_1)\psi(\mathbf{p}_3)\right)\left(\psi^\dagger(\mathbf{p}_2)\psi(\mathbf{p}_4)\right) V(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)\delta^3(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4). \tag{2.11}$$

   This scales as $s^{-1}\, s^{4-4/2} = s$ times the scaling of the δ-function. For a generic situation the δ-function does not scale (see Fig. 7). However consider a scattering process $1+2 \to 3+4$ and decompose the momenta as follows:

$$\mathbf{p}_3 = \mathbf{p}_1 + \delta\mathbf{k}_3 + \delta\mathbf{l}_3 , \tag{2.12}$$
$$\mathbf{p}_4 = \mathbf{p}_2 + \delta\mathbf{k}_4 + \delta\mathbf{l}_4 . \tag{2.13}$$

   This gives rise to

$$\delta^3(\delta\mathbf{k}_3 + \delta\mathbf{k}_4 + \delta\mathbf{l}_3 + \delta\mathbf{l}_4) . \tag{2.14}$$

   When $\mathbf{p}_1 = -\mathbf{p}_2$ and $\mathbf{p}_3 = -\mathbf{p}_4$ we see that the δ-function factorizes

$$\delta^2(\delta\mathbf{k}_3 + \delta\mathbf{k}_4)\delta(\delta\mathbf{l}_3 + \delta\mathbf{l}_4) \tag{2.15}$$

   scaling as s^{-1}. Therefore, in this kinematical situation the term (2.11) is marginal (does not scale). This means that its scaling properties should be looked at the level of quantum corrections.

3. Higher order terms Terms with $2n$ fermions ($n > 2$) scale as s^{n-1} times the scaling of the δ-function and therefore they are irrelevant.

We see that the only potentially dangerous term is the quartic interaction with the particular kinematical configuration corresponding to a Cooper pair. We will discuss the one-loop corrections to this term a bit later. Before doing that let us study the free case.

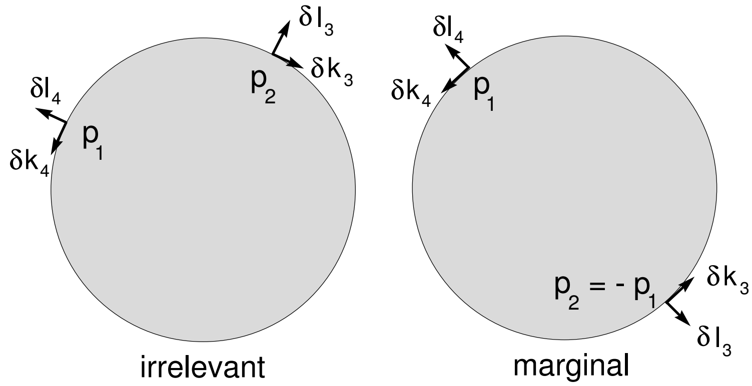

FIG. 7 *The kinematics for the quartic coupling is shown in the generic (left) and in the special (right) situations discussed in the text*

B. Free fermion gas

The statistical properties of free fermions were discussed by Landau who, however, preferred to talk about fermion liquids. The reason, as quoted in (Ginzburg and Andryushin, 1994), is that Landau thought that "Nobody has abrogated Coulomb's law".

Let us consider the free fermion theory we have discussed before. The fermions are described by the equation of motion

$$(i\partial_t - \ell v_F)\psi_\sigma(\mathbf{p}, t = 0. \tag{2.16}$$

The Green function, or the propagator of the theory is defined by

$$(i\partial_t - \ell v_F)G_{\sigma\sigma'}(\mathbf{p}, t) = \delta_{\sigma\sigma'}\delta(t). \tag{2.17}$$

It is easy to verify that a solution is given by

$$G_{\sigma\sigma'}(\mathbf{p}, t) = \delta_{\sigma\sigma'}G(\mathbf{p}, t) = -i\delta_{\sigma\sigma'}\left[\theta(t)\theta(\ell) - \theta(-t)\theta(-\ell)\right]e^{-i\ell v_F t}. \tag{2.18}$$

By using the integral representation for the step function

$$\theta(t) = \frac{i}{2\pi}\int d\omega \frac{e^{-i\omega t}}{\omega + i\epsilon}, \tag{2.19}$$

we get

$$G(\mathbf{p}, t) = \frac{1}{2\pi}\int d\omega \frac{e^{-i\ell v_F t}}{\omega + i\epsilon}\left[e^{-i\omega t}\theta(\ell) - e^{i\omega t}\theta(-\ell)\right]. \tag{2.20}$$

By changing the variable $\omega \to \omega' = \omega \pm \ell v_F$ in the two integrals and sending $\omega' \to -\omega'$ in the second integral we obtain

$$G(\mathbf{p}, t) = \frac{1}{2\pi}\int d\omega e^{-i\omega t}\left[\frac{\theta(\ell)}{\omega - \ell v_F + i\epsilon} + \frac{\theta(-\ell)}{\omega - \ell v_F - i\epsilon}\right]. \tag{2.21}$$

We may also write

$$G(\mathbf{p}, t) \equiv \frac{1}{2\pi}\int dp_0 G(p_0, \mathbf{p})e^{-ip_0 t}, \tag{2.22}$$

with

$$G(p) = \frac{1}{(1 + i\epsilon)p_0 - \ell v_F}. \tag{2.23}$$

Notice that this definition of $G(p)$ corresponds to the standard Feynman propagator since it propagates ahead in time positive energy solutions $\ell > 0$ ($p > p_F$) and backward in time negative energy solutions $\ell < 0$ ($p < p_F$) corresponding to holes in the Fermi sphere. In order to have contact with the usual formulation of field quantum theory we introduce Fermi fields

$$\psi_\sigma(x) = \sum_{\mathbf{p}} b_\sigma(\mathbf{p}, t) e^{i\mathbf{p}\cdot\mathbf{x}} = \sum_{\mathbf{p}} b_\sigma(\mathbf{p}) e^{-ip\cdot x}, \tag{2.24}$$

where $x^\mu = (t, \mathbf{x})$, $p^\mu = \ell v_F, \mathbf{p})$ and

$$p \cdot x = \ell v_F t - \mathbf{p}\cdot\mathbf{x}. \tag{2.25}$$

Notice that within this formalism fermions have no antiparticles, however the fundamental state is described by the following relations

$$\begin{array}{ll} b_\sigma(\mathbf{p})|0\rangle = 0 & \text{for} \quad |\mathbf{p}| > p_F \\ b_\sigma^\dagger(\mathbf{p})|0\rangle = 0 & \text{for} \quad |\mathbf{p}| < p_F. \end{array} \tag{2.26}$$

One could, as usual in relativistic field theory, introduce a re-definition for the creation operators for particles with $p < p_F$ as annihilation operators for holes but we will not do this here. Also we are quantizing in a box, but we will shift freely from this normalization to the one in the continuous according to the circumstances. The fermi operators satisfy the usual anticommutation relations

$$[b_\sigma(\mathbf{p}), b_{\sigma'}^\dagger(\mathbf{p}')]_+ = \delta_{\mathbf{p}\mathbf{p}'}\delta_{\sigma\sigma'} \tag{2.27}$$

from which

$$[\psi_\sigma(\mathbf{x}, t), \psi_{\sigma'}^\dagger(\mathbf{y}, t)]_+ = \delta_{\sigma\sigma'}\delta^3(\mathbf{x} - \mathbf{y}). \tag{2.28}$$

We can now show that the propagator is defined in configurations space in terms of the usual T-product for Fermi fields

$$G_{\sigma\sigma'}(x) = -i\langle 0|T(\psi_\sigma(x)\psi_{\sigma'}(0))|0\rangle. \tag{2.29}$$

In fact we have

$$G_{\sigma\sigma'}(x) = -i\delta_{\sigma\sigma'}\sum_{\mathbf{p}}\langle 0|T(b_\sigma(\mathbf{p}, t)b_\sigma^\dagger(\mathbf{p}, 0))|0\rangle e^{i\mathbf{p}\cdot\mathbf{x}} \equiv \delta_{\sigma\sigma'}\sum_{\mathbf{p}} G(\mathbf{p}, t), \tag{2.30}$$

where we have used

$$\langle 0|T(b_\sigma(\mathbf{p}, t){b'_\sigma}^\dagger(\mathbf{p}', 0))|0\rangle = \delta_{\sigma\sigma'}\delta_{\mathbf{p}\mathbf{p}'}\langle 0|T(b_\sigma(\mathbf{p}, t)b_\sigma^\dagger(\mathbf{p}, 0))|0\rangle. \tag{2.31}$$

Since

$$\begin{aligned} \langle 0|b_\sigma^\dagger(\mathbf{p})b_\sigma(\mathbf{p})|0\rangle &= \theta(p_F - p) = \theta(-\ell), \\ \langle 0|b_\sigma(\mathbf{p})b_\sigma^\dagger(\mathbf{p})|0\rangle &= 1 - \theta(p_F - p) = \theta(p - p_F) = \theta(\ell), \end{aligned} \tag{2.32}$$

we get

$$G(\mathbf{p}, t) = \begin{cases} -i\theta(\ell)e^{-i\ell v_F t} & t > 0 \\ i\theta(-\ell)e^{-i\ell v_F t} & t < 0. \end{cases} \tag{2.33}$$

We can also write

$$G(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} G(p), \tag{2.34}$$

with $G(p)$ defined in Eq. (2.23). It is interesting to notice that the fermion density can be obtained from the propagator. In fact, in the limit $\delta \to 0$ for $\delta > 0$ we have

$$G_{\sigma\sigma'}(\mathbf{0}, -\delta) = -i\langle 0|T(\psi_\sigma(\mathbf{0}, -\delta)\psi_{\sigma'}^\dagger(0)|\rangle \Rightarrow i\langle 0|\psi_{\sigma'}^\dagger\psi_\sigma|\rangle \equiv i\rho_F. \tag{2.35}$$

Therefore

$$\rho_F = -i \lim_{\delta\to 0^+} G_{\sigma\sigma}(\mathbf{0}, -\delta) = -2i \int \frac{d^4p}{(2\pi)^4} e^{ip_0\delta} \frac{1}{(1+i\epsilon)p_0 - \ell v_F}. \tag{2.36}$$

The exponential is convergent in the upper plane of p_0, where we pick up the pole for $\ell < 0$ at

$$p_0 = \ell v_F + i\epsilon. \tag{2.37}$$

Therefore

$$\rho_F = 2\int \frac{d^3\mathbf{p}}{(2\pi)^3}\theta(-\ell) = 2\int \frac{d^3\mathbf{p}}{(2\pi)^3}\theta(p_F - |\mathbf{p}|) = \frac{p_F^3}{3\pi^2}. \tag{2.38}$$

C. One-loop corrections

We now evaluate the one-loop corrections to the four-fermion scattering. These are given in Fig. 8, and we get

$$G(E) = G - G^2 \int \frac{dE'\, d^2\mathbf{k}\, dl}{(2\pi)^4} \frac{1}{((E+E')(1+i\epsilon) - v_F(\mathbf{k})l)((E-E')(1+i\epsilon) - v_F(\mathbf{k})l)}, \tag{2.39}$$

where we have assumed the vertex V as a constant G. The poles of the integrand are shown in Fig. 9

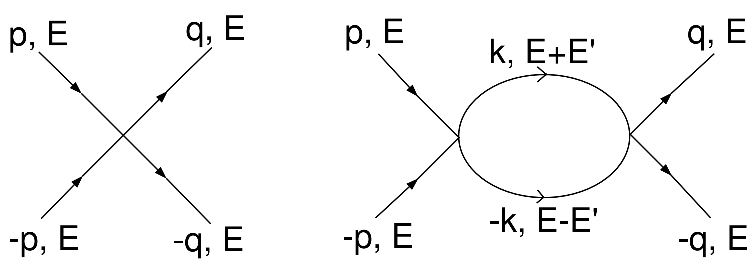

FIG. 8 *The two diagrams contributing to the one-loop four-fermi scattering amplitude*

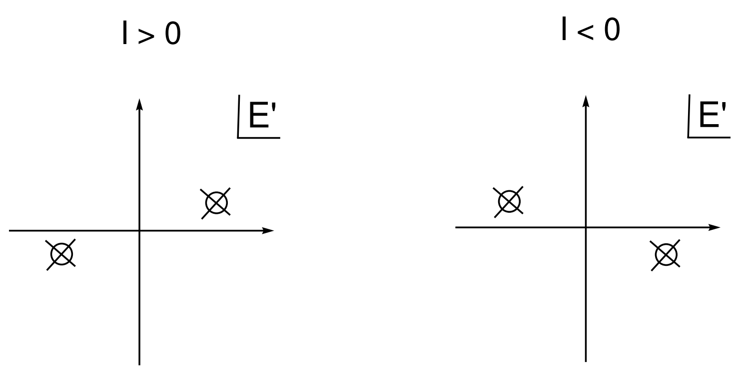

FIG. 9 *The position of the poles in the complex plane of E' in the one-loop amplitude, in the two cases $\ell \gtrless 0$*

The integrand of Eq. (2.39) can be written as

$$\frac{1}{2(E-\ell v_F)}\left[\frac{1}{E'+E-(1-i\epsilon)\ell v_F}-\frac{1}{E'-E+(1-i\epsilon)\ell v_F}\right]. \tag{2.40}$$

Therefore closing the integration path in the upper plane we find

$$iG(E)=iG-G^2\int\frac{d^2\mathbf{k}d\ell}{(2\pi)^4}\frac{1}{2(E-\ell v_F)}\left[(-2\pi i)\theta(\ell)+(2\pi i)\theta(-\ell)\right]. \tag{2.41}$$

By changing $\ell\to-\ell$ in the second integral we find

$$iG(E)=iG+iG^2\int\frac{d^2\mathbf{k}d\ell}{(2\pi)^4}\frac{\ell v_F}{E^2-(\ell v_F)^2}\theta(\ell). \tag{2.42}$$

By putting an upper cutoff E_0 on the integration over ℓ we get

$$G(E)=G-\frac{1}{2}G^2\rho\log(\delta/E), \tag{2.43}$$

where δ is a cutoff on $v_F l$ and

$$\rho=2\int\frac{d^2\mathbf{k}}{(2\pi)^3}\frac{1}{v_F(\mathbf{k})} \tag{2.44}$$

is the density of states at the Fermi surface for for the two paired fermions. For a spherical surface

$$\rho=\frac{p_F^2}{\pi^2 v_F}, \tag{2.45}$$

where the Fermi momentum is defined by

$$\epsilon(p_F)=\epsilon_F=\mu. \tag{2.46}$$

From the renormalization group equation (or just at the same order of approximation) we get easily

$$G(E)\approx\frac{G}{1+\dfrac{\rho G}{2}\log(\delta/E)}, \tag{2.47}$$

showing that for $E\to 0$ we have

- $G>0$ (repulsive interaction), $G(E)$ becomes weaker (irrelevant interaction)
- $G<0$ (attractive interaction), $G(E)$ becomes stronger (relevant interaction)

This is illustrated in Fig 10.

Therefore an attractive four-fermi interaction is unstable and one expects a rearrangement of the vacuum. This leads to the formation of Cooper pairs. In metals the physical origin of the four-fermi interaction is the phonon interaction. If it happens that at some intermediate scale E_1, with

$$E_1\approx\left(\frac{m}{M}\right)^{1/2}\delta, \tag{2.48}$$

with m the electron mass and M the nucleus mass, the phonon interaction is stronger than the Coulomb interaction, then we have the superconductivity, otherwise we have a normal metal. In a superconductor we have a non-vanishing expectation value for the difermion condensate

$$\langle\psi_\sigma(\mathbf{p})\psi_{-\sigma}(-\mathbf{p})\rangle. \tag{2.49}$$

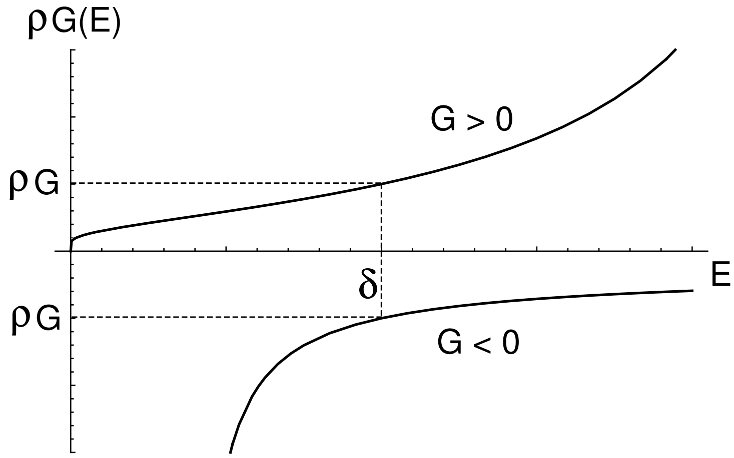

FIG. 10 *The behavior of $G(E)$ for $G > 0$ and $G < 0$.*

D. Renormalization group analysis

RG analysis indicates the possible existence of instabilities at the scale where the couplings become strong. A complete study for QCD with 3-flavors has been done in (Evans *et al.*, 1999a,b). One has to look at the four-fermi coupling with bigger coefficient C in the RG equation

$$\frac{dG(E)}{d\log E} = CG^2 \to G(E) = \frac{G}{1 - CG\log(E/E_0)}. \tag{2.50}$$

The scale of the instability is set by the corresponding Landau pole.

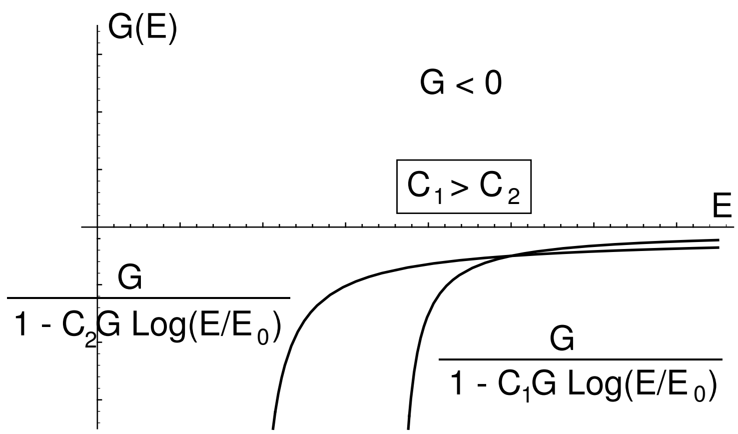

FIG. 11 *The figure shows that the instability is set in correspondence with the bigger value of the coefficient of G^2 in the renormalization group equation.*

In the case of 3-flavors QCD one has 8 basic four-fermi operators originating from one-gluon exchange

$$O^0_{LL} = (\bar\psi_L\gamma_0\psi_L)^2,\ O^0_{LR} = (\bar\psi_L\gamma_0\psi_L)(\bar\psi_R\gamma_0\psi_R), \tag{2.51}$$

$$O^i_{LL} = (\bar\psi_L\gamma_i\psi_L)^2,\ O^i_{LR} = (\bar\psi_L\gamma_i\psi_L)(\bar\psi_R\gamma_i\psi_R), \tag{2.52}$$

in two different color structures, symmetric and anti-symmetric

$$(\bar{\psi}^a \psi^b)(\bar{\psi}^c \psi^d)(\delta_{ab}\delta_{cd} \pm \delta_{ad}\delta_{bc}). \tag{2.53}$$

The coupling with the biggest C coefficient in the RG equations is given by the following operator (using Fierz)

$$(\bar{\psi}_L \gamma_0 \psi_L)^2 - (\bar{\psi}_L \vec{\gamma} \psi_L)^2 = 2(\psi_L C \psi_L)(\bar{\psi}_L C \bar{\psi}_L). \tag{2.54}$$

This shows that the dominant operator corresponds to a scalar diquark channel. The subdominant operators lead to vector diquark channels. A similar analysis can be done for 2-flavors QCD. This is somewhat more involved since there are new operators

$$\det{}_{flavor}(\bar{\psi}_R \psi_L), \quad \det{}_{flavor}(\bar{\psi}_R \vec{\Sigma} \psi_L). \tag{2.55}$$

The result is that the dominant coupling is (after Fierz)

$$\det{}_{flavor}[(\bar{\psi}_R \psi_L)^2 - (\bar{\psi}_R \vec{\Sigma} \psi_L)^2] = 2(\psi_L^{i\alpha} C \psi_L^{j\beta} \epsilon_{ij})\epsilon_{\alpha\beta I}(\psi_R^{k\gamma} C \psi_R^{l\delta} \epsilon_{kl})\epsilon_{\gamma\delta I}. \tag{2.56}$$

The dominant operator corresponds to a flavor singlet and to the antisymmetric color representation $\bar{3}$.

III. THE GAP EQUATION

In this Section we will study in detail the gap equation deriving it within the BCS approach. We will show also how to get it from the Nambu Gor'kov equations and the functional approach. A Section will be devoted to the determination of the critical temperature.

A. A toy model

The physics of fermions at finite density and zero temperature can be treated in a systematic way by using Landau's idea of quasi-particles. An example is the Landau theory of Fermi liquids. A conductor is treated as a gas of almost free electrons. However these electrons are dressed by the interactions. As we have seen, according to Polchinski (Polchinski, 1993), this procedure just works because the interactions can be integrated away in the usual sense of the effective theories. Of course, this is a consequence of the special nature of the Fermi surface, which is such that there are practically no relevant or marginal interactions. In fact, all the interactions are irrelevant except for the four-fermi couplings between pairs of opposite momentum. Quantum corrections make the attractive ones relevant, and the repulsive ones irrelevant. This explains the instability of the Fermi surface of almost free fermions against any attractive four-fermi interactions, but we would like to understand better the physics underlying the formation of the condensates and how the idea of quasi-particles comes about. To this purpose we will make use of a toy model involving two Fermi oscillators describing, for instance, spin up and spin down. Of course, in a finite-dimensional system there is no spontaneous symmetry breaking, but this model is useful just to illustrate many points which are common to the full treatment, but avoiding a lot of technicalities. We assume our dynamical system to be described by the following Hamiltonian containing a quartic coupling between the oscillators

$$H = \epsilon(a_1^\dagger a_1 + a_2^\dagger a_2) + G a_1^\dagger a_2^\dagger a_1 a_2 = \epsilon(a_1^\dagger a_1 + a_2^\dagger a_2) - G a_1^\dagger a_2^\dagger a_2 a_1. \tag{3.1}$$

We will study this model by using a variational principle. We start introducing the following normalized trial wave-function $|\Psi\rangle$

$$|\Psi\rangle = \left(\cos\theta + \sin\theta \, a_1^\dagger a_2^\dagger\right)|0\rangle. \tag{3.2}$$

The di-fermion operator, $a_1 a_2$, has the following expectation value

$$\Gamma \equiv \langle\Psi| \, a_1 a_2 |\Psi\rangle = -\sin\theta\cos\theta. \tag{3.3}$$

Let us write the hamiltonian H as the sum of the following two pieces

$$H = H_0 + H_{\rm res}, \tag{3.4}$$

with

$$H_0 = \epsilon(a_1^\dagger a_1 + a_2^\dagger a_2) - G\Gamma(a_1 a_2 - a_1^\dagger a_2^\dagger) + G\Gamma^2, \tag{3.5}$$

and

$$H_{\rm res} = G(a_1^\dagger a_2^\dagger + \Gamma)\,(a_1 a_2 - \Gamma)\,, \tag{3.6}$$

Our approximation will consist in neglecting $H_{\rm res}$. This is equivalent to the mean field approach, where the operator $a_1 a_2$ is approximated by its mean value Γ. Then we determine the value of θ by looking for the minimum of the expectation value of H_0 on the trial state

$$\langle\Psi|H_0|\Psi\rangle = 2\epsilon\sin^2\theta - G\Gamma^2. \tag{3.7}$$

We get

$$2\epsilon\sin 2\theta + 2G\Gamma\cos 2\theta = 0 \;\longrightarrow\; \tan 2\theta = -\frac{G\Gamma}{\epsilon}. \tag{3.8}$$

By using the expression (3.3) for Γ we obtain the gap equation

$$\Gamma = -\frac{1}{2}\sin 2\theta = \frac{1}{2}\frac{G\Gamma}{\sqrt{\epsilon^2 + G^2\Gamma^2}}, \tag{3.9}$$

or

$$1 = \frac{1}{2}\frac{G}{\sqrt{\epsilon^2 + \Delta^2}}, \tag{3.10}$$

where $\Delta = G\Gamma$. Therefore the gap equation can be seen as the equation determining the ground state of the system, since it gives the value of the condensate. We can now introduce the idea of quasi-particles in this particular context. The idea is to look for for a transformation on the Fermi oscillators such that H_0 acquires a canonical form (Bogoliubov transformation) and to define a new vacuum annihilated by the new annihilation operators. We write the transformation in the form

$$A_1 = a_1\cos\theta - a_2^\dagger\sin\theta, \quad A_2 = a_1^\dagger\sin\theta + a_2\cos\theta, \tag{3.11}$$

Substituting this expression into H_0 we find

$$\begin{aligned} H_0 &= 2\epsilon\sin^2\theta + G\Gamma\sin 2\theta + G\Gamma^2 + (\epsilon\cos 2\theta - G\Gamma\sin 2\theta)(A_1^\dagger A_1 + A_2^\dagger A_2) \\ &+ (\epsilon\sin 2\theta + G\Gamma\cos 2\theta)(A_1^\dagger A_2^\dagger - A_1 A_2). \end{aligned} \tag{3.12}$$

Requiring the cancellation of the bilinear terms in the creation and annihilation operators we find

$$\tan 2\theta = -\frac{G\Gamma}{\epsilon} = -\frac{\Delta}{\epsilon}. \tag{3.13}$$

We can verify immediately that the new vacuum state annihilated by A_1 and A_2 is

$$|0\rangle_N = (\cos\theta + a_1^\dagger a_2^\dagger\sin\theta)|0\rangle, \qquad A_1|0\rangle_N = A_2|0\rangle_N = 0. \tag{3.14}$$

The constant term in H_0 which is equal to $\langle\Psi|H_0|\Psi\rangle$ is given by

$$\langle\Psi|H_0|\Psi\rangle = 2\epsilon\sin^2\theta - G\Gamma^2 = \left(\epsilon - \frac{\epsilon^2}{\sqrt{\epsilon^2 + \Delta^2}}\right) - \frac{\Delta^2}{G}. \tag{3.15}$$

The first term in this expression arises from the kinetic energy whereas the second one from the interaction. We define the weak coupling limit by taking $\Delta \ll \epsilon$, then the first term is given by

$$\frac{1}{2}\frac{\Delta^2}{\epsilon} = \frac{\Delta^2}{G}, \tag{3.16}$$

where we have made use of the gap equation at the lowest order in Δ. We see that in this limit the expectation value of H_0 vanishes, meaning that the normal vacuum and the condensed one lead to the same energy. However we will see that in the realistic case of a 3-dimensional Fermi sphere the condensed vacuum has a lower energy by an amount which is proportional to the density of states at the Fermi surface. In the present case there is no condensation since there is no degeneracy of the ground state contrarily to the realistic case. Nevertheless this case is interesting due to the fact that the algebra is simpler than in the full discussion of the next Section.

Therefore we get

$$H_0 = \left(\epsilon - \frac{\epsilon^2}{\sqrt{\epsilon^2+\Delta^2}}\right) - \frac{\Delta^2}{G} + \sqrt{\epsilon^2+\Delta^2}(A_1^\dagger A_1 + A_2^\dagger A_2). \tag{3.17}$$

The gap equation is recovered by evaluating Γ

$$\Gamma = {}_N\langle 0|a_1 a_2|0\rangle_N = -\frac{1}{2}\sin 2\theta \tag{3.18}$$

and substituting inside Eq. (3.13). We find again

$$\Gamma = -\frac{1}{2}\sin 2\theta = \frac{1}{2}\frac{G\Gamma}{\sqrt{\epsilon^2+\Delta^2}}, \tag{3.19}$$

or

$$1 = \frac{1}{2}\frac{G}{\sqrt{\epsilon^2+\Delta^2}}. \tag{3.20}$$

From the expression of H_0 we see that the operators $A_i^\dagger$ create out of the vacuum quasi-particles of energy

$$E = \sqrt{\epsilon^2+\Delta^2}. \tag{3.21}$$

The condensation gives rise to the fermionic energy gap, Δ. The Bogoliubov transformation realizes the dressing of the original operators a_i and $a_i^\dagger$ to the quasi-particle ones A_i and $A_i^\dagger$. Of course, the interaction is still present, but part of it has been absorbed in the dressing process getting a better starting point for a perturbative expansion. As we have said this point of view has been very fruitful in the Landau theory of conductors.

B. The BCS theory

We now proceed to the general case. We start with the following hamiltonian containing a four-fermi interaction term of the type giving rise to one-loop relevant contribution

$$\tilde H = H - \mu N = \sum_{\mathbf{k}\sigma}\xi_{\mathbf{k}} b_\sigma^\dagger(\mathbf{k}) b_\sigma(\mathbf{k}) + \sum_{\mathbf{kq}} V_{\mathbf{kq}} b_1^\dagger(\mathbf{k}) b_2^\dagger(-\mathbf{k}) b_2(-\mathbf{q}) b_1(\mathbf{q}), \tag{3.22}$$

where

$$\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - E_F = \epsilon_{\mathbf{k}} - \mu. \tag{3.23}$$

Here the indices 1 and 2 refer to spin up and dow respectively. As before we write

$$\tilde H = H_0 + H_{\rm res}, \tag{3.24}$$

where

$$H_0 = \sum_{\mathbf{k}\sigma}\xi_{\mathbf{k}} b_\sigma^\dagger(\mathbf{k}) b_\sigma(\mathbf{k}) + \sum_{\mathbf{kq}} V_{\mathbf{kq}}\left[b_1^\dagger(\mathbf{k}) b_2^\dagger(-\mathbf{k})\Gamma_{\mathbf{q}} + b_2(-\mathbf{q}) b_1(\mathbf{q})\Gamma_{\mathbf{k}}^* - \Gamma_{\mathbf{k}}^*\Gamma_{\mathbf{q}}\right] \tag{3.25}$$

and

$$H_{\rm res} = \sum_{\mathbf{kq}} V_{\mathbf{kq}}\left(b_1^\dagger(\mathbf{k}) b_2^\dagger(-\mathbf{k}) - \Gamma_{\mathbf{k}}^*\right)\left(b_2(-\mathbf{q}) b_1(\mathbf{q}) - \Gamma_{\mathbf{q}}\right), \tag{3.26}$$

with

$$\Gamma_{\mathbf{k}} = \langle b_2(-\mathbf{k}) b_1(\mathbf{k})\rangle \tag{3.27}$$

the expectation value of the difermion operator $b_2(-\mathbf{k})b_1(\mathbf{k})$ in the BCS ground state, which will be determined later. We will neglect $H_{\rm res}$ as in the toy model. We then define

$$\Delta_{\mathbf{k}} = -\sum_{\mathbf{q}} V_{\mathbf{kq}}\Gamma_{\mathbf{q}}, \tag{3.28}$$

from which

$$H_0 = \sum_{\mathbf{k}\sigma}\xi_{\mathbf{k}} b^\dagger_\sigma(\mathbf{k})b_\sigma(\mathbf{k}) - \sum_{\mathbf{k}}\left[\Delta_k b^\dagger_1(\mathbf{k})b^\dagger_2(-\mathbf{k}) + \Delta^*_{\mathbf{k}} b_2(-\mathbf{k})b_1(\mathbf{k}) - \Delta_{\mathbf{k}}\Gamma^*_{\mathbf{k}}\right]. \tag{3.29}$$

Then, we look for new operators $A_i(\mathbf{k})$

$$\begin{aligned} b_1(\mathbf{k}) &= u^*_{\mathbf{k}} A_1(\mathbf{k}) + v_{\mathbf{k}} A^\dagger_2(\mathbf{k}), \\ b^\dagger_2(-\mathbf{k}) &= -v^*_{\mathbf{k}} A_1(\mathbf{k}) + u_{\mathbf{k}} A^\dagger_2(\mathbf{k}), \end{aligned}$$

with

$$|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1, \tag{3.30}$$

in order to get canonical anticommutation relations among the $A_i(\mathbf{k})$ oscillators. Expressing H_0 through the new operators we obtain

$$\begin{aligned} H_0 &= \sum_{\mathbf{k}\sigma}\xi_{\mathbf{k}}\left[(|u_{\mathbf{k}}|^2 - |v_{\mathbf{k}}|^2)A^\dagger_\sigma(\mathbf{k})A_\sigma(\mathbf{k})\right] \\ &+ 2\sum_{\mathbf{k}}\xi_{\mathbf{k}}\left[|v_{\mathbf{k}}|^2 + u_{\mathbf{k}}v_{\mathbf{k}}A^\dagger_1(\mathbf{k})A^\dagger_2(\mathbf{k}) - u^*_{\mathbf{k}}v^*_{\mathbf{k}}A_1(\mathbf{k})A_2(\mathbf{k})\right] \\ &+ \sum_{\mathbf{k}}\Big[(\Delta_{\mathbf{k}}u_{\mathbf{k}}v^*_{\mathbf{k}} + \Delta^*_{\mathbf{k}}u^*_{\mathbf{k}}v_{\mathbf{k}})\left(A^\dagger_1(\mathbf{k})A_1(\mathbf{k}) + A^\dagger_2(\mathbf{k})A_2(\mathbf{k}) - 1\right) \\ &+ \left(\Delta^*_{\mathbf{k}}u^{*2}_{\mathbf{k}} - \Delta_{\mathbf{k}}v^{*2}_{\mathbf{k}}\right)A_1(\mathbf{k})A_2(\mathbf{k}) - \left(\Delta_{\mathbf{k}}u^2_{\mathbf{k}} - \Delta^*_{\mathbf{k}}v^2_{\mathbf{k}}\right)A^\dagger_1(\mathbf{k})A^\dagger_2(\mathbf{k}) + \Delta_{\mathbf{k}}\Gamma^*_{\mathbf{k}}\Big]. \end{aligned} \tag{3.31}$$

In order to bring H_0 to a canonical form we must cancel the terms of the type $A^\dagger_1(\mathbf{k})A^\dagger_2(\mathbf{k})$ and $A_1(\mathbf{k})A_k(\mathbf{2})$. This can be done by choosing

$$2\xi_{\mathbf{k}}u_{\mathbf{k}}v_{\mathbf{k}} - (\Delta_{\mathbf{k}}u^2_{\mathbf{k}} - \Delta^*_{\mathbf{k}}v^2_{\mathbf{k}}) = 0. \tag{3.32}$$

Multiplying this Equation by $\Delta^*_{\mathbf{k}}/u^2_{\mathbf{k}}$ we get

$$\Delta^{*2}_{\mathbf{k}}\frac{v^2_{\mathbf{k}}}{u^2_{\mathbf{k}}} + 2\xi_{\mathbf{k}}\Delta^*_{\mathbf{k}}\frac{v_{\mathbf{k}}}{u_{\mathbf{k}}} - |\Delta_{\mathbf{k}}|^2 = 0, \tag{3.33}$$

or

$$\left(\Delta^*_{\mathbf{k}}\frac{v_{\mathbf{k}}}{u_{\mathbf{k}}} + \xi_{\mathbf{k}}\right)^2 = \xi^2_{\mathbf{k}} + |\Delta_{\mathbf{k}}|^2. \tag{3.34}$$

Introducing

$$E_{\mathbf{k}} = \sqrt{\xi^2_{\mathbf{k}} + |\Delta_{\mathbf{k}}|^2}, \tag{3.35}$$

which, as we shall see, is the energy of the quasiparticles we find

$$\Delta^*_{\mathbf{k}}\frac{v_{\mathbf{k}}}{u_{\mathbf{k}}} = E_{\mathbf{k}} - \xi_{\mathbf{k}}, \tag{3.36}$$

or

$$\left|\frac{v_{\mathbf{k}}}{u_{\mathbf{k}}}\right| = \frac{E_{\mathbf{k}} - \xi_{\mathbf{k}}}{|\Delta_{\mathbf{k}}|}. \tag{3.37}$$

This equation together with

$$|v_{\mathbf{k}}|^2 + |u_{\mathbf{k}}|^2 = 1, \tag{3.38}$$

gives

$$|v_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right), \qquad |u_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right). \tag{3.39}$$

Using these relations we can easily evaluate the coefficients of the other terms in H_0. As far as the bilinear term in the creation and annihilation operators we get

$$\begin{aligned} &\xi_{\mathbf{k}}\left(|u_{\mathbf{k}}|^2 - |v_{\mathbf{k}}|^2\right) + \Delta_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}}^* + \Delta_{\mathbf{k}}^* u_{\mathbf{k}}^* v_{\mathbf{k}} \\ &= \xi_{\mathbf{k}}\left(|u_{\mathbf{k}}|^2 - |v_{\mathbf{k}}|^2\right) + 2|u_{\mathbf{k}}|^2\left(E_{\mathbf{k}} - \xi_{\mathbf{k}}\right) = E_{\mathbf{k}}, \end{aligned} \tag{3.40}$$

showing that $E_{\mathbf{k}}$ is indeed the energy associated to the new creation and annihilation operators. Therefore we get

$$H_0 = \sum_{\mathbf{k}\sigma} E_{\mathbf{k}} A_\sigma^\dagger(\mathbf{k}) A_\sigma(\mathbf{k}) + \langle H_0 \rangle, \tag{3.41}$$

with

$$\langle H_0 \rangle = \sum_{\mathbf{k}} \left[2\,\xi_{\mathbf{k}} |v_{\mathbf{k}}|^2 - \Delta_{\mathbf{k}}^* u_{\mathbf{k}}^* v_{\mathbf{k}} - \Delta_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}}^* + \Delta_{\mathbf{k}} \Gamma_{\mathbf{k}}^*\right]. \tag{3.42}$$

We now need the BCS ground state. This is obtained by asking for a state annihilated by the operators $A_\sigma(\mathbf{k})$:

$$\begin{aligned} A_1(\mathbf{k}) &= u_{\mathbf{k}} b_1(\mathbf{k}) - v_{\mathbf{k}} b_2^\dagger(-\mathbf{k}), \\ A_2(\mathbf{k}) &= v_{\mathbf{k}} b_1^\dagger(\mathbf{k}) + u_{\mathbf{k}} b_2(-\mathbf{k}). \end{aligned} \tag{3.43}$$

It is easy to check that the required state is

$$|0\rangle_{BCS} = \prod_{\mathbf{k}} \left(u_{\mathbf{k}} + v_{\mathbf{k}} b_1^\dagger(\mathbf{k}) b_2^\dagger(-\mathbf{k})\right)|0\rangle. \tag{3.44}$$

Let us check for $A_1(\mathbf{k})$

$$\begin{aligned} &A_1(\mathbf{q})|0\rangle_{BCS} = \\ &= \prod_{\mathbf{k}\neq\mathbf{q}} \left(u_{\mathbf{k}} + v_{\mathbf{k}} b_1^\dagger(\mathbf{k}) b_2^\dagger(-\mathbf{k})\right)\left(u_{\mathbf{q}} b_1(\mathbf{q}) - v_{\mathbf{q}} b_2^\dagger(-\mathbf{q})\right)\left(u_{\mathbf{q}} + v_{\mathbf{q}} b_1^\dagger(\mathbf{q}) b_2^\dagger(-\mathbf{q})\right)|0\rangle = \\ &= \prod_{\mathbf{k}\neq\mathbf{q}} \left(u_{\mathbf{k}} + v_{\mathbf{k}} b_1^\dagger(\mathbf{k}) b_2^\dagger(-\mathbf{k})\right)\left(u_{\mathbf{q}} v_{\mathbf{q}} b_2^\dagger(-\mathbf{q}) - v_{\mathbf{q}} u_{\mathbf{q}} b_2^\dagger(-\mathbf{q})\right)|0\rangle = 0. \end{aligned} \tag{3.45}$$

We can now evaluate $\Gamma_{\mathbf{k}}$. We have

$$\begin{aligned} \Gamma_{\mathbf{k}} &= \langle b_2(-\mathbf{k}) b_1(\mathbf{k})\rangle = \left\langle\left(-v_{\mathbf{k}} A_1^\dagger(\mathbf{k}) + u_{\mathbf{k}}^* A_2(\mathbf{k})\right)\left(u_{\mathbf{k}}^* A_1(\mathbf{k}) + v_{\mathbf{k}} A_2^\dagger(\mathbf{k})\right)\right\rangle \\ &= u_{\mathbf{k}}^* v_{\mathbf{k}} \left\langle\left(1 - A_1^\dagger(\mathbf{k}) A_1(\mathbf{k}) - A_2^\dagger(\mathbf{k}) A_2(\mathbf{k})\right)\right\rangle, \end{aligned} \tag{3.46}$$

from which

$$\Gamma_{\mathbf{k}} = u_{\mathbf{k}}^* v_{\mathbf{k}}. \tag{3.47}$$

Therefore we can write Eq. (3.42) as

$$\langle H_0 \rangle = \sum_{\mathbf{k}} \left[2\,\xi_{\mathbf{k}} |v_{\mathbf{k}}|^2 - \Delta_{\mathbf{k}}^* u_{\mathbf{k}}^* v_{\mathbf{k}}\right]. \tag{3.48}$$

By Eq. (3.39) we have

$$\langle H_0 \rangle = \sum_{\mathbf{k}} \left[\xi_{\mathbf{k}} - \frac{\xi_{\mathbf{k}}^2}{E_{\mathbf{k}}} - \Delta_{\mathbf{k}}^* u_{\mathbf{k}}^* v_{\mathbf{k}}\right]. \tag{3.49}$$

Before proceeding we now derive the gap equation. Starting from the complex conjugated of Eq. (3.36) we can write

$$\Delta_{\mathbf{k}} \frac{u_{\mathbf{k}} v^*_{\mathbf{k}}}{|u_{\mathbf{k}}|^2} = E_{\mathbf{k}} - \xi_{\mathbf{k}}, \tag{3.50}$$

and using (3.39) we get

$$u_{\mathbf{k}} v^*_{\mathbf{k}} = \frac{1}{2} \frac{\Delta^*_{\mathbf{k}}}{E_{\mathbf{k}}} \tag{3.51}$$

and

$$\Gamma_{\mathbf{k}} = \frac{1}{2} \frac{\Delta_{\mathbf{k}}}{E_{\mathbf{k}}}. \tag{3.52}$$

By the definition of $\Delta_{\mathbf{k}}$ given in Eq. (3.28) we finally obtain the **gap equation**

$$\Delta_{\mathbf{k}} = -\frac{1}{2} \sum_{\mathbf{q}} V_{\mathbf{kq}} \frac{\Delta_{\mathbf{q}}}{E_{\mathbf{q}}}. \tag{3.53}$$

We can now proceed to the evaluation of the expectation value of H_0. Notice that if the interaction matrix $V_{\mathbf{kq}}$ is invertible we can write

$$\langle H_0 \rangle = \sum_{\mathbf{k}} \left[\xi_{\mathbf{k}} - \frac{\xi^2_{\mathbf{k}}}{E_{\mathbf{k}}} + \sum_{\mathbf{q}} \Delta_{\mathbf{k}} V^{-1}_{\mathbf{kq}} \Delta^*_{\mathbf{q}} \right]. \tag{3.54}$$

By choosing $V_{\mathbf{kq}}$ as in the discussion of the Cooper pairs:

$$V_{\mathbf{k},\mathbf{k}'} = \begin{cases} -G & |\xi_{\mathbf{k}}|,\ |\xi_{\mathbf{q}}| < \delta \\ 0, & \text{otherwise} \end{cases} \tag{3.55}$$

with $G > 0$, we find

$$\langle H_0 \rangle = \sum_{\mathbf{k}} \left(\xi_{\mathbf{k}} - \frac{\xi^2_{\mathbf{k}}}{E_{\mathbf{k}}} \right) - \frac{\Delta^2}{G}, \tag{3.56}$$

since the gap equation has now solutions for $\Delta_{\mathbf{k}}$ independent on the momentum. In a more detailed way the sum can be written as

$$\langle H_0 \rangle = \sum_{|\mathbf{k}|>k_F} \left(\xi_{\mathbf{k}} - \frac{\xi^2_{\mathbf{k}}}{E_{\mathbf{k}}} \right) + \sum_{|\mathbf{k}|<k_F} \left(-\xi_{\mathbf{k}} - \frac{\xi^2_{\mathbf{k}}}{E_{\mathbf{k}}} \right) - \frac{\Delta^2}{G}, \tag{3.57}$$

or

$$\langle H_0 \rangle = 2 \sum_{|\mathbf{k}|>k_F} \left(\xi_{\mathbf{k}} - \frac{\xi^2_{\mathbf{k}}}{E_{\mathbf{k}}} \right) - \frac{\Delta^2}{G}. \tag{3.58}$$

Converting the sum in an integral we get

$$\begin{aligned} \langle H_0 \rangle &= 2 \frac{p_F^2}{2\pi^2 v_F} \int_0^\delta d\xi \left(\xi - \frac{\xi^2}{\sqrt{\xi^2 + \Delta^2}} \right) - \frac{\Delta^2}{G} \\ &= \rho \left[\delta^2 - \delta \sqrt{\delta^2 + \Delta^2} + \Delta^2 \log \frac{\delta + \sqrt{\delta^2 + \Delta^2}}{\Delta} \right] - \frac{\Delta^2}{G}. \end{aligned} \tag{3.59}$$

Let us now consider the gap equation

$$\Delta = \frac{1}{2} \frac{p_F^2}{2\pi^2 v_F} 2G \int_0^\delta d\xi \frac{\Delta}{\sqrt{\xi^2 + \Delta^2}} = \frac{1}{2} \rho G \Delta \log \frac{\delta + \sqrt{\delta^2 + \Delta^2}}{\Delta}, \tag{3.60}$$

from which

$$1 = \frac{1}{2}\rho G \log \frac{\delta + \sqrt{\delta^2 + \Delta^2}}{\Delta}. \tag{3.61}$$

Using this equation in Eq. (3.59) we find

$$\langle H_0 \rangle = \frac{\rho}{2}\left[\delta^2 - \delta\sqrt{\delta^2 + \Delta^2} + \frac{2\Delta^2}{\rho G}\right] - \frac{\Delta^2}{G}. \tag{3.62}$$

The first term in this expression arises from the kinetic energy whereas the second one from the interaction. Simplifying the expression we find

$$\langle H_0 \rangle = \frac{\rho}{2}\left[\delta^2 - \delta\sqrt{\delta^2 + \Delta^2}\right]. \tag{3.63}$$

By taking the weak limit, that is $\rho G \ll 1$, or $\Delta \ll \delta$, we obtain from the gap equation

$$\Delta = 2\delta e^{-2/G\rho} \tag{3.64}$$

and

$$\langle H_0 \rangle = -\frac{1}{4}\rho\Delta^2. \tag{3.65}$$

All this calculation can be easily repeated at $T \neq 0$. In fact the only point where the temperature comes in is in evaluating $\Gamma_{\mathbf{k}}$ which must be taken as a thermal average

$$\langle \mathcal{O} \rangle_T = \frac{Tr\left[e^{-H/T}\mathcal{O}\right]}{Tr\left[e^{-H/T}\right]}. \tag{3.66}$$

The thermal average of a Fermi oscillator of hamiltonian $H = Eb^\dagger b$ is obtained easily since

$$Tr[e^{-E\,b^\dagger b/T}] = 1 + e^{-E/T} \tag{3.67}$$

and

$$Tr[b^\dagger b e^{-Eb^\dagger b/T}] = e^{-E/T}. \tag{3.68}$$

Therefore

$$\langle b^\dagger b \rangle_T = f(E) = \frac{1}{e^{E/T} + 1}. \tag{3.69}$$

It follows from Eq. (10.27)

$$\Gamma_{\mathbf{k}}(T) = \langle b_2(-\mathbf{k})b_1(\mathbf{k})\rangle_T = u^*_{\mathbf{k}}v_{\mathbf{k}}\left\langle\left(1 - A_1^\dagger(\mathbf{k})A_1(\mathbf{k}) - A_2^\dagger(\mathbf{k})A_2(\mathbf{k})\right)\right\rangle_T = u^*_{\mathbf{k}}v_{\mathbf{k}}(1 - 2f(E_{\mathbf{k}})). \tag{3.70}$$

Therefore the gap equation is given by

$$\Delta_{\mathbf{k}} = -\sum_{\mathbf{q}} V_{\mathbf{kq}}u^*_{\mathbf{q}}v_{\mathbf{q}}(1 - 2f(E_{\mathbf{q}})) = -\sum_{\mathbf{q}} V_{\mathbf{kq}}\frac{\Delta_{\mathbf{q}}}{2E_{\mathbf{q}}}\tanh\frac{E_{\mathbf{q}}}{2T}, \tag{3.71}$$

and in the BCS approximation

$$1 = \frac{1}{4}\rho G\int_{-\delta}^{+\delta}\frac{d\xi_{\mathbf{p}}}{E_{\mathbf{p}}}\tanh\frac{E_{\mathbf{p}}}{2T}, \quad E_{\mathbf{p}} = \sqrt{\xi^2_{\mathbf{p}} + \Delta^2}. \tag{3.72}$$

C. The functional approach to the gap equation

We will now show how to derive the gap equation by using the functional approach to field theory. We start assuming the following action

$$S[\psi,\psi^\dagger]=\int d^4x\left[\psi^\dagger(i\partial_t-\epsilon(|\boldsymbol{\nabla}|)+\mu)\psi+\frac{G}{2}\left(\psi^\dagger(x)\psi(x)\right)\left(\psi^\dagger(x)\psi(x)\right)\right]. \tag{3.73}$$

We can transform the interaction term in a more convenient way (Fierzing):

$$\psi^\dagger_a\psi_a\psi^\dagger_b\psi_b=-\ \psi^\dagger_a\psi^\dagger_b\psi_a\psi_b=-\frac{1}{4}\epsilon_{ab}\epsilon_{ab}\psi^\dagger C\psi^*\psi^T C\psi=-\frac{1}{2}\psi^\dagger C\psi^*\psi^T C\psi, \tag{3.74}$$

with

$$C=i\sigma_2 \tag{3.75}$$

the charge conjugation matrix. We obtain

$$S[\psi,\psi^\dagger]\equiv S_0+S_I=\int d^4x\left[\psi^\dagger(i\partial_t-\epsilon(|\boldsymbol{\nabla}|)+\mu)\psi-\frac{G}{4}\left(\psi^\dagger(x)C\psi^*(x)\right)\left(\psi^T(x)C\psi(x)\right)\right], \tag{3.76}$$

. The quantum theory is defined in terms of the functional integral

$$Z=\int\mathcal{D}(\psi,\psi^\dagger)e^{iS[\psi,\psi^\dagger]}. \tag{3.77}$$

The four-fermi interaction can be eliminated by inserting inside the functional integral the following identity

$$\text{const}=\int\mathcal{D}(\Delta,\Delta^*)e^{-\frac{i}{G}\int d^4x\left[\Delta-\frac{G}{2}(\psi^T C\psi)\right]\left[\Delta^*+\frac{G}{2}(\psi^\dagger C\psi^*)\right]}. \tag{3.78}$$

Normalizing at the free case ($G=0$) we get

$$\frac{Z}{Z_0}=\frac{1}{Z_0}\int\mathcal{D}(\psi,\psi^\dagger)\mathcal{D}(\Delta,\Delta^*)e^{iS_0[\psi,\psi^\dagger]+i\int d^4x\left[-\frac{|\Delta|^2}{G}-\frac{1}{2}\Delta(\psi^\dagger C\psi^*)+\frac{1}{2}\Delta^*(\psi^T C\psi)\right]}. \tag{3.79}$$

It is convenient to introduce the Nambu-Gorkov basis

$$\chi=\frac{1}{\sqrt{2}}\begin{pmatrix}\psi\\ C\psi^*\end{pmatrix}, \tag{3.80}$$

in terms of which the exponent appearing in Eq. (3.79) can be written as

$$S_0+\cdots=\int d^4x\left(\chi^\dagger S^{-1}\chi-\frac{|\Delta|^2}{G}\right), \tag{3.81}$$

where in momentum space

$$S^{-1}(p)=\begin{bmatrix}p_0-\xi_{\mathbf{p}} & -\Delta\\ -\Delta^* & p_0+\xi_{\mathbf{p}}\end{bmatrix}. \tag{3.82}$$

We can now perform the functional integral over the Fermi fields. Clearly it is convenient to perform this integration over the Nambu-Gorkov field, but this corresponds to double the degrees of freedom, since inside χ we count already once the fields ψ^*. To cover this aspect we can use the "replica trick" by integrating also over $\chi^\dagger$ as an independent field and taking the square root of the result.We obtain

$$\frac{Z}{Z_0}=\frac{1}{Z_0}\left[\det S^{-1}\right]^{1/2}e^{-i\int d^4x\frac{|\Delta|^2}{G}}\equiv e^{iS_{\rm eff}}, \tag{3.83}$$

where

$$S_{\rm eff}(\Delta,\Delta^*) = -\frac{i}{2}Tr[\log(S_0 S^{-1})] - \int d^4x \frac{|\Delta|^2}{G}, \tag{3.84}$$

with S_0 the free propagator ($\Delta = 0$). The saddle point equation for Δ^* gives

$$\frac{\delta S_{\rm eff}}{\delta \Delta^*} = -\frac{\Delta}{G} - \frac{i}{2}Tr\left[S\frac{\delta S^{-1}}{\delta\Delta^*}\right] = -\frac{\Delta}{G} + \frac{i}{2}Tr\left(\begin{bmatrix} \Delta & 0 \\ p_0+\xi_{\bf p} & 0\end{bmatrix}\frac{1}{p_0^2-\xi_{\bf p}^2-|\Delta|^2}\right), \tag{3.85}$$

where we have used

$$S = \frac{1}{p_0^2-\xi_{\bf p}^2-|\Delta|^2}\begin{bmatrix} p_0+\xi_{\bf p} & \Delta \\ \Delta^* & p_0-\xi_{\bf p}\end{bmatrix}. \tag{3.86}$$

Therefore we get the gap equation (the trace gives a factor 2 from the spin)

$$\Delta = iG\int\frac{d^4p}{(2\pi)^4}\frac{\Delta}{p_0^2-\xi_{\bf p}^2-|\Delta|^2}, \tag{3.87}$$

and performing the integration over p_0 we obtain

$$\Delta = \frac{G}{2}\int\frac{d^3p}{(2\pi)^3}\frac{\Delta}{\sqrt{\xi_{\bf p}^2+|\Delta|^2}}, \tag{3.88}$$

in agreement with Eq. (3.53). By considering the case $T\neq 0$ we have only to change the integration over p_0 to a sum over the Matsubara frequencies

$$\omega_n = (2n+1)\pi T, \tag{3.89}$$

obtaining

$$\Delta = GT\sum_{n=-\infty}^{+\infty}\int\frac{d^3p}{(2\pi)^3}\frac{\Delta}{\omega_n^2+\xi_{\bf p}^2+|\Delta|^2}. \tag{3.90}$$

The sum can be easily done with the result

$$\sum_{n=-\infty}^{+\infty}\frac{1}{\omega_n^2+\xi_{\bf p}^2+|\Delta|^2} = \frac{1}{2E_{\bf p}T}\left(1-2f(E_{\bf p}\right), \tag{3.91}$$

where $f(E)$ is the Fermi distribution defined in Eq. (3.69). From

$$1-2f(E) = \tanh\frac{E}{2T}, \tag{3.92}$$

we get the gap equation for $T\neq 0$

$$\Delta = \frac{G}{2}\int\frac{d^3p}{(2\pi)^3}\frac{\Delta}{E_{\bf p}}\tanh(E_{\bf p}/2T), \tag{3.93}$$

which is the same as Eq. (3.71).

If we consider the functional Z as given by Eq. (3.79) as a functional integral over ψ, $\psi^\dagger$, Δ and Δ^*, by its saddle point evaluation we see that the classical value of Δ is given by

$$\Delta = \frac{G}{2}\langle\psi^T C\psi\rangle. \tag{3.94}$$

Also, if we introduce the em interaction in the action (3.76) we see that Z, as given by Eq. (3.77) is gauge invariant under

$$\psi \to e^{i\alpha(x)}\psi \tag{3.95}$$

Therefore the way in which the em field appear in $S_{\rm eff}(\Delta,\Delta^*)$ must be such to make it gauge invariant. On the other side we see from Eq. (3.94) that Δ must transform as

$$\Delta\to e^{2i\alpha(x)}\Delta, \tag{3.96}$$

meaning that Δ has charge $-2e$ and that the effective action for Δ has to contain space-time derivatives in the form

$$D^\mu=\partial^\mu+2ieA^\mu. \tag{3.97}$$

This result was achieved for the first time by (Gor'kov, 1959) who derived the Ginzburg-Landau expansion of the free energy from the microscopic theory. This calculation can be easily repeated by inserting the em interaction and matching the general form of the effective action against the microscopic calculation. We will see an example of this kind of calculations later. In practice one starts from the form (3.79) for Z and, after established the Feynman rules, one evaluate the diagrams of Fig. 12 which give the coefficients of the terms in $|\Delta|^2$, $|\Delta|^4$, $|\Delta|^2A$ and $|\Delta|^2A^2$ in the effective lagrangian.

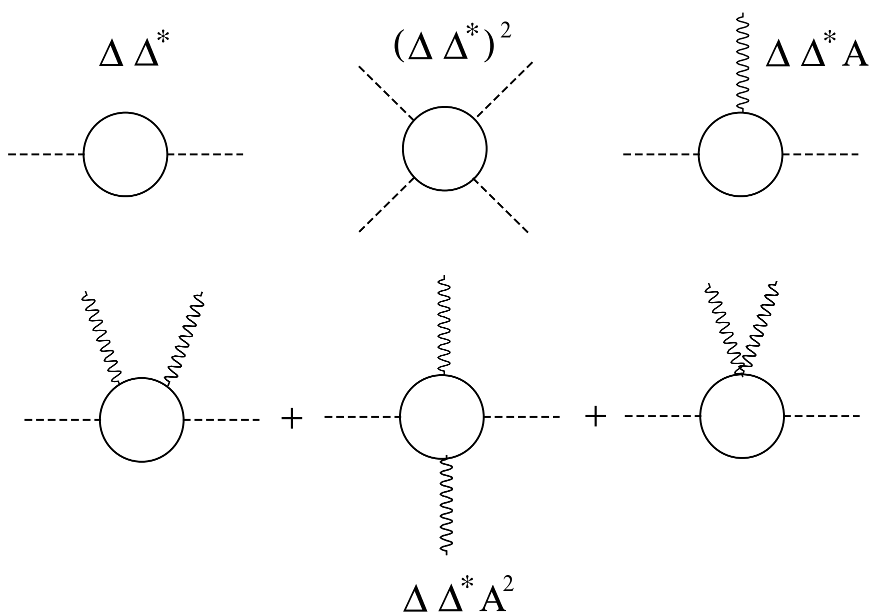

FIG. 12 *The diagrams contributing to the Ginzburg-Landau expansion. The dashed lines represent the fields Δ and Δ^*, the solid lines the Fermi fields and the wavy lines the photon field.*

An explicit evaluation of these diagrams in the static case $\dot{\mathbf{A}}=0$ can be found, for instance, in the book of (Sakita, 1985). One gets an expression of the type

$$H=\int d^3\mathbf{r}\left(-c\,\frac{1}{4m}\Delta^*(\mathbf{r})|(\boldsymbol{\nabla}+2ie\mathbf{A})|^2\Delta(\mathbf{r})+a|\Delta(\mathbf{r})|^2+\frac{1}{2}b|\Delta(\mathbf{r})|^4\right). \tag{3.98}$$

By defining $\psi=\sqrt{c}\Delta$ we obtain

$$H=\int d^3\mathbf{r}\left(-\frac{1}{4m}\psi^*(\mathbf{r})|(\boldsymbol{\nabla}+2ie\mathbf{A})\psi(\mathbf{r})|^2\psi(\mathbf{r})+\alpha|\psi(\mathbf{r})|^2+\frac{1}{2}\beta|\psi(\mathbf{r})|^4\right), \tag{3.99}$$

with

$$\alpha=\frac{a}{c},\quad \beta=\frac{b}{c^2}. \tag{3.100}$$

This expression is the same as the original proposal made by Ginzburg and Landau (see Eq. (1.47)) with

$$e^*=2e,\qquad m^*=2m. \tag{3.101}$$

However, notice that contrarily to e^* the value of m^* depends on the normalization chosen for ψ. Later we will evaluate the coefficients a and b directly from the gap equation.

D. The Nambu-Gor'kov equations

We will present now a different approach, known as Nambu-Gor'kov equations (Gor'kov, 1959; Nambu, 1960) which is completely equivalent to the previous ones and strictly related to the effective action approach of the previous Section. We start again from the action (3.73) in three-momentum space

$$
\begin{aligned}
S &= S_0 + S_{BCS} \ , && (3.102)\\
S_0 &= \int dt \, \frac{d\mathbf{p}}{(2\pi)^3} \, \psi^\dagger(\mathbf{p}) \left(i\partial_t - E(\mathbf{p}) + \mu\right) \psi(\mathbf{p}) \ , && (3.103)\\
S_{BCS} &= \frac{G}{2} \int dt \prod_{k=1}^{4} \frac{d\mathbf{p}_k}{(2\pi)^3} \left(\psi^\dagger(\mathbf{p}_1)\psi(\mathbf{p}_4)\right) \left(\psi^\dagger(\mathbf{p}_2)\psi(\mathbf{p}_3)\right) \\
&\times \ (2\pi)^3 \, \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \, . && (3.104)
\end{aligned}
$$

Here and below, unless explicitly stated, $\psi(\mathbf{p})$ denotes the 3D Fourier transform of the Pauli spinor $\psi(\mathbf{r}, t)$, i.e. $\psi(\mathbf{p}) \equiv \psi_\sigma(\mathbf{p}, t)$. For non relativistic particles the functional dependence of the energy is $E(\mathbf{p}) = \mathbf{p}^2/2m$, but we prefer to leave it in the more general form (3.103).

The BCS interaction (3.104) can be written as follows

$$
S_{BCS} = S_{cond} + S_{int} \, , \tag{3.105}
$$

with

$$
\begin{aligned}
S_{cond} &= -\frac{G}{4} \int dt \prod_{k=1}^{4} \frac{d\mathbf{p}_k}{(2\pi)^3} \Big[\tilde{\Xi}(\mathbf{p}_3,\, \mathbf{p}_4) \psi^\dagger(\mathbf{p}_1) C \psi^\dagger(\mathbf{p}_2) \\
&- \ \tilde{\Xi}^*(\mathbf{p}_1,\, \mathbf{p}_2) \psi(\mathbf{p}_3) C \psi(\mathbf{p}_4) \Big] (2\pi)^3 \, \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \ , \\
S_{int} &= -\frac{G}{4} \int dt \prod_{k=1}^{4} \frac{d\mathbf{p}_k}{(2\pi)^3} \left[\psi^\dagger(\mathbf{p}_1) C \psi^\dagger(\mathbf{p}_2) + \tilde{\Xi}^*(\mathbf{p}_1,\, \mathbf{p}_2) \right] \times \\
&\times \ \left[\psi(\mathbf{p}_3) C \psi(\mathbf{p}_4) - \tilde{\Xi}(\mathbf{p}_3,\, \mathbf{p}_4) \right] (2\pi)^3 \, \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \, , && (3.106)
\end{aligned}
$$

where $C = i\sigma_2$ and

$$
\tilde{\Xi}(\mathbf{p},\, \mathbf{p}') = < \psi(\mathbf{p}) C \psi(\mathbf{p}') > \ . \tag{3.107}
$$

In the mean field approximation the interaction term can be neglected while the gap term S_{cond} is added to S_0. Note that the spin 0 condensate $\tilde{\Xi}(\mathbf{p},\, \mathbf{p}')$ is simply related to the condensate wave function

$$
\Xi(\mathbf{r}) = < \psi(\mathbf{r}, t) C \psi(\mathbf{r}, t) > \tag{3.108}
$$

by the formula

$$
\Xi(\mathbf{r}) = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{d\mathbf{p}'}{(2\pi)^3} \, e^{-i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{r}} \, \tilde{\Xi}(\mathbf{p},\, \mathbf{p}') \ . \tag{3.109}
$$

In general the condensate wavefunction can depend on $\mathbf{r}$; only for homogeneous materials it does not depend on the space coordinates; therefore in this case $\tilde{\Xi}(\mathbf{p},\, \mathbf{p}')$ is proportional to $\delta(\mathbf{p} + \mathbf{p}')$.

In order to write down the Nambu-Gor'kov (NG) equations we define the NG spinor

$$
\chi(\mathbf{p}) = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi(\mathbf{p}) \\ \psi^c(-\mathbf{p}) \end{pmatrix} \, , \tag{3.110}
$$

where we have introduced the charge-conjugate field

$$
\psi^c = C\psi^* \ . \tag{3.111}
$$

We also define

$$
\Delta(\mathbf{p}, -\mathbf{p}') = \frac{G}{2} \int \frac{d\mathbf{p}''}{(2\pi)^6} \tilde{\Xi}(\mathbf{p}'', \mathbf{p} + \mathbf{p}' - \mathbf{p}'') \ . \tag{3.112}
$$

Therefore the free action can be written as follows:

$$S_0 = \int dt \, \frac{d\mathbf{p}}{(2\pi)^3} \frac{d\mathbf{p}'}{(2\pi)^3} \, \chi^\dagger(\mathbf{p}) \, S^{-1}(\mathbf{p}, \, \mathbf{p}')\chi(\mathbf{p}'), \tag{3.113}$$

with

$$S^{-1}(\mathbf{p}, \, \mathbf{p}') = (2\pi)^3 \begin{pmatrix} (i\partial_t - \xi_{\mathbf{p}})\delta(\mathbf{p} - \mathbf{p}') & -\Delta(\mathbf{p}, \mathbf{p}') \\ -\Delta^*(\mathbf{p}, \mathbf{p}') & (i\partial_t + \xi_{\mathbf{p}})\delta(\mathbf{p} - \mathbf{p}') \end{pmatrix} . \tag{3.114}$$

Here

$$\xi_{\mathbf{p}} = E(\mathbf{p}) - \mu \approx \mathbf{v}_F \cdot (\mathbf{p} - \mathbf{p}_F) \, , \tag{3.115}$$

where

$$\mathbf{v}_F = \frac{\partial E(\mathbf{p})}{\partial \mathbf{p}}\Big|_{\mathbf{p} = \mathbf{p}_F} \tag{3.116}$$

is the Fermi velocity. We have used the fact that we are considering only degrees of freedom near the Fermi surface, i.e.

$$p_F - \delta < p < p_F + \delta \ , \tag{3.117}$$

where δ is the ultraviolet cutoff, of the order of the Debye frequency. In particular in the non relativistic case

$$\xi_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m} - \frac{p_F^2}{2m}, \quad \mathbf{v}_F = \frac{\mathbf{p}_F}{m} \, . \tag{3.118}$$

S^{-1} in (9.5) is the 3D Fourier transform of the inverse propagator. We can make explicit the energy dependence by Fourier transforming the time variable as well. In this way we get for the inverse propagator, written as an operator:

$$S^{-1} = \begin{pmatrix} (\mathbf{G_0^+})^{-\mathbf{1}} & -\mathbf{\Delta} \\ -\mathbf{\Delta}^* & -(\mathbf{G_0^-})^{-\mathbf{1}} \end{pmatrix} , \tag{3.119}$$

and

$$\begin{aligned} [\mathbf{G_0^+}]^{-\mathbf{1}} &= E - \xi_{\mathbf{P}} + i\,\epsilon\,\mathrm{sign}\,E \, , \\ [\mathbf{G_0^-}]^{-\mathbf{1}} &= -E - \xi_{\mathbf{P}} - i\,\epsilon\,\mathrm{sign}\,E \, , \end{aligned} \tag{3.120}$$

with $\epsilon = 0^+$ and $\mathbf{P}$ the momentum operator. The $i\epsilon$ prescription is the same discussed in Section II.B. As for the NG propagator S, one gets

$$S = \begin{pmatrix} \mathbf{G} & -\tilde{\mathbf{F}} \\ -\mathbf{F} & \tilde{\mathbf{G}} \end{pmatrix} \, . \tag{3.121}$$

S has both spin, σ, σ', and a, b NG indices, i.e. $S^{ab}_{\sigma\sigma'}$[3]. The NG equations in compact form are

$$S^{-1} S = 1 \ , \tag{3.122}$$

or, explicitly,

$$\begin{aligned} [\mathbf{G_0^+}]^{-1}\mathbf{G} + \mathbf{\Delta F} &= \mathbf{1} \ , \\ -[\mathbf{G_0^-}]^{-1}\mathbf{F} + \mathbf{\Delta}^*\mathbf{G} &= \mathbf{0} \, . \end{aligned} \tag{3.123}$$

[3] We note that the presence of the factor $1/\sqrt{2}$ in (3.110) implies an extra factor of 2 in the propagator: $S(x, x') = 2 < T \;\; \chi(x)\chi^\dagger(x') \;\; >$, as it can be seen considering e.g. the matrix element S^{11}: $< T \;\; \psi(x)\psi^\dagger(x') \;\; >= \;\; i\partial_t - \xi_{-i\vec{\nabla}} - \delta\mu\sigma_3 \;\; ^{-1} \delta(x - x')$, with $(x \equiv (t, \mathbf{r}))$.

Note that we will use

$$< \mathbf{r}\,|\boldsymbol{\Delta}|\mathbf{r}' >= \frac{G}{2}\,\Xi(\mathbf{r})\,\delta(\mathbf{r}-\mathbf{r}') = \Delta(\mathbf{r})\,\delta(\mathbf{r}-\mathbf{r}')\ , \tag{3.124}$$

or

$$< \mathbf{p}\,|\boldsymbol{\Delta}|\mathbf{p}' >= \Delta(\mathbf{p},\mathbf{p}') \tag{3.125}$$

depending on our choice of the coordinate or momenta representation. The formal solution of the system (3.123) is

$$\begin{aligned} \mathbf{F} &= \mathbf{G_0^-}\boldsymbol{\Delta}^*\mathbf{G}\ , \\ \mathbf{G} &= \mathbf{G_0^+} - \mathbf{G_0^+}\boldsymbol{\Delta}\mathbf{F}\ , \end{aligned} \tag{3.126}$$

so that **F** satisfies the equation

$$\mathbf{F} = \mathbf{G_0^-}\boldsymbol{\Delta}^*\left(\mathbf{G_0^+} - \mathbf{G_0^+}\boldsymbol{\Delta}\mathbf{F}\right) \tag{3.127}$$

and is therefore given by

$$\mathbf{F} = \frac{\mathbf{1}}{\boldsymbol{\Delta}^*[\mathbf{G_0^+}]^{-1}[\boldsymbol{\Delta}^*]^{-1}[\mathbf{G_0^-}]^{-1}\ +\ \boldsymbol{\Delta}^*\boldsymbol{\Delta}}\,\boldsymbol{\Delta}^*\ . \tag{3.128}$$

In the configuration space the NG Eqs. (3.123) are as follows

$$\begin{aligned} (E - E(-i\boldsymbol{\nabla}) + \mu)\,G(\mathbf{r},\mathbf{r}',E) + \Delta(\mathbf{r})F(\mathbf{r},\mathbf{r}',E) &= \delta(\mathbf{r}-\mathbf{r}')\ , \\ (-E - E(-i\boldsymbol{\nabla}) + \mu)\,F(\mathbf{r},\mathbf{r}',E) - \Delta^*(\mathbf{r})G(\mathbf{r},\mathbf{r}',E) &= 0\,. \end{aligned} \tag{3.129}$$

The gap equation at $T = 0$ is the following consistency condition

$$\Delta^*(\mathbf{r}) = -i\,\frac{G}{2}\int\frac{dE}{2\pi}\,\mathrm{T}rF(\mathbf{r},\mathbf{r},E)\,, \tag{3.130}$$

where F is given by (3.128). To derive the gap equation we observe that

$$\begin{aligned} \Delta^*(\mathbf{r}) &= \frac{G}{2}\,\Xi^*(\mathbf{r}) = \frac{G}{2}\int\frac{d\mathbf{p}_1}{(2\pi)^3}\frac{d\mathbf{p}_2}{(2\pi)^3}\,e^{i(\mathbf{p}_1+\mathbf{p}_2)\cdot\mathbf{r}}\,\tilde{\Xi}^*(\mathbf{p}_1,\,\mathbf{p}_2) \\ &= -\frac{G}{2}\int\frac{dE}{2\pi}\frac{d\mathbf{p}_1}{(2\pi)^3}\frac{d\mathbf{p}_2}{(2\pi)^3}\,e^{i(\mathbf{p}_1+\mathbf{p}_2)\cdot\mathbf{r}}\ <\psi^\dagger(\mathbf{p}_1,E)\psi^c(\mathbf{p}_2,E)> \\ &= +i\,\frac{G}{2}\sum_\sigma\int\frac{dE}{2\pi}\frac{d\mathbf{p}_1}{(2\pi)^3}\frac{d\mathbf{p}_2}{(2\pi)^3}e^{i(\mathbf{p}_1-\mathbf{p}_2)\cdot\mathbf{r}}S^{21}_{\sigma\sigma}(\mathbf{p}_2,\mathbf{p}_1) \\ &= +i\,\frac{G}{2}\sum_\sigma\int\frac{dE}{2\pi}S^{21}_{\sigma\sigma}(\mathbf{r},\mathbf{r})\ , \end{aligned} \tag{3.131}$$

which gives (3.130).

At finite temperature, introducing the Matsubara frequencies $\omega_n = (2n+1)\pi T$, the gap equation reads

$$\Delta^*(\mathbf{r})\ =\ \frac{G}{2}\,T\sum_{n=-\infty}^{+\infty}\mathrm{T}rF(\mathbf{r},\mathbf{r},E)\Big|_{E=i\omega_n}\ . \tag{3.132}$$

It is useful to specialize these relations to the case of homogeneous materials. In this case we have

$$\Xi(\mathbf{r})\ =\ \text{const.} \equiv \frac{2\Delta}{G}\ , \tag{3.133}$$

$$\tilde{\Xi}(\mathbf{p}_1,\,\mathbf{p}_2)\ =\ \frac{2\Delta}{G}\frac{\pi^2}{p_F^2\delta}\,(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2)\ . \tag{3.134}$$

Therefore one gets

$$\Delta(\mathbf{p}_1,\mathbf{p}_2) = \Delta\,\delta(\mathbf{p}_1-\mathbf{p}_2) \tag{3.135}$$

and from (3.124) and (3.133)

$$\Delta(\mathbf{r}) = \Delta^*(\mathbf{r}) = \Delta\ . \tag{3.136}$$

Therefore $F(\mathbf{r}, \mathbf{r}, E)$ is independent of $\mathbf{r}$ and, from Eq. (3.128), one gets

$$TrF(\mathbf{r}, \mathbf{r}, E) = -2\,\Delta \int \frac{d^3p}{(2\pi)^3} \frac{1}{E^2 - \xi_{\mathbf{p}}^2 - \Delta^2} \tag{3.137}$$

which gives the gap equation at $T = 0$:

$$\Delta = i\, G\Delta \int \frac{dE}{2\pi} \frac{d^3p}{(2\pi)^3} \frac{1}{E^2 - \xi_{\mathbf{p}}^2 - \Delta^2}\ , \tag{3.138}$$

and at $T \neq 0$:

$$\Delta \ = \ GT \sum_{n=-\infty}^{+\infty} \int \frac{d^3p}{(2\pi)^3} \frac{\Delta}{\omega_n^2 + \epsilon(\mathbf{p}, \Delta)^2}\ , \tag{3.139}$$

where

$$\epsilon(\mathbf{p}, \Delta) = \sqrt{\Delta^2 + \xi_{\mathbf{p}}^2}\,. \tag{3.140}$$

is the same quantity that we had previously defined as $E_{\mathbf{p}}$. We now use the identity

$$\frac{1}{2}\left[1 - n_u - n_d\right] = \epsilon(\mathbf{p}, \Delta)T \sum_{n=-\infty}^{+\infty} \frac{1}{\omega_n^2 + \epsilon^2(\mathbf{p}, \Delta)}\ , \tag{3.141}$$

where

$$n_u(\mathbf{p}) = n_d(\mathbf{p} = \frac{1}{e^{\epsilon/T} + 1}\ . \tag{3.142}$$

The gap equation can be therefore written as

$$\Delta = \frac{G\,\Delta}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\epsilon(\mathbf{p}, \Delta)} \left(1 - n_u(\mathbf{p}) - n_d(\mathbf{p})\right)\ . \tag{3.143}$$

In the Landau theory of the Fermi liquid n_u, n_d are interpreted as the equilibrium distributions for the quasiparticles of type u, d. It can be noted that the last two terms act as blocking factors, reducing the phase space, and producing eventually $\Delta \to 0$ when T reaches a critical value T_c (see below).

E. The critical temperature

We are now in the position to evaluate the critical temperature. This can be done by deriving the Ginzburg-Landau expansion, since we are interested to the case of $\Delta \to 0$. The free energy (or rather in this case the grand potential), as measured from the normal state, near a second order phase transition is given by

$$\Omega = \frac{1}{2}\alpha\Delta^2 + \frac{1}{4}\beta\Delta^4\,. \tag{3.144}$$

Minimization gives the gap equation

$$\alpha\Delta + \beta\Delta^3 = 0\,. \tag{3.145}$$

Expanding the gap equation (9.7) up to the third order in the gap, Δ, we can obtain the coefficients α and β up to a normalization constant. One gets

$$\Delta = 2\,G\,\rho\,T\,Re \sum_{n=0}^{\infty} \int_0^{\delta} d\xi \left[\frac{\Delta}{(\omega_n^2 + \xi^2)} - \frac{\Delta^3}{(\omega_n^2 + \xi^2)^2} + + \cdots\right]\ , \tag{3.146}$$

with

$$\omega_n = (2n+1)\pi T\,. \tag{3.147}$$

The grand potential can be obtained, up to a normalization factor, integrating in Δ the gap equation. The normalization can be obtained by the simple BCS case, considering the grand potential as obtained, in the weak coupling limit, from Eqs. (3.65)

$$\Omega = -\frac{\rho}{4}\Delta^2\,. \tag{3.148}$$

The same result can be obtained multiplying the gap equation (3.61) in the weak coupling limit

$$1 - \frac{G\rho}{2}\log\frac{2\delta}{\Delta} = 0 \tag{3.149}$$

by Δ and integrating over Δ starting from $\Delta = 0$, that is the normal state. We find

$$\frac{1}{2}\Delta^2 - \frac{G\rho}{8}\Delta^2 - \frac{G\rho}{4}\Delta^2\log\frac{2\delta}{\Delta} = -\frac{G\rho}{8}\Delta^2 + \frac{1}{2}\Delta^2\left(1 - \frac{G\rho}{2}\log\frac{2\delta}{\Delta}\right). \tag{3.150}$$

Using again the gap equation to cancel the second term, we see that the grand potential is recovered if we multiply the result of the integration by $2/G$. Therefore the coefficients α and β appearing the grand potential are obtained by multiplying by $2/G$ the coefficients in the expansion of the gap equation. We get

$$\alpha \;=\; \frac{2}{G}\left(1 - 2\,G\,\rho\,T\,Re\sum_{n=0}^{\infty}\int_0^{\delta}\frac{d\xi}{(\omega_n^2+\xi^2)}\right), \tag{3.151}$$

$$\beta \;=\; 4\rho\,T\,Re\sum_{n=0}^{\infty}\int_0^{\infty}\frac{d\xi}{(\omega_n^2+\xi^2)^2}\,, \tag{3.152}$$

In the coefficient β we have extended the integration in ξ up to infinity since both the sum and the integral are convergent. To evaluate α is less trivial. One can proceed in two different ways. One can sum over the Matsubara frequencies and then integrate over ξ or one can perform the operations in the inverse order. Let us begin with the former method. We get

$$\alpha = \frac{2}{G}\left[1 - \frac{gG\,\rho}{2}\int_0^{\delta}\frac{d\xi}{\xi}\tanh\left(\frac{\xi}{2T}\right)\right]. \tag{3.153}$$

Performing an integration by part we can extract the logarithmic divergence in δ. This can be eliminated using the result (3.149) valid for $\delta\mu = T = 0$ in the weak coupling limit (Δ_0 is the gap at $T = 0$)

$$1 = \frac{G\,\rho}{2}\log\frac{2\delta}{\Delta_0}\,. \tag{3.154}$$

We find

$$\alpha = \rho\left[\log\frac{2T}{\Delta_0} + \frac{1}{2}\int_0^{\infty}dx\ln x\,\frac{1}{\cosh^2\frac{x}{2}}\right]. \tag{3.155}$$

Defining

$$\log\frac{\Delta_0}{2T_c} = \frac{1}{2}\int_0^{\infty}dx\ln x\,\frac{1}{\cosh^2\frac{x}{2}}\,, \tag{3.156}$$

we get

$$\alpha(T) = \rho\log\frac{T}{T_c}\,, \tag{3.157}$$

Performing the calculation in the reverse we first integrate over ξ obtaining a divergent series which can be regulated cutting the sum at a maximal value of n determined by

$$\omega_N = \delta \Rightarrow N \approx \frac{\delta}{2\pi T}\,. \tag{3.158}$$

We obtain

$$\alpha = \frac{2}{G}\left(1 - \pi\, G\,\rho\, T \sum_{n=0}^{N} \frac{1}{\omega_n}\right)\,. \tag{3.159}$$

The sum can be performed in terms of the Euler's function $\psi(z)$:

$$\sum_{n=0}^{N} \frac{1}{\omega_n} = \frac{1}{2\pi T}\left[\psi\left(\frac{3}{2} + N\right) - \psi\left(\frac{1}{2}\right)\right] \approx \frac{1}{2\pi T}\left(\log\frac{\delta}{2\pi T} - \psi\left(\frac{1}{2}\right)\right)\,. \tag{3.160}$$

Eliminating the cutoff as we did before we get

$$\alpha(T) = \rho\left(\log\left(4\pi\frac{T}{\Delta_0}\right) + \psi\left(\frac{1}{2}\right)\right)\,. \tag{3.161}$$

By comparing with Eq. (3.155) we get the following identity

$$\psi\left(\frac{1}{2}\right) = -\log(2\pi) + \frac{1}{2}\int_0^\infty dx\,\ln x\left(\frac{1}{\cosh^2\frac{x}{2}}\right)\,. \tag{3.162}$$

The equation (3.156) can be re-written as

$$\log\frac{\Delta_0}{4\pi T_c} = \psi\left(\frac{1}{2}\right)\,. \tag{3.163}$$

Using (C the Euler-Mascheroni constant)

$$\psi\left(\frac{1}{2}\right) = -\log(4\gamma), \qquad \gamma = e^C, \qquad C = 0.5777\ldots\,, \tag{3.164}$$

we find

$$\alpha(T) = \rho\log\frac{\pi T}{\gamma\Delta_0}\,, \tag{3.165}$$

Therefore the critical temperature, that is the value of T at which $\alpha = 0$, is

$$T_c = \frac{\gamma}{\pi}\Delta_0 \approx 0.56693\,\Delta_0\,. \tag{3.166}$$

The other terms in the expansion of the gap equation are easily evaluated integrating over ξ and summing over the Matsubara frequencies. We get

$$\beta(T) = \pi\,\rho\,T\sum_{n=0}^{\infty}\frac{1}{\omega_n^3} = -\frac{\rho}{16\,\pi^2\,T^2}\,\psi^{(2)}\left(\frac{1}{2}\right)\,, \tag{3.167}$$

where

$$\psi^{(n)}(z) = \frac{d^n}{dz^n}\psi(z)\,. \tag{3.168}$$

Using

$$\psi^{(2)}\left(\frac{1}{2}\right) = -14\zeta(3), \tag{3.169}$$

where $\zeta(3)$ is the function zeta of Riemann

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s} \tag{3.170}$$

we get

$$\beta(T) = \frac{7}{8}\frac{\rho}{\pi^2 T^2}\zeta(3) \tag{3.171}$$

Close to the critical temperature we have

$$\alpha(T) \approx -\rho\left(1 - \frac{T}{T_c}\right), \quad \beta(T) \approx \frac{7\rho}{8\pi^2 T_c^2}\zeta(3). \tag{3.172}$$

From the grand potential (or from the gap equation) we obtain

$$\Delta^2(T) = -\frac{\alpha(T)}{\beta(T)} \to \Delta(T) = \frac{2\sqrt{2}\pi T_c}{\sqrt{7\zeta(3)}}\left(1 - \frac{T}{T_c}\right)^{1/2}, \tag{3.173}$$

in agreement with the results of Section I.B.4.

IV. THE ROLE OF THE BROKEN GAUGE SYMMETRY

Superconductivity appears to be a fundamental phenomenon and therefore we would like to understand it from a more fundamental way than doing a lot of microscopical calculations. This is in fact the case if one makes the observation that the electromagnetic $U(1)$ symmetry is spontaneously broken. We will follow here the treatment given in (Weinberg, 1996). We have seen that in the ground state of a superconductor the following condensate is formed

$$\langle \epsilon_{\alpha\beta}\psi^\alpha\psi^\beta \rangle. \tag{4.1}$$

This condensate breaks the em $U(1)$ since the difermion operator has charge $-2e$. As a matter of fact this is the only thing that one has to assume, i.e., the breaking of $U(1)$ by an operator of charge $-2e$. Thinking in terms of an order parameter one introduces a scalar field, Φ, transforming as the condensate under a gauge transformation

$$A_\mu \to A_\mu + \partial_\mu\Lambda, \quad \psi \to e^{ie\Lambda}\psi \Rightarrow \Phi \to e^{2ie\Lambda}\Phi, \tag{4.2}$$

where ψ is the electron field. We then introduce the Goldstone field ϕ as the phase of the field Φ

$$\Phi = \rho\, e^{2ie\phi}. \tag{4.3}$$

Therefore ϕ transforms as the phase of the condensate under a gauge transformation

$$\phi \to \phi + \Lambda. \tag{4.4}$$

In the case of constant Λ this implies that the theory may depend only on $\partial_\mu\phi$. Notice also that the gauge invariance is broken but a subgroup Z_2 remains unbroken, the one corresponding to $\Lambda = 0$ and $\Lambda = \pi/e$. In particular ϕ and $\phi + \pi/e$ should be identified.

It is also convenient to introduce gauge invariant Fermi fields

$$\tilde{\psi} = e^{-ie\phi}\,\psi. \tag{4.5}$$

The system will be described by a gauge invariant lagrangian depending on $\tilde{\psi}$, A_μ and $\partial_\mu\phi$. Integrating out the Fermi fields one is left with a gauge invariant lagrangian depending on A_μ and $\partial_\mu\phi$. Gauge invariance requires that these fields should appear only in the combinations

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad A_\mu - \partial_\mu\phi. \tag{4.6}$$

Therefore the lagrangian has the form

$$L = -\frac{1}{4}\int d^3\mathbf{x}\, F_{\mu\nu}F^{\mu\nu} + L_s(A_\mu - \partial_\mu\phi). \tag{4.7}$$

The equation of motion for the scalar field is

$$0 = \partial_\mu \frac{\delta L_s}{\delta \partial_\mu \phi} = -\partial_\mu \frac{\delta L_s}{\delta A_\mu} = -\partial_\mu J^\mu, \tag{4.8}$$

where J_μ is the em current defined as

$$J_\mu = \frac{\delta L_s}{\delta A^\mu}. \tag{4.9}$$

Therefore the equation of motion for ϕ is nothing but the conservation of the current. The only condition on L_s is that it gives rise to a stable state of the system in the absence of A_μ and ϕ. In particular this amounts to say that the point $A_\mu = \partial_\mu\phi$ is a local minimum of the theory. Therefore the second derivative of L_s with respect to its argument should not vanish at that point.

The Meissner effect follows easily from the previous considerations. In fact, if we go deep inside the superconductor we will be in the minimum $A_\mu = \partial_\mu\phi$, implying that A_μ is a pure gauge since

$$F_{\mu\nu}(\partial_\lambda\phi) = 0. \tag{4.10}$$

In particular the magnetic field inside the superconductor is vanishing, $\mathbf{B} = 0$. We may refine this analysis by doing some considerations about the energy. Close to the minimum we have

$$L_s(A_\mu - \partial_\mu\phi) \approx L_s(0) + \frac{1}{2}\frac{\delta^2 L_s}{\delta(A_\mu - \partial_\mu\phi)^2}(A_\mu - \partial_\mu\phi)^2. \tag{4.11}$$

Notice that the dimensions of the second derivative are $[E \times E^{-2}] = [E^{-1}] = [L]$. Therefore, in the static case, up to a constant

$$L_s \approx \frac{L^3}{\lambda^2}|\mathbf{A} - \boldsymbol{\nabla}\phi|^2, \tag{4.12}$$

with L^3 the volume of the superconductor and λ some length typical of the material. If a magnetic field $\mathbf{B}$ penetrates inside the material, we expect

$$|\mathbf{A} - \boldsymbol{\nabla}\phi| \approx BL, \tag{4.13}$$

from which

$$L_s \approx \frac{B^2L^5}{\lambda^2}. \tag{4.14}$$

For the superconductor to remain in that state the magnetic field must be expelled with an energy cost of

$$B^2L^3. \tag{4.15}$$

Therefore there will be convenience in expelling $\mathbf{B}$ if

$$\frac{B^2L^5}{\lambda^2} \gg B^2L^3, \tag{4.16}$$

or

$$L \gg \lambda. \tag{4.17}$$

λ is the penetration depth, in fact from its definition it follows that it is the region over which the magnetic field is non zero. Repeating the same reasoning made in the Introduction one can see the existence of a critical magnetic field. Notice that a magnetic field smaller than the critical one penetrates inside the superconductor up to a depth λ and in that region the electric current will flow, since

$$\mathbf{J} \propto \boldsymbol{\nabla} \wedge \mathbf{B}. \tag{4.18}$$

Consider now a thick superconductor with a shape of a torus. Along the internal line C (see Fig. 13) the quantity $|\mathbf{A} - \boldsymbol{\nabla}\phi|$ vanishes but the two fields do not need to be zero. However, going around the path C ϕ has to go to an equivalent value, $\phi + n\pi/e$. Therefore

$$\int_{\mathcal{A}} \mathbf{B} \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\mathbf{x} = \oint_C \boldsymbol{\nabla}\phi \cdot d\mathbf{x} = \frac{n\pi}{e}, \tag{4.19}$$

where $\mathcal{A}$ is an area surrounded by C. We see that the flux of $\mathbf{B}$ inside the torus is quantized.

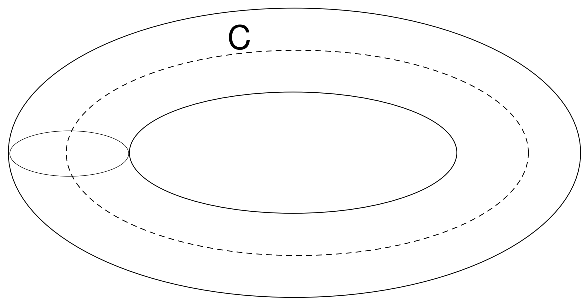

FIG. 13 *Inside the toroidal superconductor the line C of integration is shown*

Notice also that the electric current sustaining $\mathbf{B}$ flows in a layer of thickness λ below the surface of the torus. It follows that the current cannot decay smoothly but it must jump in such a way that the variation of the magnetic flux is a multiple of π/e. Therefore the electric resistance of a superconductor is rather peculiar. In order to understand better the resistance let us consider the following equation

$$\frac{\delta L_s}{\delta \dot{\phi}} = -\frac{\delta L_s}{\delta A_0} = -J_0. \tag{4.20}$$

This shows that $-J_0$ is the canonical momentum density conjugated to ϕ. Therefore the Hamilton equations of motion give

$$\dot{\phi}(x) = \frac{\delta H_s}{\delta(-J_0(x))} = -V(x), \tag{4.21}$$

where $V(x)$, being the variation of energy per change in the current density, is the "voltage" at x. If in the superconductor there is a stationary current (that is with time-independent fields), the previous equation shows that the voltage is zero. But a current with zero voltage means that the electric resistance is zero.

We are now in the position of explaining the Josephson effect (Josephson, 1962, 1965). The effect arises at the junction of two superconductors separated by a thin insulating barrier. At zero voltage difference between the two superconductors a continuous current flows, depending on the phase difference due to the two different Goldstone fields. Furthermore, if a constant voltage difference is maintained between the two superconductor an alternate current flows. These two effects are known as the *dc* and the *ac Josephson effects*. Consider first the case of zero voltage difference. By gauge invariance the lagrangian at the junction may depend only on the phase difference

$$L_{\rm junction} = F(\Delta\phi). \tag{4.22}$$

The function F must be periodic since the Goldstone fields in the two superconductors are defined mod π/e, that is

$$F(\Delta\phi) = F(\Delta\phi + n\pi/e). \tag{4.23}$$

To evaluate the current let us introduce a vector potential $\mathbf{A}$. Then

$$\Delta_{\mathbf{A}}\phi = \int_\ell d\mathbf{x} \cdot (\boldsymbol{\nabla}\phi - \mathbf{A}), \tag{4.24}$$

where the line ℓ is taken across the junction. Therefore we get

$$J^k = \frac{\delta L_{\rm junction}}{\delta A_k} = n^k F'(\Delta_{\mathbf{A}}\phi), \tag{4.25}$$

where n^k is the normal unit vector at the junction surface. By putting $\mathbf{A} = 0$ we get

$$\mathbf{J} = \mathbf{n}F'(\Delta\phi), \tag{4.26}$$

showing the *dc* Josephson effect. To get the second one, consider a constant voltage difference. From

$$\dot{\phi} = -V, \tag{4.27}$$

we get

$$\Delta\phi(t) = |\Delta V|t + \Delta\phi(0). \tag{4.28}$$

Since F has a period π/e it follows that the current oscillates with a frequency

$$\nu = \frac{e|\Delta V|}{\pi}. \tag{4.29}$$

Using this relation one can get a very accurate measure of $e/\hbar$ (going back to standard units, one has $\nu = e|\Delta V|/\pi\hbar$).

The current in the *dc* Josephson effect can be of several milliamperes for conventional superconductors. In the case of the *ac* effect for voltages of the order of millivolts, the frequency can rise to hundreds and thousands of gigahertz.

When close to the phase transition the description of the theory in terms of the Goldstone boson is not enough. In fact there is another long wave-length mode associate to the order parameter. This is because the $U(1)$ symmetry gets restored and its minimal description is in terms of a complex field. Therefore one introduces

$$\Phi = \rho\, e^{2ie\phi}. \tag{4.30}$$

Expanding L_s for small values of Φ we get (with an appropriate normalization for Φ)

$$\begin{aligned} L_s &\approx \int d^3\mathbf{x}\left[-\frac{1}{2}\Phi^*|(\boldsymbol{\nabla} - 2ie\mathbf{A})|^2\Phi - \frac{1}{2}\alpha|\Phi|^2 - \frac{1}{4}\beta|\Phi|^4\right] \\ &= \int d^3\mathbf{x}\left[-2e^2\rho^2(\boldsymbol{\nabla}\phi - e\mathbf{A})^2 - \frac{1}{2}(\boldsymbol{\nabla}\rho)^2 - \frac{1}{2}\alpha\rho^2 - \frac{1}{4}\beta\rho^4\right]. \end{aligned} \tag{4.31}$$

In this Section we have defined the penetration depth as the inverse square root of the coefficient of $-(\boldsymbol{\nabla}\phi - \mathbf{A})^2/2$. Therefore

$$\lambda = \frac{1}{\sqrt{4e^2\langle\rho^2\rangle}}. \tag{4.32}$$

Using

$$\langle\rho^2\rangle = -\frac{\alpha}{\beta}, \tag{4.33}$$

we get

$$\lambda = \frac{1}{2e}\sqrt{-\frac{\beta}{\alpha}}, \tag{4.34}$$

in agreement with Eq. (1.52) (notice that α and β are not normalized in the same way). Another length is obtained by studying the behavior of the fluctuations of the field ρ. Defining

$$\rho = \langle\rho\rangle + \rho', \tag{4.35}$$

we get

$$\boldsymbol{\nabla}^2\rho' = -2\alpha\rho'. \tag{4.36}$$

This allows us to introduce the coherence length ξ as

$$\xi = \frac{1}{\sqrt{-2\alpha}}, \tag{4.37}$$

in agreement with Eq. (1.59). Using the definitions of λ and ξ we get

$$\alpha = -\frac{1}{2\xi^2}, \quad \beta = 2\frac{e^2\lambda^2}{\xi^2}. \tag{4.38}$$

Therefore the energy density of the superconducting state is lower than the energy density of the normal state by

$$\frac{1}{4}\frac{\alpha^2}{\beta} = \frac{1}{32}\frac{1}{e^2\lambda^2\xi^2}. \tag{4.39}$$

The relative size of ξ and λ is very important, since vortex lines can be formed inside the superconductor and their stability depends on this point. More precisely inside the vortices the normal state is realized, meaning $\rho = 0$ and a flux-quantized magnetic field. The superconductors are therefore classified according to the following criterium:

- Type I superconductors: $\xi > \lambda$. The vortices are not stable since the penetration of the magnetic field is very small.

- Type II superconductors: $\xi < \lambda$. The vortices are stable and the magnetic field penetrates inside the superconductor exactly inside the vortices. This may happen since the core of the vortex is much smaller than the region where the magnetic field goes to zero. In this cases there are two critical magnetic fields, H_{c1}, where for $H < H_{c1}$ the state is superconducting, whereas for $H > H_{c_1}$ vortices are formed. Increasing the magnetic field, more and more vortex lines are formed, up to a value H_{c_2} where the magnetic field penetrates all the superconductor and the transition to the normal state arises.

V. COLOR SUPERCONDUCTIVITY

Ideas about color superconductivity go back to almost 25 years ago (Bailin and Love, 1984; Barrois, 1977; Collins and Perry, 1975; Frautschi, 1978), but only recently this phenomenon has received a lot of attention (for recent reviews see ref. (Alford, 2001; Hong, 2001; Hsu, 2000; Nardulli, 2002; Rajagopal and Wilczek, 2001; Schafer, 2003)). The naive expectation is that at very high density, due to the asymptotic freedom, quarks would form a Fermi sphere of almost free fermions. However, as we know, Bardeen, Cooper and Schrieffer proved that the Fermi surface of free fermions is unstable in presence of an attractive, arbitrary small, interaction. Since in QCD the gluon exchange in the $\bar{3}$ channel is attractive one expects the formation of a coherent state of Cooper pairs. The phase structure of QCD at high density depends on the number of flavors and there are two very interesting cases, corresponding to two massless flavors (2SC) (Alford *et al.*, 1998; Barrois, 1977; Rapp *et al.*, 1998) and to three massless flavors (CFL) (Alford *et al.*, 1999; Schafer and Wilczek, 1999a) respectively. The two cases give rise to very different patterns of symmetry breaking. If we denote left- and right-handed quark fields by $\psi^{\alpha}_{iL(R)} = \psi^{\alpha}_{ia(\dot{a})}$ by their Weyl components, with $\alpha = 1, 2, 3$, the $SU(3)_c$ color index, $i = 1, \cdots, N_f$ the flavor index (N_f is the number of massless flavors) and $a(\dot{a}) = 1, 2$ the Weyl indices, the structure of the condensate at very high density can be easily understood on the basis of the following considerations. Consider the matrix element

$$\langle 0|\psi^{\alpha}_{ia}\psi^{\beta}_{jb}|0\rangle. \tag{5.1}$$

its color, spin and flavor structure is completely fixed by the following considerations:

- The condensate should be antisymmetric in color indices (α, β) in order to have attraction;

- The condensate should be antisymmetric in spin indices (a, b) in order to get a spin zero condensate. The isotropic structure of the condensate is generally favored since it allows a better use of the Fermi surface (Brown *et al.*, 2000c; Evans *et al.*, 1999a; Schafer, 2000c; Schafer and Wilczek, 1999b);

- given the structure in color and spin, Pauli principles requires antisymmetry in flavor indices.

Since the momenta in a Cooper pair are opposite, as the spins of the quarks (the condensate has spin 0), it follows that the left(right)-handed quarks can pair only with left(right)-handed quarks. In the case of 3 flavors the favored condensate is

$$\langle 0|\psi^\alpha_{iL}\psi^\beta_{jL}|0\rangle = -\langle 0|\psi^\alpha_{iR}\psi^\beta_{jR}|0\rangle = \Delta \sum_{C=1}^{3} \epsilon^{\alpha\beta C}\epsilon_{ijC}. \tag{5.2}$$

This gives rise to the so-called color–flavor–locked (CFL) phase (Alford *et al.*, 1999; Schafer and Wilczek, 1999a). The reason for the name is that simultaneous transformations in color and in flavor leave the condensate invariant as shown in Fig. 14.

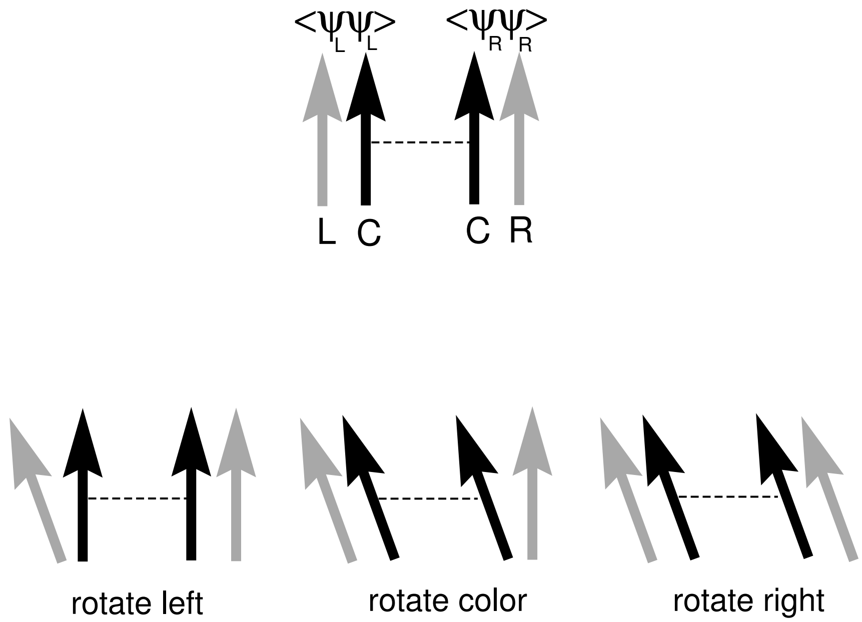

FIG. 14 *Color and flavor indices are represented by black and grey arrows respectively. In the lower part it is shown that we can leave the left-handed condensate invariant if we perform a flavor rotation followed by a related color rotation. However this rotates the color index in the right-handed condensate, so in order to have both condensates invariant we have to rotate also the right- flavor index. In conclusion we have invariance under the diagonal product of the three* $SU(3)$ *groups.*

More generally the following condensate is formed (Alford *et al.*, 1999; Schafer and Wilczek, 1999a)

$$\langle q^\alpha_{iL(R)} C q^\beta_{jL(R)}\rangle \propto \sum_{C=1}^{3} \epsilon^{ijC}\epsilon_{\alpha\beta C} + \kappa(\delta^i_\alpha\delta^j_\beta + \delta^i_\beta\delta^j_\alpha). \tag{5.3}$$

Due to the Fermi statistics, the condensate must be symmetric in color and flavor. As a consequence the two terms appearing in Eq. (5.3) correspond to the $(\bar{\mathbf{3}},\bar{\mathbf{3}})$ and $(\mathbf{6},\mathbf{6})$ channels of $SU(3)_c \otimes SU(3)_{L(R)}$. It turns out that κ is small (Alford *et al.*, 1999; Schafer, 2000b; Shovkovy and Wijewardhana, 1999) and therefore the condensation occurs mainly in the $(\bar{\mathbf{3}},\bar{\mathbf{3}})$ channel. Also in this case the ground state is left invariant by a simultaneous transformation of $SU(3)_c$ and $SU(3)_{L(R)}$. The symmetry breaking pattern is

$$\begin{array}{c} SU(3)_c \otimes SU(3)_L \otimes SU(3)_R \otimes U(1)_B \otimes U(1)_A \\ \downarrow \\ SU(3)_{c+L+R} \otimes Z_2 \otimes Z_2. \end{array} \tag{5.4}$$

The $U(1)_A$ symmetry is broken at the quantum level by the anomaly, but it gets restored at very high density since the instanton contribution is suppressed (Rapp *et al.*, 2000; Schafer, 2000b; Son and Stephanov, 2000a,b). The Z_2

symmetries arise since the condensate is left invariant by a change of sign of the left- and/or right-handed fields. The electric charge is broken but a linear combination with the broken color generator T_8 annihilates the ground state. On the contrary the baryon number is broken. Therefore there are $8 + 2$ broken global symmetries giving rise to 10 Goldstone bosons. The one associated to $U(1)_A$ gets massless only at very high density. The color group is completely broken and all the gauge particles acquire mass. Also all the fermions are gapped. We will show in the following how to construct an effective lagrangian describing the Goldstone bosons, and how to compute their couplings in the high density limit where the QCD coupling gets weaker.

The previous one is the typical situation when the chemical potential is much bigger than the quark masses m_u, m_d and m_s (here the masses to be considered are in principle density depending). However we may ask what happens decreasing the chemical potential. At intermediate densities we have no more the support of asymptotic freedom, but all the model calculations show that one still has a sizeable color condensation. In particular if the chemical potential μ is much less than the strange quark mass one expects that the strange quark decouples, and the corresponding condensate should be

$$\langle 0|\psi^{\alpha}_{iL}\psi^{\beta}_{jL}|0\rangle = \Delta\epsilon^{\alpha\beta 3}\epsilon_{ij}. \tag{5.5}$$

In fact, due to the antisymmetry in color the condensate must necessarily choose a direction in color space. Notice that now the symmetry breaking pattern is completely different from the three-flavor case:

$$SU(3)_c \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_B \to SU(2)_c \otimes SU(2)_L \otimes SU(2)_R \otimes U(1) \otimes Z_2.$$

The condensate breaks the color group $SU(3)_c$ down to the subgroup $SU(2)_c$ but it does not break any flavor symmetry. Although the baryon number, B, is broken, there is a combination of B and of the broken color generator, T_8, which is unbroken in the 2SC phase. Therefore no massless Goldstone bosons are present in this phase. On the other hand, five gluon fields acquire mass whereas three are left massless. We notice also that for the electric charge the situation is very similar to the one for the baryon number. Again a linear combination of the broken electric charge and of the broken generator T_8 is unbroken in the 2SC phase. The condensate (5.5) gives rise to a gap, Δ, for quarks of color 1 and 2, whereas the two quarks of color 3 remain un-gapped (massless). The resulting effective low-energy theory has been described in (Casalbuoni *et al.*, 2000, 2001a; Rischke *et al.*, 2001).

A final problem we will discuss has to do with the fact that when quarks (in particular the strange quark) are massive, their chemical potentials cannot be all equal. This situation has been modelled out in (Alford *et al.*, 2001). If the Fermi surfaces of different flavors are too far apart, BCS pairing does not occur. However it might be favorable for different quarks to pair each of one lying at its own Fermi surface and originating a pair of non-zero total momentum. This is the LOFF state first studied by the authors of ref. (Fulde and Ferrell, 1964; Larkin and Ovchinnikov, 1964) in the context of electron superconductivity in the presence of magnetic impurities. Since the Cooper pair has non-zero momentum the condensate breaks space symmetries and we will show that in the low-energy spectrum a massless particle, a phonon, the Goldstone boson of the broken translational symmetry, is present. We will construct the effective lagrangian also for this case (for a general review of the LOFF phase see (Bowers, 2003; Casalbuoni and Nardulli, 2003).

Of course it would be very nice if we could test all these ideas on the lattice. However the usual sampling method, which is based on a positive definite measure, does not work in presence of a chemical potential since the fermionic determinant turns out to be complex in euclidean space. The argument is rather simple. We define euclidean variables through the following substitutions

$$x_0 \to -ix^4_E, \qquad x^i \to x^i_E, \tag{5.6}$$

$$\gamma_0 \to \gamma^4_E, \qquad \gamma^i \to -i\gamma^i_E. \tag{5.7}$$

Then the Dirac operator, in presence of a chemical potential, becomes[4]

$$D(\mu) = \gamma^{\mu}_E D^{\mu}_E + \mu\gamma^4_E, \tag{5.8}$$

where $D^{\mu}_E = \partial^{\mu}_E + iA^{\mu}_E$ is the euclidean covariant derivative. In absence of a chemical potential the operator has the following properties

$$D(0)^{\dagger} = -D(0), \quad \gamma_5 D(0)\gamma_5 = -D(0). \tag{5.9}$$

[4] We neglect the mass term, since it correspond to an operator multiple of the identity, and all the following considerations can be trivially extended.

Therefore the eigenvalues of $D(0)$ are pure imaginary. Also if $|\lambda\rangle$ is an eigenvector of $D(0)$, the same is for $\gamma_5|\lambda\rangle$ but with eigenvalue $-\lambda$. This follows from

$$\gamma_5 D(0)|\lambda\rangle = \lambda\gamma_5|\lambda\rangle = -D(0)\gamma_5|\lambda\rangle. \tag{5.10}$$

Therefore

$$\det[D(0)] = \prod_\lambda(\lambda)(-\lambda) > 0. \tag{5.11}$$

For $\mu \neq 0$ this argument does not hold and the determinant is complex. Notice that this argument depends on the kind of chemical potential one has to do. For instance in the case of two degenerate flavored quarks, u and d, if we consider the isospin chemical potential, which is coupled to the conserved charge τ_3 in flavor space, we may still prove the positivity by using, for instance, $\tau_1\gamma_5$ instead of γ_5. Therefore these cases can be treated on the lattice.

A. Hierarchies of effective lagrangians

QCD at high density is conveniently studied through a hierarchy of effective field theories. The starting point is the fundamental QCD lagrangian. The way of obtaining a low energy effective lagrangian is to integrate out high-energy degrees of freedom. As we have seen Polchinski (Polchinski, 1993) has shown that the physics is particularly simple for energies close to the Fermi energy. He has shown that all the interactions are irrelevant except for a four-fermi interaction coupling pair of fermions with opposite momenta. This is nothing but the interaction giving rise to the BCS condensation. This physics can be described using the High Density Effective Theory (HDET) (Beane *et al.*, 2000; Casalbuoni *et al.*, 2001c,d; Hong, 2000a,b).

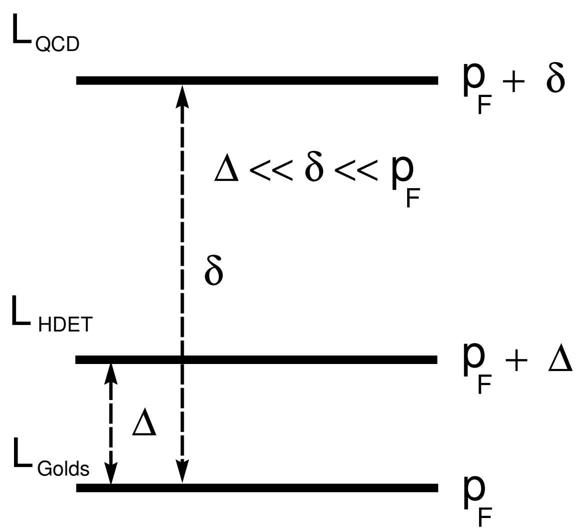

FIG. 15 *The hierarchy of effective lagrangians entering in the discussion of high density QCD.*

In this theory the condensation effects are taken into account through the introduction of a Majorana mass term. The degrees of freedom are quasi-particles (dressed fermions), holes and gauge fields. This description is supposed to hold up to a cutoff $p_F + \delta$, with δ smaller than the Fermi momentum but bigger than the gap Δ, $\Delta \ll \delta \ll p_F$. Going at momenta much smaller than the gap energy Δ all the gapped particles decouple and one is left with the low energy modes as Goldstone bosons, ungapped fermions and holes and massless gauged fields according to the breaking scheme. The corresponding effective theory in the Goldstone sector can be easily formulated using standard techniques. In the case of CFL and 2SC such effective lagrangians have been given in refs. (Casalbuoni and Gatto, 1999) and (Casalbuoni *et al.*, 2000; Rischke *et al.*, 2001). The parameters of the effective lagrangian can be evaluated at each step of the hierarchy by matching the Greens functions with the ones evaluated at the upper level.

B. The High Density Effective Theory (HDET)

We will present here the High Density Effective Theory (Beane *et al.*, 2000; Casalbuoni *et al.*, 2001c,d; Hong, 2000a,b) in the context of QCD with N_f massless flavors. As already discussed we will integrate out all the fermionic degrees of freedom corresponding to momenta greater than $p_F+\delta$ with δ a cutoff such that $\Delta \ll \delta \ll p_F$. The QCD lagrangian at finite density is given by

$$\mathcal{L}_{QCD} = \bar\psi i \not{D}\psi - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \mu\bar\psi\gamma_0\psi, \quad a = 1, \cdots, 8, \tag{5.12}$$

$$D_\mu = \partial_\mu + i g_s A^a_\mu T^a, \qquad \not{D} = \gamma_\mu D^\mu, \tag{5.13}$$

where $T^a = \lambda^a/2$, with λ^a the Gell-Mann matrices. At asymptotic values of $\mu \gg \Lambda_{QCD}$ quarks can be considered as almost free particles due to the asymptotic freedom. The corresponding Dirac equation in momentum space is

$$(\not{p} + \mu\gamma_0)\psi(p) = 0, \tag{5.14}$$

or

$$(p^0 + \mu)\psi(p) = \boldsymbol{\alpha}\cdot\mathbf{p}\psi(p), \tag{5.15}$$

where $\boldsymbol{\alpha} = \gamma_0\boldsymbol{\gamma}$. From this we get immediately the dispersion relation

$$(p^0 + \mu)^2 = |\mathbf{p}|^2, \tag{5.16}$$

or

$$p^0 = E_\pm = -\mu \pm |\mathbf{p}|. \tag{5.17}$$

For the following it is convenient to write the Dirac equation in terms of the following projectors

$$P_\pm = \frac{1 \pm \boldsymbol{\alpha}\cdot\mathbf{v}_F}{2}, \tag{5.18}$$

where

$$\mathbf{v} \equiv \mathbf{v}_F = \frac{\partial E(\mathbf{p})}{\partial \mathbf{p}} = \frac{\partial |\mathbf{p}|}{\partial \mathbf{p}} = \hat{\mathbf{p}}. \tag{5.19}$$

Decomposing $\mathbf{p} = \mu\mathbf{v}_F + \boldsymbol{\ell}$ we get

$$H\psi_+ = \boldsymbol{\alpha}\cdot\boldsymbol{\ell}\,\psi_+, \qquad H\psi_- = (-2\mu + \boldsymbol{\alpha}\cdot\boldsymbol{\ell})\ \psi_-. \tag{5.20}$$

Therefore only the states ψ_+ with energies close to the Fermi surface, $|\mathbf{p}| \approx \mu$ can be easily excited ($E_+ \approx 0$). On the contrary, the states ψ_- have $E_- \approx -2\mu$. They are well inside the Fermi sphere and decouple for high values of μ. In the limit $\mu \to \infty$ the only physical degrees of freedom are ψ_+ and the gluons. After this quantum mechanical introduction, let us consider the field theoretical version of the previous argument. The main idea of the effective theory is the observation that the quarks participating in the dynamics have large ($\sim \mu$) momenta. Therefore one can introduce velocity dependent fields by extracting the large part $\mu\mathbf{v}$ of this momentum. We start with the Fourier decomposition of the quark field $\psi(x)$:

$$\psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-i\,p\cdot x}\psi(p), \tag{5.21}$$

and then we introduce the Fermi velocity as

$$p^\mu = \mu v^\mu + \ell^\mu\ , \tag{5.22}$$

where

$$v^\mu = (0, \mathbf{v}), \tag{5.23}$$

with $|\mathbf{v}| = 1$. The four-vector

$$\ell^\mu = (\ell^0, \vec{\ell}), \tag{5.24}$$

is called the residual momentum. We define also

$$\boldsymbol{\ell} = \mathbf{v}\ell_\parallel + \boldsymbol{\ell}_\perp, \tag{5.25}$$

with

$$\boldsymbol{\ell}_\perp = \boldsymbol{\ell} - (\boldsymbol{\ell}\cdot\mathbf{v})\mathbf{v}. \tag{5.26}$$

Since we can always choose the velocity $\mathbf{v}$ parallel to $\mathbf{p}$, we have $\boldsymbol{\ell}_\perp = 0$. We will now separate in Eq. (5.21) the light and heavy fermion degrees of freedom. These are defined by the following restrictions in momentum space:

$$\begin{array}{ll} \text{light d.o.f.} & \mu - \delta \le |\mathbf{p}| \le \mu + \delta, \\ \text{heavy d.o.f.} & |\mathbf{p}| \le \mu - \delta, \quad |\mathbf{p}| \ge \mu + \delta, \end{array} \tag{5.27}$$

or, in terms of the residual momentum

$$\begin{array}{ll} \text{light d.o.f.} & -\delta \le \ell_\parallel \le +\delta, \\ \text{heavy d.o.f.} & \ell_\parallel \le -\delta, \quad \ell_\parallel \ge +\delta. \end{array} \tag{5.28}$$

Here δ is the cutoff that we choose for defining the effective theory around the Fermi sphere. We will assume that $\delta \gg \Delta$, with Δ the gap. We will assume also that the shell around the Fermi sphere is a narrow one and therefore $\delta \ll \mu$. Let us start our discussion with the light fields. We will discuss how to integrate out the heavy d.o.f later on. In this case we can write the integration in momentum space in the form

$$\int \frac{d^4p}{(2\pi)^4} = \frac{\mu^2}{(2\pi)^4}\int d\Omega \int_{-\delta}^{+\delta} d\ell_\parallel \int_{-\infty}^{+\infty} d\ell_0 = \int \frac{d\mathbf{v}}{4\pi}\frac{\mu^2}{\pi}\int \frac{d^2\ell}{(2\pi)^2}, \tag{5.29}$$

where we have taken into account that we are interested in the degrees of freedom in a shell of amplitude 2δ around the Fermi sphere, and that in the shell we can assume a constant radius μ. We have also substituted the angular integration with the integration over the Fermi velocity taking into account that it is a unit vector. That is

$$\int d\mathbf{v} \equiv \int d^3\mathbf{v}\,\delta(|\mathbf{v}| - 1) = \int |\mathbf{v}|^2\, d|\mathbf{v}|\, d\Omega\, \delta(|\mathbf{v}| - 1) = \int d\Omega. \tag{5.30}$$

In this way the original 4-dimensional integration in momentum has been factorized in the product of two 2-dimensional integrations. In particular

$$\int_{|\mathbf{p}|\in\text{shell}} \frac{d^3\mathbf{p}}{(2\pi)^3} = \int \frac{d\mathbf{v}}{4\pi}\frac{\mu^2}{\pi}\int d\ell_\parallel. \tag{5.31}$$

The Fourier decomposition (5.21) for the light d.o.f. takes the form

$$\psi(x) = \int \frac{d\mathbf{v}}{4\pi} e^{-i\mu v\cdot x}\psi_{\mathbf{v}}(x)\ , \tag{5.32}$$

where

$$\psi_{\mathbf{v}}(x) = \frac{\mu^2}{\pi}\int \frac{d^2\ell}{(2\pi)^2} e^{-i\ell\cdot x}\,\psi_{\mathbf{v}}(\ell), \tag{5.33}$$

with $\psi_{\mathbf{v}}(\ell) \equiv \psi(p)$. Notice that the fields $\psi_{\mathbf{v}}(x)$ are velocity-dependent and they include only the degrees of freedom corresponding to the shell around the Fermi sphere. Projecting with the operators $P_\pm$ we get

$$\psi(x) = \int \frac{d\mathbf{v}}{4\pi} e^{-i\mu v\cdot x}\left[\psi_+(x) + \psi_-(x)\right]\ , \tag{5.34}$$

where

$$\psi_\pm(x) = P_\pm\psi_{\mathbf{v}}(x) = P_\pm\frac{\mu^2}{\pi}\int \frac{d^2\ell}{(2\pi)^2} e^{-i\ell\cdot x}\psi_{\mathbf{v}}(\ell). \tag{5.35}$$

Let us now define

$$V^\mu = (1,\,\mathbf{v})\,, \qquad \tilde V^\mu = (1,\,-\mathbf{v})\,,$$
$$\gamma^\mu_\parallel = (\gamma^0,\,(\mathbf{v}\cdot\boldsymbol{\gamma})\,\mathbf{v})\,, \qquad \gamma^\mu_\perp = \gamma^\mu - \gamma^\mu_\parallel . \tag{5.36}$$

We can then prove the following relations

$$\begin{aligned}\bar\psi_+\gamma^\mu\psi_+ &= V^\mu\bar\psi_+\gamma^0\psi_+\,,\\ \bar\psi_-\gamma^\mu\psi_- &= \tilde V^\mu\bar\psi_-\gamma^0\psi_-\,,\\ \bar\psi_+\gamma^\mu\psi_- &= \bar\psi_+\gamma^\mu_\perp\psi_-\,,\\ \bar\psi_-\gamma^\mu\psi_+ &= \bar\psi_-\gamma^\mu_\perp\psi_+ .\end{aligned} \tag{5.37}$$

Now we want to evaluate the effective action in terms of the velocity-dependent fields defined in Eqs. (5.32) and (5.33). Substituting the expression (5.33) in terms of the type $\int d^4x\psi^\dagger\psi$ we find

$$\int d^4x\,\psi^\dagger\psi =$$
$$\left(\frac{\mu^2}{\pi}\right)^2\int\frac{d\vec v_F}{4\pi}\frac{d\vec v_F'}{4\pi}\frac{d^2\ell}{(2\pi)^2}\frac{d^2\ell'}{(2\pi)^2}(2\pi)^4\delta^4(\ell'-\ell+\mu v'-\mu v)\psi^\dagger_{\mathbf{v}'}(\ell')\psi_{\mathbf{v}}(\ell). \tag{5.38}$$

For large values of μ the integral is different from zero only if $v = v'$ and the δ-function factorizes in the product of two δ-functions one for the velocity and one for the residual momenta. Both these δ-functions are two-dimensional. Therefore we obtain

$$\int d^4x\,\psi^\dagger\psi = \frac{\mu^2}{\pi}\int\frac{d\mathbf{v}}{4\pi}\frac{d^2\ell}{(2\pi)^2}\psi^\dagger_{\mathbf{v}}(\ell)\psi_{\mathbf{v}}(\ell). \tag{5.39}$$

Analogously one could start from (5.32) finding

$$\int d^4x\,\psi^\dagger\psi = \int d^4x\int\frac{d\mathbf{v}}{4\pi}\psi^\dagger_{\mathbf{v}}(x)\psi_{\mathbf{v}}(x). \tag{5.40}$$

In the following we will need also the expression $\int d^4x\psi^TC\psi$. We find

$$\int d^4x\psi^TC\psi = \int d^4x\int\frac{d\mathbf{v}}{4\pi}\psi^T_{\mathbf{v}}(x)C\psi_{-\mathbf{v}}(x) = \frac{\mu^2}{\pi}\int\frac{d\mathbf{v}}{4\pi}\frac{d^2\ell}{(2\pi)^2}\psi^T_{\mathbf{v}}(\ell)C\psi_{-\mathbf{v}}(-\ell). \tag{5.41}$$

These expressions show that there is a *superselection rule for the Fermi velocity.*

Now we are in the position to evaluate the effective lagrangian for the fields ψ_+ which should be the relevant degrees of freedom in the high density limit. Since we have

$$i\partial_\mu e^{-i\mu v\cdot x} = \mu v_\mu e^{-i\mu v\cdot x}, \tag{5.42}$$

we get

$$\int d^4x\,\bar\psi\left(i\not{D}+\mu\gamma_0\right)\psi = \int d^4x\int\frac{d\mathbf{v}}{4\pi}(\bar\psi_++\bar\psi_-)\left[\mu(\not{v}+\gamma_0)+i\not{D}\right](\psi_++\psi_-). \tag{5.43}$$

Expanding and using Eqs. (5.37) we find

$$\int d^4x\,\bar\psi\left(i\not{D}+\mu\gamma_0\right)\psi =$$
$$\int d^4x\left(\psi^\dagger_+iV\cdot D\psi_++\psi^\dagger_-i\tilde V\cdot D\psi_-+2\mu\psi^\dagger_-\psi_-+\bar\psi_+i\not{D}_\perp\psi_-+\bar\psi_-i\not{D}_\perp\psi_+\right). \tag{5.44}$$

The lagrangian is given by

$$\mathcal{L}_D = \int\frac{d\mathbf{v}}{4\pi}\left[\psi^\dagger_+iV\cdot D\psi_++\psi^\dagger_-(2\mu+i\tilde V\cdot D)\psi_-+(\bar\psi_+i\not{D}_\perp\psi_-+h.c.)\right]. \tag{5.45}$$

From which we get the equations of motion

$$iV\cdot D\psi_++i\gamma^0\not{D}_\perp\psi_-=0,$$

$$(2\mu + i\tilde{V}\cdot D)\psi_- + i\gamma^0 \not{D}_\perp \psi_+ = 0. \tag{5.46}$$

At the leading order in $1/\mu$

$$\psi_- = 0, \quad iV\cdot D\psi_+ = 0, \tag{5.47}$$

proving the decoupling of ψ_-. At this order the effective lagrangian is simply

$$\mathcal{L}_D = \int \frac{d\mathbf{v}}{4\pi}\psi_+^\dagger iV\cdot D\psi_+. \tag{5.48}$$

Therefore the free propagator, $\langle T(\psi_+\psi_+^\dagger)\rangle$, is given by

$$\frac{1}{V\cdot\ell}. \tag{5.49}$$

This can be seen directly starting from the Dirac propagator

$$\frac{1}{p^0\gamma^0 - \vec{p}\cdot\vec{\gamma} + \mu\gamma^0} = \frac{(p^0+\mu)\gamma^0 - \vec{p}\cdot\vec{\gamma}}{(p^0+\mu)^2 - |\vec{p}|^2}. \tag{5.50}$$

By putting $p = \mu v + \ell$ and expanding at the leading order in $1/\mu$ we find

$$\frac{1}{\not{p} + \mu\gamma_0} \approx \frac{\not{V}}{2}\frac{1}{V\cdot\ell}, \tag{5.51}$$

where $V\cdot\ell = \ell^0 - \boldsymbol{\ell}\cdot\mathbf{v}$. Therefore the propagator depends (at the leading order) only on the energy ℓ^0 and on the momentum perpendicular to the Fermi surface $\ell_\| = \boldsymbol{\ell}\cdot\mathbf{v}$. Notice also that

$$\frac{\not{V}}{2} = \frac{1}{2}\gamma^0(1 - \boldsymbol{\alpha}\cdot\mathbf{v}) = P_+\gamma_0. \tag{5.52}$$

Recalling that the Dirac propagator is the $\psi\bar{\psi}$ T-product, we see that the propagator $\langle T(\psi_+\psi_+^\dagger)\rangle$ at this order is just $1/V\cdot\ell$.

1. Integrating out the heavy degrees of freedom

All the steps leading to Eq. (5.45) can be formally performed both for light and heavy degrees of freedom. However, in order to integrate out the heavy fields we need to write the effective lagrangian at all orders in μ in the following non-local form :

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \mathcal{L}_D \ , \tag{5.53}$$

where

$$\mathcal{L}_D = \int \frac{d\mathbf{v}}{4\pi}\left[\psi_+^\dagger iV\cdot D\psi_+ - \psi_+^\dagger \frac{1}{2\mu + i\tilde{V}\cdot D}\not{D}_\perp^2\psi_+\right] \ . \tag{5.54}$$

Using the identity

$$\psi_+^\dagger\gamma_\perp^\mu\gamma_\perp^\nu\psi_+ = \psi_+^\dagger P^{\mu\nu}\psi_+, \tag{5.55}$$

where

$$P^{\mu\nu} = g^{\mu\nu} - \frac{1}{2}\left[V^\mu\tilde{V}^\nu + V^\nu\tilde{V}^\mu\right] \ , \tag{5.56}$$

we can write (5.54) as:

$$\mathcal{L}_D = \int \frac{d\mathbf{v}}{4\pi}\left[\psi_+^\dagger iV\cdot D\psi_+ - P^{\mu\nu}\psi_+^\dagger\frac{1}{2\mu + i\tilde{V}\cdot D}D_\mu D_\nu\psi_+\right]\ . \tag{5.57}$$

Notice that the Eq. (5.33) does not hold in this case. We need now to decompose the fields in light and heavy contributions

$$\psi_+ = \psi_+^l + \psi_+^h. \tag{5.58}$$

Substituting inside Eq. (5.57) we get

$$\mathcal{L}_D = \mathcal{L}_D^l + \mathcal{L}_D^{lh} + \mathcal{L}_D^h, \tag{5.59}$$

where $\mathcal{L}_D^l$ is nothing but (5.48) at the leading order in μ:

$$\mathcal{L}_D^l = \int \frac{d\mathbf{v}}{4\pi}\psi_+^{l\dagger} iV\cdot D\psi_+^l, \tag{5.60}$$

whereas

$$\mathcal{L}_D^{lh} = \int \frac{d\mathbf{v}}{4\pi}\left(\psi_+^{l\dagger} iV\cdot D\psi_+^h - P^{\mu\nu}\psi_+^{l\dagger}\frac{1}{2\mu + i\tilde{V}\cdot D}D_\mu D_\nu\psi_+^h \quad + \quad (l\leftrightarrow h)\right) \tag{5.61}$$

and

$$\mathcal{L}_D^h = \int \frac{d\mathbf{v}}{4\pi}\left(\psi_+^{h\dagger} iV\cdot D\psi_+^h - P^{\mu\nu}\psi_+^{h\dagger}\frac{1}{2\mu + i\tilde{V}\cdot D}D_\mu D_\nu\psi_+^h\right). \tag{5.62}$$

When we integrate out the heavy fields, the terms arising from $\mathcal{L}_D^{lh}$ produce diagrams with two light fermions external lines and a bunch of external gluons with a heavy propagator (see, for instance Fig. 16). Due to the momentum

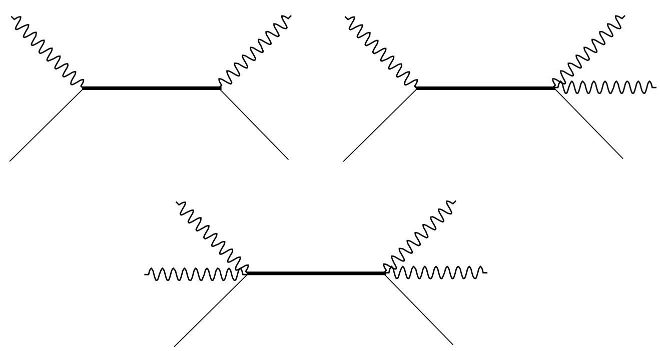

FIG. 16 *Some of the operators arising from integrating out the heavy degrees of freedom in* $\mathcal{L}_D^{lh}$. *The wavy lines represent the gluons, the thin lines the light fermionic d.o.f. and the thick lines the heavy ones*

conservation they can contribute only if some of the gluon momenta are harder than δ. However the hard gluons are suppressed by asymptotic freedom. Notice that these terms may also give rise to pure gluon terms by closing the light line in the loop, see Fig. 17.

On the other hand the second term in $\mathcal{L}_D^h$ can contribute to an operator containing only soft gluon external lines and a loop of heavy fermions at zero momentum (see for instance in Fig. 18 the contribution to the two gluon operator). Notice that the heavy fermion propagator comes from the first term in $\mathcal{L}_D^h$ and it coincides with the propagator evaluated for the free Fermi gas in Section II.A. This is so also in case of condensation since we are assuming $\delta \gg \Delta$. Furthermore, from Eq. (2.38) we know that the propagator at zero momentum is nothing but the density of states

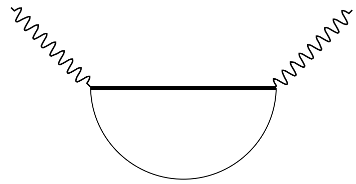

FIG. 17 *A contribution to hard gluon terms obtained by closing the light fermion lines in the diagrams of Fig. 16*

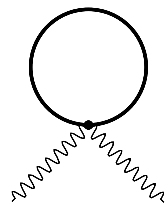

FIG. 18 *The two gluon operator arising from integrating out the heavy degrees of freedom in $\mathcal{L}_D^h$. The wavy lines represent the gluons, the thin lines the light fermionic d.o.f. and the thick lines the heavy oneslight*

inside the Fermi surface. Therefore we expect an order μ^3 contribution[5]. At the leading order in μ this gives a term of the type $\mu^2 A^2$, where A is the gluon field. We will evaluate this contribution later in Section VIII.A. Since this is a mass term for the gluons and, as we shall see, gives a contribution only to the spatial gluons, it will be referred to as **bare Meissner mass**. The result of this discussion is that the effective lagrangian for the light fields is simply given by $\mathcal{L}_D^l$, plus terms containing powers of soft gluon fields.

2. The HDET in the condensed phase

Let us now see what happens in the case of condensation after having integrated out the heavy fields. We will omit from now on the superscript identifying the light fields since these are the only fields we will deal with from now on. We will describe color and flavor with a collective index $A = 1, \cdots, N$ ($N = N_c N_f$). The general structure of the condensate, both for left- and right- handed fields is (we will neglect from now on the Weyl indices since we assume the difermions in the spin 0 state)

$$\langle \psi^A C \psi^B \rangle \approx \Delta_{AB}, \tag{5.63}$$

with Δ_{AB} a complex symmetric matrix.

We will now consider a four-fermi interaction of the BCS type

$$\mathcal{L}_I = -\frac{G}{4} \epsilon_{ab} \epsilon_{\dot{a}\dot{b}} V_{ABCD} \psi_a^A \psi_b^B \psi_{\dot{a}}^{C\dagger} \psi_{\dot{b}}^{D\dagger}. \tag{5.64}$$

We require $\mathcal{L}_I$ to be hermitian, therefore

$$V_{ABCD} = V^*_{CDAB} \tag{5.65}$$

and furthermore

$$V_{ABCD} = V_{BACD} = V_{ABDC}. \tag{5.66}$$

As we have done in Section III.D we write

$$\mathcal{L}_I = \mathcal{L}_{cond} + \mathcal{L}_{int}, \tag{5.67}$$

---

[5] To be precise we should take out the volume of the portion of the shell inside the Fermi surface that goes in the light fields definition, but since we are taking the limit of large μ this is negligible.

with

$$\mathcal{L}_{cond} = \frac{G}{4} V_{ABCD} \Gamma^{CD*} \psi^{AT} C \psi^{B} - \frac{G}{4} V_{ABCD} \Gamma^{AB} \psi^{C\dagger} C \psi^{D*} \tag{5.68}$$

and

$$\mathcal{L}_{int} = -\frac{G}{4} V_{ABCD} \left(\psi^{AT} C \psi^{B} - \Gamma^{AB}\right) \left(\psi^{C\dagger} C \psi^{D*} + \Gamma^{CD*}\right). \tag{5.69}$$

We now define

$$\Delta_{AB} = \frac{G}{2}\, V_{CDAB} \Gamma^{CD}, \quad \Delta^{*}_{AB} = \frac{G}{2} V^{*}_{CDAB} \Gamma^{CD*} = \frac{G}{2}\, V_{ABCD} \Gamma^{CD*}. \tag{5.70}$$

Clearly Δ_{AB} is a symmetric matrix. We will assume also that it can be diagonalized, meaning that

$$[\mathbf{\Delta}, \mathbf{\Delta}^{\dagger}] = 0, \tag{5.71}$$

where $\mathbf{\Delta}_{AB} = \Delta_{AB}$. Therefore ($C$ is the charge conjugation matrix, $C = i\sigma_2$)

$$\mathcal{L}_{cond} = \frac{1}{2} \Delta^{*}_{AB} \psi^{AT} C \psi^{B} - \frac{1}{2} \Delta_{AB} \psi^{A\dagger} C \psi^{B*}. \tag{5.72}$$

For the following it is convenient to introduce the following notation for the positive energy fields:

$$\psi_{\pm}(x) = \psi_{+}(\pm\mathbf{v}, x). \tag{5.73}$$

One should be careful in not identifying ψ_{-} with the negative energy solution with velocity $\mathbf{v}$. The condensation is taken into account by adding $\mathcal{L}_{cond}$ to the effective lagrangian of the previous Section (here we consider only the leading term), see Sections III.C and III.D. Therefore we will assume the following Lagrangian

$$\mathcal{L}_{D} = \int \frac{d\mathbf{v}}{4\pi} \sum_{A,B} \left[\psi_{+}^{A\dagger} (iV \cdot D)_{AB} \psi_{+}^{B} + \frac{1}{2} \psi_{-}^{A} C \psi_{+}^{B} \Delta^{*}_{AB} - \frac{1}{2} \psi_{+}^{A\dagger} C \psi_{-}^{B*} \Delta_{AB} \right]. \tag{5.74}$$

Using the symmetry $\mathbf{v} \to -\mathbf{v}$ of the velocity integration we may write the previous expression in the form

$$\mathcal{L}_{D} = \int \frac{d\mathbf{v}}{4\pi} \sum_{A,B} \frac{1}{2} \left[\psi_{+}^{A\dagger} (iV \cdot D)_{AB} \psi_{+}^{B} + \psi_{-}^{A\dagger} (i\tilde{V} \cdot D)_{AB} \psi_{-}^{B} + \psi_{-}^{A} C \psi_{+}^{B} \Delta^{*}_{AB} - \psi_{+}^{A\dagger} C \psi_{-}^{B*} \Delta_{AB} \right]. \tag{5.75}$$

We then introduce the Nambu-Gor'kov basis

$$\chi^{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_{+}^{A} \\ C \psi_{-}^{A*} \end{pmatrix} \tag{5.76}$$

in terms of which

$$\mathcal{L}_{D} = \int \frac{d\mathbf{v}}{4\pi} \chi^{A\dagger} \begin{bmatrix} iV \cdot D_{AB} & -\Delta_{AB} \\ -\Delta^{*}_{AB} & i\tilde{V} \cdot D^{*}_{AB} \end{bmatrix} \chi^{B}. \tag{5.77}$$

The lagrangian we have derived here coincides with the one that we obtained in Section III.D, that is the lagrangian giving rise to the Nambu-Gor'kov equations.

The inverse free propagator in operator notations is (notice that since $\mathbf{\Delta}$ is symmetric we have $\mathbf{\Delta}^{*} = \mathbf{\Delta}^{\dagger}$)

$$S^{-1}(\ell) = \begin{pmatrix} V \cdot \ell & -\mathbf{\Delta} \\ -\mathbf{\Delta}^{\dagger} & \tilde{V} \cdot \ell \end{pmatrix}. \tag{5.78}$$

From which

$$S(\ell) = \frac{1}{(V \cdot \ell)(\tilde{V} \cdot \ell) - \mathbf{\Delta}\mathbf{\Delta}^{\dagger}} \begin{pmatrix} \tilde{V} \cdot \ell & \mathbf{\Delta} \\ \mathbf{\Delta}^{\dagger} & V \cdot \ell \end{pmatrix}. \tag{5.79}$$

Let us now consider the relation

$$\Delta^{*}_{AB} = -\frac{G}{2} V_{ABCD} \langle \psi^{C\dagger} C \psi^{D*} \rangle. \tag{5.80}$$

This Equation is the analogous of Eq. (3.112) in configuration space. Repeating the same steps leading to Eq. (3.130) we find

$$\Delta^*_{AB} = i\,2\times\frac{G}{2}V_{ABCD}\int\frac{d\mathbf{v}}{4\pi}\frac{\mu^2}{\pi}\int\frac{d^2\ell}{(2\pi)^2}\Delta^*_{CE}\frac{1}{D_{ED}}, \tag{5.81}$$

where we have made use of Eq. (5.31) and

$$\frac{1}{D_{AB}} = \left(\frac{1}{(V\cdot\ell)(\tilde{V}\cdot\ell)-\mathbf{\Delta}\mathbf{\Delta}^\dagger}\right)_{AB}. \tag{5.82}$$

Notice that the factor 2 arises from the sum over the Weyl indices. The final result is

$$\Delta^*_{AB} = iGV_{ABCD}\int\frac{d\mathbf{v}}{4\pi}\frac{\mu^2}{\pi}\int\frac{d^2\ell}{(2\pi)^2}\Delta^*_{CE}\frac{1}{D_{ED}}, \tag{5.83}$$

in agreement with the result we found from the Nambu-Gor'kov equations by identifying E with ℓ_0 and $d^3\mathbf{p}$ with the integration over the velocity and $\ell_\parallel$. The same result can be found via the functional approach, see Appendix A.

Let us apply this formalism to the case of an effective four-fermi interaction due to one gluon exchange

$$\mathcal{L}_I = \frac{3}{16}G\bar\psi\gamma_\mu\lambda^a\psi\bar\psi\gamma^\mu\lambda^a\psi, \tag{5.84}$$

where λ^a are the Gell-Mann matrices. Using the following identities

$$\sum_{a=1}^{8}(\lambda^a)_{\alpha\beta}(\lambda^a)_{\delta\gamma} = \frac{2}{3}\left(3\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\beta}\delta_{\gamma\delta}\right) \tag{5.85}$$

and

$$(\sigma_\mu)_{\dot{a}b}(\tilde\sigma^\mu)_{d\dot{c}} = 2\epsilon_{\dot{a}\dot{c}}\epsilon_{bd} \tag{5.86}$$

where

$$\sigma^\mu = (1,\boldsymbol{\sigma}), \qquad \tilde\sigma^\mu = (1,-\boldsymbol{\sigma}), \tag{5.87}$$

with $\boldsymbol{\sigma}$ the Pauli matrices, we find

$$\mathcal{L}_I = -\frac{G}{4}V_{(\alpha i)(\beta j)(\gamma k)(\delta\ell)}\psi_i^\alpha\psi_j^\beta\psi_k^{\gamma\dagger}\psi_\ell^{\delta\dagger} \tag{5.88}$$

with

$$V_{(\alpha i)(\beta j)(\gamma k)(\delta\ell)} = -(3\delta_{\alpha\delta}\delta_{\beta\gamma}-\delta_{\alpha\gamma}\delta_{\beta\delta})\delta_{ik}\delta_{j\ell}. \tag{5.89}$$

Let us now consider the simple case of the 2SC phase. Then, as we have seen,

$$\Delta_{(\alpha i)(\beta j)} = \epsilon_{\alpha\beta 3}\epsilon_{ij}\Delta. \tag{5.90}$$

We get at once

$$\Delta = 4iG\int\frac{d\mathbf{v}}{4\pi}\frac{\mu^2}{\pi}\int\frac{d^2\ell}{(2\pi)^2}\frac{\Delta}{\ell_0^2-\ell_\parallel^2-\Delta^2}, \tag{5.91}$$

with $\ell_\parallel = \mathbf{v}\cdot\boldsymbol{\ell}$. Performing the integration over ℓ_0 we obtain

$$\Delta = \frac{G}{2}\rho\int_0^\delta d\xi\frac{\Delta}{\sqrt{\xi^2+\Delta^2}}. \tag{5.92}$$

Here we have defined the density of states as

$$\rho = \frac{4\mu^2}{\pi^2}, \tag{5.93}$$

which is the appropriate one for this case. In fact remember that in the BCS case the density is defined as $p_F^2/v_F\pi^2$. In the actual case $p_F = E_F = \mu$ and $v_F = 1$. The factor 4 arises since there are 4 fermions, ψ_i^α with $\alpha = 1, 2$, which are pairing.

It is worth to note that an alternative approach is to work directly with the Schwinger-Dyson equation (Rajagopal and Wilczek, 2001). In that case there is no necessity to Fierz transform the interaction term. The reason we have not adopted this scheme, although technically more simple, is just to illustrate more the similarities with the condensed matter treatment.

In order to make an evaluation of the gap one can fix the coupling by the requirement that this theory reproduces the chiral phenomenology in the limit of zero density and temperature. It is not difficult to show that the chiral gap equation is given by

$$1 = 8G\int_0^\Lambda \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{p^2+M^2}}. \tag{5.94}$$

Here Λ is the Nambu Jona-Lasinio cutoff and M is the constituent mass. Correspondingly one chooses $\delta = \Lambda - \mu$ in the gap equation at finite density. By choosing typical values of $\Lambda = 800\ MeV$, $M = 400\ MeV$ and $\mu = 400 \div 500\ MeV$ one finds respectively $\Delta = 39 \div 88\ MeV$[6]. Similar values are also found in the CFL case.

C. The gap equation in QCD

Having discussed the gap equation in the context of a four-fermi interaction we will now discuss the real QCD case. However this calculation has real meaning only at extremely high densities much larger than $10^8\ MeV$, see (Rajagopal and Shuster, 2000). Therefore we will give only a brief sketch of the main results. We will work in the simpler case of the 2SC phase.

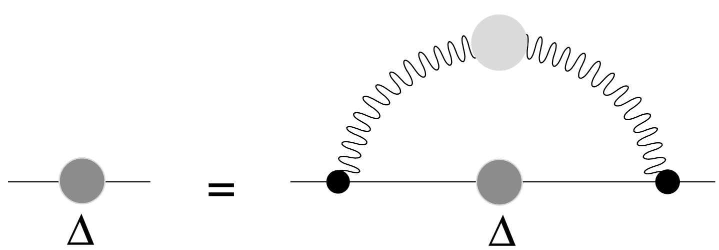

FIG. 19 *The gap equation in QCD at finite density. The light grey circle denotes the gluon self-energy insertion. The dark grey ones a gap insertions. At the leading order there is no need of vertex corrections*

As shown in Appendix A the gap equation can be obtained simply by writing down the Schwinger-Dyson equation, as shown in Fig. 19. The diagram has been evaluated in (Brown *et al.*, 2000b; Hong *et al.*, 2000b; Pisarski and Rischke, 2000a; Schafer and Wilczek, 1999d; Son, 1999). The result in euclidean space is

$$\begin{aligned}\Delta(p_0) &= \frac{g^2}{12\pi^2}\int dq_0 \int d\cos\theta \left(\frac{\frac{3}{2}-\frac{1}{2}\cos\theta}{1-\cos\theta+G/(2\mu^2)}\right.\\ &+ \left.\frac{\frac{1}{2}+\frac{1}{2}\cos\theta}{1-\cos\theta+F/(2\mu^2)}\right)\frac{\Delta(q_0)}{\sqrt{q_0^2+\Delta(q_0)^2}}.\end{aligned} \tag{5.95}$$

Here, $\Delta(p_0)$ is the energy dependent gap, g is the QCD coupling constant and G and F are the self energies of magnetic and electric gluons. The terms in the curly brackets arise from the magnetic and electric components of the gluon propagator. The numerators are the on-shell matrix elements $\mathcal{M}_{ii,00} = [\bar{u}_h(p_1)\gamma_{i,0}u_h(p_3)][\bar{u}_h(p_2)\gamma_{i,0}u_h(p_4)]$ for

[6] For a different way of choosing the cutoff in NJL models, see (Casalbuoni *et al.*, 2003)

the scattering of back-to-back fermions on the Fermi surface. The scattering angle is $\cos\theta = \mathbf{p}_1 \cdot \mathbf{p}_3$. In the case of a spin zero order parameter, the helicity h of all fermions is the same (see (Schafer and Wilczek, 1999d)).

The important difference between Eq. (5.95) and the case of a contact four-fermi interaction is due to the fact that the self-energy of the magnetic gluons vanishes at zero energy. Therefore the gap equation contains a collinear $\cos\theta \sim 1$ divergence. One makes use of the hard-loop approximation (Le Bellac, 1996) and for $q_0 \ll |\mathbf{q}| \to 0$ and to leading order in perturbation theory we have

$$F = m_D^2, \qquad G = \frac{\pi}{4} m_D^2 \frac{q_0}{|\vec{q}|}, \tag{5.96}$$

with

$$m_D^2 = N_f \frac{g^2 \mu^2}{2\pi^2}. \tag{5.97}$$

In the electric part, m_D^2 is the Debye screening mass. In the magnetic part, there is no screening of static modes, but non-static modes are dynamically screened due to Landau damping. For small energies dynamic screening of magnetic modes is much weaker than Debye screening of electric modes. As a consequence, perturbative color superconductivity is dominated by magnetic gluon exchanges. We are now able to perform the angular integral in Eq. (5.95) finding

$$\Delta(p_0) = \frac{g^2}{18\pi^2} \int dq_0 \log\left(\frac{b\mu}{|p_0 - q_0|}\right) \frac{\Delta(q_0)}{\sqrt{q_0^2 + \Delta(q_0)^2}}, \tag{5.98}$$

with

$$b = 256\pi^4 (2/N_f)^{5/2} g^{-5}. \tag{5.99}$$

This equation has been derived for the first time in QCD in Ref. (Son, 1999). In ordinary superconductivity it was realized by (Eliashberg, 1960) that the effects of retardation of phonons (taking the place of the gluons) are important and that they produce the extra logarithmic term in the gap equation. The integral equation we have obtained can be converted to a differential equation (Son, 1999) and in the weak coupling limit an approximate solution is (Brown *et al.*, 2000a,b,c; Pisarski and Rischke, 1999, 2000a,b; Schafer and Wilczek, 1999d; Son, 1999)

$$\Delta(p_0) \approx \Delta_0 \sin\left(\frac{g}{3\sqrt{2}\pi} \log\left(\frac{b}{p_0}\right)\right), \tag{5.100}$$

with

$$\Delta_0 = 2b\mu \exp\left(-\frac{3\pi^2}{\sqrt{2}g}\right). \tag{5.101}$$

This result shows why it is important to keep track of the energy dependence of Δ. In particular neglecting the energy dependence would give a wrong coefficient in the exponent appearing in Δ_0. Also we see that the collinear divergence leads to a gap equation with a double-log behavior. Qualitatively

$$1 \sim \frac{g^2}{18\pi^2} \left[\log\left(\frac{\mu}{\Delta}\right)\right]^2, \tag{5.102}$$

from which we conclude that $\Delta \sim \exp(-c/g)$. The prefactor in the expression for Δ_0 is not of easy evaluation. By writing

$$\Delta_0 \simeq 512\pi^4 (2/N_f)^{5/2} b_0' \mu g^{-5} \exp\left(-\frac{3\pi^2}{\sqrt{2}g}\right). \tag{5.103}$$

we have $b_0' = 1$ in the previous case, whereas with different approximations in (Brown *et al.*, 2000c; Wang and Rischke, 2002) it has been found

$$b_0' = \exp\left(-\frac{4+\pi^2}{8}\right). \tag{5.104}$$

Numerically one finds at $\mu = 10^{10}\ MeV$, $g = .67$ and $b_0' = 2/5$, $\Delta_0 \approx 40\ MeV$. Extrapolating at $\mu = 400\ MeV$, $g = 3.43$, one finds $\Delta_0 \approx 90\ MeV$. It turns out that Δ_0 decreases from 90 MeV to about 10 MeV for μ increasing from 400 MeV to $10^6\ MeV$. Continuing to increase μ Δ increases as it should be according to its asymptotic value. In fact we see that for increasing μ, Δ increases although $\Delta/\mu \to 0$. In particular we notice that neither the four-fermi interaction approach and the present one from first principles can be trusted at phenomenologically interesting chemical potentials of the order $400 \div 500\ MeV$. Still it is of some interest to observe that both methods lead to gaps of the same order of magnitude.

D. The symmetries of the superconductive phases

1. The CFL phase

As we have already discussed we expect that at very high density the condensate $\langle\psi^\alpha_{iL}C\psi^\beta_{jL}\rangle$ is antisymmetric in color and in flavor. Also, if we require parity invariance we have

$$\langle\psi^\alpha_{iL}\psi^\beta_{jL}\rangle = -\langle\psi^\alpha_{iR}\psi^\beta_{jR}\rangle. \tag{5.105}$$

In fact in Dirac notation we notice that

$$\langle\psi^T C\gamma_5\psi\rangle \tag{5.106}$$

is parity invariant ($\psi \to \eta_P\gamma_0\psi$) and its L and R components are precisely $\langle\psi_L\psi_L\rangle$ and $\langle\psi_R\psi_R\rangle$. The required antisymmetry implies

$$\langle\psi^\alpha_{iL}\psi^\beta_{jL}\rangle = \epsilon^{\alpha\beta\gamma}\epsilon_{ijk}A^k_\gamma, \tag{5.107}$$

with A a 3×3 matrix. In (Evans *et al.*, 2000) it has been shown that A can be diagonalized by using an $SU(3)_c\otimes SU(3)_L$ global rotation

$$A \to g_{SU(3)_c}Ag_{SU(3)_L} = A_D, \tag{5.108}$$

with

$$A_D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}. \tag{5.109}$$

Studying the gap equation (this calculation has been done in full QCD) it can be seen that three cases are possible

$$\begin{aligned}
(1,1,1)\ :\quad & A_D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
(1,1,0)\ :\quad & A_D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
(1,0,0)\ :\quad & A_D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{5.110}$$

The corresponding gaps satisfy

$$\Delta_{(1,1,1)} < \Delta_{(1,1,0)} < \Delta_{(1,0,0)}, \tag{5.111}$$

but for the free energies the result is

$$F_{(1,1,1)} < F_{(1,1,0)} < F_{(1,0,0)}. \tag{5.112}$$

The reason is that although the gaps for the less symmetric solution is bigger, there are more fermions paired in the more symmetric configurations.

The analysis of the gap equation shows that there is also a component from the color channel **6**. In fact, under the group $SU(3)_c\otimes SU(3)_L\otimes SU(3)_R$ we have

$$\psi^\alpha_{iL(R)} \in (\mathbf{3}_c, \mathbf{3}_{L(R)}). \tag{5.113}$$

Therefore

$$\left[(\mathbf{3}_c, \mathbf{3}_{L(R)})\otimes(\mathbf{3}_c, \mathbf{3}_{L(R)})\right]_S = (\mathbf{3}^*_c, \mathbf{3}^*_{L(R)})\oplus(\mathbf{6}_c, \mathbf{6}_{L(R)}). \tag{5.114}$$

Here the index S means that we have to take the symmetric combination of the tensor product since $\langle\psi^T_L C\psi_L\rangle$ is already antisymmetric in spin. From the previous argument and in the absence of the $(\mathbf{6}_c, \mathbf{6}_{L(R)})$ component we have

$$\langle\psi^\alpha_{iL}\psi^\beta_{jL}\rangle = \Delta\epsilon^{\alpha\beta I}\epsilon_{ijI}. \tag{5.115}$$

The presence of the $(\mathbf{6}_c, \mathbf{6}_{L(R)})$ implies

$$\langle \psi_{iL}^\alpha \psi_{jL}^\beta \rangle = \Delta \epsilon^{\alpha\beta I} \epsilon_{ijI} + \Delta_6 (\delta_i^\alpha \delta_j^\beta + \delta_i^\beta \delta_j^\alpha), \tag{5.116}$$

or

$$\langle \psi_{iL}^\alpha \psi_{jL}^\beta \rangle = \Delta (\delta_i^\alpha \delta_j^\beta - \delta_i^\beta \delta_j^\alpha) + \Delta_6 (\delta_i^\alpha \delta_j^\beta + \delta_i^\beta \delta_j^\alpha) = (\Delta + \Delta_6) \delta_i^\alpha \delta_j^\beta + (\Delta_6 - \Delta) \delta_i^\beta \delta_j^\alpha. \tag{5.117}$$

Notice that the $(\mathbf{6}_c, \mathbf{6}_{L(R)})$ term does not break any further symmetry other than the ones already broken by the $(\mathbf{3}_c^*, \mathbf{3}_{L(R)}^*)$. Numerically Δ_6 turns out to be quite small. The analysis has been done in (Alford *et al.*, 1999) using the parameters Δ_8 and Δ_1, defined as

$$\Delta + \Delta_6 = \frac{1}{3}\left(\Delta_8 + \frac{1}{8}\Delta_1\right), \quad \Delta - \Delta_6 = -\frac{1}{8}\Delta_1. \tag{5.118}$$

Therefore

$$\Delta = \frac{1}{3}\left(\Delta_8 - \Delta_1\right), \quad \Delta_6 = \frac{1}{6}\left(\Delta_8 + \frac{1}{2}\Delta_1\right). \tag{5.119}$$

The absence of the sextet is equivalent to require $\Delta_1 = -2\Delta_8$. The result found in (Alford *et al.*, 1999), for the choice of parameters, $\Lambda = 800\ MeV$, $M = 400\ MeV$ and $\mu = 400\ MeV$ is

$$\Delta_8 = 80\ MeV, \quad \Delta_1 = -176\ MeV, \tag{5.120}$$

implying

$$\Delta = 85.3\ MeV, \quad \Delta_6 = -1.3\ MeV. \tag{5.121}$$

As already discussed the original symmetry of the theory is

$$G_{QCD} = SU(3)_c \otimes SU(3)_L \otimes SU(3)_R \otimes U(1)_B. \tag{5.122}$$

The first factor is local whereas the other three are global since, for the moment, we are neglecting the em interaction which makes local a $U(1)$ subgroup of $SU(3)_L \otimes SU(3)_R$. The condensates lock together the transformations of $SU(3)_c$, $SU(3)_L$ and $SU(3)_R$, therefore the symmetry of the CFL phase is

$$G_{\rm CFL} = SU(3)_{c+L+R} \otimes Z_2. \tag{5.123}$$

In fact, also the $U(1)_B$ group is broken leaving a Z_2 symmetry corresponding to the multiplication of the quark fields by -1. Also, locking $SU(3)_L$ and $SU(3)_R$ to $SU(3)_c$ makes $SU(3)_L$ and $SU(3)_R$ lock together producing the breaking of the chiral symmetry. The breaking of G_{QCD} to $G_{\rm CFL}$ gives rise to

$$3 \times 8 + 1 - 8 = 8 + 8 + 1 \tag{5.124}$$

Nambu-Goldstone (NB) bosons. However 8 of the NG bosons disappear from the physical spectrum through the Higgs mechanism, giving masses to the 8 gluons, whereas 8+1 massless NG bosons are left in the physical spectrum. Of course, these NG bosons take mass due to the explicit breaking of the symmetry produced by the quark masses. Since all the local symmetries are broken (but see later as far the electromagnetism is concerned) all the gauge bosons acquire a mass.

Although the baryon number is broken no dramatic event takes place. The point is that we are dealing with a finite sample of superconductive matter. In fact, applying the Gauss' law to a surface surrounding the sample we find that changes of the baryon number inside the superconductor must be accompanied by compensating fluxes. In other words, inside the sample there might be large fluctuations and transport of baryonic number. Things are not different from what happens in ordinary superconductors where the quantum number, number of electrons (or lepton number), is not conserved. The connection between the violation of quantum numbers and phenomena of supertransport as superfluidity and superconductivity are very strictly related.

The condensate $\langle \psi_{iL}^\alpha \psi_{jL}^\beta \rangle$ is not gauge invariant and we may wonder if it is possible to define gauge invariant order parameters. To this end let us introduce the matrices

$$X_\gamma^k = \langle \psi_{iL}^\alpha \psi_{jL}^\beta \rangle^* \epsilon_{\alpha\beta\gamma} \epsilon^{ijk}, \quad Y_\gamma^k = \langle \psi_{iR}^\alpha \psi_{jR}^\beta \rangle^* \epsilon_{\alpha\beta\gamma} \epsilon^{ijk}. \tag{5.125}$$

The conjugation has been introduced for convenience reasons (see in the following). The matrix

$$(Y^\dagger X)^i_j = \sum_\alpha (Y^j_\alpha)^* X^i_\alpha \tag{5.126}$$

is gauge invariant, since color indices are saturated and breaks $SU(3)_L \otimes SU(3)_R$ to $SU(3)_{L+R}$. Analogously the 6-fermion operators

$$\det(X), \quad \det(Y) \tag{5.127}$$

are gauge invariant flavor singlet and break $U(1)_B$.

So far we have neglected $U(1)_A$. This group is broken by the anomaly. However, the anomaly is induced by a 6-fermion operator (the 't Hooft determinant) which becomes irrelevant at the Fermi surface. On the other hand this operator is qualitatively important since it is the main cause of the breaking of $U(1)_A$. If there was no such operator we would have a further NG boson associated to the spontaneous breaking of $U(1)_A$ produced by the di-fermion condensate. Since the instanton contribution is parametrically small at high density, we expect the NG boson to be very light. Notice that the $U(1)_A$ approximate symmetry is broken by the condensate to a discrete group Z_2.

The spectrum of the CFL phase

Let us spend some word about the spectrum of QCD in the CFL phase. We start with the fermions. Since all the fermions are paired they are all gapped. To understand better this point let us notice that under the symmetry group of the CFL phase $SU(3)_{c+L+R}$ quarks transform as $\mathbf{1} \oplus \mathbf{8}$. Therefore it is useful to introduce the basis

$$\psi^\alpha_i = \frac{1}{\sqrt{2}} \sum_{A=1}^{9} (\lambda_A)^\alpha_i \psi^A, \tag{5.128}$$

where λ_A, $A = 1, \cdots, 8$ are the Gell-Mann matrices and

$$\lambda_9 = \lambda_0 = \sqrt{\frac{2}{3}} \times \mathbf{1}, \tag{5.129}$$

with $\mathbf{1}$ the identity matrix in the 3×3 space. With this normalization

$$\mathrm{Tr}(\lambda_A \lambda_B) = 2\delta_{AB}. \tag{5.130}$$

Inverting Eq. (5.128)

$$\psi^A = \frac{1}{\sqrt{2}} \sum_{\alpha i} (\lambda_A)^i_\alpha \psi^\alpha_i = \frac{1}{\sqrt{2}} \mathrm{Tr}(\lambda_A \psi), \tag{5.131}$$

we obtain

$$\langle \psi^A \psi^B \rangle = \frac{1}{2} \sum (\lambda_A)^i_\alpha (\lambda_B)^j_\beta \Delta \epsilon^{\alpha\beta I} \epsilon_{ijI} = \frac{\Delta}{2} \mathrm{Tr} \sum_I \left(\lambda_A \epsilon_I \lambda_B^T \epsilon_I \right), \tag{5.132}$$

where we have defined the following three, 3×3, matrices

$$(\epsilon_I)_{\alpha\beta} = \epsilon_{\alpha\beta I}, \tag{5.133}$$

which have the following property valid for any 3×3 matrix g:

$$\sum_I \epsilon_I g^T \epsilon_I = g - \mathrm{Tr}[g]. \tag{5.134}$$

This is a simple consequence of the definition of the ϵ_I matrices. Using this equation it follows

$$\langle \psi^A \psi^B \rangle = \Delta_A \delta_{AB}, \tag{5.135}$$

with

$$\Delta_A = \begin{cases} A = 1, \cdots, 8 & \Delta_A = \Delta, \\ A = 9 & \Delta_9 = -2\Delta, \end{cases} \tag{5.136}$$

There are two gaps, one for the octet ($A = 1, \cdots, 8$) and a larger one for the singlet ($A = 9$). In each of these cases the dispersion relation for the quasi-fermions is

$$\epsilon_A(\mathbf{p}) = \sqrt{(\mathbf{v} \cdot \boldsymbol{\ell})^2 + \Delta_A^2}. \tag{5.137}$$

The next category of particles are the gluons which acquire a mass through the Higgs mechanism. Therefore we expect

$$m_g^2 \approx g_s^2 F^2, \tag{5.138}$$

with g_s the coupling constant of QCD and F the coupling constant of the NG bosons of the CFL phase. However the situation is more complicated than this due to two different effects. First there is a very large wave-function renormalization making the mass of the gluons proportional to Δ. Second, the theory is not relativistically invariant. In fact being at finite density implies a breaking of the Lorentz group down to $O(3)$, the invariance under spatial rotations.

The last category of particles are the NG bosons. As already discussed we expect 9 massless NG bosons from the breaking of G_{QCD} to $G_{\rm CFL}$, plus a light NG boson from the breaking of $U(1)_A$. Again, this is not the end of the story, since quarks are not massless. As a consequence the NG bosons of the octet acquire mass. The NG boson associated to the breaking of $U(1)_B$ remain massless since this symmetry is not broken by quark masses. However the masses of the NG bosons of the octet are parametrically small since their square turns out to be quadratic in the quark masses. The reason comes from the approximate symmetry $(Z_2)_L \otimes (Z_2)_R$ defined by

$$\begin{array}{ll} (Z_2)_L & \psi_L \to -\psi_L \\ (Z_2)_R & \psi_R \to -\psi_R. \end{array} \tag{5.139}$$

This is an approximate symmetry since only the diagonal Z_2 is preserved by the axial anomaly. However the quark mass term

$$\bar{\psi}_L M \psi_R + \text{h.c.} \tag{5.140}$$

is such that $M \to -M$ (here we are treating M as a spurion field) under the previous symmetry. Therefore the mass square for the NG bosons in the CFL phase must be quadratic in M. The anomaly breaks $(Z_2)_L \otimes (Z_2)_R$ through the instantons. As a result a condensation of the type $\langle \bar{\psi}_L \psi_R \rangle$ is produced. This is because 4 of the 6 fermions in the 't Hooft determinant ('t Hooft, 1976; Schafer and Shuryak, 1998; Shifman *et al.*, 1980) condensate through the operators X and Y (see Eq. (5.125)), leaving a condensate of the form $\langle \bar{\psi}_L \psi_R \rangle$ (Alford *et al.*, 1999; Rapp *et al.*, 2000; Schafer, 2000b). However it has been shown that this contribution is very small, of the order $(\Lambda_{QCD}/\mu)^8$ (Manuel and Tytgat, 2000; Schafer, 2000a).

In-medium electric charge

The em interaction is included noticing that $U(1)_{\rm em} \subset SU(3)_L \otimes SU(3)_R$ and extending the covariant derivative

$$D_\mu \psi = \partial_\mu \hat{\psi} - i g_\mu^a T_a \psi - i \psi Q A_\mu, \tag{5.141}$$

where $T_a = \lambda_a/2$ with λ_a the Gell-Mann matrices. The condensate breaks $U(1)_{\rm em}$ but leaves invariant a combination of Q and of the color generator. The result can be seen immediately in terms of the matrices X and Y introduced before. In fact the CFL vacuum is defined by

$$X_\alpha^i = Y_\alpha^i = \delta_\alpha^i. \tag{5.142}$$

Defining ($T_8 = (1, 1, -2)/2\sqrt{3}$)

$$Q_{SU(3)_c} \equiv -\frac{2}{\sqrt{3}} T_8 = \text{diag}(-1/3, -1/3, +2/3) = Q, \tag{5.143}$$

we see that the combination

$$Q_{SU(3)_c} \otimes \mathbf{1} - \mathbf{1} \otimes Q = Q \otimes \mathbf{1} - \mathbf{1} \otimes Q \tag{5.144}$$

leaves invariant the condensates

$$Q \langle \hat{X} \rangle - \langle \hat{X} \rangle Q \to Q_{\alpha\beta} \delta_{\beta i} - \delta_{\alpha j} Q_{ji} = 0. \tag{5.145}$$

Therefore the in-medium conserved electric charge is

$$\tilde{Q} = \mathbf{1} \otimes Q - Q \otimes \mathbf{1}. \tag{5.146}$$

The eigenvalues of $\tilde{Q}$ are $0, \pm 1$ as in the old Han-Nambu model.

The in-medium em field A_μ and the gluon field g_μ^8 get rotated to new fields $\tilde{A}_\mu$ and $\tilde{G}_\mu$

$$\begin{aligned} A_\mu &= \tilde{A}_\mu \cos\theta - \tilde{G}_\mu \sin\theta, \\ g_\mu^8 &= \tilde{A}_\mu \sin\theta + \tilde{G}_\mu \cos\theta, \end{aligned} \tag{5.147}$$

with new interactions

$$g_s g_\mu^8 T_8 \otimes 1 + e A_\mu 1 \otimes Q \;\; \rightarrow \;\; \tilde{e}\tilde{Q}\tilde{A}_\mu + g_s' \tilde{G}\tilde{T}, \tag{5.148}$$

where

$$\begin{aligned} \tan\theta &= \frac{2}{\sqrt{3}}\frac{e}{g_s}, \quad \tilde{e} = e\cos\theta, \quad g_s' = \frac{g_s}{\cos\theta}, \\ \tilde{T} &= -\frac{\sqrt{3}}{2}\left[(\cos^2\theta)\, Q \otimes 1 + (\sin^2\theta)\, 1 \otimes Q\right]. \end{aligned} \tag{5.149}$$

Therefore the photon associated with the field $\tilde{A}_\mu$ remains massless, whereas the gluon associated to g_μ^8 becomes massive due to the Meissner effect.

We shall show that also gluons and NG bosons have integer charges, $0, \pm 1$. Therefore all the elementary excitations are integrally charged.

It is interesting to consider a sample of CFL material. If quarks were massless there would be charged massless NG bosons and the low-energy em response would be dominated by these modes. This would look as a "bosonic metal". However quarks are massive and so the charged NG bosons. Therefore, for quarks of the same mass, the CFL material would look like as a transparent insulator with no charged excitations at zero temperature (transport properties of massive excitations are exponentially suppressed at zero temperature). However making quark masses different makes the story somewhat complicated since for the equilibrium one needs a non zero density of electrons or a condensate of charged kaons (Schafer, 2000a). In both cases there are massless or almost massless excitations.

Quark-Hadron Continuity

The main properties we have described so far of the CFL phase are: confinement (integral charges), chiral symmetry breaking to a diagonal subgroup and baryon number superfluidity (due to the massless NG boson). If not for the $U(1)_B$ NG boson these properties are the same as the hadronic phase of three-flavor QCD:

$$\begin{aligned} &\text{CFL phase:} \quad U(1)_B \quad \text{broken} \rightarrow NGB \\ &\text{hadr. phase at } \; T = \mu = 0: \; U(1)_B \text{ unbroken} \end{aligned}$$

The NGB makes the CFL phase a superfluid. For 3-flavors a dibaryon condensate, H, of the type $(udsuds) \approx \det(X)$ is possible (Jaffe, 1977). This may arise at μ such that the Fermi momenta of the baryons in the octet are similar allowing pairing in strange, isosinglet dibaryon states of the type ($p\Xi^-$, $n\Xi^0$, $\Sigma^+\Sigma^-$, $\Sigma^0\Sigma^0$, $\Lambda\Lambda$) (all of the type *udsuds*). This would be again a superfluid phase. The symmetries of this phase, called hypernuclear matter phase, are the same as the ones in CFL. Therefore there is no need of phase transition between hypernuclear matter and CFL phase (Schafer and Wilczek, 1999a). This is strongly suggested by complementarity idea. Complementarity refers to gauge theories with a one-to-one correspondence between the spectra of the physical states in the Higgs and in the confined phases, see (Banks and Rabinovici, 1979; Fradkin and Shenker, 1979) for $U(1)$ theories and (Abbott and Fahri, 1981; Dimopoulos *et al.*, 1980a,b; 't Hooft, 1980) for $SU(2)$. Specific examples (Fradkin and Shenker, 1979) show that the two phases are rigorously indistinguishable. No phase transition but a smooth variation of the parameters characterizes the transition between the two phases.

A way to implement complementarity is the following (Casalbuoni and Gatto, 1981). Suppose to have the following spectrum of fields

$$\begin{aligned} &\psi_i \in R \text{ of } G, \quad \text{elementary states} \\ &Q_\alpha \in \tilde{R} \text{ of } \tilde{G}, \quad \text{composite states} \end{aligned}$$

CFL phase	Hypernuclear phase
$\psi^i_\alpha\langle D^\alpha_k\rangle$	$B^i_k = \psi^i_\alpha D^\alpha_k$
$\langle (D^*)^i_\alpha\rangle g^\alpha_\beta \langle D^\beta_k\rangle$	$(D^*)^i_\alpha g^\alpha_\beta D^\beta_k$
Mesons = phases of $(D^*)^i_{\alpha L} D^\alpha_{jR}$	Mesons = phases of $\bar{\psi}^\alpha_{jL}\psi^i_{\alpha R}$

TABLE I *This table shows the complementarity of the CFL phase and of the hypernuclear phase. The diquark fields D^α_i are defined as $D^\alpha_i = \epsilon_{ijk}\epsilon^{\alpha\beta\gamma}\psi^j_\beta\psi^k_\gamma$.*

ψ^i_α	u	d	s
R	2/3	-1/3	-1/3
B	2/3	-1/3	-1/3
W	2/3	-1/3	-1/3

TABLE II *The electric charges of the quark fields ψ^i_α.*

with G the gauge group in the unbroken phase and $\tilde{G}$ the global symmetry group of the broken phase. We assume that R and G are isomorphic to $\tilde{R}$ and $\tilde{G}$. We assume also that the breaking is such that the the effective Higgs fields (ϕ^i_α) are such to map the two set of states

$$Q_\alpha(x) = \psi_i(x)\phi^i_\alpha(x), \alpha \in \tilde{G}, i \in G. \tag{5.150}$$

In the broken phase, $\langle\phi^i_\alpha\rangle \propto \delta^i_\alpha$ implying that the states in the two phases are the same, except for a necessary redefinition of the conserved quantum numbers following from the requirement that the Higgs fields should be neutral in the broken vacuum The gauge fields $(g_\mu)^\beta_\alpha$ go into the vector mesons of the confined phase

$$(Z_\mu)^i_j = -\phi^{*\beta}_j \left[\partial_\mu - (g_\mu)^\alpha_\beta\right]\phi^i_\alpha. \tag{5.151}$$

In the case of CFL phase and hypernuclear matter, we have $G = SU(3)_c$ and $\tilde{G} = SU(3)$ with the fermion fields in three copies in both phases. The effective Higgs field is given by the diquark field

$$D^\gamma_k = \epsilon_{ijk}\epsilon^{\alpha\beta\gamma}\psi^i_\alpha\psi^j_\beta, \tag{5.152}$$

, with the property

$$\langle D^\gamma_k\rangle \propto \delta^\gamma_k. \tag{5.153}$$

The two phases, as shown in Table I are very similar but there are also several differences (Schafer and Wilczek, 1999a)

- In the hypernuclear phase there is a nonet of vector bosons. However if the dibaryon H exists the singlet vector becomes unstable and does not need to appear in the effective theory
- In the CFL phase there are nine ($\mathbf{8}\oplus\mathbf{1}$) quark states, but the gap of the singlet is bigger than for the octet. The baryonic singlet has the structure

$$\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}\psi^\alpha_i\psi^\beta_j\psi^\gamma_k, \tag{5.154}$$

  and it is precisely the condensation of this baryon with itself which may produce the state H discussed before.

The CFL phase is a concrete example of complementarity. In the tables II, III, IV, V we show the electric charges of the various states. In the CFL phase the charge $\tilde{Q}$ of diquarks is zero whereas for quarks, ψ^i_α, and gluons, $g_{\alpha\beta}$, coincides with the charge Q of baryons, B^i_k, and of vector mesons, G^i_k.

D_k^γ	R	B	W
u	-2/3	-2/3	-2/3
d	1/3	1/3	1/3
s	1/3	1/3	1/3

TABLE III *The electric charges of the diquark fields* $D_k^\gamma = \epsilon_{ijk}\epsilon^{\alpha\beta\gamma}\psi_\alpha^i\psi_\beta^j$. *However their* $\tilde{Q}$ *charges are integers and equal t0* $0, \pm 1$. *This follows since the charges* $\tilde{Q}$ *of the fermions enjoy the same property.*

B_k^i	u	d	s
u	0	-1	-1
d	1	0	0
s	1	0	0

TABLE IV *The electric charges of the baryon fields* $B_k^i = \psi_\gamma^i D_k^\gamma = \psi_\gamma^i\ \ \epsilon_{rsk}\epsilon^{\alpha\beta\gamma}\psi_\alpha^r\psi_\beta^s$.

2. The 2SC phase

We remember that in this case

$$\langle \psi_{iL}^\alpha \psi_{jL}^\beta \rangle = \Delta \epsilon^{\alpha\beta 3}\epsilon_{ij}. \tag{5.155}$$

Only 4 out of the 6 quarks are gapped, the ones with color 1 and 2 whereas the 2 quarks of color 3 remain ungapped. The symmetry breaking pattern is

$$SU(3)_c \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_B \rightarrow SU(2)_c \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{\tilde{B}}.$$

In this phase the baryon number is not broken but there is a combination of B and of the color generator T_8, acting upon the color indices, defined as

$$\tilde{B} = B - \frac{2}{\sqrt{3}}T_8 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) - \frac{1}{3}(1, 1, -2) = (0, 0, 1), \tag{5.156}$$

which is conserved by the condensate. In fact, although both B and T_8 are broken, the condensate, involving only quarks of color 1 and 2, is neutral under $\tilde{B}$. Also the electric charge is rotated. In fact consider the following combination

$$\tilde{Q} = Q \otimes \mathbf{1} - \frac{1}{\sqrt{3}}\mathbf{1} \otimes T_8 = \left(\frac{2}{3}, -\frac{1}{3}\right) \otimes \mathbf{1} - \mathbf{1} \otimes \frac{1}{6}(1, 1, -2). \tag{5.157}$$

The $\tilde{B}$ and $\tilde{Q}$ quantum numbers of the quarks are given in Table VI We see that the condensate is neutral under $\tilde{Q}$ since it pairs together up and down quarks of color 1 and 2. Notice that quarks u^3 and d^3 have integer values of both $\tilde{B}$ and $\tilde{Q}$. They look like proton and neutron respectively. We can understand why these quarks are ungapped by looking at the 't Hooft anomaly condition ('t Hooft, 1980). In fact, it has been shown in (Hsu *et al.*, 2001; Sannino, 2000) that the anomaly coefficient does not change at finite density. The theory in the confined phase at zero density has an anomaly $SU(2)_{L(R)} \otimes U(1)_B$ which is given by

$$\frac{1}{4} \times \frac{1}{3} \times 3 = \frac{1}{4}, \tag{5.158}$$

whereas in the broken CFL phase at finite density there is an anomaly $SU(2)_{L(R)} \otimes U(1)_{\tilde{B}}$ given by

$$\frac{1}{4} \times 1 = \frac{1}{4} \tag{5.159}$$

This anomaly is due entirely to the states ψ_i^3, therefore they should remain massless as the quarks in the zero density phase. In the 2SC phase there are no broken global symmetries therefore we expect a first order phase transition

G^i_k	u	d	s
u	0	-1	-1
d	1	0	0
s	1	0	0

TABLE V *The electric charges of the vector meson fields* $G^i_k = (D^*)^i_\alpha g^\alpha_\beta D^\beta_k$.

	$\tilde{Q}$	$\tilde{B}$
$u^\alpha,\ \ \alpha = 1,2$	$\frac{1}{2}$	0
$d^\alpha,\ \ \alpha = 1,2$	$-\frac{1}{2}$	0
u^3	1	1
d^3	0	1

TABLE VI The electric charge ($\tilde{Q}$) and the baryon number ($\tilde{B}$) for the quarks in the CFL phase

to the nuclear matter phase with a competition between chiral and di-fermion condensates. Notice that there is no superfluidity in the 2SC phase.

The spectrum of the 2SC phase

We have already discussed the fermionic part, the modes ψ^α_i with $\alpha = 1,2$ are gapped, whereas ψ^3_i remain massless. Since the gauge group $SU(3)_c$ is broken down to $SU(2)_c$ we get

$$8 - 3 = 5 \tag{5.160}$$

massive gluons. However we have still 3 massless gluons belonging to the confining gauge group $SU(2)_c$. With respect to this group the gapped fermions ψ^α_i with $\alpha = 1,2$ are confined, whereas the massless fermions ψ^3_i are un-confined. Notice also that the electric charges of the confined states are not integers whereas the ones of the un-confined states are integers.

As in 2SC no global symmetries are broken, there are no massless NG bosons. In conclusion the only light degrees of freedom are 3 gluons and 2 fermions.

3. The case of 2+1 flavors

In nature the strange quark is much heavier than the other two and it may happen that $\mu \approx m_s$ with $\mu \gg m_{u,d}$. In this situation neither of the discussions above applies. In practice we expect that decreasing μ from $\mu \gg m_{u,d,s}$ the system undergoes a phase transition from CFL to 2SC for values of μ in between m_s and $m_{d,u}$. This is because the Fermi momenta of the different Fermi spheres get separated. In fact remember that for massive quarks the Fermi momentum is defined by the equation

$$E_F = \mu = \sqrt{p_F^2 + M^2} \to p_F = \sqrt{\mu^2 - M^2}. \tag{5.161}$$

As a consequence the radius of the Fermi sphere of a given quark decreases increasing its mass. To see why such a transition is expected, let us consider a simplified model with two quarks, one massless and the other one with mass m_s at the same chemical potential μ. The Fermi momenta are

$$p_{F_1} = \sqrt{\mu^2 - m_s^2}, \qquad p_{F_2} = \mu. \tag{5.162}$$

The grand potential for the two unpaired fermions is (factor 2 from the spin degrees of freedom)

$$\Omega_{\text{unpair.}} = 2\int_0^{p_{F_1}} \frac{d^3p}{(2\pi)^3}\left(\sqrt{\vec{p}^{\,2} + m_s^2} - \mu\right) + 2\int_0^{p_{F_2}} \frac{d^3p}{(2\pi)^3}\left(|\vec{p}| - \mu\right). \tag{5.163}$$

In fact, the grand potential is given in general by the expression

$$\Omega = -T \sum_{\mathbf{k}} \log \sum_{n_{\mathbf{k}}} \left(e^{(\mu-\epsilon_{\mathbf{k}})/T}\right)^{n_{\mathbf{k}}} = -T \sum_{\mathbf{k}} \log\left(1 + e^{(\mu-\epsilon_{\mathbf{k}})/T}\right), \tag{5.164}$$

where $\epsilon_{\mathbf{k}}$ is the energy per particle and $n_{\mathbf{k}}$ is the occupation number of the mode $\mathbf{k}$. The expression above refers to fermions. Also

$$\sum_{\mathbf{k}} \to gV \int \frac{d^3\mathbf{p}}{(2\pi)^3}, \tag{5.165}$$

with g the degeneracy factor. In the limit of $T \to 0$ and in the continuum we get

$$\Omega = gV \int \frac{d^3\mathbf{p}}{(2\pi)^3}(\epsilon_{\mathbf{p}} - \mu)\theta(\mu - \epsilon_{\mathbf{p}}). \tag{5.166}$$

In order to pair the two fermions must reach some common momentum $p^F_{\rm comm}$, and the corresponding grand potential can be written as

$$\begin{aligned} \Omega_{\rm pair.} &= 2\int_0^{p^F_{\rm comm}} \frac{d^3p}{(2\pi)^3}\left(\sqrt{\mathbf{p}^{\,2} + m_s^2} - \mu\right) + 2\int_0^{p^F_{\rm comm}} \frac{d^3p}{(2\pi)^3}\left(|\mathbf{p}\,| - \mu\right) \\ &- \frac{\mu^2\Delta^2}{4\pi^2}, \end{aligned} \tag{5.167}$$

where the last term is the energy necessary for the condensation of a fermion pair, that we recall from Eq. (3.65) to be given by

$$-\frac{1}{4}\rho\Delta^2 = -\frac{1}{4}\frac{p_F^2}{\pi^2 v_F}\Delta^2 = -\frac{\mu^2\Delta^2}{4\pi^2}. \tag{5.168}$$

This expression can be adapted to the present case by neglecting terms of order $m_s^2\Delta^2$. The common momentum $p^F_{\rm comm}$ can be determined by minimizing $\Omega_{\rm pair.}$ with respect to $p^F_{\rm comm}$, with the result

$$p^F_{\rm comm} = \mu - \frac{m_s^2}{4\mu}. \tag{5.169}$$

It is now easy to evaluate the difference $\Omega_{\rm unpair.} - \Omega_{\rm pair.}$ at the order m_s^4, with the result

$$\Omega_{\rm pair.} - \Omega_{\rm unpair.} \approx \frac{1}{16\pi^2}\left(m_s^4 - 4\Delta^2\mu^2\right). \tag{5.170}$$

We see that in order to have condensation the condition

$$\mu > \frac{m_s^2}{2\Delta} \tag{5.171}$$

must be realized. Therefore if m_s is too large the pairing does not occur. It is easy to see that the transition must be first order. In fact if it were second order the gap Δ should vanish at the transition, but in order this to happen we must have $\Delta > m_s^2/2\mu$. It should be noticed that the value of m_s appearing in this equations should be considered density dependent. For $m_s \approx 200 \div 300\ MeV$ and chemical potentials interesting for compact stellar objects (see later) the situation is shown in Fig. 20. We see that the gap should be larger than $40 \div 110\ MeV$ in order to get the CFL phase, whereas for smaller value we have the 2SC phase. These values of Δ are very close to the ones that one gets from the gap equation. Therefore we are not really able to judge if quark matter with the values of the parameters is in the CFL or in the 2SC phase. The situation is far more complicated, because differences in the Fermi momenta can be generated also from different chemical potentials (arising from the requirement of weak equilibrium, see later) and/or from the requirement of electrical and color neutrality as appropriated for compact stars. Furthermore, at the border of this transition a crystalline phase, the so-called LOFF phase, can be formed. We will discuss all these questions later on.

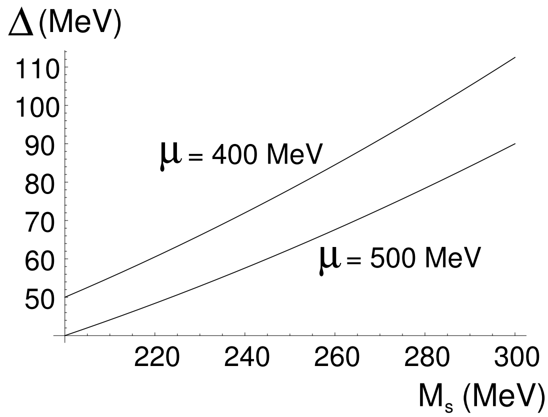

FIG. 20 *The minimal values of the gap in order to get pairing for two given values of μ vs. m_s.*

4. Single flavor and single color

According to the values of the parameters single flavor and/or single color condensation could arise. A complete list of possibilities has been studied in (Alford *et al.*, 2003) (for a review see (Bowers, 2003)). In general a single flavor condensate will occur in a state of angular momentum $J = 1$. In this case rotational invariance is not broken since the condensate is in an antisymmetric state of color $\mathbf{3}^*$ and and rotations and color transformations are locked together

$$\langle s^\alpha \sigma^\beta s^\gamma \rangle = \Delta \epsilon^{\alpha\beta\gamma}, \tag{5.172}$$

where σ^α is a spin matrix. This condensate breaks $SU(3) \otimes O(3)$ down to $O(3)$. However, it is a general statement that condensation in higher momentum states than $J = 0$ gives rise to smaller gaps, mainly due to a less efficient use of the Fermi surface. The typical gaps may range from 10 to 100 keV. We will not continue here this discussion.

In (Rajagopal and Wilczek, 2001) it is possible to find a discussion of many other possibilities as $N_f > 3$, the case of two colors (of interest because it can be discussed on the lattice) and the limit $N_f \to \infty$.

VI. EFFECTIVE LAGRANGIANS

In this Section we will derive the effective lagrangians relevant to the two phases CFL and 2SC. The approach will be the classical one, that is looking at the relevant degrees of freedom at low energy (with respect to the Fermi energy) and constructing the corresponding theory on the basis of the symmetries.

A. Effective lagrangian for the CFL phase

We will now derive the effective lagrangian for the light modes of the CFL phase. As we have seen such modes are the NG bosons. We will introduce the Goldstone fields as the phases of the condensates in the $(\bar{\mathbf{3}}, \bar{\mathbf{3}})$ channel (Casalbuoni and Gatto, 1999; Hong *et al.*, 1999) (remember also the discussion made in Section V.D)

$$X^i_\alpha \approx \epsilon^{ijk} \epsilon_{\alpha\beta\gamma} \langle \psi^j_{\beta L} \psi^k_{\gamma L} \rangle^*, \quad Y^i_\alpha \approx \epsilon^{ijk} \epsilon_{\alpha\beta\gamma} \langle \psi^j_{\beta R} \psi^k_{\gamma R} \rangle^*. \tag{6.1}$$

Since quarks belong to the representation $(\mathbf{3}, \mathbf{3})$ of $SU(3)_c \otimes SU(3)_{L(R)}$ they transform as ($g_c \in SU(3)_c$, $g_{L(R)} \in SU(3)_{L(R)}$)

$$\psi_L \to e^{i(\alpha+\beta)} g_c \psi_L g_L^T, \quad \psi_R \to e^{i(\alpha-\beta)} g_c \psi_R g_R^T, \quad e^{i\alpha} \in U(1)_B, \quad e^{i\beta} \in U(1)_A, \tag{6.2}$$

whereas the transformation properties of the fields X and Y under the total symmetry group $G = SU(3)_c \otimes SU(3)_L \otimes SU(3)_R \otimes U(1)_B \otimes U(1)_A$ are

$$X \to g_c X g_L^T e^{-2i(\alpha+\beta)}, \quad Y \to g_c Y g_R^T e^{-2i(\alpha-\beta)}. \tag{6.3}$$

The fields X and Y are $U(3)$ matrices and as such they describe $9+9=18$ fields. Eight of these fields are eaten up by the gauge bosons, producing eight massive gauge particles. Therefore we get the right number of Goldstone bosons, $10 = 18-10$. In this Section we will treat the field associated to the breaking of $U(1)_A$ as a true NG boson. However, remember that this is a massive particle with a light mass at very high density. These fields correspond to the breaking of the global symmetries in $G = SU(3)_L \otimes SU(3)_L \otimes U(1)_L \otimes U(1)_R$ (18 generators) to the symmetry group of the ground state $H = SU(3)_{c+L+R} \otimes Z_2 \otimes Z_2$ (8 generators). For the following it is convenient to separate the $U(1)$ factors in X and Y defining fields, $\hat{X}$ and $\hat{Y}$, belonging to $SU(3)$

$$X = \hat{X} e^{2i(\phi+\theta)}, \quad Y = \hat{Y} e^{2i(\phi-\theta)}, \quad \hat{X}, \hat{Y} \in SU(3). \tag{6.4}$$

The fields ϕ and θ can also be described through the determinants of X and Y

$$d_X = \det(X) = e^{6i(\phi+\theta)}, \quad d_Y = \det(Y) = e^{6i(\phi-\theta)}, \tag{6.5}$$

The transformation properties under G are

$$\hat{X} \to g_c \hat{X} g_L^T, \quad \hat{Y} \to g_c \hat{Y} g_R^T, \quad \phi \to \phi - \alpha, \quad \theta \to \theta - \beta. \tag{6.6}$$

The breaking of the global symmetry can be discussed in terms of gauge invariant fields given by d_X, d_Y and

$$\Sigma^i_j = \sum_\alpha (\hat{Y}^j_\alpha)^* \hat{X}^i_\alpha \to \Sigma = \hat{Y}^\dagger \hat{X}. \tag{6.7}$$

The Σ field describes the 8 Goldstone bosons corresponding to the breaking of the chiral symmetry $SU(3)_L \otimes SU(3)_R$, as it is made clear by the transformation properties of Σ^T, $\Sigma^T \to g_L \Sigma^T g_R^\dagger$. That is Σ^T transforms as the usual chiral field. The other two fields d_X and d_Y provide the remaining Goldstone bosons related to the breaking of the $U(1)$ factors.

In order to build up an invariant lagrangian, it is convenient to define the following currents

$$\begin{aligned} J^\mu_X &= \hat{X} D^\mu \hat{X}^\dagger = \hat{X}(\partial^\mu \hat{X}^\dagger + \hat{X}^\dagger g^\mu) = \hat{X} \partial^\mu \hat{X}^\dagger + g^\mu, \\ J^\mu_Y &= \hat{Y} D^\mu \hat{Y}^\dagger = \hat{Y}(\partial^\mu \hat{Y}^\dagger + \hat{Y}^\dagger g^\mu) = \hat{Y} \partial^\mu \hat{Y}^\dagger + g^\mu, \end{aligned} \tag{6.8}$$

with

$$g_\mu = i g_s g^a_\mu T^a \tag{6.9}$$

the gluon field and

$$T^a = \frac{\lambda_a}{2} \tag{6.10}$$

the $SU(3)_c$ generators. These currents have simple transformation properties under the full symmetry group G:

$$J^\mu_{X,Y} \to g_c J^\mu_{X,Y} g_c^\dagger. \tag{6.11}$$

The most general lagrangian, up to two derivative terms, invariant under G, the rotation group $O(3)$ (Lorentz invariance is broken by the chemical potential term) and the parity transformation, defined as:

$$P: \quad \hat{X} \leftrightarrow \hat{Y}, \quad \phi \to \phi, \quad \theta \to -\theta, \tag{6.12}$$

is (Casalbuoni and Gatto, 1999)

$$\begin{aligned} \mathcal{L} &= -\frac{F_T^2}{4} \mathrm{Tr}\left[(J^0_X - J^0_Y)^2)\right] - \alpha_T \frac{F_T^2}{4} \mathrm{Tr}\left[(J^0_X + J^0_Y)^2)\right] + \frac{1}{2}(\partial_0 \phi)^2 + \frac{1}{2}(\partial_0 \theta)^2 \\ &+ \frac{F_S^2}{4} \mathrm{Tr}\left[|\mathbf{J}_X - \mathbf{J}_Y|^2\right] + \alpha_S \frac{F_S^2}{4} \mathrm{Tr}\left[|\mathbf{J}_X + \mathbf{J}_Y|^2\right] - \frac{v_\phi^2}{2} |\boldsymbol{\nabla}\phi|^2 - \frac{v_\theta^2}{2} |\boldsymbol{\nabla}\theta|^2. \end{aligned}$$

or

$$\begin{aligned} \mathcal{L} &= -\frac{F_T^2}{4} \mathrm{Tr}\left[\left(\hat{X}\partial_0 \hat{X}^\dagger - \hat{Y}\partial_0 \hat{Y}^\dagger)^2\right)\right] - \alpha_T \frac{F_T^2}{4} \mathrm{Tr}\left[\left(\hat{X}\partial_0 \hat{X}^\dagger + \hat{Y}\partial_0 \hat{Y}^\dagger + 2g_0)^2\right)\right] \\ &+ \frac{F_S^2}{4} \mathrm{Tr}\left[\left|\hat{X}\boldsymbol{\nabla}\hat{X}^\dagger - \hat{Y}\boldsymbol{\nabla}\hat{Y}^\dagger\right|^2\right] + \alpha_S \frac{F_S^2}{4} \mathrm{Tr}\left[\left|\hat{X}\boldsymbol{\nabla}\hat{X}^\dagger + \hat{Y}\boldsymbol{\nabla}\hat{Y}^\dagger + 2\mathbf{g}\right|^2\right] \\ &+ \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\partial_0\theta)^2 - \frac{v_\phi^2}{2}|\boldsymbol{\nabla}\phi|^2 - \frac{v_\theta^2}{2}|\boldsymbol{\nabla}\theta|^2. \end{aligned} \tag{6.13}$$

Using $SU(3)_c$ color gauge invariance we can choose $\hat{X} = \hat{Y}^\dagger$, making 8 of the Goldstone bosons disappear and giving mass to the gluons. The properly normalized Goldstone bosons, Π^a, are given in this gauge by

$$\hat{X} = \hat{Y}^\dagger = e^{i\Pi^a T^a/F_T}, \tag{6.14}$$

and expanding Eq. (6.13) at the lowest order in the fields we get

$$\mathcal{L} \approx \frac{1}{2}(\partial_0 \Pi^a)^2 + \frac{1}{2}(\partial_0 \phi)^2 + \frac{1}{2}(\partial_0 \theta)^2 - \frac{v^2}{2}|\boldsymbol{\nabla}\Pi^a|^2 - \frac{v_\phi^2}{2}|\boldsymbol{\nabla}\phi|^2 - \frac{v_\theta^2}{2}|\boldsymbol{\nabla}\theta|^2, \tag{6.15}$$

with

$$v = \frac{F_S}{F_T}. \tag{6.16}$$

The gluons g_0^a and g_i^a acquire Debye and Meissner masses given by

$$m_D^2 = \alpha_T g_s^2 F_T^2, \quad m_M^2 = \alpha_S g_s^2 F_S^2 = \alpha_S g_s^2 v^2 F_T^2. \tag{6.17}$$

It should be stressed that these are not the true rest masses of the gluons, since there is a large wave function renormalization effect making the gluon masses of order of the gap Δ, rather than μ (see later) (Casalbuoni *et al.*, 2001c,d). Since this description is supposed to be valid at low energies (we expect much below the gap Δ), we could also decouple the gluons solving their classical equations of motion neglecting the kinetic term. The result from Eq. (6.13) is

$$g_\mu = -\frac{1}{2}\left(\hat{X}\partial_\mu \hat{X}^\dagger + \hat{Y}\partial_\mu \hat{Y}^\dagger\right). \tag{6.18}$$

It is easy to show that substituting this expression in Eq. (6.13) one gets

$$\mathcal{L} = \frac{F_T^2}{4}\left(\mathrm{Tr}[\dot{\Sigma}\dot{\Sigma}^\dagger] - v^2 \mathrm{Tr}[\vec{\nabla}\Sigma \cdot \vec{\nabla}\Sigma^\dagger]\right) + \frac{1}{2}\left(\dot{\phi}^2 - v_\phi^2|\vec{\nabla}\phi|^2\right) + \frac{1}{2}\left(\dot{\theta}^2 - v_\phi^2|\vec{\nabla}\theta|^2\right). \tag{6.19}$$

Notice that the first term is nothing but the chiral lagrangian except for the breaking of the Lorentz invariance. This is a way of seeing the quark-hadron continuity, that is the continuity between the CFL and the hypernuclear matter phase in three flavor QCD discussed previously. Furthermore one has to identify the NG ϕ associated to the breaking of $U(1)_B$ with the meson H of the hypernuclear phase (Schafer and Wilczek, 1999a).

B. Effective lagrangian for the 2SC phase

In Section V.D.2 we have seen that the only light degrees of freedom in the 2SC phase are the u and d quarks of color 3 and the gluons belonging to the unbroken $SU(2)_c$. The fermions are easily described in terms of ungapped quasi-particles at the Fermi surface and we will not elaborate about these modes further. In this Section we will discuss only the massless gluons. An effective lagrangian describing the 5 would-be Goldstone bosons and their couplings to the gluons has been given in (Casalbuoni *et al.*, 2000). The effective lagrangian for the massless gluons has been given in (Rischke *et al.*, 2001) and it can be obtained simply by noticing that it should be gauge invariant. Therefore it depends only on the field strengths

$$E_i^a = F_{0i}^a, \quad B_i^a = \frac{1}{2}\epsilon_{ijk}F_{jk}^a, \tag{6.20}$$

where $F_{\mu\nu}^a$ is the usual non-abelian curvature. As we know, Lorentz invariance is broken, but rotations are good symmetries, therefore the most general effective lagrangian we can write at the lowest order in the derivatives of the gauge fields is

$$\mathcal{L}_{\rm eff} = \frac{1}{g^2}\sum_{a=1}^{3}\left(\frac{\epsilon}{2}\mathbf{E}^a \cdot \mathbf{E}^a - \frac{1}{2\lambda}\mathbf{B}^a \cdot \mathbf{B}^a\right). \tag{6.21}$$

The constants ϵ and λ have the meaning of the dielectric constant and of the magnetic permeability. The speed of the gluons turns out to be given by

$$v = \frac{1}{\sqrt{\epsilon\lambda}}. \tag{6.22}$$

We will see later how to evaluate these constants starting from the HDET but we can see what are the physical consequences of this modification of the usual relativistic lagrangian. The most interesting consequence has to do with the Coulomb law and the effective gauge coupling. In fact, as we shall see, at the lowest order in $1/\mu$ expansion, $\lambda = 1$. The Coulomb potential between two static charges get modified

$$V_{\rm Coul} = \frac{g^2}{\epsilon r} = \frac{g^2_{eff}}{r}, \quad g^2_{eff} = \frac{g^2}{\epsilon}. \tag{6.23}$$

Furthermore, the value of α_s^{eff}, reintroducing the velocity of light c, and keeping $\hbar = 1$ is

$$\alpha_s = \frac{g^2}{4\pi c} \to \alpha_s' = \frac{g^2_{eff}}{4\pi v} = \frac{g^2}{4\pi c\sqrt{\epsilon}}. \tag{6.24}$$

If, as it is the case, ϵ is much bigger than 1, the gluons move slowly in the superconducting medium and

$$\frac{\alpha_s'}{\alpha_s} = \frac{1}{\sqrt{\epsilon}} \ll 1. \tag{6.25}$$

Therefore the effect of the medium is to weaken the residual strong interaction. The same result can be obtained by performing the following transformation in time, fields and coupling:

$$x^{0'} = \frac{x^0}{\sqrt{\epsilon}}, \quad A_0^{a'} = \sqrt{\epsilon} A_0^a, \quad g' = \frac{g}{\epsilon^{1/4}}. \tag{6.26}$$

The effective lagrangian takes the usual form for a relativistic gauge theory in terms of the new quantities

$$\mathcal{L}_{\rm eff} = -\frac{1}{4g'^2} F^{a'}_{\mu\nu} F^{\mu\nu a'}, \tag{6.27}$$

with

$$F^{a'}_{\mu\nu} = \partial'_\mu A^{a'}_\nu - \partial'_\nu A^{a'}_\mu + f^{abc} A^{b'}_\mu A^{c'}_\nu. \tag{6.28}$$

The calculation shows (Casalbuoni *et al.*, 2002d; Rischke, 2000; Rischke *et al.*, 2001) that

$$\epsilon = 1 + \frac{g^2\mu^2}{18\pi^2\Delta^2} \approx \frac{g^2\mu^2}{18\pi^2\Delta^2}, \tag{6.29}$$

since typically $\Delta \ll \mu$. We recall also that at asymptotic values of μ the calculations from QCD show that (see Section V.C)

$$\Delta = c\mu g^{-5} e^{-3\pi^2/\sqrt{2}g}. \tag{6.30}$$

Using Eqs. (6.29) and (6.24) we obtain

$$\alpha_s' = \frac{3}{2\sqrt{2}} \frac{g\Delta}{\mu}. \tag{6.31}$$

This is the way in which the coupling gets defined at the matching scale, that is at Δ. Since $SU(2)$ is an asymptotically free gauge theory, going at lower energies makes the coupling to increase. The coupling gets of order unity at a scale Λ'_{QCD}. Since the coupling at the matching scale is rather small it takes a long way before it gets of order one. As a consequence Λ'_{QCD} is expected to be small. From one-loop beta function we get

$$\Lambda'_{QCD} \approx \Delta e^{-2\pi/\beta_0\alpha_s'} \approx \Delta \exp\left(-\frac{2\sqrt{2}\pi}{11}\frac{\mu}{g\Delta}\right), \tag{6.32}$$

where β_0 is the first coefficient of the beta function. In SU(2) we have $\beta_0 = 22/3$. Notice that it is very difficult to give a good estimate of Λ'_{QCD} since it depends crucially on the value of c which is very poorly known. In fact different approximations give different values of c. In (Pisarski and Rischke, 2000a; Schafer and Wilczek, 1999c) it has been found

$$c = 512\pi^4, \tag{6.33}$$

whereas in (Brown *et al.*, 2000c; Wang and Rischke, 2002)

$$c = 512\pi^4 \exp\left(-\frac{4+\pi^2}{8}\right). \tag{6.34}$$

Using $\Lambda_{QCD} = 200\ MeV$ to get g and $\mu = 600\ MeV$ one finds $\Lambda'_{QCD} = 10\ MeV$ in the first case and $\Lambda'_{QCD} = 0.3\ keV$ in the second case. Although a precise determination of Λ'_{QCD} is lacking, it is quite clear that

$$\Lambda'_{QCD} \ll \Lambda_{QCD}. \tag{6.35}$$

Notice also that increasing the density, that is μ, Λ'_{QCD} decreases exponentially. Therefore the confinement radius $1/\Lambda'_{QCD}$ grows exponentially with μ. This means that looking at physics at some large, but fixed, distance and increasing the density, there is a crossover density when the color degrees of freedom become deconfined.

VII. NGB AND THEIR PARAMETERS

In this Section we will evaluate the parameters appearing in the effective lagrangian for the NG bosons in the CFL phase. Furthermore we will determine the properties of the gluons in the 2SC phases. A nice way of organizing this calculation is to make use of the HDET which holds for residual momenta such that $\Delta \ll \ell \ll \delta$ and to match, at the scale Δ, with the effective lagrangian supposed to hold for momenta $\ell \ll \Delta$. This calculation can be extended also to massive gluons both in CFL and in 2SC. However the expansion in momenta is not justified here, since the physical masses turn out to be of order Δ. On the other hand by a numerical evaluation we can show that the error is not more than 30%, and therefore this approach gives sensible results also in this case. The way of evaluating the propagation parameters for NG bosons and for gluons is rather simple, one needs to evaluate the self-energy in both cases. The self-energy for the gluons is not a problem since we know the couplings of the gluons to fermions. However we need the explicit expression of the interaction between NG bosons and quarks. This interaction can be easily obtained since we know the coupling of the fermions to the gap and the phases of the gap are the NG boson fields. Therefore we need only to generalize the Majorana mass terms (the coupling with the gap) to an interaction with the NG bosons such to respect the total symmetry of the theory. We will illustrate the way of performing this calculation in the specific example of the NG boson corresponding to the breaking of $U(1)_B$ and we will illustrate the results for the NG bosons. Before doing that we will derive first the lagrangians for the HDET in the CFL and in the 2SC phases. We have also to mention that the parameters of the NG bosons have been derived by many authors (Beane *et al.*, 2000; Hong *et al.*, 2000a; Manuel and Tytgat, 2000, 2001; Rho *et al.*, 2000a,b; Son and Stephanov, 2000a,b), although using different approaches from the one considered here (Casalbuoni *et al.*, 2002d, 2001c,d).

A. HDET for the CFL phase

In the CFL phase the symmetry breaking is induced by the condensates

$$\langle \psi^{L\,T}_{\alpha i} C \psi^L_{\beta j}\rangle = -\langle \psi^{R\,T}_{\alpha i} C \psi^R_{\beta j}\rangle \approx \Delta\, \epsilon_{\alpha\beta I}\epsilon_{ijI}\ , \tag{7.1}$$

where $\psi^{L,\,R}$ are Weyl fermions and $C = i\sigma_2$. The corresponding Majorana mass term is given by ($\psi \equiv \psi_L$):

$$\frac{\Delta}{2}\sum_{I=1}^{3} \psi_-^T C \epsilon_I \psi_+ \epsilon_I\ + (L\to R) + h.c., \tag{7.2}$$

with the 3×3 matrices ϵ_I defined as in Eq. (5.133)

$$(\epsilon_I)_{ab} = \epsilon_{Iab}\ . \tag{7.3}$$

Expanding the quark fields in the basis (5.128)

$$\psi_\pm = \frac{1}{\sqrt{2}}\sum_{A=1}^{9}\lambda_A \psi_\pm^A\ . \tag{7.4}$$

and following Section V.B we get the effective lagrangian at the Fermi surface

$$\mathcal{L}_D = \int \frac{d\mathbf{v}}{4\pi}\sum_{A,B=1}^{9}\chi^{A\dagger}\begin{pmatrix} \frac{i}{2}\mathrm{Tr}[\lambda_A\, V\cdot D\,\lambda_B] & -\Delta_{AB} \\ -\Delta_{AB} & \frac{i}{2}\mathrm{Tr}[\lambda_A\, \tilde V\cdot D^*\,\lambda_B]\end{pmatrix}\chi^B + (L\to R), \tag{7.5}$$

where

$$\Delta_{AB} = \frac{1}{2}\sum_{I=1}^{3}\Delta\,\mathrm{Tr}[\epsilon_I\lambda_A^T\epsilon_I\lambda_B], \tag{7.6}$$

and we recall from Section V.B that

$$\Delta_{AB} \;=\; \Delta_A\delta_{AB}, \tag{7.7}$$

with

$$\Delta_1 = \cdots = \Delta_8 = \Delta, \tag{7.8}$$

$$\Delta_9 = -2\Delta. \tag{7.9}$$

The CFL free fermionic lagrangian assumes therefore the form:

$$\mathcal{L}_D = \int\frac{d\mathbf{v}}{4\pi}\sum_{A=1}^{9}\chi^{A\dagger}\begin{pmatrix} iV\cdot\partial & -\Delta_A \\ -\Delta_A & i\tilde{V}\cdot\partial\end{pmatrix}\chi^A \;+\; (L\to R)\;. \tag{7.10}$$

From this equation one can immediately obtain the free fermion propagator in momentum space

$$S_{AB}(p) = \frac{\delta_{AB}}{V\cdot\ell\,\tilde{V}\cdot\ell - \Delta_A^2}\begin{pmatrix}\tilde{V}\cdot\ell & \Delta_A \\ \Delta_A & V\cdot\ell\end{pmatrix}. \tag{7.11}$$

We recall that the NG bosons have been described in terms of the fields X and Y of Eq. (6.1). Since we want to couple the NG bosons with fermions in a G invariant way let us look at the transformation properties of the gap term. We have

$$\sum_{I=1}^{3}\mathrm{Tr}[\psi_L^T C\epsilon_I\psi_L\epsilon_I] \to \sum_{I=1}^{3}\mathrm{Tr}[g_L\psi_L^T g_c^T\epsilon_I g_c\psi_L g_L^T\epsilon_I]. \tag{7.12}$$

Using the following property, holding for any unitary 3×3 matrix , g

$$g^T\epsilon_I g = \sum_{J=1}^{3}\epsilon_J g_{JI}^{\dagger}\det[g], \tag{7.13}$$

which follows from

$$\epsilon_{ijk}g_{ii'}g_{jj'}g_{kk'} = \epsilon_{i'j'k'}\det[g], \tag{7.14}$$

we get

$$\sum_{I=1}^{3}\mathrm{Tr}[\psi_L^T C\epsilon_I\psi_L\epsilon_I] \to \sum_{I,J,K=1}^{3}(g_c)_{JI}^{\dagger}(g_L)_{KI}^{\dagger}\mathrm{Tr}[\psi_L^T\epsilon_J\psi_L\epsilon_K]. \tag{7.15}$$

This expression shows not only that the gap term is invariant under the locked transformations $g_L^* = g_c$, but also that it could be made invariant modifying the transformation properties of the fermions by changing g_L^T to $g_c^{\dagger}$. This is simply done by using the matrices X and Y respectively for the left- and right- handed quarks

$$(\psi_L X^{\dagger}) \to g_c(\psi_L X^{\dagger})g_c^{\dagger}, \qquad (\psi_R Y^{\dagger}) \to g_c(\psi_R Y^{\dagger})g_c^{\dagger}. \tag{7.16}$$

Therefore, the coupling of the left-handed Weyl spinors ψ's to the octet of NG boson fields is:

$$\frac{\Delta}{2}\sum_{I=1,3}Tr[(\psi_- X^{\dagger})^T C\epsilon_I(\psi_+ X^{\dagger})\epsilon_I] + \;(L\to R) + h.c., \tag{7.17}$$

Both X and Y have v.e.v. given by

$$\langle X\rangle = \langle Y\rangle = 1 \tag{7.18}$$

and we shall use the gauge

$$X = Y^\dagger \ . \tag{7.19}$$

The lagrangian giving the the coupling of the quarks to the external fields can be obtained from (7.5) and is given by

$$\mathcal{L}_D = \int \frac{d\mathbf{v}}{4\pi} \sum_{A,B=1}^{9} \chi^{A\dagger} \begin{pmatrix} i\,V\cdot\partial\delta_{AB} & -\Xi^*_{BA} \\ -\Xi_{AB} & i\,\tilde V\cdot\partial\delta_{AB} \end{pmatrix} \chi^B + (L\to R), \tag{7.20}$$

where

$$\begin{aligned}\Xi_{AB} &= \sum_{I=1}^{3} \frac{1}{2}\Delta\, \mathrm{Tr}\left[\epsilon_I \left(\lambda_A X^\dagger\right)^T \epsilon_I \left(\lambda_B X^\dagger\right)\right] = \\ &= \sum_{I=1}^{3} \frac{1}{2}\Delta \left(\mathrm{Tr}[\lambda_A X^\dagger \lambda_B X^\dagger] - Tr[\lambda_A X^\dagger]\,\mathrm{Tr}[\lambda_B X^\dagger]\right) \ .\end{aligned} \tag{7.21}$$

One can now expand X in terms of the NGB fields

$$X = \exp i \left(\frac{\lambda_a \Pi^a}{2F}\right), \qquad a = 1,\cdots,8 \ , \tag{7.22}$$

and obtain the 3-point $\chi\chi\Pi$, and the 4-point $\chi\chi\Pi\Pi$ couplings. The result is, for the 3-point coupling

$$\begin{aligned}\mathcal{L}_{\chi\chi\Pi} &= -i \int \frac{d\mathbf{v}}{4\pi}\frac{\Delta}{F}\Big\{ \sum_{a=1}^{8} \frac{\Pi^a}{\sqrt{6}}\left[\chi^{9\dagger}\Gamma_0\,\chi^a + \chi^{a\dagger}\Gamma_0\,\chi^9\right] - \\ &\quad - \sum_{a,b,c=1}^{8} d_{abc}\chi^{a\dagger}\Gamma_0\chi^b\Pi^c \Big\} \ .\end{aligned} \tag{7.23}$$

and for the 4-point coupling

$$\begin{aligned}\mathcal{L}_{\chi\chi\Pi\Pi} &= \int \frac{d\mathbf{v}}{4\pi}\Big\{ \frac{4}{3}\sum_{a=1}^{8} \frac{\Delta}{8F^2}\chi^{9\dagger}\Gamma_1\,\chi^9\ \Pi^a\Pi^a \\ &\quad +3\sqrt{\frac{2}{3}} \sum_{a,b,c=1}^{8} \left(\frac{\Delta}{8F^2} d_{abc}\chi^{c\dagger}\Gamma_1\,\chi^9\ \Pi^a\Pi^b + \mathrm{h.c.}\right) \\ &\quad + \sum_{a,b,c,d=1}^{8} \frac{\Delta}{8F^2} h_{abcd}\chi^{c\dagger}\Gamma_0\,\chi^d\Pi^a\Pi^b \Big\} \ ,\end{aligned} \tag{7.24}$$

where we have defined

$$h_{abcd} = 2\sum_{p=1}^{8} \left(g_{cap}g_{dbp} + d_{cdp}d_{abp}\right) - \frac{8}{3}\delta_{ac}\delta_{db} + \frac{4}{3}\delta_{cd}\delta_{ab}, \tag{7.25}$$

with

$$g_{abc} = d_{abc} + i f_{abc} \tag{7.26}$$

and

$$\Gamma_0 = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \ , \qquad \Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{7.27}$$

In the CFL case we have also the NG bosons associated to the breaking of the $U(1)$ factors. Let us consider the following fields

$$U = e^{i\sigma/f_\sigma}, \quad V = e^{i\tau/f_\tau}, \tag{7.28}$$

with

$$\sigma = f_\sigma \phi, \qquad \tau = f_\tau \theta, \tag{7.29}$$

where ϕ and θ are the dimensionless $U(1)$ fields transforming according to Eq. (6.6)[7]. Clearly under $U(1)_B$ and $U(1)_A$ groups we have

$$U \to e^{-i\alpha} U, \quad V \to e^{-i\beta} V, \tag{7.30}$$

whereas

$$\psi_L \to e^{i(\alpha+\beta)} \psi_L, \quad \psi_R \to e^{i(\alpha-\beta)} \psi_R. \tag{7.31}$$

We can make invariant couplings with the fermions simply by taking the combinations

$$UV\psi_L, \quad UV^\dagger \psi_R. \tag{7.32}$$

As a result we get full invariant couplings to the NG bosons modifying the quantity Ξ_{AB} appearing in Eq. (7.20) as follows:

$$\Xi_{AB} \to \Xi_{AB} U^2 V^2. \tag{7.33}$$

B. HDET for the 2SC phase

For the two flavour case, which encompasses both the 2SC model and the existing calculation in the LOFF phase (see later) we follow a similar approach.

The symmetry breaking is induced by the condensates

$$\langle \psi^{L\,T}_{\alpha i} C \psi^L_{\beta j} \rangle = -\langle \psi^{R\,T}_{\alpha i} C \psi^R_{\beta j} \rangle \approx \Delta\, \epsilon_{\alpha\beta 3} \epsilon_{ij3} \ , \tag{7.34}$$

corresponding to the invariant coupling ($\psi \equiv \psi^L$):

$$\frac{\Delta}{2}\, \psi_-^T C \epsilon\, \psi_+ \epsilon \ - (L \to R) + \text{h.c.} \ , \tag{7.35}$$

where

$$\epsilon = i\sigma_2 \ . \tag{7.36}$$

As in the previous Section we use a different basis for the fermion fields by writing

$$\begin{aligned} \psi_{+,\alpha i} &= \sum_{A=0}^{3} \frac{(\sigma_A)_{\alpha i}}{\sqrt{2}} \psi_+^A \qquad (i,\, \alpha = 1,\, 2) \\ \psi_{+,31} &= \psi_+^4 \\ \psi_{+,32} &= \psi_+^5 \ , \end{aligned} \tag{7.37}$$

where σ_A are the Pauli matrices for $A = 1, 2, 3$ and $\sigma_0 = 1$.

A different, but also convenient notation for the fields $\psi_{+,\,\alpha i}$, makes use of the following combination of λ matrices, as follows

$$\psi_{+,\alpha i} = \sum_{A=0}^{5} \frac{(\tilde\lambda_A)_{\alpha i}}{\sqrt{2}} \psi_+^A \ . \tag{7.38}$$

The $\tilde\lambda_A$ matrices are defined in terms of the usual λ matrices as follows:

$$\tilde\lambda_0 = \frac{1}{\sqrt{3}}\lambda_8 + \sqrt{\frac{2}{3}}\lambda_0, \quad \tilde\lambda_A = \lambda_A \quad (A = 1, 2, 3), \quad \tilde\lambda_4 = \frac{\lambda_{4-i5}}{\sqrt{2}}, \quad \tilde\lambda_5 = \frac{\lambda_{6-i7}}{\sqrt{2}}. \tag{7.39}$$

---

[7] The couplings f_σ and f_τ have dimension 1 in units of mass.

Proceeding as before the 2SC fermionic lagrangian assumes the form:

$$\mathcal{L}_D = \int \frac{d\mathbf{v}}{4\pi} \sum_{A,B=0}^{5} \chi^{A\dagger} \begin{pmatrix} \frac{i}{2}\mathrm{Tr}[\tilde\lambda_A\, V\cdot D\,\tilde\lambda_B] & -\Delta_{AB} \\ -\Delta_{AB} & \frac{i}{2}\mathrm{Tr}[\tilde\lambda_A\, \tilde V\cdot D^*\,\tilde\lambda_B] \end{pmatrix} \chi^B + (L \to R)\ . \tag{7.40}$$

Here

$$\begin{aligned} \Delta_{AB} &= \frac{\Delta}{2}\,\mathrm{Tr}[\,\epsilon\sigma_A^T\epsilon\sigma_B] \qquad && (A,B=0,...3), \\ \Delta_{AB} &= 0 && (A,B=4,5)\ . \end{aligned} \tag{7.41}$$

We now use the identity (g any 2×2 matrix), analogous to (5.134):

$$\epsilon g^T\epsilon \,=\, g \,-\, \mathrm{Tr}[g]\ ; \tag{7.42}$$

and we obtain

$$\Delta_{AB} \,=\, \Delta_A\delta_{AB} \tag{7.43}$$

where Δ_A is defined as follows:

$$\Delta_A = (-\,\Delta,\, +\Delta,\, +\Delta,\, +\Delta,\, 0,\, 0)\ . \tag{7.44}$$

Therefore the effective lagrangian for free quarks in the 2SC model can be written as follows

$$\mathcal{L}_D = \int \frac{d\mathbf{v}}{4\pi} \sum_{A=0}^{5} \chi^{A\dagger} \begin{pmatrix} iV\cdot\partial & -\Delta_A \\ -\Delta_A & i\tilde V\cdot\partial \end{pmatrix} \chi^A \;+\; (L\to R)\ . \tag{7.45}$$

From this equation one can immediately obtain the free fermion propagator that in momentum space is still given by (7.11), with the Δ_A given by (7.44).

C. Gradient expansion for the $U(1)$ NGB in the CFL model and in the 2SC model

In order to illustrate the procedure of evaluating the self-energy of the NG bosons we will consider here the case of the NG boson associated with the breaking of $U(1)_B$. In particular, we will get the first terms in the effective action for the NG bosons by performing an expansion in momenta, $\ell \ll \Delta$. An expansion of this kind is also called gradient expansion (see (Eguchi, 1976) for a very clear introduction).

To start with we consider the $U(1)_B$ NG boson within the CFL model. Therefore we can put to zero all the NG fields in the HDET lagrangian except for the field $U = \exp i\sigma/f_\sigma$. Correspondingly the quark lagrangian of Eq. (7.20)with the modification (7.33) becomes

$$\mathcal{L}_D = \int \frac{d\mathbf{v}}{4\pi} \sum_{A=1}^{9} \chi^{A\dagger} \begin{pmatrix} iV\cdot\partial & -\Delta_A\,U^{\dagger\,2} \\ -\Delta_A\,U^2 & i\tilde V\cdot\partial \end{pmatrix} \chi^A \;+\; (L\to R)\ . \tag{7.46}$$

At the lowest order in the field σ we have

$$\mathcal{L}_\sigma \approx \int \frac{d\mathbf{v}}{4\pi} \sum_{A=1}^{9} \chi^{A\dagger}\Delta_A \begin{pmatrix} 0 & \dfrac{2i\sigma}{f_\sigma} + \dfrac{2\sigma^2}{f_\sigma^2} \\ -\dfrac{2i\sigma}{f_\sigma} + \dfrac{2\sigma^2}{f_\sigma^2} & \end{pmatrix} \chi^A \;+\; (L\to R)\ , \tag{7.47}$$

which contains the couplings $\sigma\chi\chi$ and $\sigma\sigma\chi\chi$. Notice that σ does not propagate at tree level, however a non trivial kinetic term is generated by quantum corrections. To show this we consider the generating functional where we take only left-handed fields for simplicity. Also in this case we will make use of the replica trick to take into account the fact that χ and $\chi^\dagger$ are not independent variables. Therefore we will have to take the square root of the fermion determinant. In (Casalbuoni *et al.*, 2001c,d) this was taken into account by dividing by a factor 2 the sum over the velocities appearing in the fermionic loops

$$\int \frac{d\mathbf{v}}{4\pi} \to \int \frac{d\mathbf{v}}{8\pi}. \tag{7.48}$$

The generating functional is

$$Z[\sigma] \;=\; \int \mathcal{D}\chi\mathcal{D}\chi^\dagger \exp\left\{ i\int \chi^\dagger A \chi \right\} \tag{7.49}$$

where we have introduced (omit the indices of the fields for simplicity)

$$A = S^{-1} + \frac{2i\sigma\Delta}{f_\sigma}\,\Gamma_0 \,+\, \frac{2\sigma^2\Delta}{f_\sigma^2}\,\Gamma_1 \tag{7.50}$$

and

$$\Gamma_0 = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\,, \qquad \Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,. \tag{7.51}$$

with S^{-1} the free propagator. Performing the integration over the Fermi fields we get

$$Z[\sigma] = (\det[A])^{1/2} = e^{\frac{1}{2}\mathrm{Tr}[\log A]}. \tag{7.52}$$

Therefore

$$S_{eff}(\sigma) = -\frac{i}{2}\mathrm{Tr}[\log A]. \tag{7.53}$$

Evaluating the trace we get

$$\begin{aligned} -\,i\,Tr\log A \;&=\; -i\,Tr\log S^{-1}\left(1 \,+\, S\,\frac{2i\sigma\Delta}{f_\sigma}\Gamma_0 \,+\, S\,\frac{2\sigma^2\Delta}{f_\sigma^2}\Gamma_1\right) \\ &= -i\,Tr\log S^{-1} - i\sum_{n=1}^{\infty}\frac{(-1)^{2n-1}}{n}\left(iS\,\frac{2i\sigma\Delta}{f_\sigma}i\Gamma_0 \,+\, iS\frac{2\,\sigma^2\Delta}{f_\sigma^2}i\Gamma_1\right)^n \,. \end{aligned} \tag{7.54}$$

This is a loop expansion. At the lowest order it produces the effective action

$$\begin{aligned} \mathcal{S}_{eff} \;&=\; \frac{i}{4}\,Tr\int dxdy\,\left[\frac{i\,S(y,x)2i\sigma(x)\Delta}{f_\sigma}\,i\,\Gamma_0\,\frac{i\,S(x,y)2i\sigma(y)\Delta}{f_\sigma}\,i\,\Gamma_0\right] \\ &+\; \frac{i}{2}\,Tr\int dx\,\left[\frac{i\,S(x,x)\,2\,\Delta\,\sigma^2(x)}{f_\sigma^2}\,i\,\Gamma_1\right]\,. \end{aligned} \tag{7.55}$$

The two terms correspond to the diagrams in Fig. 21, i.e. the self-energy, Fig. 21 a), and the tadpole, Fig. 21 b). They can be computed by the following set of Feynman rules to provide $S_{eff}(\sigma)$ In momentum space the Feynman rules are as follows:

1. For each fermionic internal line with momentum p, associate the propagator

$$iS_{AB}(p) \;=\; i\delta_{AB}S(p) \;=\; \frac{i\,\delta_{AB}}{V\cdot\ell\,\tilde V\cdot\ell - \Delta_A^2 + i\epsilon}\begin{pmatrix} \tilde V\cdot\ell & \Delta_A \\ \Delta_A & V\cdot\ell \end{pmatrix}\,; \tag{7.56}$$

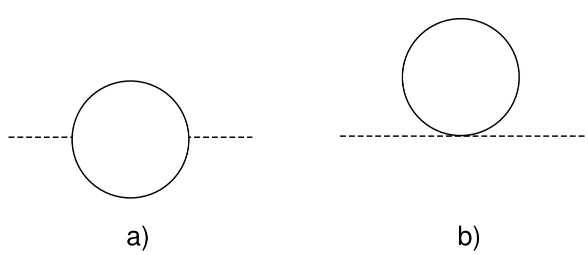

FIG. 21 *One-loop diagrams. External lines represent the NG boson field σ. Full lines are fermion propagators.*

2. Each vertex introduces a term $i\mathcal{L}_{int}$ that can be obtained from the effective lagrangian; for example the σ couplings to quarks can be derived from (7.47);

3. For each internal momentum not constrained by the momentum conservation perform the integration

$$\frac{4\pi\mu^2}{(2\pi)^4}\int d^2\ell \;=\; \frac{\mu^2}{4\pi^3}\int_{-\delta}^{+\delta} d\ell_\parallel \int_{-\infty}^{+\infty} d\ell_0\ ; \tag{7.57}$$

4. A factor (-1) and a factor of 2 for each fermion loop to take into account the spin $(L+R)$. Also a factor $1/2$ is necessary due to the replica trick. This $1/2$ will be associated to the sum over the velocities

5. A statistical factor arising from the Wick theorem if needed.

The result of the calculation of the effective lagrangian in momentum space is as follows:

$$\begin{aligned} i\mathcal{L}_I &= -2\frac{1}{2}\frac{\mu^2}{4\pi^3}\frac{1}{2}\int\frac{d\mathbf{v}}{4\pi}\sum_{A,B}\int d^2\ell \\ &\times\ Tr\left[iS_{AB}(\ell+p)\frac{2i\Delta_B\sigma}{f_\sigma}i\Gamma_0 iS_{BA}(\ell)\frac{2i\Delta_A\sigma}{f_\sigma}i\Gamma_0\right]\ , \\ i\,\mathcal{L}_{II} &= -2\,\frac{\mu^2}{4\pi^3}\frac{1}{2}\int\frac{d\mathbf{v}}{4\pi}\sum_{A,B}\int d^2\ell\, Tr\left[iS_{AB}(\ell)\frac{2\,\Delta_B\sigma^2}{f_\sigma^2}\,i\,\Gamma_1\right], \end{aligned} \tag{7.58}$$

corresponding to the two diagrams of Figs. 21 a) and 21 b) respectively. After some computation one has

$$\begin{aligned} i\,\mathcal{L}_{eff}(p) \;=\; i\,\mathcal{L}_I(p) \;+\; i\,\mathcal{L}_{II}(p) \;=\; -\frac{1}{2}\int\frac{d\mathbf{v}}{4\pi}\sum_A\frac{\mu^2\Delta_A^2}{\pi^3 f_\sigma^2} \\ \times\ \int d^2\ell\,\Big[\frac{\tilde{V}\cdot(\ell+p)\sigma V\cdot\ell\,\sigma + V\cdot(\ell+p)\sigma\tilde{V}\cdot\ell\sigma - 2\Delta_A^2\sigma^2}{D_A(\ell+p)D_A(\ell)} \;-\; \frac{2\sigma^2}{D_A(\ell)}\Big], \end{aligned} \tag{7.59}$$

where we have defined

$$D_A(p) \;=\; V\cdot p\,\tilde{V}\cdot p - \Delta_A^2 + i\epsilon\ . \tag{7.60}$$

One can immediately notice that

$$\mathcal{L}_I(p=0) \;+\; \mathcal{L}_{II}(p=0) \;=\; 0\ . \tag{7.61}$$

This result implies that the scalar σ particle has no mass, in agreement with Goldstone's theorem. To get the effective lagrangian in the CFL model at the lowest order in the σ momentum we expand the function in p ($|p| \ll |\Delta$) to get, in momentum space

$$i\,\mathcal{L}_{eff}(p) \;=\; -\frac{1}{2}\int\frac{d\mathbf{v}}{4\pi}\sum_A\frac{2\mu^2\Delta_A^4}{\pi^3 f_\sigma^2}(V\cdot p)\sigma(\tilde{V}\cdot p)\sigma\ I_2\ , \tag{7.62}$$

where we have defined

$$I_2 = \int\frac{d^2\ell}{D_A^3(\ell)}. \tag{7.63}$$

This and other integrals can be found in Appendix B. In getting this result we have used also the fact

$$\int\frac{(V\cdot\ell)^2}{D_A(\ell)^3} = \int\frac{(\tilde{V}\cdot\ell)^2}{D_A(\ell)^3} = 0. \tag{7.64}$$

In fact, since the integral is convergent we can send the cutoff δ to ∞ and to use the Lorentz invariance in 2 dimensions to prove that these integrals are proportional to $V^2 = \tilde{V}^2 = 0$. We have

$$I_2 = -\frac{i\,\pi}{2\Delta_A^4}\ . \tag{7.65}$$

Therefore we get, in configuration space,

$$\mathcal{L}_{eff}(x) \;=\; \frac{9\mu^2}{\pi^2 f_\sigma^2}\frac{1}{2}\int \frac{d\mathbf{v}}{4\pi}(V\cdot\partial\sigma)(\tilde{V}\cdot\partial\sigma)\ . \tag{7.66}$$

Since

$$\frac{1}{2}\int\frac{d\mathbf{v}}{4\pi}V^\mu\tilde{V}^\nu = \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -\frac{1}{3} & 0 & 0\\ 0 & 0 & -\frac{1}{3} & 0\\ 0 & 0 & 0 & -\frac{1}{3}\end{pmatrix}_{\mu\nu}\ , \tag{7.67}$$

we obtain

$$\mathcal{L}_{eff}(x) \;=\; \frac{9\mu^2}{\pi^2 f_\sigma^2}\frac{1}{2}\left((\partial_0\sigma)^2 \;-\; v_\sigma^2(\vec{\nabla}\sigma)^2\right)\ , \tag{7.68}$$

with

$$v_\sigma^2 \;=\; \frac{1}{3}\ . \tag{7.69}$$

This kinetic lagrangian has the canonical normalization factor provided

$$f_\sigma^2 = \frac{9\mu^2}{\pi^2} \qquad\qquad (\text{CFL})\ , \tag{7.70}$$

Therefore the effective lagrangian for the NGB σ particle is:

$$\mathcal{L}_{eff} = \frac{1}{2}\left((\partial_0\sigma)^2 \;-\; \frac{1}{3}(\vec{\nabla}\sigma)^2\right) = \frac{1}{2}f_\sigma^2\left(\dot{U}\dot{U}^\dagger - \frac{1}{3}\boldsymbol{\nabla}U\cdot\boldsymbol{\nabla}U^\dagger\right)\ , \tag{7.71}$$

We note that the value of the velocity (7.69) is a consequence of the average over the Fermi velocities and reflects the number of the space dimensions, i.e. 3. Therefore it is universal and we expect the same value in all the calculations of this type.

In the case of the 2SC model there is no σ field since the baryon number is not broken. However, neglecting the mass of the τ field one can perform the same kind of calculation by using the invariant coupling derived for the CFL case. The final result differs from the CFL case only in the coefficient in front of (7.70) which reflects the number of color-flavor gapped degrees of freedom, 9 in the CFL case and 4 in the 2SC case; therefore one has

$$f_\tau^2 = \frac{4\mu^2}{\pi^2} \qquad\qquad (\text{2SC})\ , \tag{7.72}$$

whereas the result (7.69), being universal, holds also in this case. The NGB effective lagrangian is still of the form (7.71). The NGB boson is, in this case, only a would-be NGB because the axial $U(1)$ is explicitly broken, though this breaking is expected to be small at high μ since the instanton density vanishes for increasing μ. (Son *et al.*, 2001).

D. The parameters of the NG bosons of the CFL phase

Using the Feynman rules given above and the interaction lagrangians (7.23) and (7.24) we get the effective lagrangian as follows:

$$\mathcal{L}_{eff}^{kin} = \frac{\mu^2(21-8\ln 2)}{36\pi^2 F^2}\,\frac{1}{2}\sum_{a=1}^{8}\left(\dot{\Pi}^a\dot{\Pi}^a - \frac{1}{3}|\vec{\nabla}\Pi_a|^2\right)\ . \tag{7.73}$$

Comparing with the effective lagrangian in Eq. (6.19) we see that

$$F_T^2 = F^2 = \frac{\mu^2(21-8\ln 2)}{36\pi^2},\qquad v^2 = \frac{1}{3}. \tag{7.74}$$

Therefore the pion satisfy the dispersion relation

$$E = \frac{1}{\sqrt{3}}|\mathbf{p}|. \tag{7.75}$$

We notice that the evaluation of F_T could be done just computing the coupling of the pions to their currents. To this end one has to compute the diagram of Fig. 22 (Casalbuoni *et al.*, 2001c,d). The result is

$$\langle 0|J^a_\mu|\Pi^b\rangle = iF\delta_{ab}\tilde{p}_\mu, \quad \tilde{p}_\mu = \left(p^0, \frac{1}{3}\mathbf{p}\right), \tag{7.76}$$

with F the same evaluated before. This result is particularly interesting since it shows how the conservation of the color currents is realized through the dispersion relation of the pions. In fact

$$p\cdot\tilde{p} = E^2 - \frac{1}{3}|\mathbf{p}|^2 = 0. \tag{7.77}$$

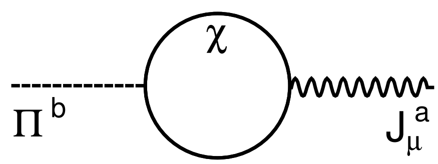

FIG. 22 *The one-loop contribution to the coupling of the NG bosons (dotted line) to their currents (wavy line). The fermions in the loop are represented by a solid line.*

E. The masses of the NG bosons in the CFL phase

Up to now we have considered massless quarks. In this Section we want to determine the influence of mass terms at the level of NG bosons. Remember that the QCD mass terms have the structure

$$\bar{\psi}_L M\psi_R \quad + \quad h.c. \tag{7.78}$$

where M is the quark mass matrix. In order to determine the structure of the NG boson masses it is convenient to think to M as a field (usually called a spurion field) transforming in such a way that the expression (7.78) is invariant. Recalling the transformation properties (6.2) of quark fields

$$\psi_L \to e^{i(\alpha+\beta)}g_c\psi_L g_L^T, \quad \psi_R \to e^{i(\alpha-\beta)}g_c\psi_R g_R^T, \quad e^{i\alpha}\in U(1)_B, \quad e^{i\beta}\in U(1)_A, \tag{7.79}$$

it follows that M transforms as

$$M \to e^{2i\beta}g_L M g_R^\dagger. \tag{7.80}$$

In this Section we will make use of the $U(3)$ fields

$$X \to g_c X g_L^T e^{-2i(\alpha+\beta)}, \quad Y \to g_c Y g_R^T e^{-2i(\alpha-\beta)}. \tag{7.81}$$

In this discussion the $U(1)_A$ symmetry will play an important role. We recall again that this symmetry is broken by the anomaly but this breaking goes to zero in the high density limit. If, in particular, we consider the subgroup $(Z_2)_A$ we see that NG boson mass terms can be only even in the matrix M. Linear terms in M may arise only from this breaking which is suppressed by $(\Lambda_{QCD}/\mu)^8$ (see discussion in Section V.D). In order to build up the most general mass term for the NG bosons let us introduce the gauge invariant $U(3)$ field

$$\tilde{\Sigma} = (Y^\dagger X)^T = e^{4i\theta}\Sigma^T, \tag{7.82}$$

where θ is the $U(1)_A$ field and Σ was defined in (6.7). We have

$$\tilde{\Sigma} \to e^{-4i\beta}g_L\tilde{\Sigma}g_R^\dagger. \tag{7.83}$$

We recall also the other two gauge invariant fields defined in (6.5)

$$d_X = \det(X) = e^{6i(\phi+\theta)}, \quad d_Y = \det(Y) = e^{6i(\phi-\theta)}, \tag{7.84}$$

and transforming as

$$d_X \to e^{-6i(\alpha+\beta)} d_X, \quad d_Y \to e^{-6i(\alpha-\beta)} d_Y. \tag{7.85}$$

We have also

$$\det(M) \to e^{6i\beta} \det(M), \quad \det(\tilde{\Sigma}) \to e^{-12i\beta} \det(\tilde{\Sigma}). \tag{7.86}$$

Notice that since d_X and d_Y are the only terms transforming under $U(1)_V$, invariance requires that the effective lagrangian should depend only on the combination

$$d_X d_{Y^\dagger} = \det(\tilde{\Sigma}). \tag{7.87}$$

Other quantities that can be considered are

$$\begin{aligned} M\Sigma^\dagger, &\to e^{6i\beta} g_L M \Sigma^\dagger g_L^\dagger, \quad M^\dagger \Sigma \to e^{-6i\beta} g_R M^\dagger \Sigma g_R^\dagger \\ M^{-1}\Sigma &\to e^{-6i\beta} g_R M \Sigma^\dagger g_R^\dagger, \quad M^{-1\dagger}\Sigma^\dagger \to e^{6i\beta} g_L M^{-1\dagger} \Sigma^\dagger g_L^\dagger. \end{aligned} \tag{7.88}$$

One can build terms transforming only with respecto to $U(1)_A$ by taking traces of these quantities. However, taking into account the Cayley identity for 3×3 matrices

$$A^3 - Tr(A)\, A^2 + \frac{1}{2}((Tr(A))^2 - Tr(A^2))\, A - \det(A) = 0, \tag{7.89}$$

we see that multiplying this expression by A^{-1} and taking the trace we can express the trace of terms in the second line of (7.88) as a combination of the trace of the terms in the first line, their second power and the determinants of M and $\tilde{\Sigma}$. In the same way we see that it is enough to consider only the first and the second power of the terms in the first line of (7.88). Summarizing, it is enough to take into considerations the following quantities

$$\begin{gathered} \det(\tilde{\Sigma}), \quad \det(M), \quad \det(M^\dagger), \\ \mathrm{Tr}[M\tilde{\Sigma}^\dagger], \quad \mathrm{Tr}[M^\dagger\tilde{\Sigma}], \quad \mathrm{Tr}[(M\tilde{\Sigma}^\dagger)^2], \quad \mathrm{Tr}[(M^\dagger\tilde{\Sigma})^2]. \end{gathered} \tag{7.90}$$

Therefore the most general invariant term involving the quark mass matrix M is of the form

$$\begin{gathered} I = (\det(\tilde{\Sigma}))^{a_1} (\det(M))^{a_2} (\det(M^\dagger))^{\bar{a}_2} \\ \times (\mathrm{Tr}[M\tilde{\Sigma}^\dagger])^{a_3} (\mathrm{Tr}[M^\dagger\tilde{\Sigma}])^{\bar{a}_3} (\mathrm{Tr}[(M\tilde{\Sigma}^\dagger)^2])^{a_4} (\mathrm{Tr}[(M^\dagger\tilde{\Sigma})^2])^{\bar{a}_4} \quad . \end{gathered} \tag{7.91}$$

Requiring analyticity all the exponents must be integers. The term I is invariant under the full group except for $U(1)_A$. Requiring also $U(1)_A$ invariance we get the equation

$$-2a_1 + (a_2 - \bar{a}_2) + (a_3 - \bar{a}_3) + 2(a_4 - \bar{a}_4) = 0. \tag{7.92}$$

If we ask I to be of a given order n in the mass matrix we have also the condition

$$3(a_2 + \bar{a}_2) + (a_3 + \bar{a}_3) + 2(a_4 + \bar{a}_4) = n. \tag{7.93}$$

If we subtract these two equations one by another we find

$$2a_1 + 4a_2 + 2\bar{a}_2 + 2a_3 + 4a_4 = n, \tag{7.94}$$

which implies that n must be even, as it should be by $(Z_2)_A$ invariance alone. This argument shows also that the it would be enough to require the invariance under the discrete axial group. If we now select $n = 2$ (the lowest power in the mass), Eq. (7.93) implies

$$a_2 = \bar{a}_2 = 0. \tag{7.95}$$

Therefore the only solutions to our conditions are

$$\begin{aligned} &a_1 = 1, \quad a_3 = 2, \quad \bar{a}_3 = 0, \quad a_4 = 0, \quad \bar{a}_4 = 0 \\ &a_1 = 1, \quad a_3 = 0, \quad \bar{a}_3 = 0, \quad a_4 = 1, \quad \bar{a}_4 = 0 \\ &a_1 = 0, \quad a_3 = 1, \quad \bar{a}_3 = 1, \quad a_4 = 0, \quad \bar{a}_4 = 0 \ . \end{aligned} \tag{7.96}$$

If the matrix M has no zero eigenvalues we can use the Cayley identity to write a linear combination of the first two solutions in the following form

$$\mathrm{Tr}[M^{-1}\tilde{\Sigma}]\det(M) = \frac{1}{2}\det(\tilde{\Sigma})\left\{\left(\mathrm{Tr}[M\tilde{\Sigma}^\dagger]\right)^2 - \mathrm{Tr}[(M\tilde{\Sigma}^\dagger)^2]\right\}. \tag{7.97}$$

In conclusion the most general invariant term is given by

$$\begin{aligned} \mathcal{L}_{masses} = &-c\left(\det(M)\mathrm{Tr}[M^{-1}\tilde{\Sigma}] + h.c.\right) - c'\left(\det(\tilde{\Sigma})\mathrm{Tr}[(M\tilde{\Sigma}^\dagger)^2] + h.c.\right) \\ &-c''\left(\mathrm{Tr}[M\tilde{\Sigma}^\dagger]\mathrm{Tr}[M^\dagger\tilde{\Sigma}]\right). \end{aligned} \tag{7.98}$$

The coefficients appearing in this expression can be evaluated by using the matching technique (Beane *et al.*, 2000; Hong *et al.*, 2000a; Rho *et al.*, 2000b; Son and Stephanov, 2000a,b) (see also the review paper (Schafer, 2003)). The idea is to evaluate the contribution of $\mathcal{L}_{masses}$ to the vacuum energy. One starts evaluating an effective four-fermi interaction due to a hard gluon exchange and performing a matching between QCD and HDET as illustrated in Fig. 23. The contribution from QCD arises from a double chirality violating process producing a contribution proportional to the square of the masses (Schafer, 2002).

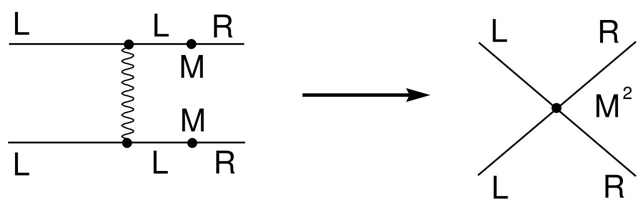

FIG. 23 *The effective four-fermi interaction produced by a chirality violating process in QCD. There are other contributions obtained by exchanging the points where the gluon line is attached with the mass insertions.*

Then one uses this effective interaction in HDET to evaluate the contribution to the vacuum energy to be matched against the contribution from $\mathcal{L}_{masses}$ obtained by putting $\tilde{\Sigma} = 1$ (the vacuum state). This is illustrated in Fig. 24.

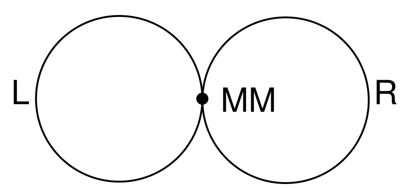

FIG. 24 *The effective four-fermi interaction of Fig. 23 is used to evaluate the contribution to the vacuum energy of the terms proportional to* M^2.

The result of this computation is

$$c = \frac{3\Delta^2}{2\pi^2}, \quad c' = c'' = 0. \tag{7.99}$$

There is a simple way of understanding why only the c contribution survives at the leading order. In fact, this is the only term which arises working at the second order in perturbation theory and approximating the product of Fermi fields with the corresponding NG fields X and Y. We find

$$(\bar{\psi}_L M\psi_R)^2 = (\psi^{i\dagger}_{L\alpha}M^j_i\psi^\alpha_{Rj})(\psi^{k\dagger}_{L\beta}M^l_k\psi^\beta_{Rl}) \approx \epsilon^{ikm}\epsilon_{\alpha\beta\gamma}X^\gamma_m M^j_i M^l_k\epsilon_{jlp}\epsilon^{\alpha\beta\delta}Y^{p*}_\delta$$
$$\approx \epsilon^{ikm}\epsilon_{jlp}M^j_i M^l_k\tilde{\Sigma}^p_m = \epsilon^{ikm}\epsilon_{jlp}M^j_i M^l_k M^a_m(M^{-1})^b_a\tilde{\Sigma}^p_b \approx \det(M)\mathrm{Tr}[M^{-1}\tilde{\Sigma}]. \tag{7.100}$$

There is also another interesting point about the quark masses. Starting from the complete QCD lagrangian

$$\mathcal{L}_{QCD} = \bar{\psi}(i\not{D} + \mu\gamma_0)\psi - \bar{\psi}_L M\psi_R - \bar{\psi}_R M^\dagger\psi_L - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu a} \tag{7.101}$$

and repeating what we did in Section V.B for deriving the HDET expansion we find that at the leading order the expression for the negative energy left-handed fields is

$$\psi_{-,L} = \frac{1}{2\mu}\left(-i\gamma_0\not{D}_\perp\psi_{+,L} + \gamma_0 M\psi_{+,R}\right). \tag{7.102}$$

Substituting inside the lagrangian one finds (we neglect here the condensate terms for simplicity)

$$\mathcal{L}_D = \psi^\dagger_{+,L}(iV\cdot D)\psi_{+,L} - \frac{1}{2\mu}\psi^\dagger_{+,L}\left[(\not{D}_\perp)^2 + MM^\dagger\right]\psi_{+,L} + (L\leftrightarrow R, M\leftrightarrow M^\dagger) + \cdots. \tag{7.103}$$

The new term that we have obtained is just what expected from the expansion of the kinetic energy of the quark. We see that the mass terms in the effective lagrangian behave as effective chemical potentials. Notice that in presence of a chemical potential a Dirac lagrangian has the form

$$\bar{\psi}i\gamma_\nu(\partial^\nu - i\mu g^{\nu 0})\psi, \tag{7.104}$$

therefore the chemical potential acts as the fourth component of an abelian gauge field. This implies that the lagrangian is invariant under the gauge transformation

$$\psi \to e^{i\alpha(t)}\psi, \qquad \mu \to \mu + \dot{\alpha}(t). \tag{7.105}$$

This argument implies that the HDET lagrangian has the invariance[8]

$$\psi_L \to L(t)\psi_L, \qquad \psi_R \to R(t)\psi_R, \tag{7.106}$$

where $L(t)$ and $R(t)$ are time dependent flavor transformation and with

$$X_L = \frac{1}{2\mu}MM^\dagger, \quad X_R = \frac{1}{2\mu}M^\dagger M \tag{7.107}$$

transforming as left- and right- handed gauge fields (Bedaque and Schafer, 2002). But then also the effective lagrangian for the filed Σ has to satisfy this symmetry, meaning that we should substitute the time derivative with the covariant derivative

$$\partial_0\Sigma \to \nabla_0\Sigma = \partial_0\Sigma + i\Sigma\left(\frac{MM^\dagger}{2\mu}\right)^T - i\left(\frac{M^\dagger M}{2\mu}\right)^T\Sigma, \tag{7.108}$$

where we have taken into account that

$$MM^\dagger \to g_L MM^\dagger g_L^\dagger, \quad M^\dagger M \to g_R M^\dagger M g_L^\dagger, \quad \Sigma \to g_R^*\Sigma g_L^T. \tag{7.109}$$

This result has been confirmed by a microscopic calculation done in (Bedaque and Schafer, 2002). We can now see that a generic term in the expansion of the effective lagrangian has the form

$$F^2\Delta^2\left(\frac{\partial_0 - iMM^\dagger/(2\mu)}{\Delta}\right)^n\left(\frac{\boldsymbol{\nabla}}{\Delta}\right)^m\left(\frac{M^2}{F^2}\right)^p(\Sigma)^q(\Sigma^\dagger)^r. \tag{7.110}$$

---

[8] Notice that $\not{D}_\perp$ does not involve time derivatives.

The various terms are easily understood on the basis of the fact that the momentum expansion is a series in p/Δ. The factor Δ^2 in front is just to adjust the normalization of the kinetic terms arising for $n = 2$ and $m = 0$ and $n = 0$, $m = 2$, with $q = r = 1$. The term quadratic in the mass matrix, M, has also the correct normalization. We now see that the contribution to the square masses of the mesons of the masses originating from the kinetic energy expansion and the ones coming from the explicit quark mass terms might be of the same order of magnitude. In fact, the contribution to the NG bosons square masses are (the extra $1/F^2$ terms originate from the expansion of Σ)

$$\begin{aligned} \text{kinetic energy}: \quad & F^2\Delta^2\frac{m^4}{\mu^2\Delta^2}\frac{1}{F^2} \approx \frac{m^4}{\mu^2}, \\ \text{quark masses}: \quad & F^2\Delta^2\frac{m^2}{F^2}\frac{1}{F^2} \approx \frac{\Delta^2 m^2}{F^2}, \end{aligned} \tag{7.111}$$

and the two contributions are of the same order of magnitude for $m \approx \Delta$, since $F \approx \mu$. Before studying the mass spectrum we will now introduce another subject which is relevant to the physics of the NG bosons of the CFL phase, that is the subject of Boson-Einstein condensation.

1. The role of the chemical potential for scalar fields: Bose-Einstein condensation

Let us consider a relativistic complex scalar field (however similar considerations could be done for the non-relativistic case) described by the lagrangian density

$$\partial_\mu\phi^\dagger\partial^\mu\phi - m^2\psi^\dagger\phi - \lambda(\phi^\dagger\phi)^2. \tag{7.112}$$

The quartic term gives rise to a repulsive interaction for $\lambda > 0$ as it is required by the stability of theory. The lagrangian has a phase symmetry $U(1)$ giving rise to a conserved current. A simple way of introducing the chemical potential is to notice that we can promote the global $U(1)$ to a local one by adding a gauge field. However, by definition

$$\frac{\partial\mathcal{L}}{\partial A_\mu} = -j^\mu, \tag{7.113}$$

and therefore the charge density is obtained by varying the lagrangian with respect to the fourth component of the gauge potential. But in a system at finite density the variation with respect to the chemical potential is just the corresponding conserved charge, therefore the chemical potential must enter in the lagrangian exactly as the fourth component of a gauge field. Therefore we get

$$\begin{aligned} \mathcal{L} &= (\partial_0 + i\mu)\phi^\dagger(\partial_0 - i\mu)\phi - (\boldsymbol{\nabla}\phi^\dagger)\cdot(\boldsymbol{\nabla}\phi) - m^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \\ &= \partial_\mu\phi^\dagger\partial^\mu\phi - (m^2-\mu^2)\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 + i\mu(\phi^\dagger\partial_0\phi - \partial_0\phi^\dagger\phi). \end{aligned} \tag{7.114}$$

Notice that the last term breaks the charge conjugation symmetry of the theory since μ multiplies the charge density. As a result the mass spectrum is given by

$$p^2 - (m^2 - \mu^2) + 2\mu Q p_0 = 0, \tag{7.115}$$

where $Q = \pm 1$ is the charge for particles and antiparticles. The dispersion relation can be written as

$$(E + \mu Q)^2 = m^2 + |\mathbf{p}|^2 \tag{7.116}$$

with $E = p_0$. We see that the mass of particles and antiparticles are different and given by

$$m_{p,\bar{p}} = \mp\mu + m. \tag{7.117}$$

This is what happens if $\mu^2 < m^2$. At $\mu^2 = m^2$ the system undergoes a second order phase transition and a condensate is formed. The condensate is obtained by minimizing the potential

$$V(\phi) = (m^2 - \mu^2)\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2, \tag{7.118}$$

from which

$$\langle\phi^\dagger\phi\rangle = \frac{\mu^2 - m^2}{2\lambda}. \tag{7.119}$$

Notice also that at $\mu = m$ the mass of the particle mode goes to zero. In correspondence of the condensation the system has a charge density given by

$$\rho = -\frac{\partial V}{\partial \mu} = 2\mu\langle\phi^\dagger\phi\rangle = \frac{\mu}{\lambda}(\mu^2 - m^2). \tag{7.120}$$

Therefore the ground state of the system is a Bose-Einstein condensate. Defining

$$\langle\phi\rangle = \frac{v}{\sqrt{2}}, \quad v^2 = \frac{\mu^2 - m^2}{\lambda}, \tag{7.121}$$

we can derive the physical spectrum of the system through the replacement

$$\phi(x) = \frac{1}{\sqrt{2}}(v + h(x))e^{i\theta(x)/v}. \tag{7.122}$$

The quadratic part of the lagrangian is

$$\mathcal{L}_2 = \frac{1}{2}\partial_\mu\theta\partial^\mu\theta + \frac{1}{2}\partial_\mu h\partial^\mu h - \lambda v^2 h^2 - 2\mu h\partial_0\theta. \tag{7.123}$$

The mass spectrum is given by the condition

$$\det\begin{pmatrix} p^2 - 2\lambda v^2 & 2i\mu E \\ -2i\mu E & p^2 \end{pmatrix} = 0. \tag{7.124}$$

At zero momentum we get ($p^\mu = (M, \mathbf{0})$)

$$M^2(M^2 - 2\lambda v^2 - 4\mu^2) = 0. \tag{7.125}$$

Therefore the antiparticle remains massless after the transition, whereas the particle gets a mass given by

$$M^2 = 6\mu^2 - 2m^2. \tag{7.126}$$

At the transition point these two masses agree with the ones in the unbroken phase, Eq. (7.117). The masses in the two phases are illustrated in Fig. 25.

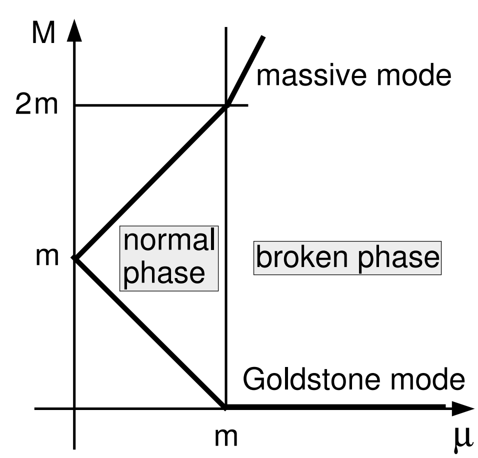

FIG. 25 *This figure shows the evolution of the particle and antiparticle masses from the normal phase to the broken one.*

It is interesting to find out the energies of these particles in the small momentum limit. We find for the Goldstone mode

$$E_{NG} \approx \sqrt{\frac{\mu^2 - m^2}{3\mu^2 - m^2}}|\mathbf{p}|, \tag{7.127}$$

whereas for the massive one

$$E_{massive} \approx \sqrt{6\mu^2 - 2m^2 + \frac{9\mu^2 - m^2}{6\mu^2 - 2m^2}|\mathbf{p}|^2}. \tag{7.128}$$

The velocity of the Goldstone mode is particularly interesting since it is zero at the transition point and goes to $1/\sqrt{3}$ at large densities. Remember that this is just the velocity we have found in our effective lagrangian describing the NG bosons in the CFL phase.

2. Kaon condensation

Starting from the mass terms for the NG bosons of the CFL phase obtained in Section VII.E one can easily understand the results for the masses of the flavored NG bosons

$$\begin{aligned} m_{\pi^\pm} &= \mp\frac{m_d^2 - m_u^2}{2\mu} + \sqrt{\frac{2c}{F^2}(m_u + m_d)m_s}, \\ m_{K^\pm} &= \mp\frac{m_s^2 - m_u^2}{2\mu} + \sqrt{\frac{2c}{F^2}(m_u + m_s)m_d}, \\ m_{K^0,\bar{K}^0} &= \mp\frac{m_s^2 - m_d^2}{2\mu} + \sqrt{\frac{2c}{F^2}(m_d + m_s)m_u}. \end{aligned} \tag{7.129}$$

The two terms in this expression have their counterpart in Eq. (7.117). The first term arises from the ”chemical potential” terms $MM^\dagger/2\mu$ whereas the second term comes from the ”true” mass term. Also in this case a Bose-Einstein condensate might be formed. For simplicity consider the case $m_s \gg m_u, m_d$. Then we get

$$\begin{aligned} m_{\pi^\pm} &\approx \frac{1}{F}\sqrt{2cm_s(m_u + m_d)}, \\ m_{K^\pm} &\approx \mp\frac{m_s^2}{2\mu} + \frac{1}{F}\sqrt{2cm_s m_d}, \\ m_{K^0,\bar{K}^0} &\approx \mp\frac{m_s^2}{2\mu} + \frac{1}{F}\sqrt{2cm_s m_u}. \end{aligned} \tag{7.130}$$

The pion masses are independent on the chemical potential terms, however the masses of K^+ and K^0 are pushed down (whereas the ones of K^- and $\bar{K}^0$ are pushed up) and therefore they become massless at

$$m_s\Big|_{\text{crit}} = \left(\frac{12\mu^2}{\pi^2 F^2}\right)^{1/3} \sqrt[3]{m_{u,d}\Delta^2} = 6\left(\frac{2}{21 - 8\log 2}\right)^{1/3} \sqrt[3]{m_{u,d}\Delta^2} \approx 3.03\sqrt[3]{m_u\Delta}. \tag{7.131}$$

The critical value of m_s can vary between 41 MeV for $m_u = 1.5$ MeV and $\Delta = 40$ MeV and 107 MeV for $m_u = 4.5$ MeV and $\Delta = 100$ MeV. For larger values of m_s the modes K^+ and K^0 become unstable. This is the signal for condensation (Bedaque and Schafer, 2002; Schafer, 2000a). In fact if we look for a kaon condensed ground state of the type[9]

$$\Sigma = e^{i\alpha\lambda_4} = 1 + (\cos\alpha - 1)\lambda_4^2 + i\lambda_4 \sin\alpha, \tag{7.132}$$

we obtain from our effective lagrangian, in the limit of exact isospin symmetry, (subtracting the term for $\alpha = 0$) the potential

$$V(\alpha) = F^2\left(-\frac{1}{2}\left(\frac{m_s^2}{2\mu}\right)^2 \sin^2\alpha + (m_K^0)^2(1 - \cos\alpha)\right), \tag{7.133}$$

---

[9] The most general ansatz would be to assume in the exponential a linear combination of λ^a, $a = 4, 5, 6, 7$, but it turns out that the effective potential depends only on the coefficient of λ_4. This is related to the fact that, as we shall see, there are three broken symmetries and therefore there must be three flat directions in the potential.

where m_K^0 is the lowest order square mass of the kaon in m_s, that is

$$(m_K^0)^2 = \frac{2\,c\,m_s\,m}{F^2} \tag{7.134}$$

and $m = m_u = m_d$. As in the example of a complex scalar field we see that the "chemical potential" terms give a negative contribution, whereas the "mass" terms gives a positive one. Therefore it is convenient to introduce the effective chemical potential

$$\mu_{eff} = \frac{m_s^2}{2\mu}. \tag{7.135}$$

We see that

$$V(\alpha) = F^2\left(-\frac{1}{2}\mu_{eff}^2 \sin^2\alpha + (m_K^0)^2(1-\cos\alpha)\right). \tag{7.136}$$

Minimizing the potential we find a solution with $\alpha \neq 0$, given by

$$\cos\alpha = \frac{(m_K^0)^2}{\mu_{eff}^2} \tag{7.137}$$

if

$$\mu_{eff} \geq m_K^0. \tag{7.138}$$

One can also derive the hypercharge density

$$n_Y = -\frac{\partial V}{\partial \mu_{eff}} = \mu_{eff} F^2\left(1 - \frac{(m_K^0)^4}{\mu_{eff}^4}\right). \tag{7.139}$$

The mass terms break the original $SU(3)_{c+L+R}$ symmetry of the CFL ground state to $SU(2)_I \otimes U(1)_Y$. The kaon condensation breaks this symmetry to the diagonal $U(1)$ group generated by

$$Q = \frac{1}{2}\left(\lambda_3 - \frac{1}{\sqrt{3}}\lambda_8\right). \tag{7.140}$$

In fact, one can easily verify that

$$[Q, \Sigma] = 0. \tag{7.141}$$

This result can be simply understood by the observation that under $SU(2)_I \otimes U(1)_Y$ the NG bosons in Σ decompose as a triplet, the pions, a complex doublet $(\bar{K}_0, K^-)$ and its complex conjugate (K^+, K^0) and a singlet, η. The $\alpha \neq 0$ solution for Σ gives rise to an expectation value for the doublets. We see that the symmetry breaking mechanism is the same as for the electroweak sector of the Standard Model (SM). Therefore $SU(2) \otimes U(1)$ is broken down to $U(1)$

Notice that by the usual counting of NG bosons one would expect 3 massless modes. However we have seen from the unbroken phase that only two modes become massless at the transition. This comes from the breaking of Lorentz invariance due to the presence of the chemical potential in the original QCD lagrangian. A theorem due to Chada and Nielsen (Nielsen and Chada, 1976) gives the key for the right counting of the NG physical bosons. The essence of the theorem is that the number depends on the dispersion relation of the NG bosons. If the energy is linear in the momentum (or a odd power) the counting is normal. If the energy depends on the momentum quadratically (or through an even power), then there are two broken generators associated to a single NG bosons. Notice that in relativistic theories the dispersion relations are always of the first type. A more recent discussion of this topics is in (Miransky and Shovkovy, 2002; Schafer *et al.*, 2001). In particular in (Schafer *et al.*, 2001) it is proved a theorem which helps to show algebraically when the dispersion relations are odd or even in the momenta. The behavior of the masses with the effective chemical potential is represented in Fig. 26.

The effective lagrangian for the NG bosons in the CFL phase has been useful for the study of many phenomenological interesting questions in the realm of compact stellar objects (Buckley and Zhitnitsky, 2002; Jaikumar *et al.*, 2002; Kaplan and Reddy, 2002a,b; Reddy *et al.*, 2003; Shovkovy and Ellis, 2002).

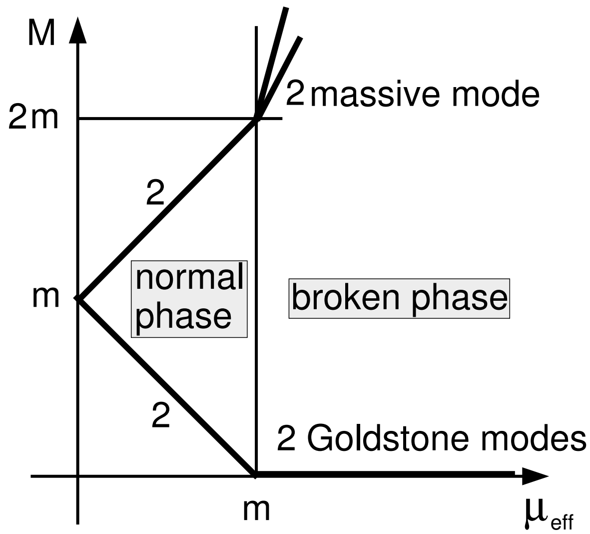

FIG. 26 *This figure shows the evolution of the masses of the K modes in going from the normal to the broken phase.*

VIII. THE DISPERSION LAW FOR THE GLUONS

In this Section will be evaluate the dispersion relations both for the gluons in the 2SC and in the CFL phase. In the 2SC phase we will be able, in this way, of getting the parameters of the effective theory introduced in Section VI.B. We will also show that the low-energy expansion is accurate at about 30% when compared with the numerical calculations.

A. Evaluating the bare gluon mass

In deriving the HDET we have so far neglected the heavy fields contribution to the leading correction in $1/\mu$ given by the operator

$$-P^{\mu\nu}\psi_+^\dagger\frac{D_\mu D_\nu}{2\mu+i\tilde{V}\cdot D}\psi_+. \tag{8.1}$$

Remember that in order to get the effective lagrangian we have integrated out all the fields with momenta greater that δ. However these degrees of freedom give a contribution of order μ which compensates the μ in the denominator leaving a finite contribution to the operator g^2, where g is the gluon field. This contribution is called the bare gluon mass (Son and Stephanov, 2000a,b) and it must be inserted in the HDET when one is studying the gluon properties. In fact, as we shall see, this contribution is needed, in the case of unbroken gauge symmetries, to cancel a contribution to the Meissner mass coming form the gluon polarization. To make a correct calculation of these contribution it is necessary to evaluate the tadpole contribution from the interaction (8.1) for momenta greater than δ (see Fig. 18). Since it appears a heavy propagator at zero momentum, which is essentially the density of the states (see Eq. (2.36)), it is easier to derive the result in a first-quantization approach, where we sum over all the degrees of freedom within the Fermi sphere in order to get the two-gluon contribution. Notice also that we are integrating over residual momenta $|\ell| > \delta >> \Delta$, therefore we can neglect the gap. As a consequence this calculation applies to both cases of gapped and/or ungapped fermions.

We start with the observation that in first-quantization we have $H = |\mathbf{p}|$. Coupling the particle to a gauge field

$$H = |\mathbf{p}-g\mathbf{A}| + eA_0 \approx |\mathbf{p}| + gA_0 - g\mathbf{v}\cdot\mathbf{A} + \frac{g^2}{2|\mathbf{p}|}\left(|\mathbf{A}|^2 - (\mathbf{v}\cdot\mathbf{A})^2\right). \tag{8.2}$$

The first three terms correspond to the terms considered so far in the HDET lagrangian. The fourth term is nothing but the operator sandwiched among the quark fields in Eq. (8.1) evaluated at the Fermi surface. In fact

$$P_{\mu\nu}A^\mu A^\nu \approx |\mathbf{A}|^2 - (\mathbf{v}\cdot\mathbf{A})^2. \tag{8.3}$$

What we have to do now is just to sum over all the particles inside the Fermi sphere, which is equivalent to the tadpole calculation. Notice that we should leave aside the fermions within the shell of momentum $\mu-\delta<|\mathbf{p}|<\mu+\delta$, but this give a negligible contribution since $\delta\ll\mu$. Therefore we get

$$\frac{g^2}{2}\times 2\times N_f\times\int_{|\mathbf{p}|\le\mu}\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{|\mathbf{p}|}\mathrm{Tr}\left[\mathbf{A}^2-\frac{(\mathbf{p}\cdot\mathbf{A})^2}{|\mathbf{p}|^2}\right],\tag{8.4}$$

where the factors 2 and N_f comes from the spin and the number of flavors. On the other hand the trace is over the color indices. The result is simply

$$N_f\frac{g^2\mu^2}{6\pi^2}\frac{1}{2}\sum_a\mathbf{A}^a\cdot\mathbf{A}^a.\tag{8.5}$$

Therefore in the effective lagrangian of HDET we have to introduce a term

$$-\frac{1}{2}m_D^2\sum_a\mathbf{A}^a\cdot\mathbf{A}^a\tag{8.6}$$

with

$$m_{BM}^2=N_f\frac{g^2\mu^2}{6\pi^2}.\tag{8.7}$$

One could equally well perform this evaluation by using the Feynman rules for the heavy fields, which are easily obtained, to determine the contribution to the polarization function $\Pi_{\mu\nu}^{ab}$. The contribution is the following

$$\Pi_{\mu\nu}^{ab\,\mathrm{BM}}=(-i)2\times 2\times N_f\times(-1)\int\frac{d\ell_\parallel}{(2\pi)}\frac{(\ell_\parallel+\mu)^2}{\pi}\int\frac{d\mathbf{v}}{4\pi}\int\frac{d\ell_0}{2\pi}\frac{ig^2\delta_{ab}}{2(2\mu+\tilde{V}\cdot\ell)}\frac{i}{V\cdot\ell}P_{\mu\nu}.\tag{8.8}$$

The different factors have the following origin. The first $(-i)$ is due to the definition of $\Pi_{\mu\nu}$ such to reproduce the mass term in the lagrangian. Then there is a factor 2 from the spin, a factor 2 from symmetry reasons (2 gluon fields). A factor N_f from the trace over the heavy fermions, a (-1) from the loop. We have left the residual momenta as integration variable, but the measure cannot be any more approximated as μ^2 and we have put back the original integration factors. There is no extra 2 in the velocity integration since we have now no necessity of introducing the Nambu-Gor'kov fields. The next factor arises from the vertex 2 gluons-2heavy fields and it is the result after the trace over the color indices. Finally we have the propagator of the ψ_+^h field. If we define the propagator at zero momentum as in Section II.A we get a contribution from the integral over ℓ_0 proportional to $\theta(-\ell_\parallel)$ (see Eq. (2.38)). This restricts the integration over $\ell_\parallel$ between $-\mu$ and 0. Performing the calculation one gets easily the result (8.7). In fact we have

$$\begin{aligned}\Pi_{\mu\nu}^{ab\,\mathrm{BM}}&=-2iN_fg^2\delta_{ab}\int_{-\mu}^{0}\frac{d\ell_\parallel}{2\pi}\frac{(\ell_\parallel+\mu)^2}{\pi}\frac{1}{2\pi}\frac{(2\pi i)}{2(\ell_\parallel+\mu)}\int\frac{d\mathbf{v}}{4\pi}P_{\mu\nu}=\frac{g^2}{2\pi^2}\delta_{ab}\frac{\mu^2}{2}(-\frac{2}{3}\delta_{ij})\\&=-m_{BM}^2\delta_{ij}\delta_{ab}.\end{aligned}\tag{8.9}$$

B. The parameters of the effective lagrangian for the 2SC case

We have now all the elements to evaluate the parameters appearing in the effective lagrangian for the 2SC case, which amounts to evaluate the propagation properties of the gluons belonging to the unbroken color group $SU(2)_c$. We will need to evaluate the vacuum polarization function $\Pi_{\mu\nu}^{ab}$, $a,b=1,2,3$, defined by the diagram in Fig. 27.

As usual we need the interaction term which is given by the coupling of the gauge field with the fermionic current

$$igA_\mu^aJ_a^\mu.\tag{8.10}$$

One finds

$$J_\mu^a=\int\frac{d\mathbf{v}}{4\pi}\sum_{A,B=0}^{5}\chi^{A\dagger}\begin{pmatrix}i\,V_\mu K_{AaB} & 0\\ 0 & -i\,\tilde{V}_\mu K_{AaB}^*\end{pmatrix}\chi^B+(L\to R)\ .\tag{8.11}$$

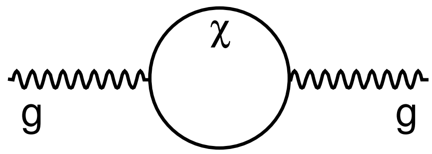

FIG. 27 *The Feynman diagram corresponding to the vacuum polarization of the gluons.*

where we have used the basis (7.38) for the quark fields, leading to the coefficients K_{AaB}

$$K_{AaB} \;=\; \frac{1}{4}\,Tr\{\tilde\lambda_A\lambda^a\tilde\lambda_B\} = (\mathbf{K})_{AB}\ . \tag{8.12}$$

Using our Feynman rules we can compute $\Pi^{ab}_{\mu\nu}$ (Casalbuoni *et al.*, 2002d)[10]. From the self energy diagram of Fig. 27 $(a,b=1,2,3)$ we find:

$$\begin{aligned}\Pi^{\mu\nu\,\rm self}_{ab}(p) \;&=\; 2\times(-i)(-1)\int\frac{d\mathbf{v}}{8\pi}\frac{\mu^2}{\pi}\int\frac{d^2\ell}{(2\pi)^2}\Big(\frac{(igV^\mu)(igV^\nu)i\tilde V\cdot\ell\, i\tilde V\cdot(\ell+p)+(V\leftrightarrow\tilde V)}{D(\ell+p)D(\ell)}+\\ &-\;\Delta^2\frac{(igV^\mu)(ig\tilde V^\nu)+(igV^\nu)(ig\tilde V^\mu)}{D(\ell+p)D(\ell)}\Big){\rm Tr}[\mathbf{K}_a\mathbf{K}_b]\ .\end{aligned} \tag{8.13}$$

where

$$Tr[\mathbf{K}_a\mathbf{K}_b]=\delta_{ab}. \tag{8.14}$$

In order to derive the effective lagrangian it is enough to expand this expression up to the second order in momenta and add the contribution from the bare Meissner mass. The result is (Casalbuoni *et al.*, 2002d)

$$\Pi^{00}_{ab}(p) \;=\; \delta_{ab}\frac{\mu^2g^2}{18\pi^2\Delta^2}|\vec p\,|^2\,, \tag{8.15}$$

$$\Pi^{kl}_{ab}(p) \;=\; \Pi^{kl\,\rm self}_{ab}+\Pi^{kl\,\rm BM}_{ab} \;=\; \delta_{ab}\delta^{kl}\frac{\mu^2g^2}{3\pi^2}\left(1+\frac{p_0^2}{6\Delta^2}\right)-\delta_{ab}\delta^{kl}\frac{\mu^2g^2}{3\pi^2}=\delta_{ab}\delta^{kl}\frac{\mu^2g^2}{18\pi^2}\frac{p_0^2}{\Delta^2}\,, \tag{8.16}$$

$$\Pi^{0k}_{ab}(p) \;=\; \delta_{ab}\frac{\mu^2g^2}{18\pi^2\Delta^2}p^0p^k\,. \tag{8.17}$$

We see that the bare Meissner mass contribution $\Pi^{kl\,\rm BM}_{ab}(p)$ just cancels the zero momentum contribution to the gluon self-energy. As a consequence the unbroken gluons remain massless. These results agree with the outcomes of (Rischke, 2000; Rischke *et al.*, 2001).

We notice also that there is no contribution to the Debye mass (the mass associated to the component Π^{00}_{ab} of the vacuum polarization). This results reflects the fact that in the 2SC model the $SU(2)_c$ color subgroup generated by the first three generators T^c $(c=1,2,3)$ remains unbroken.

One can now compute the dispersion laws for the unbroken gluons. From the previous formulas one gets:

$$\mathcal{L} \;=\; -\frac{1}{4}F^{\mu\nu}_aF^a_{\mu\nu}+\frac{1}{2}\Pi^{\mu\nu}_{ab}A^a_\mu A^b_\nu\ \ . \tag{8.18}$$

Introducing the fields $E^a_i\equiv F^a_{0i}$ and $B^a_i\equiv i\varepsilon_{ijk}F^a_{jk}$, and using (8.15), (8.16) and (8.17) these results can be written as follows (assuming gauge invariance the terms with 3 and 4 gluons are completely fixed, but see later):

$$\mathcal{L} \;=\; \frac{1}{2}(E^a_iE^a_i-B^a_iB^a_i)+\frac{k}{2}E^a_iE^a_i\ , \tag{8.19}$$

[10] For the same calculation using different methods see: (Rischke, 2000; Rischke *et al.*, 2001).

with

$$k = \frac{g^2\mu^2}{18\pi^2\Delta^2} \ . \tag{8.20}$$

These results have been first obtained in (Rischke *et al.*, 2001). As discussed in this paper, these results imply that the medium has a very high *dielectric constant* $\epsilon = k + 1$ and a *magnetic permeability* $\lambda = 1$. The gluon speed in this medium is now

$$v = \frac{1}{\sqrt{\epsilon\lambda}} \propto \frac{\Delta}{g\mu} \tag{8.21}$$

and in the high density limit it tends to zero. We have already discussed the physical consequences of these results in Section VI.B.

In (Casalbuoni *et al.*, 2002d) we have also computed the vacuum polarization of the gluons belonging to the broken part of the gauge group. That is the gluons with color index $a = 4, 5, 6, 7, 8$. In Table VII we give the results at zero momentum in all the cases. These results are in agreement with a calculation performed by (Rischke, 2000) with a

a	$\Pi^{00}(0)$	$-\Pi^{ij}(0)$
$1-3$	0	0
$4-7$	$\frac{3}{2}m_g^2$	$\frac{1}{2}m_g^2$
8	$3m_g^2$	$\frac{1}{3}m_g^2$

TABLE VII Debye and Meissner masses in the 2SC phase; $m_g^2 = \mu^2 g^2/3\pi^2$.

different method. In (Casalbuoni *et al.*, 2002d) we have also made an expansion in momenta. A priori this expansion cannot be used to derive the dispersion relation for the broken gluons. In fact their mass, taking into account the wave function renormalization (of order $g^2\mu^2/\Delta^2$), is of order Δ. Defining the mass of the gluon as the value of the energy at zero momentum one finds from the expansion

$$m_R = \sqrt{2}\Delta \tag{8.22}$$

for colors $a = 4, 5, 6, 7$, whereas the numerical calculation gives

$$m_R = 0.894\Delta \tag{8.23}$$

and we see that the expansion overestimates the value of the mass. For the color 8 we find

$$m_R = \frac{\mu g}{\pi}. \tag{8.24}$$

The large value obtained of the mass for the gluon 8 in the small momentum expansion can be shown to be valid at all orders in momentum. Except for the case $a = 8$ we see that generally there is a very large wave function renormalization making the physical gluon masses of order Δ rather than of order $g\mu$. This result was shown for the first time in (Casalbuoni *et al.*, 2001c,d) in the case of the CFL phase. In (Casalbuoni *et al.*, 2002d) the one-loop contributions to the three and four gluon vertices has been evaluated. It has been checked that the result gives rise to the correct gauge invariant contribution when added to the tree level functions.

C. The gluons of the CFL phase

We will discuss here some of the results obtained in (Casalbuoni *et al.*, 2001c,d) for the CFL case. We will skip many technical details. We will only give the coupling of the currents to fermions which are again obtained from the gauge coupling. Working in the basis χ^A we use Eq. (7.5) to write

$$\mathcal{L}_1 \ = i\, g\, A_a^\mu J_\mu^a \ , \tag{8.25}$$

where

$$J_\mu^a \;=\; \int \frac{d\mathbf{v}}{4\pi} \sum_{A,B=1}^{9} \chi^{A\dagger} \begin{pmatrix} i\, V_\mu h_{AaB} & 0 \\ 0 & -i\, \tilde{V}_\mu h^*_{AaB} \end{pmatrix} \chi^B + (L \to R), \tag{8.26}$$

and

$$h_{AaB} = \frac{1}{4}\, Tr[T_A \lambda_a T_B] \; . \tag{8.27}$$

Performing the trace we find

$$\begin{aligned} J_\mu^a \;&=\; \frac{i}{2}\sqrt{\frac{2}{3}} \sum_{\vec{v}} \left(\chi^{9\dagger} \begin{pmatrix} V_\mu & 0 \\ 0 & -\tilde{V}_\mu \end{pmatrix} \chi^a + \text{h.c.} \right) \;+ \\ &+\; \frac{i}{2} \sum_{\vec{v}} \sum_{b,c=1}^{8} \chi^{b\dagger} \begin{pmatrix} V_\mu g_{bac} & 0 \\ 0 & -\tilde{V}_\mu g^*_{bac} \end{pmatrix} \chi^c, \end{aligned} \tag{8.28}$$

where

$$g_{abc} = d_{abc} + i f_{abc} \; . \tag{8.29}$$

and d_{abc}, f_{abc} are the usual $SU(3)$ symbols. The result of the self energy diagram (see Fig. 27) can be written as follows:

$$\begin{aligned} i\,\Pi_{ab}^{\mu\nu\,\text{self}}(p) = -2 \int \frac{d\mathbf{v}}{8\pi} \sum_{A,C,D,E} (-ig)^2 \, \frac{\mu^2}{\pi} \int \frac{d^2\ell}{(2\pi)^2}\, Tr\Big[iS_{CD}(\ell + p) \times \\ \times \begin{pmatrix} V_\nu h_{DbE} & 0 \\ 0 & -\tilde{V}_\nu h^*_{DbE} \end{pmatrix} iS_{EA}(\ell) \begin{pmatrix} V_\mu h_{AaC} & 0 \\ 0 & -\tilde{V}_\mu h^*_{AaC} \end{pmatrix} \Big], \end{aligned} \tag{8.30}$$

where the propagator is given by Eq. (7.56). We note the minus sign on the r.h.s of (8.30), due to the presence of a fermion loop and the factor 2 due to the spin $(L + R)$. To this result one should add the contribution arising from the bare Meissner mass (see Eq. (8.7)):

$$\Pi_{ab}^{\mu\nu\,\text{BM}} \;=\; -\, \frac{g^2\mu^2}{2\pi^2} \delta_{ab} \delta^{jk} \delta^{\mu j} \delta^{\nu k} \; . \tag{8.31}$$

To derive the dispersion law for the gluons, we write the equations of motion for the gluon field A_μ^b in momentum space and high-density limit:

$$\left[\left(-p^2 g^{\nu\mu} + p^\nu p^\mu \right) \delta_{ab} + \Pi_{ab}^{\nu\mu} \right] A_\mu^b \;= 0 \; . \tag{8.32}$$

We define the rotational invariant quantities Π_0, Π_1, Π_2 and Π_3 by means of the following equations,

$$\Pi^{\mu\nu}(p_0, \vec{p}) = \begin{cases} \Pi^{00} = \Pi_0(p_0, \vec{p}) \\ \Pi^{0i} = \Pi^{i0} = \Pi_1(p_0, \vec{p})\, n^i \\ \Pi^{ij} = \Pi_2(p_0, \vec{p})\, \delta^{ij} \,+\, \Pi_3(p_0, \vec{p})\, n^i n^j \end{cases} \tag{8.33}$$

with $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$. It is clear that in deriving the dispersion laws we cannot go beyond momenta of order Δ, as the Fermi velocity superselection rule excludes gluon exchanges with very high momentum; it is therefore an approximation, but nevertheless a useful one, as in most cases hard gluon exchanges are strongly suppressed by the asymptotic freedom property of QCD.

By the equation

$$p_\nu \Pi_{ab}^{\nu\mu} A_\mu^b = 0 \; , \tag{8.34}$$

one obtains the relation

$$\left(p_0\, \Pi_0 \,-\, |\vec{p}|\, \Pi_1 \right) A_0 = \vec{n} \cdot \vec{A} \left(p_0\, \Pi_1 \,-\, |\vec{p}| \left(\Pi_2 + \Pi_3 \right) \right) \; , \tag{8.35}$$

between the scalar and the longitudinal component of the gluon fields. The dispersion laws for the scalar, longitudinal and transverse gluons are respectively

$$\begin{aligned}
\left(\Pi_2 \,+\, \Pi_3 \,+\, p_0^2\right)\left(|\vec{p}|^2 \,+\, \Pi_0\right) &= p_0|\vec{p}|\left(2\,\Pi_1 \,+ p_0|\vec{p}|\right)\ , \\
\left(\Pi_2 \,+\, \Pi_3 \,+\, p_0^2\right)\left(|\vec{p}|\, p_0 \,+\, \Pi_0\right) &= p_0|\vec{p}|\left(2\,\Pi_1 \,+\, p_0^2\right) + \Pi_1^2\ , \\
p_0^2 - |\vec{p}|^2 + \Pi_2 &= 0\ .
\end{aligned} \tag{8.36}$$

The analysis of these equations is rather complicated (Gusynin and Shovkovy, 2002) and we will give only the results arising from the expansion of the vacuum polarization up to the second order in momenta. The relevant expressions can be found in (Casalbuoni *et al.*, 2001c,d). The results at zero momentum are the same for all the gluons and they are summarized in the Debye mass (from Π^{00}) and in the Meissner mass (from Π^{ii}

$$m_D^2 \;=\; \frac{\mu^2 g^2}{36\pi^2}\,(21 - 8\ln 2) \;=\; g^2 F^2\ , \tag{8.37}$$

and

$$m_M^2 \;=\; \frac{\mu^2 g^2}{\pi^2}\left(-\frac{11}{36} - \frac{2}{27}\ln 2 + \frac{1}{2}\right) \;=\; \frac{m_D^2}{3}\ , \tag{8.38}$$

where the first two terms are the result of the diagram of Fig. 27, whereas the last one is the bare Meissner mass. These results agree with the findings of other authors as, for instance, (Son and Stephanov, 2000a,b; Zarembo, 2000). Recalling Eqs. (6.17)

$$m_D^2 = \alpha_T g^2 F_T^2, \quad m_M^2 = \alpha_S v^2 g^2 F_T^2 \tag{8.39}$$

derived from the effective lagrangian for the CFL phase and the results already obtained for the parameters F_T^2 and v^2 we see that

$$\alpha_S = \alpha_T = 1. \tag{8.40}$$

This completes the evaluation of the parameters of the effective lagrangian for the CFL phase.

The Debye and Meissner masses do not exhaust the analysis of the dispersion laws for the gluons in the medium. We will give here the result for the rest mass defined as the energy at zero momentum. Due to the large wave function renormalization, of order $g^2\mu^2/\Delta^2$, the rest mass turns out to be of order Δ. Precisely we find that for large values of μ one has

$$m_A^R \approx \frac{m_D}{\sqrt{3\alpha_1}}\ . \tag{8.41}$$

with

$$\alpha_1 = \frac{\mu^2 g_s^2}{216\Delta^2\pi^2}\left(7 + \frac{16}{3}\ln 2\right)\ . \tag{8.42}$$

Therefore

$$m_A^R \approx \frac{m_D}{\sqrt{3\alpha_1}} = \sqrt{6\,\frac{21 - 8\ln 2}{21 + 16\ln 2}}\,\Delta \approx 1.70\,\Delta\ . \tag{8.43}$$

It is interesting to notice that the gluons are below threshold for the decay $g \to q\bar{q}$ since in the CFL all the quarks are gapped with rest masses Δ or 2Δ. The rest mass can be also evaluated numerically without expanding in momenta. One finds

$$m \equiv m_R = \ 1.36\ \Delta\ . \tag{8.44}$$

A comparison with (8.43) shows that the relative error between the two procedures is of the order of 20% and this is also the estimated difference for the dispersion law at small $\vec{p}$. We notice that also in this case the momentum expansion approximation overestimates the correct result (Gusynin and Shovkovy, 2002).

The result about the large renormalization of the gluon fields can be easily understood by using the chiral expansion of Eq. (7.110) (Jackson and Sannino, 2003). To this end let us first notice that the renormalizion factor for the fields to be canonical is

$$g_\mu \to \frac{g\mu}{\Delta} g_\mu, \tag{8.45}$$

which implies a coupling renormalization

$$g \to \frac{\Delta}{g\mu} g = \frac{\Delta}{\mu}. \tag{8.46}$$

Therefore, after renormalization the coupling becomes Δ/μ. Now let us go back to the effective theory for the CFL phase and precisely to the equation used to decouple the gluon fields, Eq. (6.18)

$$g_\mu^a = -\frac{1}{2g}\left(\hat{X}\partial_\mu \hat{X}^\dagger + \hat{Y}\partial_\mu \hat{Y}^\dagger\right) \equiv \frac{1}{g}\omega_\mu^a. \tag{8.47}$$

Now consider the kinetic term for the gluon fields

$$\sum_a F_{\mu\nu}^a F^{\mu\nu a}, \tag{8.48}$$

with

$$F_{\mu\nu}^a = g\partial_\mu g_\nu^a - \partial_\nu g_\mu^a - g f_{abc} g_\mu^b g_\nu^c \tag{8.49}$$

and f_{abc} the structure constants of the gauge group. Substituting the expression (8.47) inside the kinetic term we find

$$\text{kinetic rem} \approx \sum_a \left[\left(\frac{1}{g}(\partial_\mu \omega_\nu^a - \partial_\nu \omega_\mu^a - g\frac{1}{g^2} f_{abc}\omega_\mu^b \omega_\nu^c\right)^2\right] \approx \frac{1}{g^2} \times \text{four derivative operator.} \tag{8.50}$$

However the chiral expansion gives a coefficient for a four derivative operator given by

$$F^2\Delta^2 \times \frac{1}{\Delta^4} = \frac{\mu^2}{\Delta^2}. \tag{8.51}$$

Comparing these two results, obtained both for canonical gluon fields, we see that the coupling must be of the order Δ/μ, recovering the result obtained from the explicit calculation.

IX. QUARK MASSES AND THE GAP EQUATION

We have discussed in Section V.D.3 the more realistic case of 2 massless flavors and a third massive one, showing that for $\mu < m_s^2/2\Delta$ the condensate of the heavy quark with the light ones may be disrupted. In this Section we will investigate more carefully this problem but limiting ourselves two the case of two flavors (one massless and one massive). We know (always from Section V.D.3) that the radii of the Fermi spheres are

$$p_{F_1} = \sqrt{\mu^2 - m_s^2}, \quad p_{F_2} = \mu. \tag{9.1}$$

In the case $m_s \ll \mu$ we can approximate the radius of the sphere of the massive fermion as

$$p_{F_1} \approx \mu - \frac{m_s^2}{2\mu}. \tag{9.2}$$

At the lowest order the effect of the mass is to split by a constant term the Fermi momenta. Also, the splitting is the ratio $m_s^2/2\mu$ that we have already encountered in Section VII.E, and we see from here in a simple way why this expression is an effective chemical potential. In this approximation we can substitute the problem with another one where we have two massless fermions but with a split chemical potential. This problem has received a lot of attention in normal superconductivity in presence of a magnetic field. The coupling of the field to the spin of the electrons produces a splitting of the Fermi surfaces related to spin up and spin down electrons. In practice this coupling is completely dominated by the coupling of the magnetic field to the orbital angular momentum, but it is possible to conceive situations where the effect is important[11]

[11] For recent reviews about this point and the LOFF phase that will be discussed later, see (Bowers, 2003; Casalbuoni and Nardulli, 2003)

Measuring the splitting from the middle point we define the chemical potentials for the two species of fermions as

$$\mu_u = \mu + \delta\mu, \qquad \mu_d = \mu - \delta\mu, \tag{9.3}$$

where we have denoted by u and d the two species of fermions under study. To describe the situation we can add the following interaction hamiltonian

$$H_I = -\delta\mu\psi^\dagger\sigma_3\psi, \tag{9.4}$$

where σ_3 is a Pauli matrix acting in the two-dimensional space corresponding to the two fermions. Notice that in the case of normal superconductivity $\delta\mu$ is proportional to the magnetic field. Let us start considering the case of normal superconductivity. In this case we have a simple modification in the diagonal terms of the Gor'kov equations, introduced in Section III.D, leading to the inverse propagator

$$S^{-1}(\mathbf{p}) = \begin{pmatrix} (i\partial_t - \xi_{\mathbf{p}} + \delta\mu\sigma_3) & -\Delta \\ -\Delta^* & (i\partial_t + \xi_{\mathbf{p}} + \delta\mu\sigma_3) \end{pmatrix} . \tag{9.5}$$

Then, evaluating the gap equation, as given in Eq. (3.130, one gets easily at $T = 0$:

$$\Delta = i\, g\Delta \int \frac{dE}{2\pi}\frac{d^3p}{(2\pi)^3}\frac{1}{(E-\delta\mu)^2 - \xi_{\mathbf{p}}^2 - \Delta^2}\,, \tag{9.6}$$

and at $T \neq 0$:

$$\Delta \;=\; gT\sum_{n=-\infty}^{+\infty}\int\frac{d^3p}{(2\pi)^3}\frac{\Delta}{(\omega_n + i\delta\mu)^2 + \epsilon(\mathbf{p},\Delta)^2}\,, \tag{9.7}$$

where, we recall that

$$\epsilon(\mathbf{p},\Delta) = \sqrt{\Delta^2 + \xi_{\mathbf{p}}^2}\,. \tag{9.8}$$

We now use the identity

$$\frac{1}{2}\left[1 - n_u - n_d\right] = \epsilon(\mathbf{p},\Delta)T\sum_{n=-\infty}^{+\infty}\frac{1}{(\omega_n + i\delta\mu)^2 + \epsilon^2(\mathbf{p},\Delta)}\,, \tag{9.9}$$

where

$$n_u(\mathbf{p}) = \frac{1}{e^{(\epsilon+\delta\mu)/T} + 1}\,, \qquad\qquad n_d(\mathbf{p} = \frac{1}{e^{(\epsilon-\delta\mu)/T} + 1}\,. \tag{9.10}$$

The gap equation can be therefore written as

$$\Delta = \frac{g\,\Delta}{2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\epsilon(\mathbf{p},\Delta)}\left(1 - n_u(\mathbf{p}) - n_d(\mathbf{p})\right)\,. \tag{9.11}$$

In the Landau theory of the Fermi liquid n_u, n_d are interpreted as the equilibrium distributions for the quasiparticles of type u, d. It can be noted that the last two terms act as blocking factors, reducing the phase space.

Before considering the solutions of the gap equations in the general case let us consider the case $\delta\mu = 0$; the corresponding gap is denoted Δ_0. At $T = 0$ there is no reduction of the phase space and we know already the solution

$$\Delta_0 = \frac{\delta}{\sinh\dfrac{2}{g\rho}}\,. \tag{9.12}$$

Here

$$\rho = \frac{p_F^2}{\pi^2 v_F} \tag{9.13}$$

is the density of states and we have used $\xi_{\mathbf{p}} \approx v_F(p - p_F)$, see Eqs. (3.115)-(3.118). In the weak coupling limit (9.12) gives

$$\Delta_0 = 2\delta\, e^{-2/\rho g} \ . \tag{9.14}$$

Let us now consider the case $\delta\mu \neq 0$. By (3.143) the gap equation is written as

$$-1 + \frac{g}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\epsilon} = \frac{g}{2}\int \frac{d^3p}{(2\pi)^3}\frac{n_u + n_d}{\epsilon} \ . \tag{9.15}$$

Using the gap equation for the BCS superconductor, the l.h.s can be written, in the weak coupling limit, as

$$\text{l.h.s} = \frac{g\rho}{2}\log\frac{\Delta_0}{\Delta} \ , \tag{9.16}$$

where we got rid of the cutoff δ by using Δ_0, the gap at $\delta\mu = 0$ and $T = 0$. Let us now evaluate the r.h.s. at $T = 0$. We get

$$\text{r.h.s.}\Big|_{T=0} = \frac{g\rho}{2}\int_0^\delta \frac{d\xi_{\mathbf{p}}}{\epsilon}\left[\theta(-\epsilon - \delta\mu) + \theta(-\epsilon + \delta\mu)\right] . \tag{9.17}$$

The gap equation at $T = 0$ can therefore be written as follows:

$$\log\frac{\Delta_0}{\Delta} = \theta(\delta\mu - \Delta)\text{arcsinh}\frac{\sqrt{\delta\mu^2 - \Delta^2}}{\Delta} \ , \tag{9.18}$$

i.e.

$$\log\frac{\Delta_0}{\delta\mu + \sqrt{\delta\mu^2 - \Delta^2}} = 0\,. \tag{9.19}$$

One can immediately see that there are no solutions for $\delta\mu > \Delta_0$. For $\delta\mu \leq \Delta_0$ one has two solutions.

$$a)\quad \Delta = \Delta_0\,, \tag{9.20}$$

$$b)\quad \Delta^2 = 2\,\delta\mu\Delta_0 - \Delta_0^2\,. \tag{9.21}$$

The first arises since for $\Delta = \Delta_0$ the l.h.s. of the Eq. (9.18) is zero. But since we may have solutions only for $\delta\mu \leq \Delta_0$ the θ-function in Eq. (9.18) makes zero also the r.h.s..

The existence of this solution can also be seen from Eq. (9.6). In fact in this equation one can shift the integration variable as follows: $E \to E + \delta\mu$, getting the result that, in the superconductive phase, *the gap* Δ *is independent of* $\delta\mu$, i.e. $\Delta = \Delta_0$. Notice that the shift is admissible only if no singularity is found. However, since the integrand has poles at $-\delta\mu \pm \sqrt{\xi_{\mathbf{p}}^2 + \Delta^2} - i\,sign(E)$, when $\delta\mu > \Delta_0$ the pole corresponding to the sign plus becomes negative and therefore it goes into the upper plane together with the other pole. In this case by closing the path in the lower half plane we find zero

We take this occasion to compute the contribution of the free energy to the grand potential again in a different way. To compute this contribution we make use of a theorem asserting that for small variations of an external parameter all the thermodynamical quantities vary in the same way (Landau and Lifshitz, 1996). We apply this to the grand potential to get

$$\frac{\partial\Omega}{\partial g} = \left\langle\frac{\partial H}{\partial g}\right\rangle . \tag{9.22}$$

From the expression of the interaction hamiltonian (see Eq. (3.104) with $G = g$) we find immediately (cfr. (Abrikosov *et al.*, 1963), cap. 7)[12]:

$$\Omega = -\int \frac{dg}{g^2}|\Delta|^2 \ . \tag{9.23}$$

[12] We will use indifferently the symbol Ω for the grand potential and its density Ω/V.

Using the result (9.14) one can trade the integration over the coupling constant g for an integration over Δ_0, the BCS gap at $\delta\mu = 0$, because $d\Delta_0/\Delta_0 = 2dg/\rho g^2$. Therefore the difference in free energy between the superconductor and the normal state is

$$\Omega_\Delta - \Omega_0 = -\frac{\rho}{2}\int_{\Delta_f}^{\Delta_0} \Delta^2 \, \frac{d\Delta_0}{\Delta_0}\,. \tag{9.24}$$

Here Δ_f is the value of Δ_0 corresponding to $\Delta = 0$. $\Delta_f = 0$ in the case a) of Eq. (9.20) and $\Delta_f = 2\delta\mu$ in the case b) of Eq. (9.21); in the latter case one sees immediately that $\Omega_\Delta - \Omega_0 > 0$ because from Eq. (9.21) it follows that $\Delta_0 < 2\delta\mu$. The free energies for $\delta\mu \neq 0$ corresponding to the cases a), b) above can be computed substituting (9.20) and (9.21) in (9.24). Before doing that let us derive the density of free energy at $T = 0$ and $\delta\mu \neq 0$ in the normal non superconducting state. Let us start from the very definition of the grand potential for free spin 1/2 particles

$$\Omega_0(0,T) \;=\; -2VT\int\frac{d^3p}{(2\pi)^3}\ln\left(1+e^{(\mu-\epsilon(\mathbf{p}))/T}\right)\,. \tag{9.25}$$

Integrating by parts this expression we get, for $T \to 0$,

$$\Omega_0(0) \;=\; -\frac{V}{12\pi^3}\int d\Omega_{\mathbf{p}}\, p^3\, d\epsilon\, \theta(\mu-\epsilon)\,. \tag{9.26}$$

From this expression we can easily evaluate the grand-potential for two fermions with different chemical potentials expanding at the first non-trivial order in $\delta\mu/\mu$. The result is

$$\Omega_0(\delta\mu) \;=\; \Omega_0(0) - \frac{\delta\mu^2}{2}\rho\,. \tag{9.27}$$

Therefore from (9.20), (9.21) and (9.24) in the cases a), b) one has

$$a)\quad \Omega_\Delta(\delta\mu) \;=\; \Omega_0(\delta\mu) - \frac{\rho}{4}\left(-2\,\delta\mu^2 + \Delta_0^2\right)\,, \tag{9.28}$$

$$b)\quad \Omega_\Delta(\delta\mu) \;=\; \Omega_0(\delta\mu) - \frac{\rho}{4}\left(-4\,\delta\mu^2 + 4\delta\mu\Delta_0 - \Delta_0^2\right)\,. \tag{9.29}$$

Comparing (9.28) and (9.29) we see that the solution $a)$ has lower Ω. Therefore, for $\delta\mu < \Delta_0/\sqrt{2}$ the BCS superconductive state is stable (Chandrasekhar, 1962; Clogston, 1962). At $\delta\mu = \Delta_0/\sqrt{2}$ it becomes metastable, as the normal state has a lower free energy. This transition is first order since the gap does not depend on $\delta\mu$.

The grand potentials for the two cases a) and b) and for the gapless phase, Eq. (9.27), are given in Fig. 28, together with the corresponding gaps.

This analysis shows that at $\delta\mu = \delta\mu_1 = \Delta_0/\sqrt{2}$ one goes from the superconducting ($\Delta \neq 0$) to the normal ($\Delta = 0$) phase. However, as we shall discuss below, the real ground state for $\delta\mu > \delta\mu_1$ turns out to be an inhomogeneous one, where the assumption (3.136) of a uniform gap is not justified.

The considerations made in this Section may be repeated for the 2SC case in color superconductivity. In fact, as we have seen, the only difference is in the density at the Fermi surface which is four time the one considered here with $v_F = 1$, $p_F = \mu$.

A. Phase diagram of homogeneous superconductors

We will now study the phase diagram of the homogeneous superconductor for small values of the gap parameter, which allows to perform a Ginzburg-Landau expansion of gap equation and grand potential. In order to perform a complete study we need to expand the grand-potential up to the 6$^{\text{th}}$ order in the gap. As a matter of fact in the plane $(\delta\mu, T)$ there is a first order transition at $(\delta\mu_1, 0)$ and a second order one at $(0, T_c)$ (the usual BCS second order transition). Therefore we expect that a second order and a first order line start from these points and meet at a tricritical point, which by definition is the meeting point of a second order and a first order transition line. A tricritical point is characterized by the simultaneous vanishing of the Δ^2 and Δ^4 coefficients in the grand-potential expansion, which is why one needs to introduce in the grand potential the 6$^{\text{th}}$ order term. For stability reasons the corresponding coefficient should be positive; if not, one should include also the Δ^8 term.

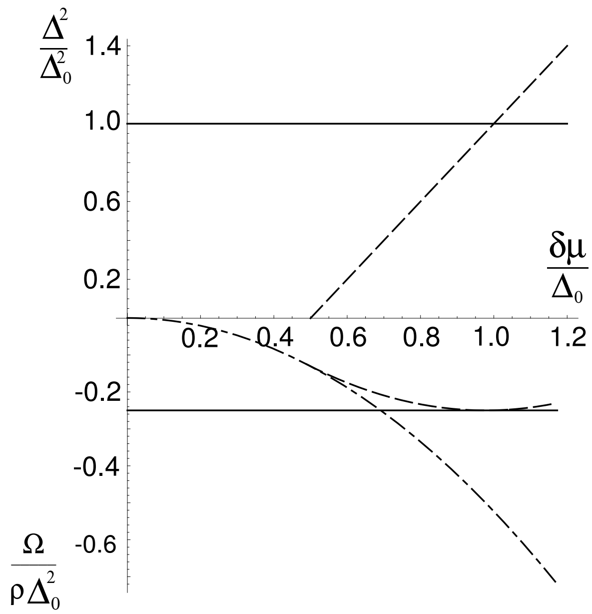

FIG. 28 *Gap and grand potential as functions of $\delta\mu$ for the two solutions a) and b) discussed in the text, see Eqs.(9.20), (9.21) and (9.28), (9.29). Upper solid (resp. dashed) line: Gap for solution a) (resp. solution b)). In the lower part we plot the grand potential for the solution a) (solid line) and solution b) (dashed line); we also plot the grand potential for the normal gapless state with $\delta\mu \neq 0$ (dashed-dotted line). All the grand potentials are referred to the value $\Omega_0(0)$ (normal state with $\delta\mu = 0$).*

We consider the grand potential, as measured from the normal state, near a second order phase transition

$$\Omega = \frac{1}{2}\alpha\Delta^2 + \frac{1}{4}\beta\Delta^4 + \frac{1}{6}\gamma\Delta^6\,. \tag{9.30}$$

Minimization gives the gap equation:

$$\alpha\Delta + \beta\Delta^3 + \gamma\Delta^5 = 0\,. \tag{9.31}$$

Expanding Eq. (9.7) up to the 5^{th} order in Δ and comparing with the previous equation one determines the coefficients α, β and γ up to a normalization constant. One gets

$$\Delta = 2\,g\,\rho T\,Re\sum_{n=0}^{\infty}\int_0^{\delta} d\xi\left[\frac{\Delta}{(\bar\omega_n^2+\xi^2)} - \frac{\Delta^3}{(\bar\omega_n^2+\xi^2)^2} + \frac{\Delta^5}{(\bar\omega_n^2+\xi^2)^3} + \cdots\right]\,, \tag{9.32}$$

with

$$\bar\omega_n = \omega_n + i\delta\mu = (2n+1)\pi T + i\delta\mu\,. \tag{9.33}$$

As we have discussed in Section III.E the grand potential can be obtained, integrating in Δ the gap equation and integrating the result provided that we multiply it by the factor $2/g$. Therefore

$$\alpha = \frac{2}{g}\left(1 - 2\,g\,\rho T\,Re\sum_{n=0}^{\infty}\int_0^{\delta}\frac{d\xi}{(\bar\omega_n^2+\xi^2)}\right)\,, \tag{9.34}$$

$$\beta = 4\rho T\,Re\sum_{n=0}^{\infty}\int_0^{\infty}\frac{d\xi}{(\bar\omega_n^2+\xi^2)^2}\,, \tag{9.35}$$

$$\gamma = -4\rho T\,Re\sum_{n=0}^{\infty}\int_0^{\infty}\frac{d\xi}{(\bar\omega_n^2+\xi^2)^3}\,. \tag{9.36}$$

Notice that for $\delta\mu = 0$ we recover the expressions of Section III.E. In the coefficients β and γ we have extended the integration in ξ up to infinity since both the sum and the integral are convergent. To evaluate α we proceed as in Section III.E first integrating over ξ and then summing over the Matsubara frequencies (Buzdin and Kachkachi, 1997). In Eq. (9.34) we first integrate over ξ obtaining a divergent series which can be regulated cutting the sum at a maximal value of n determined by

$$\omega_N = \delta \Rightarrow N \approx \frac{\delta}{2\pi T} \,. \tag{9.37}$$

We obtain

$$\alpha = \frac{2}{g}\left(1 - \pi\, g\, \rho\, T\, Re \sum_{n=0}^{N} \frac{1}{\bar{\omega}_n}\right) \,. \tag{9.38}$$

The sum can be performed in terms of the Euler's function $\psi(z)$:

$$\begin{aligned} Re \sum_{n=0}^{N} \frac{1}{\bar{\omega}_n} &= \frac{1}{2\pi T}\, Re \left[\psi\left(\frac{3}{2} + i\frac{y}{2\pi} + N\right) - \psi\left(\frac{1}{2} + i\frac{y}{2\pi}\right)\right] \\ &\approx \frac{1}{2\pi T}\left(\log\frac{\delta}{2\pi T} - Re\,\psi\left(\frac{1}{2} + i\frac{y}{2\pi}\right)\right) \,, \end{aligned} \tag{9.39}$$

where

$$y = \frac{\delta\mu}{T}. \tag{9.40}$$

Eliminating the cutoff and using the gap equation at $T = 0$ we find

$$\alpha(v,t) = \rho\left(\log(4\pi t) + Re\,\psi\left(\frac{1}{2} + i\frac{v}{2\pi t}\right)\right) \,. \tag{9.41}$$

with

$$v = \frac{\delta\mu}{\Delta_0}, \quad t = \frac{T}{\Delta_0}, \quad y = \frac{v}{t}. \tag{9.42}$$

Let us introduce the function $T_c(y)$ defined by

$$\log\frac{\Delta_0}{4\pi T_c(y)} = Re\,\psi\left(\frac{1}{2} + \frac{iy}{2\pi}\right) . \tag{9.43}$$

Then we find

$$\alpha(v,t) = \rho \log\frac{t}{t_c(v/t)}, \tag{9.44}$$

where

$$t_c(y) = \frac{T_c(y)}{\Delta_0}. \tag{9.45}$$

The line where $\alpha(v,t) = 0$, that is

$$t = t_c(v/t) \tag{9.46}$$

defines the second order phase transitions (see discussion later). In particular at $\delta\mu = 0$, using (C the Euler-Mascheroni constant)

$$\psi\left(\frac{1}{2}\right) = -\log(4\gamma), \qquad \gamma = e^C, \qquad C = 0.5777\ldots \,, \tag{9.47}$$

we find from Eq. (9.41)

$$\alpha(0, T/\Delta_0) = \rho\,\log\frac{\pi T}{\gamma\Delta_0} \,, \tag{9.48}$$

reproducing the critical temperature for the BCS case

$$T_c = \frac{\gamma}{\pi}\Delta_0 \approx 0.56693\,\Delta_0\,. \tag{9.49}$$

The other terms in the expansion of the gap equation are easily evaluated integrating over ξ and summing over the Matsubara frequencies. We get

$$\beta \;=\; \pi\,\rho\,T\,Re\sum_{n=0}^{\infty}\frac{1}{\bar\omega_n^3} = -\frac{\rho}{16\,\pi^2\,T^2}Re\,\psi^{(2)}\left(\frac{1}{2}+i\frac{\delta\mu}{2\pi T}\right)\,, \tag{9.50}$$

$$\gamma \;=\; -\frac{3}{4}\pi\,\rho\,T\,Re\sum_{n=0}^{\infty}\frac{1}{\bar\omega_n^5} = \frac{3}{4}\frac{\rho}{768\,\pi^4\,T^4}Re\,\psi^{(4)}\left(\frac{1}{2}+i\frac{\delta\mu}{2\pi T}\right)\,, \tag{9.51}$$

where

$$\psi^{(n)}(z) = \frac{d^n}{dz^n}\psi(z)\,. \tag{9.52}$$

Let us now briefly review some results on the grand potential in the GL expansion (9.30). We will assume $\gamma > 0$ in order to ensure the stability of the potential. The minimization leads to the solutions

$$\Delta = 0\,, \tag{9.53}$$

$$\Delta^2 = \Delta_\pm^2 = \frac{1}{2\gamma}\left(-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}\right)\,. \tag{9.54}$$

The discussion of the minima of Ω depends on the signs of the parameters α and β. The results are the following:

1. $\boxed{\alpha > 0, \quad \beta > 0}$

   In this case there is a single minimum given by (9.53) and the phase is symmetric.

2. $\boxed{\alpha > 0, \quad \beta < 0}$

   Here there are three minima, one is given by (9.53) and the other two are degenerate minima at

   $$\Delta = \pm\Delta_+\,. \tag{9.55}$$

   The line along which the three minima become equal is given by:

   $$\Omega(0) = \Omega(\pm\Delta_+) \quad\longrightarrow\quad \beta = -4\sqrt{\frac{\alpha\gamma}{3}}\,. \tag{9.56}$$

   Along this line there is a first order transition with a discontinuity in the gap given by

   $$\Delta_+^2 = -\frac{4\alpha}{\beta} = -\frac{3}{4}\frac{\beta}{\gamma}\,. \tag{9.57}$$

   To the right of the first order line we have $\Omega(0) < \Omega(\pm\Delta_+)$. It follows that to the right of this line there is the symmetric phase, whereas the broken phase is in the left part (see Fig. 29).

3. $\boxed{\alpha < 0, \quad \beta > 0}$

   In this case Eq. (9.53) gives a maximum, and there are two degenerate minima given by Eq. (9.55). Since for $\alpha > 0$ the two minima disappear, it follows that there is a second order phase transition along the line $\alpha = 0$. This can also be seen by noticing that going from the broken phase to the symmetric one we have

   $$\lim_{\alpha\to 0}\Delta_+^2 = 0\,. \tag{9.58}$$

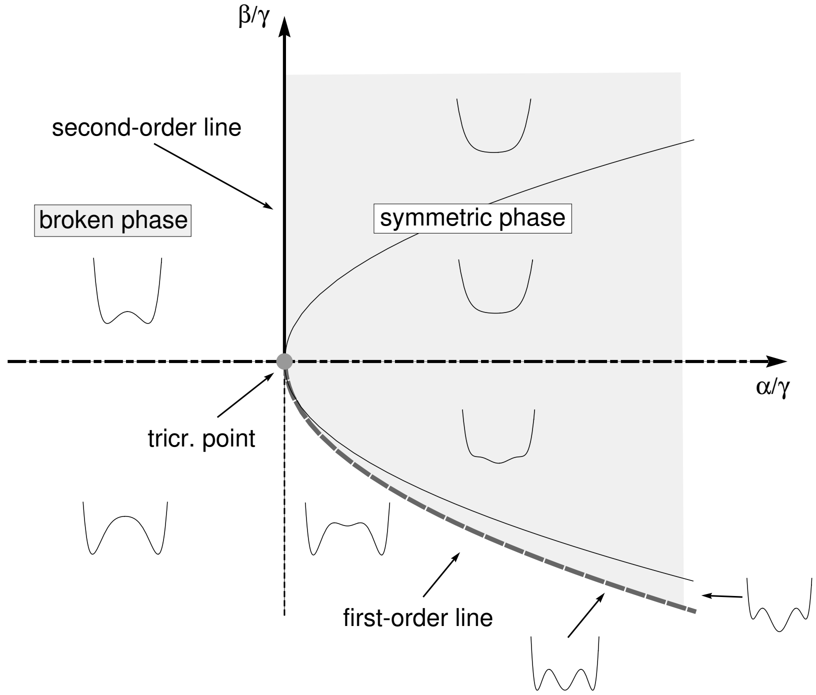

FIG. 29 *The graph shows the first order and the second order transition lines for the potential of Eq. (9.30). We show the tricritical point and the regions corresponding to the symmetric and the broken phase. Also shown is the behavior of the grand potential in the various regions. The thin solid line is the locus of the points $\beta^2 - 4\alpha\gamma = 0$. In the interior region we have $\beta^2 - 4\alpha\gamma < 0$.*

4. $\boxed{\alpha < 0, \quad \beta < 0}$

   The minima and the maximum are as in the previous case.

Notice also that the solutions $\Delta_\pm$ do not exist in the region $\beta^2 < 4\alpha\gamma$. The situation is summarized in Fig. 29. Here we show the behavior of the grand potential in the different sectors of the plane $(\alpha/\gamma, \beta/\gamma)$, together with the transition lines. Notice that in the quadrant $(\alpha > 0, \beta < 0)$ there are metastable phases corresponding to non absolute minima. In the sector included between the line $\beta = -2\sqrt{\alpha/\gamma}$ and the first order transition line the metastable phase is the broken one, whereas in the region between the first order and the $\alpha = 0$ lines the metastable phase is the symmetric one.

Using Eqs. (9.41), (9.50) and (9.51) which give the parameters α, β and γ in terms of the variables $v = \delta\mu/\Delta_0$ and $t = T/\Delta_0$, we can map the plane α and β into the plane $(\delta\mu/\Delta_0, T/\Delta_0)$. The result is shown in Fig. 30. From this mapping we can draw several conclusions. First of all the region where the previous discussion in terms of the parameters α, β and γ applies is the inner region of the triangular part delimited by the lines $\gamma = 0$. In fact, as already stressed, our expansion does not hold outside this region. This statement can be made quantitative by noticing that along the first order transition line the gap increases when going away from the tricritical point as

$$\Delta_+^2 = -\frac{4\alpha}{\beta} = \sqrt{\frac{3\alpha}{\gamma}}\,. \tag{9.59}$$

Notice that the lines $\beta(v,t) = 0$ and $\gamma(v,t) = 0$ are straight lines, since these zeroes are determined by the functions $\psi^{(2)}$ and $\psi^{(4)}$ which depend only on the ratio v/t. Calculating the first order line around the tricritical point one gets the result plotted as a solid line in Fig. 30. Since we know that $\delta\mu = \delta\mu_1 = \Delta_0/\sqrt{2}$ is a first order transition point, the first order line must end there. In Fig. 30 we have simply connected the line with the point with a grey dashed line. To get this line a numerical evaluation at all orders in Δ would be required. This is feasible but we will skip it since the results will not be necessary in the following, see (Sarma, 1963). The location of the tricritical point is determined by the intersection of the lines $\alpha = 0$ and $\beta = 0$. One finds (Buzdin and Kachkachi, 1997; Combescot and

Mora, 2002)

$$\frac{\delta\mu}{\Delta_0}\Big|_{\rm tric} = 0.60822, \qquad \frac{T}{\Delta_0}\Big|_{\rm tric} = 0.31833\,. \tag{9.60}$$

We also note that the line $\alpha = 0$ should cross the temperature axis at the BCS point. In this way one reobtains the result in Eq. (9.49) for the BCS critical temperature, and also the value for the tricritical temperature

$$\frac{T_{\rm tric}}{T_{\rm BCS}} = 0.56149\,. \tag{9.61}$$

The results given in this Section are valid as long as other possible condensates are neglected. In fact, we will see that close to the first order transition of the homogeneous phase the LOFF phase with inhomogeneous gap can be formed.

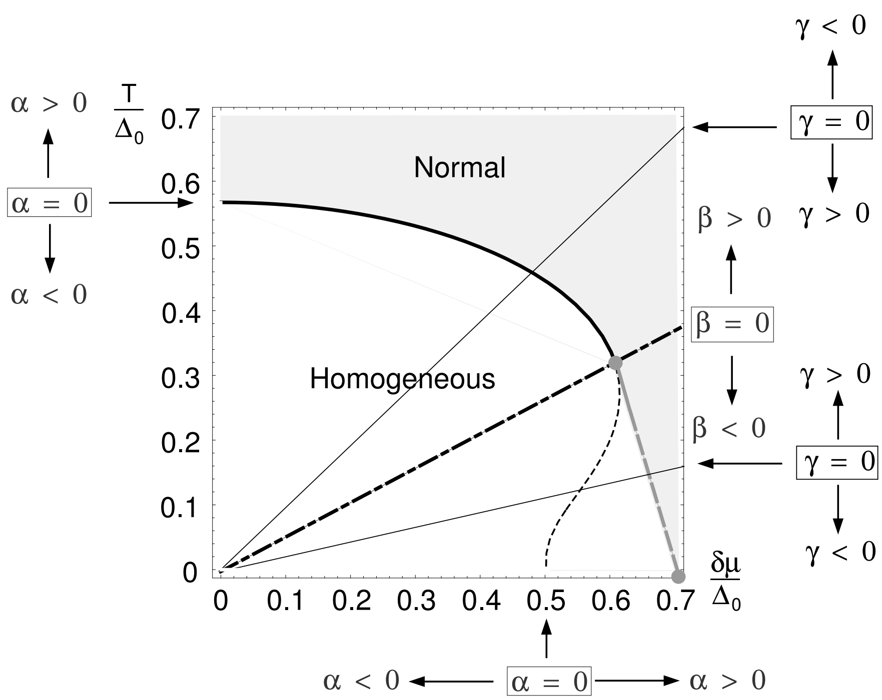

FIG. 30 *The curve shows the points solutions of the equation* $\Delta = 0$ *in the plane* $(v, t) = (\delta\mu/\Delta_0, T/\Delta_0)$. *The tricritical point at* $(\delta\mu, T) \approx (0.61, 0.32)\,\Delta_0$ *is also shown. The upper part of the curve (solid line) separates the homogeneous phase from the normal one. Along the dashed line* $\Delta = 0$ *but this is not the absolute minimum of the grand potential.*

B. Dependence of the condensate on the quark masses

In the previous Section we have ignored all the mass corrections but for the ones coming from the shift of the chemical potential. However there is a small mass dependence on the condensate itself. This is because the presence of the mass gives rise to a reduction of the density of the states at the Fermi surface weakening the gap. This problem has been studied in (Casalbuoni *et al.*, 2002a; Kundu and Rajagopal, 2002). We will follow here the approach given in (Casalbuoni *et al.*, 2002a). Let us consider the QCD lagrangian

$$\mathcal{L}_{QCD} = \bar\psi\,(i\,\not{D} + \mu\,\gamma^0 - m)\,\psi = \bar\psi\,(i\,\not{D} + \mu\,\gamma^0 - x\mu)\,\psi, \tag{9.62}$$

where we will keep

$$x = \frac{m}{\mu}\,, \qquad 0 < x < 1 \tag{9.63}$$

fixed in the $\mu \to \infty$ limit. This particular limit is convenient when one discusses problems related to compact stellar objects where the strange quark mass is not very far from the relevant chemical potential (of order of 400 MeV). This lagrangian and the one at zero quark mass can be related by making use of the Cini-Touschek transformation (Cini and Touschek, 1958) that was invented to study the ultra-relativistic limit of the Dirac equation for a massive fermion.

In order to describe the method used we need to discuss a little kinematics. In the massive case, as we have seen, the Fermi momentum is given by

$$p_F^2 = \mu^2(1 - x^2) \tag{9.64}$$

and the Fermi velocity by

$$\mathbf{v}_F = \frac{\partial E(\mathbf{p})}{\partial \mathbf{p}}\Big|_{p=p_F} = \frac{\mathbf{p}_F}{E_F} = \frac{\mathbf{p}_F}{\mu}. \tag{9.65}$$

Introducing the unit vector

$$\mathbf{n} = \frac{\mathbf{p}_F}{|\mathbf{p}_F|}, \tag{9.66}$$

we get

$$\mathbf{v}_F = \sqrt{1 - x^2}\vec{n}. \tag{9.67}$$

Now let us do again the decomposition of the momentum in the Fermi momentum plus the residual momentum

$$\mathbf{p} = \mathbf{p}_F + \boldsymbol{\ell} = \mu\sqrt{1 - x^2}\,\mathbf{n} + \boldsymbol{\ell}. \tag{9.68}$$

Substituting inside the Dirac hamiltonian

$$H = p_0 = \boldsymbol{\alpha} \cdot \mathbf{p} - \mu + m\gamma_0, \tag{9.69}$$

we get

$$H = -\mu + \mu(\sqrt{1 - x^2}\boldsymbol{\alpha} \cdot \mathbf{n} + x\gamma_0) + \boldsymbol{\alpha} \cdot \boldsymbol{\ell}. \tag{9.70}$$

We proceed now introducing the following two projection operators

$$P_\pm = \frac{1 \pm \left(x\gamma_0 - \sqrt{1 - x^2}\boldsymbol{\alpha} \cdot \mathbf{n}\right)}{2}. \tag{9.71}$$

These are projection operators since the square of the operator in parenthesis is one. In terms of the projected wave functions the Dirac equations splits into the two equations

$$H\psi_+ = \boldsymbol{\alpha} \cdot \boldsymbol{\ell}\psi_+, \quad H\psi_- = \left(-2\mu + \boldsymbol{\alpha} \cdot \boldsymbol{\ell}\right)\psi_-. \tag{9.72}$$

Therefore we reproduce exactly the same situation as in the massless case (see Section V.B) and all the formalism holds true introducing the two four-vectors

$$V^\mu = (1, \mathbf{v}_F), \qquad \tilde{V}^\mu = (1, -\mathbf{v}_F), \tag{9.73}$$

with

$$|\mathbf{v}_F|^2 = 1 - x^2. \tag{9.74}$$

Without entering in too many details we can now explain the results found in (Casalbuoni *et al.*, 2002a). One starts from the same four-fermi interaction of Section V.B.2 and writes the Schwinger-Dyson equation obtaining a gap equation which is the same obtained in Section V.B.2 with a few differences. Since the main interest of the paper was to understand how the condensate varies with quark masses the assumption was made that the chemical potentials

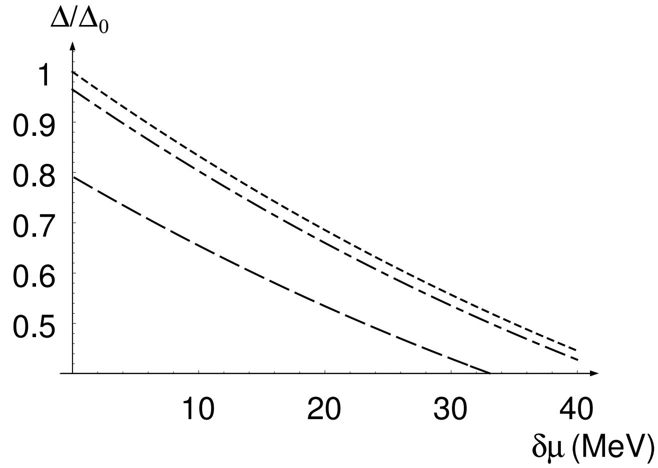

FIG. 31 *The ratio of the gap for finite masses Δ over the gap for massless quarks Δ_0 as a function of $\delta\mu$. The lines give the ratio for values $x_1 = 0$ (dotted line), $x_1 = 0.1$ (dash-dotted line), $x_1 = 0.25$ (dashed one). The values used are $\mu = 400\ MeV$, $\delta = 400\ MeV$, $G = 10.2\ GeV^{-2}$, corresponding to $\Delta_0 \approx 40\ MeV$.*

of the up and down quarks were arranged in such a way to make equal the two Fermi momentum. This gives the relation

$$p_F = p_{F_1} = p_{F_2} = \sqrt{\mu_1^2 - m_1^2} = \sqrt{\mu_2^2 - m_2^2}, \tag{9.75}$$

from which

$$\delta\mu = \frac{m_1^2 - m_2^2}{4}. \tag{9.76}$$

Then we have the main modifications come from a change of the density at the Fermi surface. Precisely

$$\mu^2 \to p_F^2 = (\mu + \delta\mu)(\mu - \delta\mu)\alpha(x_1, x_2), \tag{9.77}$$

where

$$\alpha(x_1, x_2) = \sqrt{(1 - x_1^2)(1 - x_2^2)}. \tag{9.78}$$

Also there is a factor

$$\tilde{V}_2 \cdot V_1 = \frac{1 + \alpha(x_1, x_2)}{2} \tag{9.79}$$

coming from the interaction, and finally a modification of the propagator (where the product $(\tilde{V}_2 \cdot \ell)V_1 \cdot \ell)$ appears. Putting everything together we find that the gap is given by

$$\Delta = 2\sqrt{\frac{4\alpha + \beta^2}{4}}\delta e^{-2/\rho_N G} \tag{9.80}$$

where

$$\beta(x_1, x_2) = \sqrt{1 - x_1^2} - \sqrt{1 - x_2^2} \tag{9.81}$$

and

$$\rho_N = \frac{4\mu^2}{\pi^2}\left(1 - \frac{\delta\mu^2}{\mu^2}\right)\frac{\alpha(x_1, x_2)(1 + \alpha(x_1, x_2))}{\sqrt{4\alpha(x_1, x_2) + \beta(x_1, x_2)^2}}. \tag{9.82}$$

In Fig. 31 we plot the condensate normalized at its value, Δ_0, for $m_1 = m_2 = 0$ as a function of $\delta\mu$ in different situations. We have chosen $\mu = 400\ MeV$, $\delta = 400\ MeV$ and $G = 10.3\ GeV^{-2}$ in such a way that $\Delta_0 = 40\ MeV$. The diagram refers to one massive quark and one massless. Notice also that plotting $\delta\mu$ is the same as plotting $m_2^2 - m_1^2$.

X. THE LOFF PHASE

We will review here briefly the so called LOFF phase. Many more details can be found in two recent reviews (Bowers, 2003; Casalbuoni and Nardulli, 2003). We have already discussed the fact that when the chemical potentials of two fermions are too apart the condensate may break. In particular we have shown (see Section IX) that when the difference between the two chemical potentials, $\delta\mu$, satisfies

$$\delta\mu = \frac{\Delta_0}{\sqrt{2}}, \tag{10.1}$$

where Δ_0 is the BCS gap at $\delta\mu = 0$, the system undergoes a first order phase transition, with the gap going from Δ_0 to zero. However just close at this point something different may happen. According to the authors (Fulde and Ferrell, 1964; Larkin and Ovchinnikov, 1964) when fermions belong to two different Fermi spheres, they may prefer to pair staying as much as possible close to their own Fermi surface. When they are sitting exactly at the surface, the pairing is as shown in Fig. 32. We see that the total momentum of the pair is $\mathbf{p}_1 + \mathbf{p}_2 = 2\mathbf{q}$ and, as we shall see, $|\mathbf{q}|$

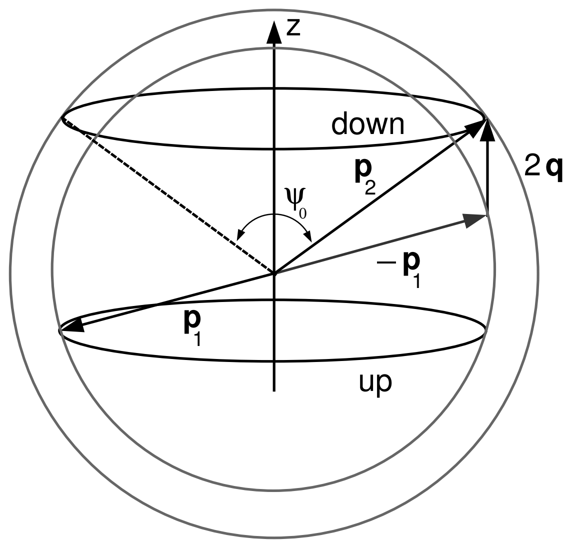

FIG. 32 *Pairing of fermions belonging to two Fermi spheres of different radii according to LOFF.*

is fixed variationally whereas the direction of $\mathbf{q}$ is chosen spontaneously. Since the total momentum of the pair is not zero the condensate breaks rotational and translational invariance. The simplest form of the condensate compatible with this breaking is just a simple plane wave (more complicated functions will be considered later)

$$\langle\psi(x)\psi(x)\rangle \approx \Delta\, e^{2i\mathbf{q}\cdot\mathbf{x}}. \tag{10.2}$$

It should also be noticed that the pairs use much less of the Fermi surface than they do in the BCS case. In fact, in the case considered in Fig. 32 the fermions can pair only if they belong to the circles drawn there. More generally there is a quite large region in momentum space (the so called blocking region) which is excluded from the pairing. This leads to a condensate smaller than the BCS one.

Let us now begin the discussion. Remember that for two fermions at different densities we have an extra term in the hamiltonian which can be written as

$$H_I = -\delta\mu\sigma_3. \tag{10.3}$$

In the original LOFF papers (Fulde and Ferrell, 1964; Larkin and Ovchinnikov, 1964) the case of ferromagnetic alloys with paramagnetic impurities was considered. The impurities produce a constant magnetic exchange field which, acting upon the electron spin, gives rise to an effective difference in the chemical potential of the opposite fields. In this case $\delta\mu$ is proportional to the exchange field. In the actual case $\delta\mu = (\mu_1 - \mu_2)/2$ and σ_3 is a Pauli matrix acting on the space of the two fermions. According to (Fulde and Ferrell, 1964; Larkin and Ovchinnikov, 1964) this favors the formation of pairs with momenta

$$\mathbf{p}_1 = \mathbf{k} + \mathbf{q}, \quad \mathbf{p}_2 = -\mathbf{k} + \mathbf{q}. \tag{10.4}$$

We will discuss in detail the case of a single plane wave (see Eq. (10.2)) and we will give some results about the general case. The interaction term of Eq. (10.3) gives rise to a shift in the variable $\xi = E(\mathbf{p}) - \mu$ due both to the non-zero momentum of the pair and to the different chemical potential

$$\xi = E(\vec{p}) - \mu \to E(\pm\vec{k} + \vec{q}) - \mu \mp \delta\mu \approx \xi \mp \bar{\mu}, \tag{10.5}$$

with

$$\bar{\mu} = \delta\mu - \vec{v}_F \cdot \vec{q}. \tag{10.6}$$

Here we have assumed $\delta\mu \ll \mu$ (with $\mu = (\mu_1 + \mu_2)/2$) allowing us to expand E at the first order in $\mathbf{q}$ (see Fig. 32). The gap equation has the same formal expression as Eq. (3.143) for the BCS case

$$1 = \frac{g}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1 - n_u - n_d}{\epsilon(\vec{p}, \Delta)}, \tag{10.7}$$

but now $n_u \neq n_d$

$$n_{u,d} = \frac{1}{e^{(\epsilon(\vec{p},\Delta) \pm \bar{\mu})/T} + 1}, \tag{10.8}$$

where Δ is the LOFF gap. In the limit of zero temperature we obtain

$$1 = \frac{g}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\epsilon(\vec{p}, \Delta)} \left(1 - \theta(-\epsilon - \bar{\mu}) - \theta(-\epsilon + \bar{\mu})\right). \tag{10.9}$$

The two step functions can be interpreted saying that at zero temperature there is no pairing when $\epsilon(\vec{p}, \Delta) < |\bar{\mu}|$. This inequality defines the so called blocking region. The effect is to inhibit part of the Fermi surface to the pairing giving rise a to a smaller condensate with respect to the BCS case where all the surface is used.

We are now in the position to show that increasing $\delta\mu$ from zero we have first the BCS phase. Then at $\delta\mu \approx \delta\mu_1$ there is a first order transition to the LOFF phase (Alford *et al.*, 2001; Larkin and Ovchinnikov, 1964), and at $\delta\mu = \delta\mu_2 > \delta\mu_1$ there is a second order phase transition to the normal phase (with zero gap) (Alford *et al.*, 2001; Larkin and Ovchinnikov, 1964). We start comparing the grand potential in the BCS phase to the one in the normal phase. Their difference, from Eq. (9.28), is given by

$$\Omega_{\rm BCS} - \Omega_{\rm normal} = -\frac{\rho}{4}\left(\Delta_{BCS}^2 - 2\delta\mu^2\right). \tag{10.10}$$

We have assumed $\delta\mu \ll \mu$. The situation is represented in Fig. 33. In order to compare with the LOFF phase we

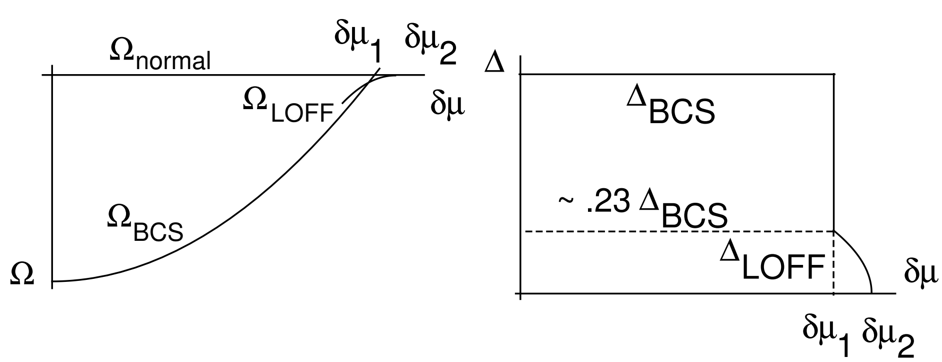

FIG. 33 *The grand potential and the condensate of the BCS and LOFF phases vs.* $\delta\mu$.

will now expand the gap equation around the point $\Delta = 0$ (Ginzburg-Landau expansion) exploring the possibility of a second order phase transition. Using the gap equation for the BCS phase in the first term on the right-hand side of Eq. (10.9) and integrating the other two terms in ξ we get

$$\rho \log \frac{\Delta_{BCS}}{\Delta} = \rho \int \frac{d\Omega}{4\pi} \,\text{arcsinh} \frac{C(\theta)}{\Delta}, \tag{10.11}$$

where

$$C(\theta)=\sqrt{(\delta\mu-qv_F\cos\theta)^2-\Delta^2}. \tag{10.12}$$

For $\Delta\to 0$ we get easily

$$-\log\frac{\Delta_{BCS}}{2\delta\mu}+\frac{1}{2}\int_{-1}^{+1}\log\left(1-\frac{u}{z}\right)=0,\quad z=\frac{\delta\mu}{qv_F}. \tag{10.13}$$

This expression is proportional to the coefficient α in the Ginzburg-Landau expansion (recall the discussion in Section IX.A). Therefore it should be minimized with respect to q. The minimum is given by

$$\frac{1}{z}\tanh\frac{1}{z}=1, \tag{10.14}$$

implying

$$qv_F\approx 1.2\,\delta\mu_2. \tag{10.15}$$

Putting this value back in eq. (10.13) we obtain

$$\delta\mu_2\approx 0.754\,\Delta_{BCS}. \tag{10.16}$$

From the expansion of the gap equation around $\delta\mu_2$ it is easy to obtain

$$\Delta^2\approx 1.76\,\delta\mu_2(\delta\mu_2-\delta\mu). \tag{10.17}$$

Recalling Eq. (9.23), we can express the grand potential of the LOFF phase relatively to the one of the normal phase as

$$\Omega_{\rm LOFF}-\Omega_{\rm normal}=-\int_0^g\frac{dg}{g^2}\Delta^2. \tag{10.18}$$

Using Eq. (3.64) for the BCS gap and Eq.(10.16) we can write

$$\frac{dg}{g^2}=\frac{\rho}{2}\frac{d\Delta_{BCS}}{\Delta_{BCS}}=\frac{\rho}{2}\frac{d\delta\mu_2}{\delta\mu_2}. \tag{10.19}$$

Noticing that Δ is zero for $\delta\mu_2=\delta\mu$ we are now able to perform the integral (10.18) obtaining

$$\Omega_{\rm LOFF}-\Omega_{\rm normal}\approx -0.44\,\rho(\delta\mu-\delta\mu_2)^2. \tag{10.20}$$

We see that in the window between the intersection of the BCS curve and the LOFF curve in Fig. 33 and $\delta\mu_2$ the LOFF phase is favored. Furthermore at the intersection there is a first order transition between the LOFF and the BCS phase. Notice that since $\delta\mu_2$ is very close to $\delta\mu_1$ the intersection point is practically given by $\delta\mu_1$. In Fig. 33 we show also the behaviour of the condensates. Altough the window $(\delta\mu_1,\delta\mu_2)\simeq(0.707,0.754)\Delta_{BCS}$ is rather narrow, there are indications that considering the realistic case of QCD (Leibovich *et al.*, 2001) the window may open up. Also, for different structures than the single plane wave there is the possibility that the window opens up (Leibovich *et al.*, 2001).

A. Crystalline structures

The ground state in the LOFF phase is a superposition of states with different occupation numbers (N even)

$$|0\rangle_{LOFF}=\sum_N c_N|N\rangle. \tag{10.21}$$

Therefore the general structure of the condensate in the LOFF ground state will be

$$\begin{aligned}\langle\psi(x)\psi(x)\rangle &= \sum_N c_N^* c_{N+2}e^{2i\mathbf{q}_N\cdot\mathbf{x}}\langle N|\psi(x)\psi(x)|N+2\rangle\\ &= \sum_N \Delta_N e^{2i\mathbf{q}_N\cdot\vec{x}}.\end{aligned} \tag{10.22}$$

The case considered previously corresponds to all the Cooper pairs having the same total momentum $2\mathbf{q}$. A more general situation, although not the most general, is when the vectors $\mathbf{q}_N$ reduce to a set $\mathbf{q}_i$ defining a regular crystalline structure. The corresponding coefficients $\Delta_{\mathbf{q}_i}$ (linear combinations of subsets of the Δ_N's) do not depend on the vectors $\mathbf{q}_i$ since all the vectors belong to the same orbit of the group. Furthermore all the vectors $\mathbf{q}_i$ have the same length (Bowers and Rajagopal, 2002) given by Eq. (10.15). In this case

$$\langle 0|\psi(x)\psi(x)|0\rangle = \Delta_q \sum_i e^{2i\mathbf{q}_i\cdot\mathbf{x}}. \tag{10.23}$$

This more general case has been considered in (Bowers and Rajagopal, 2002; Larkin and Ovchinnikov, 1964) by evaluating the grand-potential of various crystalline structures through a Ginzburg-Landau expansion, up to sixth order in the gap (Bowers and Rajagopal, 2002)

$$\Omega = \alpha\Delta^2 + \frac{\beta}{2}\Delta^4 + \frac{\gamma}{3}\Delta^6. \tag{10.24}$$

These coefficients can be evaluated microscopically for each given crystalline structure by following the procedure outlined in Sections IX.A and III.E. The general procedure is to start from the gap equation represented graphically in Fig. 34. Then, one expands the exact propagator in a series of the gap insertions as given in Fig. 35. Inserting

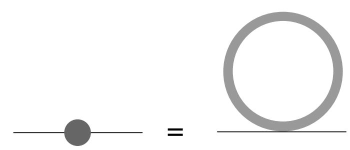

FIG. 34 *Gap equation in graphical form. The thick line is the exact propagator. The black dot the gap insertion.*

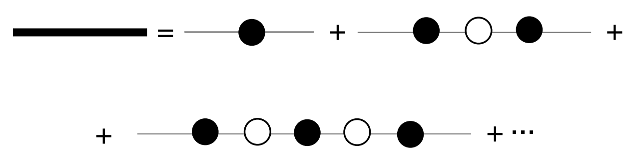

FIG. 35 *The expansion of the propagator in graphical form. The darker boxes represent a Δ^* insertion, the lighter ones a Δ insertion.*

this expression back into the gap equation one gets the expansion illustrated in Fig. 36.

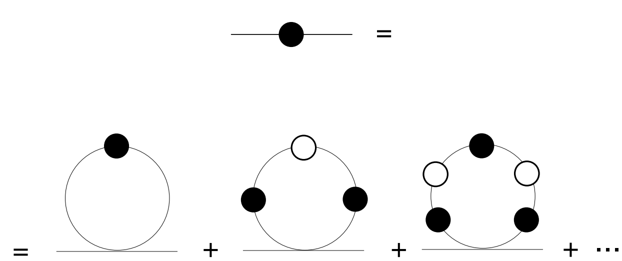

FIG. 36 *The expansion of the gap equation in graphical form. Notations as in Fig. 35.*

On the other hand the gap equation is obtained minimizing the grand-potential (10.24), i.e.

$$\alpha\Delta + \beta\Delta^3 + \gamma\Delta^5 + \cdots = 0. \tag{10.25}$$

Comparing this expression with the result of Fig. 36 one is able to derive the coefficients α, β and γ. Except for an overall coefficient (the number of plane waves) the coefficient α has the same expression for all kind of crystals. Therefore the results obtained for the single plane wave and depending on the properties of this coefficients are universal.

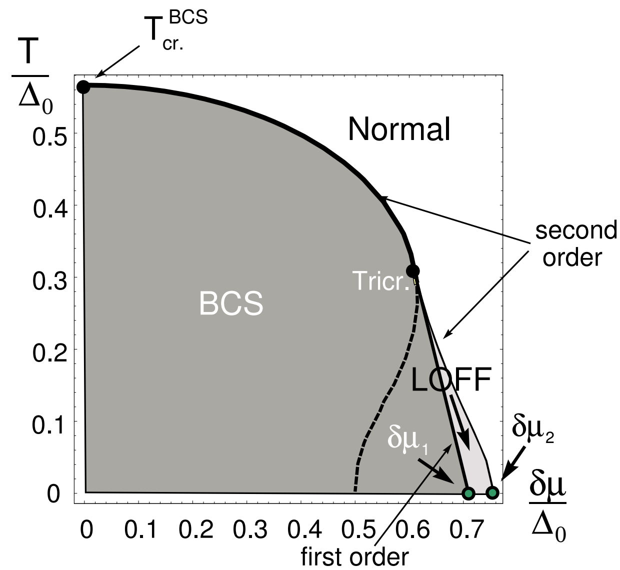

FIG. 37 *The phase diagram for the LOFF phase in the plane $(\delta\mu, T)$. It is shown the tricritical point. Here three lines meet: the second order transition line from the normal case to the BCS phase, the first order transition line from the BCS phase to the LOFF phase and the second order transition line from the LOFF phase to the normal one.*

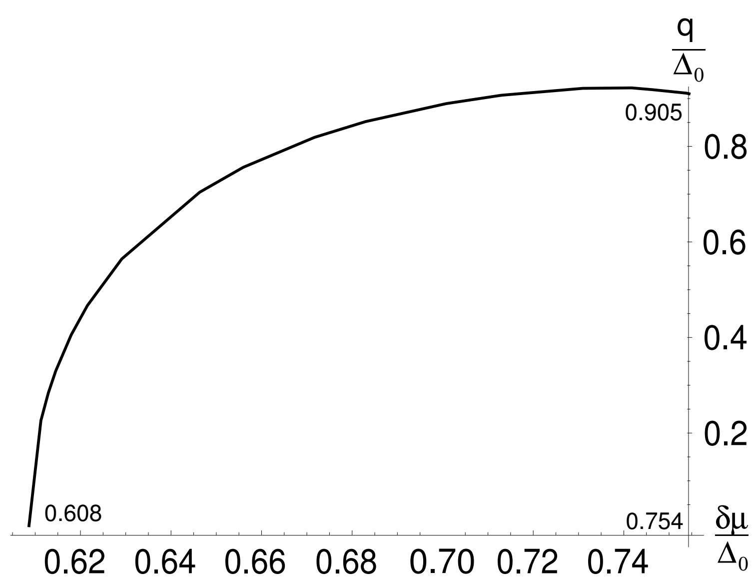

FIG. 38 *The behavior of the length of the vector* **q** *along the second order transition line from the LOFF phase to the normal phase is shown, vs.* $\delta\mu/\Delta_0$.

In ref. (Bowers and Rajagopal, 2002) more than 20 crystalline structures have been considered, evaluating for each of them the coefficients of Eq. (10.24). The result of this analysis is that the face-centered cube appears to be the favored structure among the ones considered (for more details see ref. (Bowers and Rajagopal, 2002)). However it should be noticed that this result can be trusted only at $T = 0$. In fact one knows that in the $(\delta\mu, T)$ phase space, the LOFF phase has a tricritical point and that around this point the favored crystalline phase corresponds to two antipodal waves (see (Buzdin and Kachkachi, 1997; Combescot and Mora, 2002) and for a review (Casalbuoni and Nardulli, 2003)). Therefore there could be various phase transitions going down in temperature as it happens in the two-dimensional case (Shimahara, 1998). For completeness we give here the phase diagram of the LOFF phase in the plane $(\delta\mu, T)$, in Fig. 37, and the behavior of the length of the vector $\mathbf{q}$ along the second order critical line in Fig 38.

B. Phonons

As we have seen QCD at high density is conveniently studied through a hierarchy of effective field theories, see Section V.A. By using the same procedure in the case of the LOFF phase one can derive the analogue of the HDET (Casalbuoni *et al.*, 2001b, 2002c) and the effective lagrangian for the Goldstone bosons (phonons) associated to the breaking of translational and rotational symmetries in the LOFF phase (Casalbuoni *et al.*, 2002b,e). The number and the features of the phonons depend on the particular crystalline structure. We will consider here the case of the single plane-wave (Casalbuoni *et al.*, 2001b) and of the face-centered cube (Casalbuoni *et al.*, 2002e). We will introduce the phonons as it is usual for NG bosons (Casalbuoni *et al.*, 2001b), that is as the phases of the condensate. Considering the case of a single plane-wave we introduce a scalar field $\Phi(x)$ through the replacement

$$\Delta(\vec{x}) = \exp^{2i\mathbf{q}\cdot\mathbf{x}}\Delta \to e^{i\Phi(x)}\Delta. \tag{10.26}$$

We require that the scalar field $\Phi(x)$ acquires the following expectation value in the ground state

$$\langle\Phi(x)\rangle = 2\,\mathbf{q}\cdot\mathbf{x}. \tag{10.27}$$

The phonon field is defined as

$$\frac{1}{f}\phi(x) = \Phi(x) - 2\mathbf{q}\cdot\mathbf{x}. \tag{10.28}$$

Notice that the phonon field transforms nontrivially under rotations and translations. From this it follows that non derivative terms for $\phi(x)$ are not allowed. One starts with the most general invariant lagrangian for the field $\Phi(x)$ in the low-energy limit. This cuts the expansion of Φ to the second order in the time derivative. However one may have an arbitrary number of space derivative, since from Eq. (10.27) it follows that the space derivatives do not need to be small. Therefore

$$\mathcal{L}_{\rm phonon} = \frac{f^2}{2}\left(\dot{\Phi}^2 + \sum_k c_k \Phi(\vec{\nabla}^2)^k\Phi\right). \tag{10.29}$$

Using the definition (10.28) and keeping the space derivative up to the second order (we can make this assumption for the phonon field) we find

$$\mathcal{L}_{\rm phonon} = \frac{1}{2}\left(\dot{\phi}^2 - v_\perp^2\vec{\nabla}_\perp\phi\cdot\vec{\nabla}_\perp\phi - v_\parallel^2\vec{\nabla}_\parallel\phi\cdot\vec{\nabla}_\parallel\phi\right), \tag{10.30}$$

where

$$\vec{\nabla}_\parallel = \vec{n}(\vec{n}\cdot\vec{\nabla}), \qquad \vec{\nabla}_\perp = \vec{\nabla} - \vec{\nabla}_\parallel, \qquad \vec{n} = \frac{\vec{q}}{|\vec{q}|}. \tag{10.31}$$

We see that the propagation of the phonon in the crystalline medium is anisotropic.

The same kind of considerations can be made in the case of the cube. The cube is defined by 8 vectors $\mathbf{q}_i$ pointing from the origin of the coordinates to the vertices of the cube parameterized as in Fig. 39.

The condensate is given by (Bowers and Rajagopal, 2002)

$$\Delta(x) = \Delta\sum_{k=1}^{8} e^{2i\vec{q}_k\cdot\vec{x}} = \Delta\sum_{i=1,(\epsilon_i=\pm)}^{3} e^{2i|\vec{q}|\epsilon_i x_i}. \tag{10.32}$$

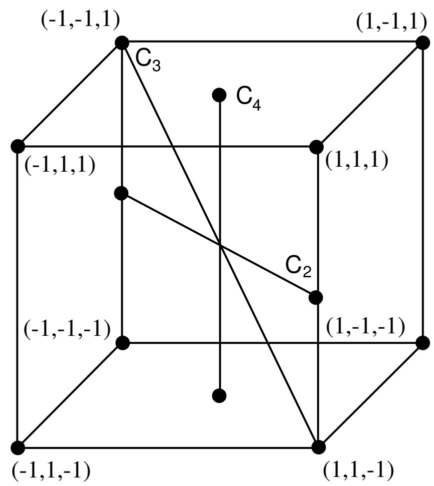

FIG. 39 *The figure shows the vertices and corresponding coordinates of the cube described in the text. Also shown are the symmetry axes.*

We introduce now three scalar fields such that

$$\langle\Phi^{(i)}(x)\rangle = 2|\vec{q}\,|x_i, \tag{10.33}$$

through the substitution

$$\Delta(x) \to \Delta \sum_{i=1,(\epsilon_i=\pm)}^{3} e^{i\epsilon_i\Phi^{(i)}(x)} \tag{10.34}$$

and the phonon fields

$$\frac{1}{f}\phi^{(i)}(x) = \Phi^{(i)}(x) - 2|\vec{q}\,|x_i. \tag{10.35}$$

Notice that the expression (10.34) is invariant under the symmetry group of the cube acting upon the scalar fields $\Phi^{(i)}(x)$. This group has three invariants for the vector representation

$$I_2(\vec{X}) = |\vec{X}|^2, \qquad I_4(\vec{X}) = X_1^2X_2^2 + X_2^2X_3^3 + X_3^2X_1^2$$
$$I_6(\vec{X}) = X_1^2X_2^2X_3^2. \tag{10.36}$$

Therefore the most general invariant lagrangian invariant under rotations, translations and the symmetry group of the cube, at the lowest order in the time derivative, is

$$\begin{aligned} L_{\rm phonon} &= \frac{f^2}{2}\sum_{i=1,2,3}(\dot{\Phi}^{(i)})^2 \\ &+ L_{\rm s}(I_2(\vec{\nabla}\Phi^{(i)}), I_4(\vec{\nabla}\Phi^{(i)}), I_6(\vec{\nabla}\Phi^{(i)})). \end{aligned} \tag{10.37}$$

Expanding this expression at the lowest order in the space derivatives of the phonon fields one finds (Casalbuoni *et al.*, 2002e)

$$\begin{aligned} L_{\rm phonos} &= \frac{1}{2}\sum_{i=1,2,3}(\dot{\phi}^{(i)})^2 - \frac{a}{2}\sum_{i=1,2,3}|\vec{\nabla}\phi^{(i)}|^2 \\ &- \frac{b}{2}\sum_{i=1,2,3}(\partial_i\phi^{(i)})^2 - c\sum_{i<j=1,2,3}\partial_i\phi^{(i)}\partial_j\phi^{(j)}. \end{aligned} \tag{10.38}$$

The parameters appearing in the phonon lagrangian can be evaluated following the strategy outlined in (Casalbuoni *et al.*, 2002b,c) which is the same used for evaluating the parameters of the NG bosons in the CFL phase. It is enough to calculate the self-energy of the phonons (or the NG bosons) through one-loop diagrams due to fermion pairs. Again the couplings of the phonons to the fermions are obtained noticing that the gap acts as a Majorana mass for the quasi-particles. Therefore the couplings originate from the substitutions (10.26) and (10.34). In this way one finds the following results: for the single plane-wave

$$v_\perp^2 = \frac{1}{2}\left(1-\left(\frac{\delta\mu}{|\vec{q}\,|}\right)^2\right), \quad v_\parallel^2 = \left(\frac{\delta\mu}{|\vec{q}\,|}\right)^2 \tag{10.39}$$

and for the cube

$$a = \frac{1}{12}, \quad b = 0, \quad c = \frac{1}{12}\left(3\left(\frac{\delta\mu}{|\vec{q}\,|}\right)^2 - 1\right). \tag{10.40}$$

XI. ASTROPHYSICAL IMPLICATIONS

Should we look for a laboratory to test color superconductivity, we would face the problem that in the high energy physics programmes aiming at new states of matter, such as the Quark Gluon Plasma, the region of the $T-\mu$ plane under investigation is that of low density and high temperature. On the contrary we need physical situations characterized by low temperature and high densities. These conditions are supposed to occur in the inner core of neutron stars, under the hypothesis that, at the center of these compact stars, nuclear matter has become so dense as to allow the transition to quark matter. A schematic view of a neutron star is in Fig. 40.

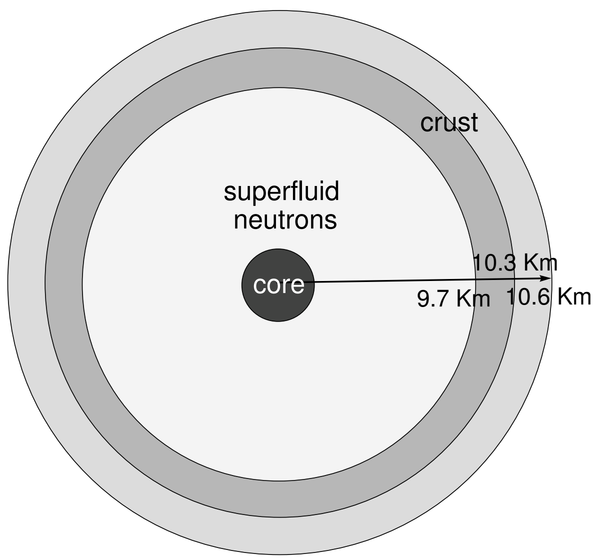

FIG. 40 *Schematic view of a neutron star as computed by an equation of state with three nucleon interaction (Shapiro and Teukolski, 1983).*

In the subsequent sections we shall give a pedagogical introduction to the physics of compact stars and we shall review some possible astrophysical implications of the color superconductivity.

A. A brief introduction to compact stars

To begin with, let us show that for a fermion gas high chemical potential means high density. To simplify the argument we assume that the fermions are massless and not interacting, so that $\mu = \epsilon_F = p_F$. From the grand potential in Eq. (5.166)

$$\Omega = gV\int \frac{d^3\mathbf{p}}{(2\pi)^3}(\epsilon_{\mathbf{p}}-\mu)\theta(\mu-\epsilon_{\mathbf{p}}), \tag{11.1}$$

one finds (g is the number of species of fermions times 2 for the spin)

$$\Omega = -\frac{gV}{24\pi^2}\mu^4. \tag{11.2}$$

It follows that

$$\rho = -\frac{\partial\Omega}{\partial\mu} = \frac{gV}{6\pi^2}\mu^3, \tag{11.3}$$

which means that the chemical potential increases as $\rho^{1/3}$: This is the reason why we should search color superconductivity in media with very high baryonic density.

In the general case, the equation of state is determined, for instance, by evaluating the pressure as a function of the other thermodynamical quantities. This can be done starting from the grand potential. For the simple case of free massless fermions we obtain

$$P = -\Omega = \frac{gV}{24\pi^2}\mu^4, \tag{11.4}$$

and therefore the equation of state has the form

$$P = K\rho^{\frac{4}{3}}. \tag{11.5}$$

Clearly this result also holds for massive fermions in the Ultra Relativistic (UR) case where the mass is negligible. In the Non Relativistic (NR) case one has

$$P = \frac{4}{15}\frac{gV}{\pi^2}\frac{m^{3/2}}{\sqrt{2}}\mu^{5/2}, \qquad \rho = \frac{2}{3}\frac{g}{\pi^2}\frac{m^{3/2}}{\sqrt{2}}\mu^{3/2} \tag{11.6}$$

from which

$$P = K\rho^{\frac{5}{3}}, \tag{11.7}$$

. More generally the equation of state can be approximated by the expression

$$P = K\rho^{\gamma}, \tag{11.8}$$

and the two cases discussed above are characterized as follows:

$$NR\left(\gamma = \frac{5}{3}\right) \quad \begin{array}{l} \rho \ll 10^6 g/cm^3 \ \{\text{electrons}\} \\ \\ \rho \ll 10^{15} g/cm^3 \ \{\text{neutrons}\} \end{array}$$

$$UR\left(\gamma = \frac{4}{3}\right) \quad \begin{array}{l} \rho \gg 10^6 g/cm^3 \ \{\text{electrons}\} \\ \\ \rho \gg 10^{15} g/cm^3 \ \{\text{neutrons}\} \ . \end{array} \tag{11.9}$$

Let us note explicitly that, at $T = 0$, $P \neq 0$. This is a quantum-mechanical effect due to the Pauli principle and the Fermi Dirac statistics (for comparison, for a classical Maxwell Boltzmann gas $P \to 0$ when $T \to 0$). In absence of other sources of outward pressure it is the pressure of the degenerate fermion gas that balances the gravity and avoids the stellar collapse.

One can see that the densities that can be reached in the compact stars are very different depending on the nature of the fermions. The two cases correspond to two classes of compact stars, the white dwarfs and the neutron stars. White dwarfs (w.d.s) are stars that have exhausted nuclear fuel; well known examples are Sirius B, or 40 Eri B. In the Hertzsprung-Russel diagram w.d.s fill in a narrow corner below the main sequence. In a w.d. stellar equilibrium is reached through a compensation between the inward pressure generated by gravity and the outward pressure of degenerate electrons. Typical values of the central density, mass and radius for a w.d. are $\rho = 10^6 g/cm^3$, $M \sim M_\odot$, $R \sim 5,000km$. Notice that the nuclear saturation density, defined as the density of a nucleon of radius 1.2 fm. is about 1.5×10^{14} g/cm^3.

Suppose now that in the star higher values of ρ are reached. If ρ increases, inverse beta decay becomes important:

$$e^- p \to n\nu \; . \tag{11.10}$$

This process fixes the chemical composition at equilibrium

$$\mu_e + \mu_p = \mu_n \; . \tag{11.11}$$

In the cases of ultrarelativistic particles we get

$$\rho_e^{1/3} + \rho_p^{1/3} = \rho_n^{1/3}. \tag{11.12}$$

On the other hand one has to enforce neutrality:

$$\rho_e = \rho_p \; , \tag{11.13}$$

implying

$$\frac{\rho_p}{\rho_n} = \frac{1}{8} \, . \tag{11.14}$$

This number should be seen as a benchmark value, as it is derived under simplifying hypotheses, most notably the absence of interactions and the neglect of masses. In any event it suggests that, for higher densities, the star tends to have a relatively larger fraction of neutrons and therefore it is named a *neutron star.* It must be stressed that one of the relevant facts about neutron stars is that the general relativity effects cannot be ignored and the relevant equilibrium equations to be used are the Oppenheimer-Volkov equations of hydrostatic equilibrium.

The following simple argument, due to Landau (1932) can be used to evaluate the relevant parameters of white dwarfs and neutron stars (see the textbook (Shapiro and Teukolski, 1983); more recent reviews of compact stars are in (Heiselberg and Pandharipande, 2000; Page, 1998; Tsuruta, 1998)). Let us consider N fermions in a sphere of radius R at $T = 0$; the number of fermion per volume unit scales as $n \sim N/R^3$; the volume per fermion is therefore $\sim 1/n$ and the uncertainty on the position is of the order of $n^{-1/3}$; the Fermi momentum is of the order of the uncertainty on the fermion momentum and therefore

$$p_F \sim n^{1/3}\hbar \; , \tag{11.15}$$

a result we obtained already under more stringent hypotheses (Fermi-Dirac distribution) and derived again here using only the uncertainty relations. The Fermi energy of the baryons is therefore

$$\epsilon_F \sim \frac{\hbar c N^{1/3}}{R} \; , \tag{11.16}$$

if N is the total number of baryons. Note that this applies both to neutron stars and to electron stars, because also in stars where the pressure mainly come from electrons there will be a considerable amount of protons and neutrons and the largest part of the energy comes from the baryons, not from the electrons. On the other hand the gravitational energy per baryon is

$$E_G \sim -\frac{GNm_B^2}{R} \; , \tag{11.17}$$

and the total energy can be estimated as

$$E = E_G + E_F \sim \frac{\hbar c N^{1/3}}{R} - \frac{GNm_B^2}{R} \; . \tag{11.18}$$

Now equilibrium can exist only if $E \geq 0$. As a matter of fact if $E < 0$ (N large) then $\lim_{R\to 0} E = -\infty$, which means that the energy is unbounded from below and the system is unstable. Therefore, $E \geq 0$ gives the maximum number of baryons as follows:

$$N \leq N_{max} = \left(\frac{\hbar c}{Gm_B^2}\right)^{3/2} \sim 2 \times 10^{57} \; . \tag{11.19}$$

As a consequence the maximum mass is

$$M_{max} = N_{max} m_B = 1.5 M_\odot \; . \tag{11.20}$$

This mass can be estimated better and its better determination ($\sim 1.4 M_\odot$) is known as the Chandrasekhar limit; for our purposes the estimate (11.20) is however sufficient. Notice that the Chandrasekhar limit is similar for compact stars where the degeneracy pressure is mainly supplied by electrons and also where it is supplied by baryons.

One can estimate the radius of a star whose mass is given by (11.20). One has

$$\epsilon_F \sim \frac{\hbar c}{R} N_{max}^{1/3} \sim \frac{\hbar c}{R} \left(\frac{\hbar c}{G m_B^2} \right)^{1/2} \tag{11.21}$$

and, therefore,

$$R \sim \frac{\hbar}{mc} \left(\frac{\hbar c}{G m_B^2} \right)^{1/2} = \begin{cases} 5 \times 10^8 \, cm \ \{m = m_e\} \\ 3 \times \, 10^5 \, cm \ \{m = m_n\}. \end{cases} \tag{11.22}$$

If a neutron star accretes its mass beyond the Chandrasekhar limit nothing can prevent the collapse and it becomes a black hole[13]. In Table VIII we summarize our discussion; notice that we report for the various stars also the value of the parameter GM/Rc^2 i.e. the ratio of the Schwarzschild radius to the star's radius. Its smallness measures the validity of the approximation of neglecting the general relativity effects; one can see that for the sun and the white dwarfs the newtonian treatment of gravity represents a fairly good approximation.

	M	R	$\rho \ \frac{g}{cm^3}$	$\frac{GM}{Rc^2}$
Sun	$M_\odot$	$R_\odot$	1	10^{-6}
White Dwarf	$\leq M_\odot$	$10^{-2} R_\odot$	$\leq 10^7$	10^{-4}
Neutron Star	$1 - 3 M_\odot$	$10^{-5} R_\odot$	$\leq 10^{15}$	10^{-1}
Black Hole	arbitrary	$\frac{2GM}{c^2}$	$\sim \frac{M}{R^3}$	1

TABLE VIII *Parameters of different stellar objects.*

Neutron stars are the most likely candidate for the theoretical description of pulsars. Pulsars are rapidly rotating stellar objects, discovered in 1967 by Hewish and collaborators and identified as rotating neutron stars by (Gold, 1969); so far about 1200 pulsars have been identified.

Pulsars are characterized by the presence of strong magnetic fields with the magnetic and rotational axis misaligned; therefore they continuously emit electromagnetic energy (in the form of radio waves) and constitute indeed a very efficient mean to convert rotational energy into electromagnetic radiation. The rotational energy loss is due to dipole radiation and is therefore given by

$$\frac{dE}{dt} = I\omega \frac{d\omega}{dt} = -\frac{B^2 R^6 \omega^4 \sin^2 \theta}{6c^3} \ . \tag{11.23}$$

Typical values in this formula are, for the moment of inertia $I \sim R^5 \rho \sim 10^{45} \mathrm{g/cm^3}$, magnetic fields $B \sim 10^{12}$ G, periods $T = 2\pi/\omega$ in the range 1.5 msec-8.5 sec; these periods increase slowly[14], with derivatives $dT/dt \sim 10^{-12}$ - 10^{-21}, and never decrease except for occasional jumps (called *glitches*).

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[13] The exact determination of the mass limit depends on the model for nuclear forces; for example in (Cameron and Canuto, 1974) the neutron star mass limit is increased beyond $1.4 M_\odot$.

[14] Rotational period and its derivative can be used to estimate the pulsar's age by the approximate formula $T/2(dT/dt)$, see e.g. (Lorimer, 1999).

Glitches were first observed in the Crab and Vela pulsars in 1969; the variations in the rotational frequency are of the order $10^{-8} - 10^{-6}$.

This last feature is the most significant phenomenon pointing to neutron stars as a model of pulsars in comparison to other form of hadronic matter, such as strange quarks. It will be discussed in more detail in Section XI.E, where we will examine the possible role played by the crystalline superconducting phase. In the subsequent three paragraphs we will instead deal with other possible astrophysical implications of color superconductivity.

B. Supernovae neutrinos and cooling of neutron stars

Neutrino diffusion is the single most important mechanism in the cooling of young neutron stars, i.e. with an age $< 10^5$ years; it affects both the early stage and the late time evolution of these compact stars. To begin with let us consider the early evolution of a Type II Supernova.

Type II supernovae are supposed to be born by collapse of massive ($M \sim 8 - 20 M_\odot$) stars[15]. These massive stars have unstable iron cores[16] with masses of the order of the Chandrasekhar mass. The explosion producing the supernova originates within the core, while the external mantle of the red giant star produces remnants that can be analyzed by different means, optical, radio and X rays. These studies agree with the hypothesis of a core explosion. The emitted energy ($\sim 10^{51} erg$) is much less than the total gravitational energy of the star, which confirms that the remnants are produced by the outer envelope of the massive star; the bulk of the gravitational energy, of the order of 10^{53} erg, becomes internal energy of the proto neutron star (PNS). The suggestion that neutron stars may be formed in supernovae explosions was advanced in 1934 by (Baade and Zwicky, 1934) and it has been subsequently confirmed by the observation of the Crab pulsar in the remnant of the Crab supernova observed in China in 1054 A.D.

We do not proceed in this description as it is beyond the scope of this review and we concentrate our attention on the cooling of the PNS[17], which is mostly realized through neutrino diffusion. By this mechanism one passes from the initial temperature $T \sim 20 - 30$ MeV to the cooler temperatures of the neutron star at subsequent stages. This phase of fast cooling lasts 10-20 secs and the neutrinos emitted during it have mean energy ~ 20 MeV. These properties, that can be predicted theoretically, are also confirmed by data from SN 1987A.

The role of quark color superconductivity at this stage of the evolution of the neutron stars has been discussed in (Carter and Reddy, 2000). In this paper the neutrino mean free path is computed in a color super-conducting medium made up by quarks in two flavor (2SC model). The results obtained indicate that the cooling process by neutrino emission slows down when the quark matter undergoes the phase transition to the superconducting phase at the critical temperature T_c, but then accelerates when T decreases below T_c. There should be therefore changes in the neutrino emission by the PNS and they might be observed in some future supernova event; this would produce an interesting test for the existence of a color superconducting phase in compact stars.

Let us now consider the subsequent evolution of the neutron star, which also depends on neutrino diffusion. The simplest processes of neutrino production are the so called direct Urca processes

$$\begin{aligned} & f_1 + \ell \to f_2 + \nu_\ell \ , \\ & f_2 + \ell \to f_1 + \ell \to f_2 + \bar{\nu}_\ell \ ; \end{aligned} \tag{11.24}$$

by these reactions, in absence of quark superconductivity, the interior temperature T of the star drops below 10^9 K (~ 100 KeV) in a few minutes and in 10^2 years to temperatures $\sim 10^7$ K. Generally speaking the effect of the formation of gaps is to slow down the cooling, as it reduces both the emissivity and the specific heat. However not only quarks, if present in the neutron star, but also other fermions, such as neutrons, protons or hyperons have gaps, as the formation of fermion pairs is unavoidable if there is an attractive interaction in any channel. Therefore, besides quark color superconductivity, one has also the phenomenon of baryon superconductivity and neutron superfluidity, which is the form assumed by this phenomenon for neutral particles. The analysis is therefore rather complicated; the thermal evolution of a late time neutron star has been discussed in (Blaschke *et al.*, 2000; Page *et al.*, 2000), but no clear signature for the presence of color superconductivity seems to emerge from the theoretical simulations and, therefore, one may tentatively conclude that the late time evolution of the neutron stars does not offer a good laboratory to test the existence of color superconductivity in compact stars.

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[15] The other supernovae, i.e. type I supernovae, result from the complete explosion of a star with $4M_\odot \le M \le 8M_\odot$ with no remnants.
[16] Fusion processes favor the formation of iron, as the binding energy per nucleon in nuclei has a maximum for $A \sim 60$.
[17] See (Burrows and Lattimer, 1986; Colgate and White, 1966; Imsshennyk and Nadyozhin, 1973; Keil and Janka, 1995; Pons and et al, 1999) for further discussions.

C. R-mode instabilities in neutron stars and strange stars

Rotating relativistic stars are in general unstable against the rotational mode (r-mode instability) (Andersson, 1998). The instability is due to the emission of angular momentum by gravitational waves from the mode. Unless it is damped by viscosity effects, this instability would spin down the star in relatively short times. More recently it has been realized (Andersson *et al.*, 2000; Bildsten and Ushomirsky, 1999) that in neutron stars there is an important viscous interaction damping the r-mode i.e. that between the external metallic crust and the neutron superfluid. The consequence of this damping is that r-modes are significant only for young neutron stars, with periods $T < 2$ msec. For larger rotating periods the damping of the $r-$mode implies that the stars slow down only due to magnetic dipole braking.

All this discussion is relevant for the nature of pulsars: Are they neutron stars or strange stars?

The existence of strange stars, i.e. compact stars made of quarks u, d, s in equal ratios would be a consequence of the existence of stable strangelets. This hypothetical form of nuclear matter, made of a large number of u, d, s quarks has been suggested by (Bodmer, 1971) and (Witten, 1984) as energetically favored in comparison to other hadronic phases when a large baryonic number is involved. The reason is that in this way the fermions, being of different flavors, could circumvent the Pauli principle and have a lower energy, in spite of the larger strange quark mass. If strangelets do exist, basically all the pulsars should be strange stars because the annihilation of strange stars, for example from a binary system, would fill the space around with strangelets that, in turn, would convert ordinary nuclear stars into strange stars.

An argument in favor of the identification of pulsars with strange stars is the scarcity of pulsars with very high frequency ($T < 2.5$ msec).

This seems to indicate that indeed the $r-$mode instability is effective in slowing down the compact stars and favors strange stars, where, differently from neutron stars, the crust can be absent. Even in the presence of the external crust, that in a quark star can be formed by the gas after the supernova explosion or subsequent accretion, the dampening of the $r - mode$ is less efficient. As a matter of fact, since electrons are only slightly bounded, in comparison with quarks that are confined, they tend to form an atmosphere having a thickness of a few hundred Fermi; this atmosphere produces a separation between the nuclear crust and the inner quark matter and therefore the viscosity is much smaller.

In quark matter with color superconductivity the presence of gaps $\Delta \gg T$ exponentially reduces the bulk and shear viscosity, which renders the $r-$mode unstable. According to (Madsen, 2000, 2002) this would rule out compact stars entirely made of quarks in the CFL case (the 2SC model would be marginally compatible, as there are ungapped quarks in this case). For example for $\Delta > 1$ MeV any star having $T < 10$ msec would be unstable, which would contradict the observed existence of pulsars with time period less than 10 msec.

However this conclusion does not rule out the possibility of neutron stars with a quark core in the color superconducting state, because as we have stressed already, for them the dampening of the $r-$mode instability would be provided by the viscous interaction between the nuclear crust and the neutron superfluid.

D. Miscellaneous results

Color superconductivity is a Fermi surface phenomenon and as such, it does not affect significantly the equation of state of the compact star. Effects of this phase could be seen in other astrophysical contexts, such as those considered in the two previous paragraphs or in relation to the pulsar glitches, which will be examined in the next paragraph. A few other investigations have been performed in the quest of possible astrophysical signatures of color superconductivity; for instance in (Ouyed and Sannino, 2001) it has been suggested that the existence of a 2SC phase might be partly responsible of the gamma ray bursts, due to the presence in the two-flavor superconducting phase of a light glueball that can decay into two photons. Another interesting possibility is related to the stability of strangelets, because, as observed in (Madsen, 2001), CFL strangelets, i.e. lumps of strange quark matter in the CFL phase may be significantly more stable than strangelets without color superconductivity.

Finally we wish to mention the observation of (Alford *et al.*, 2000) concerning the evolution of the magnetic field in the interior of neutron stars. Inside an ordinary neutron star, neutron pairs are responsible for superfluidity, while proton pairs produce BCS superconductivity. In this condition magnetic fields experience the ordinary Meissner effect and are either expelled or restricted to flux tubes where there is no pairing. In the CFL (and also 2SC) case, as we know from V.D, a particular $U(1)$ group generated by

$$\tilde{Q} = \mathbf{1} \otimes Q - Q \otimes \mathbf{1} \tag{11.25}$$

remains unbroken and plays the role of electromagnetism. Instead of being totally dragged out or confined in flux tubes, the magnetic field will partly experience Meissner effect (the component $\tilde{A}_\mu$), while the remaining part will remain free in the star. During the slowing down this component of the magnetic field should not decay because, even though the color superconductor is not a BCS conductor for the group generated by (11.25), it may be a good conductor due to the presence of the electrons in the compact star. Therefore it has been suggested (Alford *et al.*, 2000) that a quark matter core inside a neutron star may serve as an "anchor" for the magnetic field.

E. Glitches in neutron stars

Glitches, that is very sudden variations in the period, are a typical phenomenon of the pulsars, in the sense that probably all the pulsars have glitches (for a recent review see (Link *et al.*, 2000)), see Fig. 41. Several models have been proposed to explain the glitches. Their most popular explanation is based on the idea that these sudden jumps of the rotational frequency are due to the the angular momentum stored in the superfluid neutrons in the inner crust (see Fig. 40), more precisely in vortices pinned to nuclei. When the star slows down, the superfluid neutrons do not participate in the movement, until the state becomes unstable and there is a release of angular momentum to the crust, which is seen as a jump in the rotational frequency.

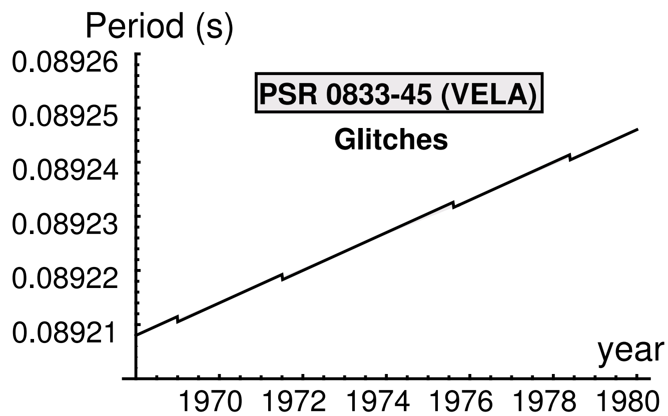

FIG. 41 *The variation of the period of the pulsar PSR 0833-45 (VELA) with the typical structure of glitches shown.*

The presence of glitches is one of the main reasons for the identification of pulsars with neutron stars; as a matter of fact neutron stars are supposed to have a dense metallic crust, differently from quark stars where the crust is absent or, if present, is much less dense ($\approx 10^{11}$ g cm^{-3}).

Since the density in the inner of a star is a function of the radius, it results that one has a sort of laboratory to study the phase diagram of QCD at zero temperature, at least in the corresponding range of densities. A possibility is that a CFL state occurs as a core of the star, then a shell in the LOFF state and eventually the exterior part made up of neutrons (see Fig. 42). Since in the CFL state the baryonic number is broken there is superfluidity. Therefore the same mechanism explained above might work with vortices in the CFL state pinned to the LOFF crystal.

This would avoid the objection raised in (Friedman and Caldwell, 1991) that excludes the existence of strange stars. This objection is based on the following observation: strange stars cannot have a metallic crust, and in that case they can hardly develop vortices, so no glitches would arise. However, if the strange matter exists, strange stars should be rather common, as we discussed in Section XI.C, in contrast with the widespread appearance of glitches in pulsars. Therefore, if the color crystalline structure is able to produce glitches, the argument in favor of the existence of strange stars would be reinforced.

Considering the very narrow range of values for $\delta\mu$ in order to be in the LOFF phase one can ask if the previous possibility has some chance to be realized (Casalbuoni and Nardulli, 2003). Notice that using the typical LOFF value $\delta\mu \approx 0.75\Delta_{BCS}$ one would need values of $\delta\mu$ around $15 \div 70\ MeV$. Let us consider a very crude model of three free quarks with $M_u = M_d = 0$, $M_s \neq 0$. Assuming at the core of the star a density around 10^{15} g/cm^3, that is from 5 to 6 times the saturation nuclear density 0.15×10^{15} g/cm^3, one finds for M_s ranging between 200 and 300 MeV corresponding values of $\delta\mu$ between 25 and 50 MeV. We see that these values are just in the right range for being

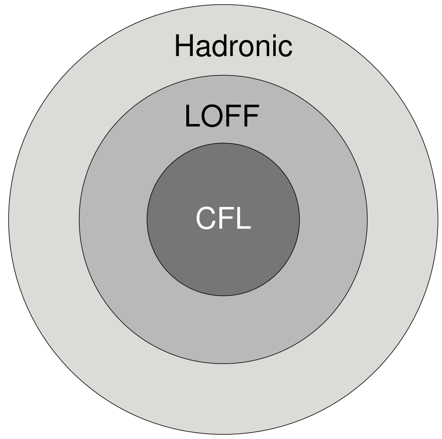

FIG. 42 *A model of a neutron star with an inner shell in the LOFF phase.*

within the LOFF window. Therefore a possible phase diagram for QCD could be of the form illustrated in Fig. 43. We see that in this case a coexistence of the CFL, LOFF and neutron matter would be possible inside the neutron star.

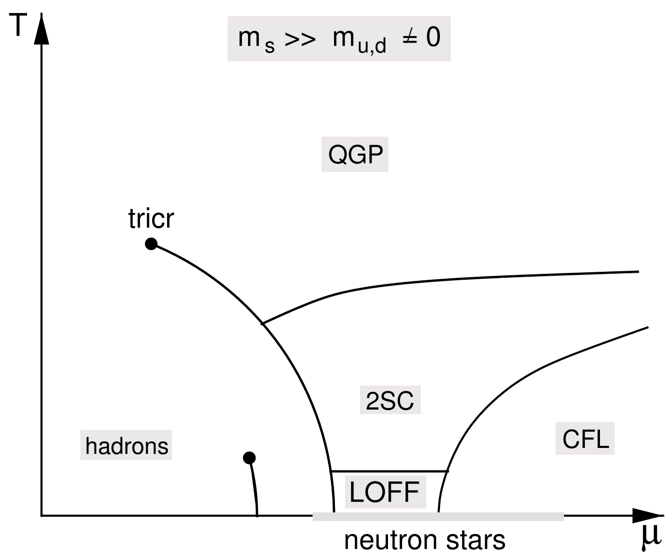

FIG. 43 *A possible QCD phase diagram showing the end point of the first order transition phase, labelled as "tricr". The hadron phase with the transition to the nuclear phase, the Quark-gluon plasma, the CFL, the 2SC and the LOFF phase are also shown.*

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APPENDIX A: The gap equation in the functional formalism from HDET

We will now derive the gap equation by using the functional formalism within the general formalism of Section V.B. The method used is a trivial generalization of the one used in Section III.C. We start again from Eq. 5.64

$$\mathcal{L}_I = -\frac{G}{4}\epsilon_{ab}\epsilon_{\dot{c}\dot{d}}V_{ABCD}\psi_a^A\psi_b^B\psi_{\dot{c}}^{C\dagger}\psi_{\dot{d}}^{D\dagger} \tag{A1}$$

For simplicity we suppress the notation for fixed velocity. We will insert inside the functional integral the following identity

$$\text{const} = \int \mathcal{D}(\Delta_{AB}, \Delta_{AB}^*)e^{-\frac{i}{G}\int d^4x F[\Delta_{AB}, \Delta_{AB}^*, \psi^A, \psi^{A\dagger}]} \tag{A2}$$

where

$$F = \left[\Delta_{AB} - \frac{G}{2}V_{GHAB}(\psi^{GT}C\psi^H)\right]W_{ABCD}\left[\Delta_{CD}^* + \frac{G}{2}V_{CDEF}(\psi^{E\dagger}C\psi^{F*})\right] \tag{A3}$$

The quantity W_{ABCD} is defined in such a way that

$$W_{ABCD}V_{CDEF} = \delta_{AE}\delta_{BF}, \quad V_{ABCD}W_{CDEF} = \delta_{AE}\delta_{DF} \tag{A4}$$

Then we get

$$\begin{aligned} F &= \frac{G}{4}V_{ABCD}\psi^{CT}C\psi^B\psi^{C\dagger}C\psi^{D*} + \frac{1}{2}\Delta_{CD}^*\psi^{CT}C\psi^D - \frac{1}{2}\Delta_{AB}\psi^{A\dagger}C\psi^{B*} \\ &- \frac{1}{g}\Delta_{AB}W_{ABCD}\Delta_{CD}^* \end{aligned} \tag{A5}$$

Normalizing at the free case ($G = 0$) we get

$$\begin{aligned} \frac{Z}{Z_0} &= \frac{1}{Z_0}\int \mathcal{D}(\psi, \psi^\dagger)\mathcal{D}(\Delta, \Delta^*)e^{iS_0[\psi, \psi^\dagger]} \\ &\times\ e^{+i\int d^4x\left[-\frac{\Delta_{AB}W_{ABCD}\Delta_{CD}^*}{G} - \frac{1}{2}\Delta_{AB}(\psi^{A\dagger}C\psi^{B*}) + \frac{1}{2}\Delta_{AB}^*(\psi^{AT}C\psi^B)\right]} \end{aligned} \tag{A6}$$

Going to the velocity formalism and introducing the Nambu-Gor'kov field we can write the fermionic part in the exponent inside the functional integral as

$$\int d^4x \int \frac{d\mathbf{v}}{4\pi}\chi^{A\dagger}S_{AB}^{-1}\chi^B \tag{A7}$$

with

$$S_{AB}^{-1} = \begin{pmatrix} V\cdot\ell\,\delta_{AB} & -\Delta_{AB} \\ -\Delta_{AB}^* & \tilde{V}\cdot\ell\,\delta_{AB} \end{pmatrix} \tag{A8}$$

Using again the replica trick as in Section III.C we perform the functional integral over the Fermi fields obtaining

$$\frac{Z}{Z_0} = \frac{1}{Z_0}\left[\det S^{-1}\right]e^{-\frac{i}{G}\int d^4x\Delta_{AB}W_{ABCD}\Delta_{CD}^*} \equiv e^{iS_{\text{eff}}} \tag{A9}$$

with

$$S_{\text{eff}} = -\frac{i}{2}\text{Tr}[\log S_0S^{-1}] - \frac{i}{G}\int d^4x\Delta_{AB}W_{ABCD}\Delta_{CD}^* \tag{A10}$$

Differentiating with respect to Δ_{AB} we get immediately the gap equation (5.83).

APPENDIX B: Some useful integrals

We list a few 2-D integrals which have been used in the text (Nardulli, 2002). Let us define

$$I_n = \int \frac{d^N \ell}{(V\cdot\ell\,\tilde V\cdot\ell - \Delta^2 + i\epsilon)^{n+1}} = \frac{i\,(-i)^{n+1}\pi^{\frac{N}{2}}}{n!}\frac{\Gamma(n+1-\frac{N}{2})}{\Delta^{2n+2-N}}\ ; \tag{B1}$$

therefore, for $N = 2-\epsilon$ and denoting by γ the Euler-Mascheroni constant, we get

$$\begin{aligned} I_0 &= -\frac{2i\pi}{\epsilon} + i\pi\ln\pi\Delta^2 + i\pi\gamma\ , \\ I_1 &= +\frac{i\pi}{\Delta^2}\ ,\ I_2 = -\frac{i\pi}{2\Delta^4}\ ,\ I_3 = +\frac{i\pi}{3\Delta^6}\ . \end{aligned} \tag{B2}$$

Moreover defining

$$I_{n,\,m} = \int \frac{d^2\ell}{(V\cdot\ell\,\tilde V\cdot\ell - \Delta^2 + i\epsilon)^n(V\cdot\ell\,\tilde V\cdot\ell - \Delta'^2 + i\epsilon)^m}\ , \tag{B3}$$

we find

$$\begin{aligned} I_{1,\,1} &= \frac{i\pi}{\Delta^2 - \Delta'^2}\ln\frac{\Delta^2}{\Delta'^2}\ , \\ I_{2,\,1} &= i\pi\left[\frac{1}{\Delta^2(\Delta^2-\Delta'^2)} - \frac{1}{(\Delta^2-\Delta'^2)^2}\ln\frac{\Delta^2}{\Delta'^2}\right]\ , \\ I_{3,\,1} &= \frac{i\pi}{2}\left[\frac{-1}{\Delta^4(\Delta^2-\Delta'^2)} - \frac{2}{\Delta^2(\Delta^2-\Delta'^2)^2} + \frac{2}{(\Delta^2-\Delta'^2)^3}\ln\frac{\Delta^2}{\Delta'^2}\right]\ , \\ I_{2,\,2} &= i\pi\left[\left(\frac{1}{\Delta^2}+\frac{1}{\Delta'^2}\right)\frac{1}{(\Delta^2-\Delta'^2)^2} + \frac{2}{(\Delta^2-\Delta'^2)^3}\ln\frac{\Delta^2}{\Delta'^2}\right]. \end{aligned} \tag{B4}$$

Of some interest is the following infrared divergent integral:

$$\tilde I_1 = \int \frac{(V\cdot\ell)^2\ d^2\ell}{(V\cdot\ell\,\tilde V\cdot\ell)^2}\ . \tag{B5}$$

We can regularize the divergence by going to finite temperature and then taking the limit $T\to 0$ ($\ell_0 = i\omega_n,\ \omega_n = \pi T(2n+1)$; $T\to 0$, $\mu\to\infty$):

$$\tilde I_1 = 2\pi T i\int_{-\mu}^{+\mu} dx\sum_{-\infty}^{+\infty}\frac{(i\omega_n - x)^2}{(x^2+\omega_n^2)^2} = 2\pi T i\left(-\frac{1}{T}\tanh\frac{\mu}{2T}\right) \to -2\pi i\ . \tag{B6}$$

Other divergent integrals, such as

$$\tilde I = \int \frac{d^2\ell}{(V\cdot\ell\,\tilde V\cdot\ell)(V\cdot\ell\,\tilde V\cdot\ell - \Delta^2)} \tag{B7}$$

are treated in a similar way.

Finally, useful angular integrations are:

$$\int\frac{d\mathbf{v}}{4\pi}v^j v^k = \frac{\delta^{jk}}{3}\ , \tag{B8}$$

$$\int\frac{d\mathbf{v}}{4\pi}v^i v^j v^k v^\ell = \frac{1}{15}(\delta^{ij}\delta^{k\ell} + \delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk})\ . \tag{B9}$$

References

Abbott, L. F., and E. Fahri, 1981, CERN-TH3015 .
Abrikosov, A. A., 1957, Zh. Exsp. teor. Fiz. **32**, 1442.
Abrikosov, A. A., L. P. Gor'kov, and I. E. Dzyaloshinski, 1963, *Methods of Quantum Field Theory in Statistical Physics* (Dover, New York).
Alford, M. G., 2001, Ann. Rev. Nucl. Part. Sci. **51**, 131.
Alford, M. G., J. Berges, and K. Rajagopal, 2000, Nucl. Phys. **B571**, 269.
Alford, M. G., J. A. Bowers, J. M. Cheyne, and G. A. Cowan, 2003, Phys. Rev. **D67**, 054018.
Alford, M. G., J. A. Bowers, and K. Rajagopal, 2001, Phys. Rev. **D63**, 074016.
Alford, M. G., K. Rajagopal, and F. Wilczek, 1998, Phys. Lett. **B422**, 247.
Alford, M. G., K. Rajagopal, and F. Wilczek, 1999, Nucl. Phys. **B537**, 443.
Andersson, N., 1998, Astrophys. J. **502**, 708.
Andersson, N., D. I. Jones, K. D. Kokkotas, and N. Stergioulas, 2000, Astrophys. J. **534**, L75.
Baade, W., and F. Zwicky, 1934, Proc. Nat. Acad. Sci. **20**.
Bailin, D., and A. Love, 1984, Phys. Rept. **107**, 325.
Banks, T., and E. Rabinovici, 1979, Nucl. Phys. **B160**, 349.
Bardeen, J., 1956, in *Handbuch der Physik. vol XV* (Springer Verlag, Berlin).
Bardeen, J., L. N. Cooper, and J. R. Schrieffer, 1957, Phys. Rev. **106**, 162.
Barrois, B. C., 1977, Nucl. Phys. **B129**, 390.
Beane, S. R., P. F. Bedaque, and M. J. Savage, 2000, Phys. Lett. **B483**, 131.
Bedaque, P. F., and T. Schafer, 2002, Nucl. Phys. **A697**, 802.
Benfatto, G., and G. Gallavotti, 1990, Phys. Rev. **B42**, 9967.
Bildsten, L., and G. Ushomirsky, 1999, eprint astro-ph/9911155.
Blaschke, D., T. Klahn, and D. N. Voskresensky, 2000, Astrophys. J. **533**, 406.
Bodmer, A. R., 1971, Phys. Rev. **D4**, 1601.
Bowers, J. A., 2003, eprint hep-ph/0305301.
Bowers, J. A., and K. Rajagopal, 2002, Phys. Rev. **D66**, 065002.
Brown, W. E., J. T. Liu, and H.-c. Ren, 2000a, Phys. Rev. **D62**, 054013.
Brown, W. E., J. T. Liu, and H.-c. Ren, 2000b, Phys. Rev. **D61**, 114012.
Brown, W. E., J. T. Liu, and H.-c. Ren, 2000c, Phys. Rev. **D62**, 054016.
Buckley, K. B. W., and A. R. Zhitnitsky, 2002, JHEP **08**, 013.
Burrows, A., and J. M. Lattimer, 1986, Astrophys. J. **307**.
Buzdin, A. I., and Kachkachi, 1997, Phys. Lett. **A225**, 341.
Cameron, A. G. W., and V. Canuto, 1974, in *Proceedings 16th Solvay Conference on Astrophysics and Gravitation* (Université Libre de Bruxelles).
Carter, G. W., and S. Reddy, 2000, Phys. Rev. **D62**, 103002.
Casalbuoni, R., F. De Fazio, R. Gatto, G. Nardulli, and M. Ruggieri, 2002a, Phys. Lett. **B547**, 229.
Casalbuoni, R., Z.-y. Duan, and F. Sannino, 2000, Phys. Rev. **D62**, 094004.
Casalbuoni, R., Z.-y. Duan, and F. Sannino, 2001a, Phys. Rev. **D63**, 114026.
Casalbuoni, R., E. Fabiano, R. Gatto, M. Mannarelli, and G. Nardulli, 2002b, Phys. Rev. **D66**, 094006.
Casalbuoni, R., and R. Gatto, 1981, Phys. Lett. **B103**, 113.
Casalbuoni, R., and R. Gatto, 1999, Phys. Lett. **B464**, 111.
Casalbuoni, R., R. Gatto, M. Mannarelli, and G. Nardulli, 2001b, Phys. Lett. **B511**, 218.
Casalbuoni, R., R. Gatto, M. Mannarelli, and G. Nardulli, 2002c, Phys. Rev. **D66**, 014006.
Casalbuoni, R., R. Gatto, M. Mannarelli, and G. Nardulli, 2002d, Phys. Lett. **B524**, 144.
Casalbuoni, R., R. Gatto, and G. Nardulli, 2001c, Phys. Lett. **B498**, 179.
Casalbuoni, R., R. Gatto, and G. Nardulli, 2001d, Phys. Lett. (B498 Erratum) **B517**, 483.
Casalbuoni, R., R. Gatto, and G. Nardulli, 2002e, Phys. Lett. **B543**, 139.
Casalbuoni, R., R. Gatto, G. Nardulli, and M. Ruggieri, 2003, eprint hep-ph/0302077.
Casalbuoni, R., and G. Nardulli, 2003, eprint hep-ph/0305069.
Chandrasekhar, B. S., 1962, App. Phys. Lett. **1**, 7.
Cini, M., and B. Touschek, 1958, Nuovo Cimento **7**, 422.
Clogston, A. M., 1962, Phys. Rev. Lett. **9**, 266.
Colgate, S. A., and R. H. White, 1966, Astrophys. J. **143**.
Collins, J. C., and M. J. Perry, 1975, Phys. Rev. Lett. **34**, 1353.
Combescot, R., and C. Mora, 2002, Eur. P. J. **B28**, 397.
Cooper, L. N., 1956, Phys. Rev. **104**, 1189.
Corak, W. S., B. B. Goodman, C. B. Satterthwaite, and A. Wexler, 1954, Phys. Rev. **96**, 1442.
Corak, W. S., B. B. Goodman, C. B. Satterthwaite, and A. Wexler, 1956, Phys. Rev. **102**, 656.
Daunt, J. G., and K. Mendelssohn, 1946, Proc. Roy. Soc. **A185**, 225.
Dimopoulos, S., S. Raby, and L. Susskind, 1980a, Nucl. Phys. **B169**, 373.
Dimopoulos, S., S. Raby, and L. Susskind, 1980b, Nucl. Phys. **B173**, 208.

Eguchi, T., 1976, Phys. Rev. **D14**, 2755.
Eliashberg, G. M., 1960, Sov. Phys. JETP **11**, 696.
Evans, N., J. Hormuzdiar, S. D. H. Hsu, and M. Schwetz, 2000, Nucl. Phys. **B581**, 391.
Evans, N., S. D. H. Hsu, and M. Schwetz, 1999a, Nucl. Phys. **B551**, 275.
Evans, N., S. D. H. Hsu, and M. Schwetz, 1999b, Phys. Lett. **B449**, 281.
Fradkin, E., and S. H. Shenker, 1979, Phys. Rev. **D19**, 3682.
Frautschi, S. C., 1978, presented at Workshop on Hadronic Matter at Extreme Energy Density, Erice, Italy, Oct 13-21, 1978.
Friedman, J. L., and R. R. Caldwell, 1991, Phys. Lett. **B264**, 143.
Frolich, H., 1950, Phys. Rev. **79**, 845.
Frolich, H., 1952, Proc. Roy. Soc. **A215**, 291.
Fulde, P., and R. A. Ferrell, 1964, Phys. Rev. **135**, A550.
de Gennes, P. G., 1989, *Superconductivity in Metals and Alloys* (Addison Wesley, Reading, MA).
Ginzburg, L., and L. D. Landau, 1950, Zh. Exsp. teor. Fiz. **20**, 1064.
Ginzburg, V. L., 1953, Fortschr. Phys. **1**, 101.
Ginzburg, V. L., and E. A. Andryushin, 1994, *Superconductivity* (World Scientific).
Glover, R. E., and M. Tinkham, 1956, Phys. Rev. **104**, 844.
Gold, T., 1969, Nature **221**, 25.
Gor'kov, L. P., 1959, Zh. Exsp. teor. Fiz. **36**, 1918.
Gorter, C. J., and H. G. B. Casimir, 1934a, Phys. Z. **35**, 963.
Gorter, C. J., and H. G. B. Casimir, 1934b, Z. Tech. Phys. **15**, 539.
Gross, D., and F. Wilczek, 1973, Phys. Rev. Lett. **30**, 1343.
Gusynin, V. P., and I. A. Shovkovy, 2002, Nucl. Phys. **A700**, 577.
Heiselberg, H., and V. Pandharipande, 2000, Ann. Rev. Nucl. Part. Sci. **50**, 481.
Hong, D. K., 2000a, Nucl. Phys. **B582**, 451.
Hong, D. K., 2000b, Phys. Lett. **B473**, 118.
Hong, D. K., 2001, Acta Phys. Polon. **B32**, 1253.
Hong, D. K., T. Lee, and D.-P. Min, 2000a, Phys. Lett. **B477**, 137.
Hong, D. K., V. A. Miransky, I. A. Shovkovy, and L. C. R. Wijewardhana, 2000b, Phys. Rev. **D61**, 056001.
Hong, D. K., M. Rho, and I. Zahed, 1999, Phys. Lett. **B468**, 261.
't Hooft, G., 1976, Phys. Rev. Lett. **37**, 8.
't Hooft, G., 1980, in *Recent Developments in Gauge Theories* (Plenum Press).
Hsu, S. D. H., 2000, eprint hep-ph/0003140.
Hsu, S. D. H., F. Sannino, and M. Schwetz, 2001, Mod. Phys. Lett. **A16**, 1871.
Imsshennyk, V. S., and D. K. Nadyozhin, 1973, Sov. Phys. JETP **36**.
Jackson, A. D., and F. Sannino, 2003, eprint hep-ph/0308182.
Jaffe, R. L., 1977, Phys. Rev. Lett. **38**, 195, 617(E).
Jaikumar, P., M. Prakash, and T. Schafer, 2002, Phys. Rev. **D66**, 063003.
Josephson, D., 1962, Phys. Lett. **1**, 251.
Josephson, D., 1965, Adv. Phys. **14**, 419.
Kamerlingh Onnes, H., 1911, Leiden Comm. **120b, 122b, 124c**.
Kaplan, D. B., and S. Reddy, 2002a, Phys. Rev. **D65**, 054042.
Kaplan, D. B., and S. Reddy, 2002b, Phys. Rev. Lett. **88**, 132302.
Keil, W., and H. T. Janka, 1995, Astron. Astrophys. **296**.
Kundu, J., and K. Rajagopal, 2002, Phys. Rev. **D65**, 094022.
Landau, L., and E. M. Lifshitz, 1996, *Statistical Physics, Part I* (Butterworth, Heinemann).
Landau, L., E. M. Lifshitz, and L. P. Pitaevskii, 1980, *Statistical Physics, Part II* (Oxford: Pergamon).
Larkin, A. J., and Y. N. Ovchinnikov, 1964, Zh. Exsp. teor. Fiz. **47**, 1136.
Le Bellac, M., 1996, *Thermal Field Theory* (Cambridge University Press, Cambridge, England).
Leibovich, A. K., K. Rajagopal, and E. Shuster, 2001, Phys. Rev. **D64**, 094005.
Link, B., R. I. Epstein, and J. M. Lattimer, 2000, eprint astro-ph/0001245.
London, F., and H. London, 1935, Proc. Roy. Soc. **A149**, 71.
Lorimer, D. R., 1999, eprint astro-ph/9911519.
Madsen, J., 2000, Phys. Rev. Lett. **85**, 10.
Madsen, J., 2001, Phys. Rev. Lett. **87**, 172003.
Madsen, J., 2002, eConf **C010815**, 155.
Manuel, C., and M. H. G. Tytgat, 2000, Phys. Lett. **B479**, 190.
Manuel, C., and M. H. G. Tytgat, 2001, Phys. Lett. **B501**, 200.
Maxwell, E., 1950, Phys. Rev. **78**, 477.
Meissner, W., and R. Ochsenfeld, 1933, Naturwissenschaften **21**, 787.
Miransky, V. A., and I. A. Shovkovy, 2002, Phys. Rev. Lett. **88**, 111601.
Nambu, Y., 1960, Phys. Rev. **117**, 648.
Nambu, Y., and G. Jona-Lasinio, 1961a, Phys. Rev. **122**, 345.
Nambu, Y., and G. Jona-Lasinio, 1961b, Phys. Rev. **124**, 246.
Nardulli, G., 2002, Riv. Nuovo Cim. **25N3**, 1.

Nielsen, H. B., and S. Chada, 1976, Nucl. Phys. **B105**, 445.
Ouyed, R., and F. Sannino, 2001, Phys. Lett. **B511**, 66.
Page, D., 1998, in *The Many Faces of Neutron Stars* (Kluwer Academic Publishers).
Page, D., M. Prakash, J. M. Lattimer, and A. Steiner, 2000, Phys. Rev. Lett. **85**, 2048.
Pines, D., 1958, Phys. Rev. **109**, 280.
Pippard, A. B., 1953, Proc. Roy. Soc. **A216**, 547.
Pisarski, R. D., and D. H. Rischke, 1999, Phys. Rev. **D60**, 094013.
Pisarski, R. D., and D. H. Rischke, 2000a, Phys. Rev. **D61**, 074017.
Pisarski, R. D., and D. H. Rischke, 2000b, Phys. Rev. **D61**, 051501.
Polchinski, J., 1993, in *Recent directions in particle theory: from superstrings and black holes to the standard model (TASI - 92)* (World Scientific), eprint hep-th/9210046.
Politzer, H. D., 1973, Phys. Rev. Lett. **30**, 1346.
Pons, J., and et al, 1999, Astrophys. J. **513**.
Rajagopal, K., and E. Shuster, 2000, Phys. Rev. **D62**, 085007.
Rajagopal, K., and F. Wilczek, 2001, in *At the frontier of particle physics, vol. 3* (World Scientific), eprint hep-ph/0011333.
Rapp, R., T. Schafer, E. V. Shuryak, and M. Velkovsky, 1998, Phys. Rev. Lett. **81**, 53.
Rapp, R., T. Schafer, E. V. Shuryak, and M. Velkovsky, 2000, Annals Phys. **280**, 35.
Reddy, S., M. Sadzikowski, and M. Tachibana, 2003, Nucl. Phys. **A714**, 337.
Reynolds, C. A., B. Serin, W. H. Wright, and L. B. Nesbitt, 1950, Phys. Rev. **78**, 487.
Rho, M., E. V. Shuryak, A. Wirzba, and I. Zahed, 2000a, Nucl. Phys. **A676**, 273.
Rho, M., A. Wirzba, and I. Zahed, 2000b, Phys. Lett. **B473**, 126.
Rischke, D. H., 2000, Phys. Rev. **D62**, 034007.
Rischke, D. H., D. T. Son, and M. A. Stephanov, 2001, Phys. Rev. Lett. **87**, 062001.
Sakita, B., 1985, *Quantum theory of many-variable systems and fields* (World Scientific).
Sannino, F., 2000, Phys. Lett. **B480**, 280.
Sarma, G., 1963, J. Phys. Chem. Solids **24**, 1029.
Schafer, T., 2000a, Phys. Rev. Lett. **85**, 5531.
Schafer, T., 2000b, Nucl. Phys. **B575**, 269.
Schafer, T., 2000c, Phys. Rev. **D62**, 094007.
Schafer, T., 2002, Phys. Rev. **D65**, 074006.
Schafer, T., 2003, eprint hep-ph/0304281.
Schafer, T., and E. V. Shuryak, 1998, Rev. Mod. Phys. **70**, 323.
Schafer, T., D. T. Son, M. A. Stephanov, D. Toublan, and J. J. M. Verbaarschot, 2001, Phys. Lett. **B522**, 67.
Schafer, T., and F. Wilczek, 1999a, Phys. Rev. Lett. **82**, 3956.
Schafer, T., and F. Wilczek, 1999b, Phys. Lett. **B450**, 325.
Schafer, T., and F. Wilczek, 1999c, Phys. Rev. **D60**, 074014.
Schafer, T., and F. Wilczek, 1999d, Phys. Rev. **D60**, 114033.
Schrieffer, J. R., 1964, *Theory of Superconductivity* (The Benjamin/Cummings Publishing Company, Inc.).
Shankar, R., 1994, Rev. Mod. Phys. **66**, 129.
Shapiro, S. L., and S. A. Teukolski, 1983, *Black holes, White Dwarfs and Neutron Stars: The Physics of Compact Objects* (Wiley-Interscience).
Shifman, M. A., A. I. Vainshtein, and V. I. Zakharov, 1980, Nucl. Phys. **B163**, 46.
Shimahara, H., 1998, J. Phys. Soc. Japan **67**, 736.
Shovkovy, I. A., and P. J. Ellis, 2002, Phys. Rev. **C66**, 015802.
Shovkovy, I. A., and L. C. R. Wijewardhana, 1999, Phys. Lett. **B470**, 189.
Son, D. T., 1999, Phys. Rev. **D59**, 094019.
Son, D. T., and M. A. Stephanov, 2000a, Phys. Rev. **D61**, 074012.
Son, D. T., and M. A. Stephanov, 2000b, Phys. Rev. (D61 Erratum) **D62**, 059902.
Son, D. T., M. A. Stephanov, and A. R. Zhitnitsky, 2001, Phys. Rev. Lett. **86**, 3955.
Tinkham, M., 1995, *Introduction to Superconductivity* (McGraw-Hill).
Tsuruta, S., 1998, Phys. Rep. **292**, 1.
Wang, Q., and D. H. Rischke, 2002, Phys. Rev. **D65**, 054005.
Weinberg, S., 1996, *The Quantum theory of Fields, Vol. II* (Cambridge University Press).
Witten, E., 1984, Phys. Rev. **D30**, 272.
Zarembo, K., 2000, Phys. Rev. **D62**, 054003.
Ziman, J. M., 1964, *Principles of the Theory of Solids* (Cambridge University Press, New York).