Cauchy principal value and hypersingular integrals

Chapter 1

Introduction

In this introduction to *Cauchy principal value and hypersingular integrals* two aspects will be presented. In the first place a general overview will be given to this work to motivate its existence. In the second place an introduction to the mathematical content will be given.

1.1 Overview

Everybody working in the field of singular integrals and integral equations will know that during the last few decades an entirely new mathematical field of Cauchy principal value integrals and hypersingular integrals has developed.

Since this is a recent mathematical development, it is not always easy for readers, including academics, engineers and researchers, to get a grasp on this field. For assistance, such readers will have to consult research articles or books, where they will be confronted by a mass of information at an advanced level. To the best of my knowledge, there presently does not exist any concise and compact book on this topic.

This led me to the conclusion that a need exists for a single reference work that addresses at least the following topics:

- a presentation of the historical development of Cauchy principal value integrals and hypersingular integrals;
- a summary of different definitions of such integrals;
- a discussion of analytical integration results;
- a discussion of available numerical integration techniques (including derivations of these methods, convergence results, error analyses, etc);
- a list of relevant tables with constants for the different numerical formulae (for example, nodes and weights);
- a theoretical example illustrating every method presented;

- practical examples that give more flesh to the bone;
- a discussion of extensions that could be made in this field (for example, to super-singular integrals), and lastly,
- a comprehensive list of bibliographic information.

As anyone would realize, the writing of a book comprising of all this would be a major task. A simpler and more practical way out of this situation would be to begin with a smaller set of ideas, starting for example with the following themes:

- a summary of different definitions of Cauchy principal value integrals and hyper-singular integrals,
- a discussion of analytical integration results, and
- a discussion of available numerical integration techniques, together with tables of nodes and weights for the different formulae.

Presenting these themes is therefore the intention of the present work.

One practical way of presenting such a work is by giving lists of results. Books like *Mathematical handbook* by MR Spiegel, *Handbook of mathematical functions* by M Abramowitz & IA Stegun and *Tables of integrals, series and products* by IS Gradshteyn & IM Ryzhik may be cited as examples. These books, however, are not mentioned here with the idea of comparing the present work to them, only to explain the idea.

Some practical aspects of the present work need to be mentioned.

In compiling an index one normally keeps in mind a typical person for whom the book is written and the same goes for the present work. The typical person for whom this work on Cauchy principal value integrals and hypersingular integrals should be accessible, is a chemical engineer looking for the solution of a singular integral.

Let us say this engineer wants to evaluate the integral

$$I = \neq\!\!\!\!\!\!\int_0^1 \frac{\sin \pi t}{(t-x)^3}\, dt,$$

for $0 < x < 1$. The index should then be such that it is easy to find the result. The index of topics at the beginning of Chapter 3 shows that Section 3.3.1(c) will give the required result. The user then should have no difficulty in finding the result and can apply it to whatever larger problem may be solved, namely:

For $0 < x < 1$, there is the expression

$$
\begin{aligned}
⨍_0^1 \frac{\sin \pi t}{(t-x)^3}\, dt \;=\; & \cos \pi x \left[\frac{\pi}{x(x-1)} + \sum_{j=0}^{\infty} (-1)^{j+1} \frac{\pi^{2j+3}}{(2j+1).(2j+3)!} \left((1-x)^{2j+1} - (-x)^{2j+1} \right) \right] \\
& + \sin \pi x \left[\frac{(1-x)^2 - x^2}{2x^2(x-1)^2} - \frac{\pi^2}{2} \ln \left(\frac{1-x}{x} \right) \right. \\
& \left. + \sum_{j=1}^{\infty} (-1)^{j+1} \frac{\pi^{2j+2}}{(2j).(2j+2)!} \left((1-x)^{2j} - (-x)^{2j} \right) \right].
\end{aligned}
$$

Some remarks on this and similar results also need to be made:

- It is clear that when $x \approx 0$ or 1, a problem will be encountered.
- It is not immediately clear how many terms in the infinite sums have to be computed.

However, on closer inspection one realizes that due to the terms $(2j+3)!$ and $(2j+2)!$ in the denominators the calculation will most probably lead to a quick convergence. Experimenting a little will produce an acceptable answer. The point is: one should not use the published results blindly but should use them with the necessary degree of discretion.

This work is organized as follows. In Chapter 2 attention is paid to definitions and characteristics of singular integrals, in Chapter 3 to the analytical calculation and in Chapter 4 to the numerical calculation of such integrals. In all three cases the subsections are then devoted to (i) Cauchy principal value integrals, (ii) hypersingular integrals and (iii) generalizations.

It is certainly clear that a work like this will never really be complete. Remarks will be welcomed, for example on the presentation of the information. Kindly email any suggestions to johan.deklerk@nwu.ac.za .

I hope that this introduction will clarify the intention with this work and will place the aim with the text in the correct perspective. Much effort has been put in this work, especially to evaluate problems and to organize them in a logical way. I trust that it will be useful to people at all levels, not in the least also to young upcoming scientists.

1.2 Mathematical considerations

In calculus an improper integral may be defined as the limit of a definite integral when an endpoint of the interval of integration approaches either (i) a specific real number or (ii) $-\infty$ or ∞. Specifically, an improper integral is a limit of either the form

$$
\lim_{b \to \infty} \int_a^b f(t)\, dt, \qquad \lim_{a \to -\infty} \int_a^b f(t)\, dt
$$

or of the form

$$\lim_{c\to b^-}\int_a^c f(t)\,dt, \qquad \lim_{c\to a^+}\int_c^b f(t)\,dt.$$

These may also include cases where the integrand is undefined at one or more interior points of the domain of integration.

It is often necessary to use an improper integral in order to compute a value for an integral that does not exist in the more conventional sense (for example, as a Riemann integral) because of a singularity in the integrand, or of an infinite endpoint of the domain of integration.

Consider the following two examples:

The integral

$$I_1 = \int_1^\infty \frac{1}{t^2}\,dt$$

does not exist as a Riemann integral (because the domain of integration is unbounded, and the Riemann integral is only well-defined over a bounded domain). I_1 may, however, be assigned a value by interpreting it instead as an improper integral, namely,

$$I_1 = \lim_{b\to\infty}\int_1^b \frac{1}{t^2}\,dt = \lim_{b\to\infty}\left[-\frac{1}{b}+1\right] = 1.$$

The integral

$$I_2 = \int_0^1 \frac{1}{\sqrt{t}}\,dt$$

also does not exist as a Riemann integral (because again, the integrand is unbounded, and the Riemann integral is only well-defined for bounded functions). I_2 may, however, also be assigned a value by interpreting it instead as an improper integral, namely,

$$I_2 = \lim_{a\to 0^+}\int_0^1 \frac{1}{\sqrt{t}}\,dt = \lim_{a\to 0^+}\left[2\sqrt{1}-2\sqrt{a}\right] = 2.$$

Consult the following reference for more detail: Improper integral (Wikipedia)

It may sometimes happen that not even an improper integral can be used to define an integral. Depending on the type of singularity in the integrand f, one can however define an improper integral in the sense of a Cauchy principal value integral. For example, suppose that $f(t)$ is unbounded at an interior point t_0 of the interval $[a,b]$. Then one can define

$$\int_a^b f(t)\,dt = \lim_{\epsilon_1\to 0^+}\int_a^{t_0-\epsilon_1} f(t)\,dt + \lim_{\epsilon_2\to 0^+}\int_{t_0+\epsilon_2}^b f(t)\,dt.$$

However, it may happen that the limits in this expression do not exist if ϵ_1 and ϵ_2 approach 0 independently. In the special case of equality, that is, $\epsilon_1 = \epsilon_2 = \epsilon$, however, it may be possible that the limit in

$$\int_a^b f(t)\,dt = \lim_{\epsilon \to 0^+} \left[\int_a^{t_0-\epsilon} f(t)\,dt + \int_{t_0+\epsilon}^b f(t)\,dt \right]$$

exists. If that is the case, the expression is known as the **Cauchy principal value** of the given integral.

Consider as an example the following two different scenarios for the same expression, I_3:

$$\begin{aligned} I_3 = \fint_0^5 \frac{1}{4-t}\,dt &= \lim_{\epsilon_1 \to 0^+} \int_0^{4-\epsilon_1} \frac{1}{4-t}\,dt + \lim_{\epsilon_2 \to 0^+} \int_{4+\epsilon_2}^5 \frac{1}{4-t}\,dt \\ &= \lim_{\epsilon_1 \to 0^+} \Big[-\ln|4-t| \Big]_0^{4-\epsilon_1} + \lim_{\epsilon_2 \to 0^+} \Big[-\ln|4-t| \Big]_{4+\epsilon_2}^5 \\ &= -\lim_{\epsilon_1 \to 0^+} \Big[\ln\epsilon_1 - \ln 4 \Big] - \lim_{\epsilon_2 \to 0^+} \Big[\ln 1 - \ln\epsilon_2 \Big], \end{aligned}$$

that does not exist if ϵ_1 and ϵ_2 approach 0 independently. On the other hand, one has, for $\epsilon_1 = \epsilon_2 = \epsilon$, that

$$\begin{aligned} I_3 = \fint_0^5 \frac{1}{4-t}\,dt &= \lim_{\epsilon \to 0^+} \int_0^{4-\epsilon} \frac{1}{4-t}\,dt + \lim_{\epsilon \to 0^+} \int_{4+\epsilon}^5 \frac{1}{4-t}\,dt \\ &= \lim_{\epsilon \to 0^+} \Big[-\ln|4-t| \Big]_0^{4-\epsilon} + \lim_{\epsilon \to 0^+} \Big[-\ln|4-t| \Big]_{4+\epsilon}^5 \\ &= \lim_{\epsilon \to 0^+} \Big[-\ln|\epsilon| + \ln 4 - \ln 1 + \ln|\epsilon| \Big] \\ &= \ln 4. \end{aligned}$$

There are different nomenclatures in use for the Cauchy principal value integral. Among these are the following:

$$C\int f(t)\,dt, \qquad PV\int f(t)\,dt, \qquad \fint f(t)\,dt.$$

Of these, the last one mentioned is used in this study, specifically in the form

$$\fint_a^b \frac{f(t)}{t-x}\,dt, \qquad a < x < b.$$

In the following chapters results on the Cauchy principal value integrals will be given in the first section of each chapter. Also consult the following reference for more detail: Cauchy principal value (Wikipedia)

It is known that if the Cauchy principal value integral is differentiated with respect to x, it gives the result

$$\frac{d}{dx}\int_a^b \frac{f(t)}{t-x}\,dt = \fint_a^b \frac{f(t)}{(t-x)^2}\,dt, \quad a<x<b.$$

This result is called the **Hadamard finite part integral** or **hypersingular integral**. For this integral there are also different nomenclatures, but the one given above will be used throughout this text.

Writing this in the reverse form, namely,

$$\fint_a^b \frac{f(t)}{(t-x)^2}\,dt = \frac{d}{dx}\int_a^b \frac{f(t)}{t-x}\,dt, \quad a<x<b,$$

one then naturally can use this expression as a definition. There are also other definitions, among them,

$$\fint_a^b \frac{f(t)}{(t-x)^2}\,dt = \lim_{\epsilon\to 0}\Big[\int_a^{x-\epsilon} \frac{f(t)}{(t-x)^2}\,dt + \int_{x+\epsilon}^b \frac{f(t)}{(t-x)^2}\,dt - \frac{2f(x)}{\epsilon}\Big].$$

Information on hypersingular integrals can be found in all of the next three chapters, each time in the second section. Also consult the following references for more detail: Hypersingular/Hadamard integral (Wikipedia) and Hypersingular integral (WT Ang)

The third sections in all three of the following chapters cover results that can be labeled as "generalizations". Among these you will find extensions of different kinds. One that can be mentioned here, by way of an example, is that for $a<x<b$ and $p\geq 3$, the result is

$$\begin{aligned}\fint_a^b \frac{f(t)}{(t-x)^{p+1}}\,dt &= \lim_{\epsilon\to 0}\Bigg[\int_a^{x-\epsilon} \frac{f(t)}{(t-x)^{p+1}}\,dt + \int_{x+\epsilon}^b \frac{f(t)}{(t-x)^{p+1}}\,dt \\ &\quad + \frac{1}{p!}\frac{d^{p-1}}{dx^{p-1}}\Big(-\frac{2f(x)}{\epsilon}\Big) + \sum_{k=2}^{p-1}\Big(\prod_{j=0}^{p-k-1}\frac{1}{p-j}\Big)\frac{d^{p-k}S_k}{dx^{p-k}} + S_p\Bigg],\end{aligned}$$

with $S_2 = -\frac{f'(x)}{\epsilon}$ and

$$S_l = \begin{cases} -\sum_{k=1,3,\ldots}^{l-1} \frac{\epsilon^{k-l}}{l}\frac{f^{(k)}(x)}{k!}, & l \text{ even} \\ -\sum_{k=0,2,\ldots}^{l-1} \frac{\epsilon^{k-l}}{l}\frac{f^{(k)}(x)}{k!}, & l \text{ odd.}\end{cases}$$

To conclude, it should again be mentioned that a work like this will never be complete. If there are any queries or suggestions, please let me know at johan.deklerk@nwu.ac.za.

In Chapter 4 a variety of algorithms may be found for the numerical evaluation of singular integrals. This chapter has not been completed yet and comments on the different techniques are still being awaited. Please use the algorithms and if necessary, make comments. Also, at the end of chapter 4 a pro forma may be found for suggestions and details of other algorithms. It would be appreciated if you would make use of it and would forward it to the above email address.

Although brief and cursory, I hope that with this introduction I have given an idea of what is to be expected in this work. Please use the information (definitions, formulae, algorithms, etc) in your own work. It would be appreciated if you would also distribute it among your colleagues and students.

Johan DeKlerk
Mathematics and Applied Mathematics
North West University
Potchefstroom
South Afrika

Last update: November 2011

Chapter 2

Definitions and characteristics

Theme	**Section**
Definitions	2.1.0
$\fint_a^b \frac{\alpha f(t)+\beta g(t)}{t-x}\,dt$	2.1.1
$\frac{d}{dx}\fint_a^b \frac{f(t)}{t-x}\,dt$	2.1.2
...	2.1.3
...	2.1.4
Change of interval	2.1.5
Definitions	2.2.0
$\fint_a^b \frac{\alpha f(t)+\beta g(t)}{(t-x)^2}\,dt$	2.2.1
$\frac{d}{dx}\fint_a^b \frac{f(t)}{(t-x)^2}\,dt$	2.2.2
Change of power	2.2.3
Change of power	2.2.4
Change of interval	2.2.5
Definitions	2.3.0
$\fint_a^b \frac{\alpha f(t)+\beta g(t)}{(t-x)^{p+1}}\,dt$	2.3.1
$\frac{d}{dx}\fint_a^b \frac{f(t)}{(t-x)^{p+1}}\,dt$	2.3.2
Change of power	2.3.3
Change of power	2.3.4
Change of interval	2.3.5

2.1 Cauchy principal value integrals

2.1.0(a) For respectively $a < x < b$, $a < x = b$ and $a = x < b$, one has the definitions

$$\begin{aligned}
\fint_a^b \frac{f(t)}{t-x}\,dt &= \lim_{\epsilon\to 0}\Big[\int_a^{x-\epsilon}\frac{f(t)}{t-x}\,dt + \int_{x+\epsilon}^b \frac{f(t)}{t-x}\,dt\Big],\\
\fint_a^x \frac{f(t)}{t-x}\,dt &= \lim_{\epsilon\to 0}\Big[\int_a^{x-\epsilon}\frac{f(t)}{t-x}\,dt - f(x)\ln\epsilon\Big],\\
\fint_x^b \frac{f(t)}{t-x}\,dt &= \lim_{\epsilon\to 0}\Big[\int_{x+\epsilon}^b \frac{f(t)}{t-x}\,dt + f(x)\ln\epsilon\Big].
\end{aligned}$$

Source: Kythe & Schäferkotter (2005) (5.4.4) and (5.4.5)

2.1.0(b) For $a < x < b$

$$\fint_a^b \frac{f(t)}{t-x}\,dt = \int_a^b \frac{f(t)-f(x)}{t-x}\,dt + f(x)\ln\frac{b-x}{x-a}.$$

Source: Special case of 2.3.0(b)

2.1.1 For $a < x < b$

$$\fint_a^b \frac{\alpha f(t)+\beta g(t)}{t-x}\,dt = \alpha\fint_a^b \frac{f(t)}{t-x}\,dt + \beta\fint_a^b \frac{g(t)}{t-x}\,dt.$$

Source: Special case of 2.3.1

2.1.2

(a) For $a < x < b$

$$\frac{d}{dx}\fint_a^b \frac{f(t)}{t-x}\,dt = \fint_a^b \frac{f(t)}{(t-x)^2}\,dt.$$

(b) For $x = a$

$$\frac{d}{dx}\fint_a^b \frac{f(t)}{t-x}\,dt = \fint_a^b \frac{f(t)}{(t-x)^2}\,dt - f'(x).$$

Source: Special case of 2.3.2

2.1.5(a1) For $a < x < b$

$$\fint_a^b \frac{f(t)}{t-x}\,dt = \fint_{-1}^1 \frac{g(u)}{u-X}\,du$$

with $X = \frac{2x-b-a}{b-a}$ and $g(u) = f\big(\frac{1}{2}(b-a)u + \frac{1}{2}(b+a)\big)$.

2.1.5(a2) For $x = a$

$$\fint_a^b \frac{f(t)}{t-a}\,dt = \fint_{-1}^{1} \frac{g(u)}{u+1}\,du + f(a)\,\ln\left[\frac{b-a}{2}\right]$$

with $g(u) = f\left(\frac{1}{2}(b-a)u + \frac{1}{2}(b+a)\right)$.

Source: Special case of 2.3.5

2.2 Hypersingular integrals

2.2.0(a0) For $a < x < b$, one has the definition

$$\oint_a^b \frac{f(t)}{(t-x)^2}\,dt = \lim_{\epsilon\to 0}\Big[\int_a^{x-\epsilon} \frac{f(t)}{(t-x)^2}\,dt + \int_{x+\epsilon}^b \frac{f(t)}{(t-x)^2}\,dt - \frac{2f(x)}{\epsilon}\Big].$$

Source: Sun & Wu (2005) (p298); Kythe & Schäferkotter (2005) (p254); Kolm & Rokhlin (2001) (p329); Hui & Shia (1999) (p206); Linz (1985) (p346); Paget (1981) (p447)

2.2.0(a1) For $a < x = b$, one has

$$\oint_a^x \frac{f(t)}{(t-x)^2}\,dt = \lim_{\epsilon\to 0}\Big[\int_a^{x-\epsilon} \frac{f(t)}{(t-x)^2}\,dt - \frac{f(x)}{\epsilon} - f'(x)\ln|x|\Big],$$

and for $x = a < b$, one has

$$\oint_x^b \frac{f(t)}{(t-x)^2}\,dt = \lim_{\epsilon\to 0}\Big[\int_{x+\epsilon}^b \frac{f(t)}{(t-x)^2}\,dt - \frac{f(x)}{\epsilon} + f'(x)\ln|x|\Big].$$

Source: Kythe & Schäferkotter (2005) (p254)

2.2.0(b) For $a < x < b$

$$\begin{aligned}\oint_a^b \frac{f(t)\,dt}{(t-x)^2} &= \int_a^b \frac{1}{(t-x)^2}\big[f(t) - f(x) - f'(x)(t-x)\big]\,dt \\ &+ f(x)\oint_a^b \frac{dt}{(t-x)^2} + f'(x)\oint_a^b \frac{dt}{t-x}.\end{aligned}$$

Source: Special case of 2.3.0(b)

2.2.1 For $a < x < b$

$$\oint_a^b \frac{\alpha f(t) + \beta g(t)}{(t-x)^2}\,dt = \alpha\oint_a^b \frac{f(t)}{(t-x)^2}\,dt + \beta\oint_a^b \frac{g(t)}{(t-x)^2}\,dt.$$

Source: Special case of 2.3.1

2.2.2

(a) For $a < x < b$

$$\frac{d}{dx}\oint_a^b \frac{f(t)}{(t-x)^2}\,dt = 2\oint_a^b \frac{f(t)}{(t-x)^3}\,dt.$$

(b) For $x = a$

$$\frac{d}{dx}\oint_a^b \frac{f(t)}{(t-x)^2}\,dt = 2\oint_a^b \frac{f(t)}{(t-x)^3}\,dt - \frac{f^{(2)}(x)}{2}.$$

Source: Special case of 2.3.2

2.2.3 For $a < x < b$

$$\fint_a^b \frac{f(t)}{(t-x)^2}\,dt = -\Big[\frac{f(b)}{b-x} - \frac{f(a)}{a-x}\Big] + \fint_a^b \frac{f'(t)}{t-x}\,dt.$$

Source: Special case of 2.3.3

2.2.4 For $a < x < b$

$$\fint_a^b \frac{f(t)}{(t-x)^2}\,dt = \frac{d}{dx}\fint_a^b \frac{f(t)}{t-x}\,dt.$$

Source: Special case of 2.3.4

2.2.5(a1) For $a < x < b$

$$\fint_a^b \frac{f(t)}{(t-x)^2}\,dt = \Big(\frac{2}{b-a}\Big)\fint_{-1}^1 \frac{g(u)}{(u-X)^2}\,du$$

with $X = \frac{2x-b-a}{b-a}$ and $g(u) = f\big(\frac{1}{2}(b-a)u + \frac{1}{2}(b+a)\big)$.

2.2.5(a2) For $x = a$

$$\fint_a^b \frac{f(t)}{(t-x)^2}\,dt = \Big(\frac{2}{b-a}\Big)\fint_{-1}^1 \frac{g(u)}{(u+1)^2}\,du + f'(a)\,\ln\left[\frac{b-a}{2}\right]$$

with $g(u) = f\big(\frac{1}{2}(b-a)u + \frac{1}{2}(b+a)\big)$.

Source: Special case of 2.3.5

2.3 Generalizations

2.3.0(a3) For $a < x < b$

$$\displaystyle{=}\!\!\!\!\!\!\int_a^b \frac{f(t)}{(t-x)^3}\,dt = \lim_{\epsilon\to 0}\left[\int_a^{x-\epsilon} \frac{f(t)}{(t-x)^3}\,dt + \int_{x+\epsilon}^b \frac{f(t)}{(t-x)^3}\,dt - \frac{2f'(x)}{\epsilon}\right].$$

Source: De Klerk

2.3.0(a4) For $a < x < b$

$$\displaystyle{=}\!\!\!\!\!\!\int_a^b \frac{f(t)}{(t-x)^4}\,dt = \lim_{\epsilon\to 0}\left[\int_a^{x-\epsilon} \frac{f(t)}{(t-x)^4}\,dt + \int_{x+\epsilon}^b \frac{f(t)}{(t-x)^4}\,dt - \frac{2}{3}\frac{f(x)}{\epsilon^3} - \frac{f''(x)}{\epsilon}\right].$$

Source: De Klerk

Generalization:

2.3.0(an) For $a < x < b, \quad p \geq 3,$

$$\begin{aligned}\displaystyle{=}\!\!\!\!\!\!\int_a^b \frac{f(t)}{(t-x)^{p+1}}\,dt &= \lim_{\epsilon\to 0}\Bigg[\int_a^{x-\epsilon} \frac{f(t)}{(t-x)^{p+1}}\,dt + \int_{x+\epsilon}^b \frac{f(t)}{(t-x)^{p+1}}\,dt \\ &\quad + \frac{1}{p!}\frac{d^{p-1}}{dx^{p-1}}\left(-\frac{2f(x)}{\epsilon}\right) + \sum_{k=2}^{p-1}\left(\prod_{j=0}^{p-k-1}\frac{1}{p-j}\right)\frac{d^{p-k}S_k}{dx^{p-k}} + S_p\Bigg],\end{aligned}$$

with $S_2 = -\frac{f'(x)}{\epsilon}$ and

$$S_l = \begin{cases} -2\sum_{k=1,3,\ldots}^{l-1} \frac{\epsilon^{k-l}}{l}\frac{f^{(k)}(x)}{k!}, & l \text{ even} \\ -2\sum_{k=0,2,\ldots}^{l-1} \frac{\epsilon^{k-l}}{l}\frac{f^{(k)}(x)}{k!}, & l \text{ odd}. \end{cases}$$

Remark: This result also holds for the case $p = 2$ taking in mind that $\sum_{k=1}^{q\,(<1)} \cdots = 0$.

Source: De Klerk

2.3.0(b) For $a < x < b$, $p \geq 0$, $r > p$, r an integer,

$$\begin{aligned}\displaystyle{=}\!\!\!\!\!\!\int_a^b \frac{f(t)}{(t-x)^{p+1}}\,dt = \int_a^b \frac{1}{(t-x)^{p+1}}\left[f(t) \quad - \quad \sum_{j=0}^{r} \frac{f^{(j)}(x)(t-x)^j}{j!}\right]dt \\ + \quad \sum_{j=0}^{r}\frac{f^{(j)}(x)}{j!}\, {=}\!\!\!\!\!\!\int_a^b \frac{dt}{(t-x)^{p+1-j}}.\end{aligned}$$

Source: Monegato (1994) (2.5)

Special case: For $a = x$, $x = 0$ and $p \geq 0$ an integer,

$$\displaystyle{\neq}\!\!\!\!\int_0^b \frac{f(t)}{t^{p+1}}\,dt = \int_0^b \frac{1}{t^{p+1}}\Big[f(t) \;-\; \sum_{j=0}^{p} \frac{f^{(j)}(0)t^j}{j!}\Big]\,dt \;+\; \sum_{j=0}^{p-1} \frac{f^{(j)}(0)}{j!}\,\frac{b^{-p+j}}{-p+j} \;+\; \frac{f^{(p)}(0)}{p!}\ln b.$$

Source: Ramm & Van der Sluis (1990) (2)

Special case: For $a = x$, $x = 0$ and $p > 0$ not an integer, $k > p$, k an integer and a free choice,

$$\displaystyle{\neq}\!\!\!\!\int_0^b \frac{f(t)}{t^{p+1}}\,dt = \int_0^b \frac{1}{t^{p+1}}\Big[f(t) \;-\; \sum_{j=0}^{k} \frac{f^{(j)}(0)t^j}{j!}\Big]\,dt \;+\; \sum_{j=0}^{k} \frac{f^{(j)}(0)}{j!}\,\frac{b^{-p+j}}{-p+j}.$$

Source: Ramm & Van der Sluis (definition) (1990) (2)

2.3.1 For $a < x < b$, $p \geq 0$,

$$\displaystyle{\neq}\!\!\!\!\int_a^b \frac{\alpha f(t) + \beta g(t)}{(t-x)^{p+1}}\,dt = \alpha\, {\neq}\!\!\!\!\int_a^b \frac{f(t)}{(t-x)^{p+1}}\,dt + \beta\, {\neq}\!\!\!\!\int_a^b \frac{g(t)}{(t-x)^{p+1}}\,dt.$$

Source: Monegato (1994) (p12)

2.3.2

For $a < x < b$ and $\alpha > 0$,

$$\frac{d}{dx}\, {\neq}\!\!\!\!\int_a^b \frac{f(t)}{(t-a)^p}\,dt = {\neq}\!\!\!\!\int_a^b \frac{\partial}{\partial x}\Big[\frac{1}{(t-x)^p}\Big] f(t)\,dt.$$

Source: Kaya & Erdogan (1987) (12)

Special cases:

(a) For $a < x < b$ and $p > 0$ an integer,

$$\frac{d}{dx}\, {\neq}\!\!\!\!\int_a^b \frac{f(t)}{(t-x)^p}\,dt = p\, {\neq}\!\!\!\!\int_a^b \frac{f(t)}{(t-x)^{p+1}}\,dt$$

or, of course also,

$${\neq}\!\!\!\!\int_a^b \frac{f(t)}{(t-x)^{p+1}}\,dt = \frac{1}{p}\,\frac{d}{dx}\, {\neq}\!\!\!\!\int_a^b \frac{f(t)}{(t-x)^p}\,dt.$$

Source: Monegato (1994) (2.6)

(b) For $x = a$ and $p > 0$ not an integer,

$$\frac{d}{dx}\, {\neq}\!\!\!\!\int_a^b \frac{f(t)}{(t-x)^p}\,dt = p\, {\neq}\!\!\!\!\int_a^b \frac{f(t)}{(t-x)^{p+1}}\,dt.$$

Source: Monegato (1994) (2.6)

(c) For $x = a$ and $p > 0$ an integer,

$$\frac{d}{dx} \fint_a^b \frac{f(t)}{(t-x)^p}\, dt = p \fint_a^b \frac{f(t)}{(t-x)^{p+1}}\, dt - \frac{f^{(p)}(x)}{p!}.$$

Source: Monegato (1994) (2.6)

2.3.3 For $a < x < b$

$$\fint_a^b \frac{f(t)}{(t-x)^{p+1}}\, dt = -\frac{1}{p}\Big[\frac{f(b)}{(b-x)^p} - \frac{f(a)}{(a-x)^p}\Big] + \frac{1}{p} \fint_a^b \frac{f'(t)}{(t-x)^p}\, dt.$$

Source: Monegato (1994) (2.8)

2.3.4 For $a < x < b$ and p an integer

$$\fint_a^b \frac{f(t)}{(t-x)^{p+1}}\, dt = \frac{1}{p!} \frac{d^p}{dx^p} \fint_a^b \frac{f(t)}{t-x}\, dt.$$

Source: Monegato (1994) (2.7); Korsunsky (1997) (8)

2.3.5(a1) For $a \le x \le b$ and p not an integer,

$$\fint_a^b \frac{f(t)}{(t-x)^{p+1}}\, dt = \Big(\frac{2}{b-a}\Big)^p \fint_{-1}^1 \frac{g(u)}{(u-X)^{p+1}}\, du$$

with $X = \frac{2x-b-a}{b-a}$ and $g(u) = f\big(\frac{1}{2}(b-a)u + \frac{1}{2}(b+a)\big)$.

2.3.5(a2) For $a < x < b$ and p an integer,

$$\fint_a^b \frac{f(t)}{(t-x)^{p+1}}\, dt = \Big(\frac{2}{b-a}\Big)^p \fint_{-1}^1 \frac{g(u)}{(u-X)^{p+1}}\, du$$

with $X = \frac{2x-b-a}{b-a}$ and $g(u) = f\big(\frac{1}{2}(b-a)u + \frac{1}{2}(b+a)\big)$.

2.3.5(a3) For $x = a$ and p an integer,

$$\fint_a^b \frac{f(t)}{(t-a)^{p+1}}\, dt = \Big(\frac{2}{b-a}\Big)^p \fint_{-1}^1 \frac{g(u)}{(u+1)^{p+1}}\, du + \frac{f^{(p)}(a)}{p!} \ln\Big[\frac{b-a}{2}\Big]$$

with $g(u) = f\big(\frac{1}{2}(b-a)u + \frac{1}{2}(b+a)\big)$.

Source: Monegato (1994) (p 13)

Chapter 3

Singular integrals: analytically calculated

Theme	**Section**
$\fint_a^b \frac{t^n}{t-x}\,dt,\quad \ffint_a^b \frac{t^n}{(t-x)^2}\,dt,\quad \ffint_a^b \frac{t^n}{(t-x)^{p+1}}\,dt$	3.1.1(a), 3.2.1(a), 3.3.1(a)
$\fint_{-1}^1 \frac{P_n(t)}{t-x}\,dt,\quad \ffint_{-1}^1 \frac{P_n(t)}{(t-x)^2}\,dt,\quad \ffint_{-1}^1 \frac{P_n(t)}{(t-x)^{p+1}}\,dt$	3.1.1(b), 3.2.1(b), 3.3.1(b)
$\fint_a^b \frac{\sin t/\cos t}{t-x}\,dt,\quad \ffint_a^b \frac{\sin t/\cos t}{(t-x)^2}\,dt,\quad \ffint_a^b \frac{\sin t/\cos t}{(t-x)^{p+1}}\,dt$	3.1.1(c), 3.2.1(c), 3.3.1(c)
$\fint_a^b \frac{t^n\sqrt{1-t}}{t-x}\,dt,\quad \ffint_a^b \frac{t^n\sqrt{1-t}}{(t-x)^2}\,dt,\quad \ffint_a^b \frac{t^n\sqrt{1-t}}{(t-x)^{p+1}}\,dt$	3.1.2(a), 3.2.2(a), 3.3.2(a)
$\fint_a^b \frac{t^n\sqrt{1-t^2}}{t-x}\,dt,\quad \ffint_a^b \frac{t^n\sqrt{1-t^2}}{(t-x)^2}\,dt,\quad \ffint_a^b \frac{t^n\sqrt{1-t^2}}{(t-x)^{p+1}}\,dt$	3.1.3(a), 3.2.3(a), 3.3.3(a)
$\fint_a^b \frac{U_n(t)\sqrt{1-t^2}}{(t-x)^p}\,dt,\quad \fint_{-1}^1 \frac{T_n(t)\,dt}{(t-x)^p\sqrt{1-t^2}}$	3.1.3(b), 3.2.3(b), 3.3.3(b)
$\fint_a^b \frac{t^n}{(t-x)\sqrt{1-t}}\,dt,\quad \ffint_a^b \frac{t^n}{(t-x)^2\sqrt{1-t}}\,dt,\quad \ffint_a^b \frac{t^n}{(t-x)^{p+1}\sqrt{1-t}}\,dt$	3.1.4(a), 3.2.4(a), 3.3.4(a)
$\fint_a^b \frac{t^n}{(t-x)\sqrt{1-t^2}}\,dt,\quad \ffint_a^b \frac{t^n}{(t-x)^2\sqrt{1-t^2}}\,dt,\quad \ffint_a^b \frac{t^n}{(t-x)^{p+1}\sqrt{1-t^2}}\,dt$	3.1.5(a), 3.2.5(a), 3.3.5(a)
$\fint_a^b \frac{T_n(t)}{(t-x)\sqrt{1-t^2}}\,dt$	3.1.5(b)
Variety of cases	3.3.6

3.1 Cauchy principal value integrals

3.1.1(a0) For $a < x < b$

$$\fint_a^b \frac{dt}{t-x} = \ln \frac{b-x}{x-a}.$$

Source: Special case of 3.3.1(a0)

Special case: For $x = 0$

$$\fint_{-1}^1 \frac{dt}{t} = 0.$$

Special case: For $0 < x < 1$

$$\fint_0^1 \frac{dt}{t-x} = \ln \frac{1-x}{x}.$$

3.1.1(a1) For $a < x < b$

$$\fint_a^b \frac{t}{t-x}\, dt = (b-a) + x \ln \frac{b-x}{x-a}.$$

Source: Special case of 3.3.1(a1)

3.1.1(a2) For $a < x < b$

$$\fint_a^b \frac{t^2}{t-x}\, dt = \frac{1}{2}[(b-x)^2 - (a-x)^2] + 2x(b-a) + x^2 \ln \frac{b-x}{x-a}$$

or

$$\fint_a^b \frac{t^2}{t-x}\, dt = \frac{1}{2}(b^2-a^2) + x(b-a) + x^2 \ln \frac{b-x}{x-a}.$$

Source: Special case of 3.3.1(a2)

3.1.1(a3) For $a < x < b$

$$\begin{aligned}\fint_a^b \frac{t^3}{t-x}\, dt &= \tfrac{1}{3}[(b-x)^3 - (a-x)^3] + \tfrac{3}{2}x[(b-x)^2 - (a-x)^2] \\ &\quad + 3x^2(b-a) + x^3 \ln \frac{b-x}{x-a}\end{aligned}$$

or

$$\fint_a^b \frac{t^3}{t-x}\, dt = \tfrac{1}{3}(b^3-a^3) + \tfrac{1}{2}x(b^2-a^2) + x^2(b-a) + x^3 \ln \frac{b-x}{x-a}.$$

Source: Special case of 3.3.1(a3)

3.1.1(an) For $a < x < b$ and $\nu > 0$ rational,

$$\fint_a^b \frac{t^\nu}{t-x}\, dt = x \fint_a^b \frac{t^{\nu-1}}{t-x}\, dt + \frac{1}{\nu}(b^\nu - a^\nu).$$

Source: Compare Kythe & Schäferkotter (2005), p249

For $\nu = n$ an integer,

$$\fint_a^b \frac{t^n}{t-x}\,dt = x^n \ln\frac{b-x}{x-a} + \sum_{k=0}^{n-1} \frac{x^k}{n-k}\Big(b^{n-k} - a^{n-k}\Big),$$

or, alternatively,

$$\fint_a^b \frac{t^n}{t-x}\,dt = x^n \ln\frac{b-x}{x-a} + \sum_{k=1}^{n} x^{n-k}\binom{n}{k}\frac{(b-x)^k - (a-x)^k}{k}.$$

Source: De Klerk

3.1.1(b) For $-1 < x < 1$

$$\fint_{-1}^1 \frac{P_n(t)}{t-x}\,dt = -2Q_n(x), \qquad n \geq 0,$$

where P_n and Q_n are respectively the Legendre polynomials of the first kind and Legendre functions of the second kind.

Source: Kaya & Erdogan (1987) (28)

3.1.1(c1) For $a < x < b$ and $\alpha = a - x, \quad \beta = b - x, \quad \alpha < 0 < \beta,$

$$\fint_a^b \frac{e^{kt}}{t-x}\,dt \;=\; e^{kx}\left[\ln\Big(-\frac{\beta}{\alpha}\Big) + \sum_{i=1}^{\infty}\frac{k^i(\beta^i - \alpha^i)}{i.i!}\right].$$

Special cases:

$$\begin{aligned}
\fint_a^b \frac{e^t}{t-x}\,dt &= e^x\left[\ln\Big(-\frac{\beta}{\alpha}\Big) + \sum_{i=1}^{\infty}\frac{\beta^i - \alpha^i}{i.i!}\right], \\
\fint_a^b \frac{e^{-t}}{t-x}\,dt &= e^{-x}\left[\ln\Big(-\frac{\beta}{\alpha}\Big) + \sum_{i=1}^{\infty}\frac{(-1)^i(\beta^i - \alpha^i)}{i.i!}\right]. \\
&= e^{-x}\left[\ln\Big(-\frac{\beta}{\alpha}\Big) - \sum_{i=1,3,\ldots}^{\infty}\frac{\beta^i - \alpha^i}{i.i!} + \sum_{i=2,4,\ldots}^{\infty}\frac{\beta^i - \alpha^i}{i.i!}\right]. \\
\fint_{-1}^1 \frac{e^t}{t}\,dt &= 2.114\,501\,751. \\
\fint_0^\infty \frac{e^{-t}}{t-x}\,dt &= e^{-x}\left[\ln\tfrac{1}{x} - \gamma - \sum_{i=1}^{\infty}\frac{x^i}{i.i!}\right],
\end{aligned}$$

where γ is the Euler constant, with value $\gamma = 0.577\ 215\ 664\ 9...$.

Source: De Klerk

3.1.1(c2) For $a < x < b$ and $\alpha = a - x, \quad \beta = b - x, \quad \alpha < 0 < \beta$,

$$\begin{aligned}
&⨍_a^b \frac{e^{it}}{t-x}\,dt = \\
&\cos x\left[\ln\left(-\frac{\beta}{\alpha}\right) + \sum_{i=1}^{\infty}(-1)^i\frac{\beta^{2i}-\alpha^{2i}}{(2i).(2i)!}\right] - \sin x\left[\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\beta^{2i-1}-\alpha^{2i-1}}{(2i-1).(2i-1)!}\right] \\
&+i\left\{\cos x\left[\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\beta^{2i-1}-\alpha^{2i-1}}{(2i-1).(2i-1)!}\right] + \sin x\left[\ln\left(-\frac{\beta}{\alpha}\right) + \sum_{i=1}^{\infty}(-1)^i\frac{\beta^{2i}-\alpha^{2i}}{(2i).(2i)!}\right]\right\} \\
&\text{Special case: } ⨍_{-1}^1 \frac{e^{it}}{t}\,dt = 2.107\ 833\ 859i.
\end{aligned}$$

Source: De Klerk

3.1.1(c3) For $a < x < b$ and $\alpha = a - x, \quad \beta = b - x, \quad \alpha < 0 < \beta$,

$$\begin{aligned}
⨍_a^b \frac{\sin t}{t-x}\,dt &= \cos x\left[\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\beta^{2i-1}-\alpha^{2i-1}}{(2i-1).(2i-1)!}\right] \\
&\quad + \sin x\left[\ln\left(-\frac{\beta}{\alpha}\right) + \sum_{i=1}^{\infty}(-1)^i\frac{\beta^{2i}-\alpha^{2i}}{(2i).(2i)!}\right] \\
\text{Special case:} \qquad ⨍_{-1}^1 \frac{\sin t}{t}\,dt &= 2.107\ 833\ 859.
\end{aligned}$$

Source: De Klerk

3.1.1(c4) For $a < x < b$ and $\alpha = a - x, \quad \beta = b - x, \quad \alpha < 0 < \beta$,

$$\begin{aligned}
⨍_a^b \frac{\cos t}{t-x}\,dt &= \cos x\left[\ln\left(-\frac{\beta}{\alpha}\right) + \sum_{i=1}^{\infty}(-1)^i\frac{\beta^{2i}-\alpha^{2i}}{(2i).(2i)!}\right] \\
&\quad - \sin x\left[\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\beta^{2i-1}-\alpha^{2i-1}}{(2i-1).(2i-1)!}\right] \\
\text{Special case:} \qquad ⨍_{-1}^1 \frac{\cos t}{t}\,dt &= 0.0.
\end{aligned}$$

Source: De Klerk

3.1.1(c5) For $a < x < b$, k a constant and $\alpha = a - x, \quad \beta = b - x, \quad \alpha < 0 < \beta$,

$$\int_a^b \frac{\sin kt}{t-x}\,dt = \cos kx \left[\sum_{i=1}^{\infty} (-1)^{i-1} k^{2i-1} \frac{\beta^{2i-1} - \alpha^{2i-1}}{(2i-1).(2i-1)!} \right] + \sin kx \left[\ln\left(-\frac{\beta}{\alpha} \right) + \sum_{i=1}^{\infty} (-1)^i k^{2i} \frac{\beta^{2i} - \alpha^{2i}}{(2i).(2i)!} \right].$$

Special case: For $0 < x < 1$

$$\begin{aligned} \int_0^1 \frac{\sin \pi t}{t-x}\,dt &= \cos \pi x \Big[\sum_{i=1}^{\infty} (-1)^{i-1} \pi^{2i-1} \frac{(1-x)^{2i-1} + x^{2i-1}}{(2i-1).(2i-1)!} \Big] \\ &\quad + \sin \pi x \Big[\ln\Big(\frac{1-x}{x} \Big) + \sum_{i=1}^{\infty} (-1)^i \pi^{2i} \frac{(1-x)^{2i} - x^{2i}}{(2i).(2i)!} \Big] \\ &= \cos \pi x \Big[Si\big(\pi(1-x)\big) + Si(\pi x) \Big] + \sin \pi x \Big[Ci(\pi x) - Ci\big(\pi(1-x)\big) \Big], \end{aligned}$$

where

$$\begin{aligned} Si(x) &= \int_0^x \frac{\sin u}{u}\,du = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^{2i-1}}{(2i-1).(2i-1)!}, \\ Ci(x) &= \int_x^{\infty} \frac{\cos u}{u}\,du = -\gamma - \ln x + \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^{2i}}{(2i).(2i)!}, \\ \gamma &= 0.577\ 215\ 664\ 9... \ . \end{aligned}$$

Source: De Klerk

3.1.1(c6) For $a < x < b$, k a constant and $\alpha = a - x, \quad \beta = b - x, \quad \alpha < 0 < \beta$,

$$\int_a^b \frac{\cos kt}{t-x}\,dt = \cos kx \left[\ln\left(-\frac{\beta}{\alpha} \right) + \sum_{i=1}^{\infty} (-1)^i k^{2i} \frac{\beta^{2i} - \alpha^{2i}}{(2i).(2i)!} \right] - \sin kx \left[\sum_{i=1}^{\infty} (-1)^{i+1} k^{2i-1} \frac{\beta^{2i-1} - \alpha^{2i-1}}{(2i-1).(2i-1)!} \right].$$

Source: De Klerk

Special case: For $0 < x < 1$

$$\begin{aligned} \int_0^1 \frac{\cos \pi t}{t-x}\,dt &= \cos \pi x \Big[\ln\Big(\frac{1-x}{x} \Big) + \sum_{i=1}^{\infty} (-1)^i \pi^{2i} \frac{(1-x)^{2i} - x^{2i}}{(2i).(2i)!} \Big] \\ &\quad - \sin \pi x \Big[\sum_{i=1}^{\infty} (-1)^{i-1} \pi^{2i-1} \frac{(1-x)^{2i-1} + x^{2i-1}}{(2i-1).(2i-1)!} \Big] \\ &= \cos \pi x \Big[Ci(\pi x) - Ci\big(\pi(1-x)\big) \Big] - \sin \pi x \Big[Si\big(\pi(1-x)\big) + Si(\pi x) \Big], \end{aligned}$$

where

$$Si(x) = \int_0^x \frac{\sin u}{u}\,du = \frac{x}{1.1!} - \frac{x^3}{3.3!} + \frac{x^5}{5.5!} - \frac{x^7}{7.7!} + \dots,$$

$$Ci(x) = \int_x^\infty \frac{\cos u}{u}\,du = -\gamma - \ln x + \frac{x^2}{2.2!} - \frac{x^4}{4.4!} + \frac{x^6}{6.6!} - \frac{x^8}{8.8!} + \dots,$$

$$\gamma = 0.5772156649\dots.$$

Source: De Klerk

3.1.2(a0) For $a < x < b \leq 1$

$$\int_a^b \frac{\sqrt{1-t}}{t-x}\,dt = 2\sqrt{1-b} - 2\sqrt{1-a} + \sqrt{1-x}\,\ln|B^*|,$$

with

$$B^* = \frac{\sqrt{-a+1}+\sqrt{1-x}}{\sqrt{-a+1}-\sqrt{1-x}}\,\frac{\sqrt{-b+1}-\sqrt{1-x}}{\sqrt{-b+1}+\sqrt{1-x}}.$$

Source: De Klerk

Special case: For $-1 < x < 1$

$$\int_{-1}^1 \frac{\sqrt{1-t}}{t-x}\,dt = \sqrt{1-x}\ln|B| - 2\sqrt{2}, \quad B = \frac{1+\sqrt{\frac{1-x}{2}}}{1-\sqrt{\frac{1-x}{2}}}.$$

Source: Kaya & Erdogan (1987) (21)

3.1.2(a1) For $a < x < b \leq 1$

$$\begin{aligned}\int_a^b \frac{t\sqrt{1-t}}{t-x}\,dt &= -\tfrac{2}{3}\Big[(1-b)^{3/2} - (1-a)^{3/2}\Big]\\ &\quad +x\Big[2\sqrt{1-b} - 2\sqrt{1-a} + \sqrt{1-x}\,\ln|B^*|\Big],\end{aligned}$$

with B^* as in 3.1.2(a0).

Source: De Klerk

3.1.2(a2) For $a < x < b \leq 1$

$$\begin{aligned}\int_a^b \frac{t^2\sqrt{1-t}}{t-x}\,dt &= -\tfrac{2}{15}\big[(3b+2)(1-b)^{3/2} - (3a+2)(1-a)^{3/2}\big]\\ &\quad -\tfrac{2}{3}x\big[(1-b)^{3/2} - (1-a)^{3/2}\big]\\ &\quad +x^2\Big[2\sqrt{1-b} - 2\sqrt{1-a} + \sqrt{1-x}\,\ln|B^*|\Big],\end{aligned}$$

with B^* as in 3.1.2(a0).

Source: De Klerk

3.1.2(a3) For $a < x < b \leq 1$

$$\fint_a^b \frac{t^3\sqrt{1-t}}{t-x}\,dt = -\tfrac{2}{105}\left[(15b^2+12b+8)(1-b)^{3/2} - (15a^2+12a+8)(1-a)^{3/2}\right] + x\fint_a^b \frac{t^2\sqrt{1-t}}{t-x}\,dt,$$

together with 3.1.2(a2).

Source: De Klerk

3.1.2(an) For $a < x < b \leq 1$ and n an integer, one has

$$\fint_a^b \frac{t^n\sqrt{1-t}}{t-x}\,dt = \sum_{i=0}^{n-1}\left(x^i \int_a^b t^{n-1-i}\sqrt{1-t}\,dt\right) + x^n\left[\sqrt{1-x}\ln|B^*| - 2\left(\sqrt{-a+1} - \sqrt{-b+1}\right)\right],$$

with B^* as in 3.1.2(a0) and

$$\int_a^b t^m\sqrt{1-t}\,dt = -\frac{2t^m(1-t)^{3/2}}{2m+3}\Big|_a^b + \frac{2m}{2m+3}\int_a^b t^{m-1}\sqrt{1-t}\,dt,$$

repeatedly.

Source: De Klerk

3.1.3(a0) For $-1 \leq a < x < b \leq 1$

$$\fint_a^b \frac{\sqrt{1-t^2}}{t-x}\,dt = \sqrt{1-b^2} - \sqrt{1-a^2} - x\sin^{-1} b + x\sin^{-1} a + C^*,$$

where

$$C^* = -\sqrt{1-x^2}\,\ln\left|\frac{2\sqrt{1-x^2}\sqrt{1-t^2} + 2(1-xt)}{t-x}\right|\Bigg|_a^b.$$

Source: De Klerk

Special case: For $-1 < x < 1$

$$\fint_{-1}^1 \frac{\sqrt{1-t^2}}{t-x}\,dt = -\pi x.$$

Source: Kaya & Erdogan (1987) (17)

3.1.3(a1) For $-1 \leq a < x < b \leq 1$

$$\fint_a^b \frac{t\sqrt{1-t^2}}{t-x}\,dt = \left[\left(\tfrac{1}{2}t + x\right)\sqrt{1-t^2} + \left(\tfrac{1}{2} - x^2\right)\sin^{-1} t\right]\Big|_a^b + xC^*$$

with C^* as in 3.1.3(a0).
Special case:

$$\fint_{-1}^{1} \frac{t\sqrt{1-t^2}}{t-x}\,dt = \frac{\pi}{2}(1-2x^2).$$

Source: De Klerk

3.1.3(a2) For $-1 \le a < x < b \le 1$

$$\fint_{a}^{b} \frac{t^2\sqrt{1-t^2}}{t-x}\,dt = \Big[-\tfrac{1}{3}(1-t^2)^{3/2} + (\tfrac{1}{2}t + x)x\sqrt{1-t^2} + (\tfrac{1}{2} - x^2)x\sin^{-1} t\Big]\Big|_a^b + x^2 C^*$$

with C^* as in 3.1.3(a0).

Source: De Klerk

3.1.3(a3) For $-1 \le a < x < b \le 1$

$$\fint_{a}^{b} \frac{t^3\sqrt{1-t^2}}{t-x}\,dt = \Big[-(\tfrac{t}{4}+\tfrac{x}{3})(1-t^2)^{3/2}+(\tfrac{t}{8}+\tfrac{tx^2}{2}+x^3)\sqrt{1-t^2}+(\tfrac{1}{8}+\tfrac{x^2}{2}-x^4)\sin^{-1} t\Big]\Big|_a^b+x^3C^*$$

with C^* as in 3.1.3(a0).

Source: De Klerk

3.1.3(an) For $-1 \le a < x < b \le 1$ and n an integer, one has

$$\fint_{a}^{b} \frac{t^n\sqrt{1-t^2}}{t-x}\,dt = \sum_{i=0}^{n-1} x^i \int_a^b t^{n-1-i}\sqrt{1-t^2}\;dt + x^n\Big[\sqrt{1-t^2} - x\sin^{-1} t\Big]\Big|_a^b + x^n C^*,$$

with C^* as in 3.1.3(a0).

Source: De Klerk

Special case: For $-1 < x < 1$

$$\fint_{-1}^{1} \frac{t^n\sqrt{1-t^2}}{t-x}\,dt = \pi\sum_{k=0}^{n+1} b_k x^k, \quad n \ge 0,$$

with,

$$b_k = \begin{cases} 0, & n-k \quad \text{even} \\ \frac{1}{2\sqrt{\pi}}\frac{\Gamma\left(\frac{n-k}{2}\right)}{\Gamma\left(\frac{n-k+3}{2}\right)}, & n-k \quad \text{odd} \end{cases}$$

and with

$$\Gamma(-\tfrac{1}{2}) = -2\Gamma(\tfrac{1}{2}) = -2\sqrt{\pi}.$$

Source: Kaya & Erdogan (1987) (34, B6)

3.1.3(b1) For $-1 < x < 1$

$$\fint_{-1}^{1} \frac{U_n(t)\sqrt{1-t^2}}{t-x}\,dt = -\pi T_{n+1}(x), \quad n \geq 0,$$

where T_n and U_n are respectively the Chebyshev polynomials of the first and second kind.

Source: Kaya & Erdogan (1987) (30); Spiegel 31.45

3.1.3(b2) For $-1 < x < 1$

$$\fint_{-1}^{1} \frac{T_n(t)}{(t-x)\sqrt{1-t^2}}\,dt = \pi U_{n-1}(x), \quad n \geq 1,$$

where T_n and U_n are respectively the Chebyshev polynomials of the first and second kind.

Source: Spiegel 30.44

3.1.3(c1) For $-1 \leq a < x < b \leq 1$

$$\fint_{a}^{b} \frac{\sqrt{1-t^2}\,e^t}{t-x}\,dt = \sum_{j=0}^{\infty} \frac{1}{j!}\left[\sum_{i=0}^{j-1}\left(x^i \int_a^b t^{j-1-i}\sqrt{1-t^2}\,dt\right) + x^j\left[\sqrt{1-t^2}\big|_a^b - x\sin^{-1} t\big|_a^b + C^*\right]\right],$$

with C^* as in 3.1.3(a0) and $\sum_{i=0}^{q\,(<0)} \cdots = 0$.

Source: De Klerk

3.1.4(a0) For $a < x < b \leq 1$

$$\fint_{a}^{b} \frac{dt}{(t-x)\sqrt{1-t}} = \frac{1}{\sqrt{1-x}} \ln|B^*|,$$

with

$$B^* = \frac{\sqrt{-a+1}+\sqrt{1-x}}{\sqrt{-a+1}-\sqrt{1-x}}\;\frac{\sqrt{-b+1}-\sqrt{1-x}}{\sqrt{-b+1}+\sqrt{1-x}}.$$

Source: De Klerk

Special case: For $-1 < x < 1$

$$\fint_{-1}^{1} \frac{dt}{(t-x)\sqrt{1-t}} = \frac{\ln|B|}{\sqrt{1-x}}, \quad B = \frac{1+\sqrt{\frac{1-x}{2}}}{1-\sqrt{\frac{1-x}{2}}}.$$

Source: Kaya & Erdogan (1987) (23)

3.1.4(a1) For $a < x < b \leq 1$

$$\fint_a^b \frac{t}{(t-x)\sqrt{1-t}}\,dt = -2\sqrt{1-t}\big|_a^b + \frac{x}{\sqrt{1-x}}\ln|B^*|,$$

with B^* as in 3.1.4(a0).

Source: De Klerk

3.1.4(a2) For $a < x < b \leq 1$

$$\fint_a^b \frac{t^2}{(t-x)\sqrt{1-t}}\,dt \quad = \quad \Big[-\tfrac{2}{3}(t+2)\sqrt{1-t} - 2x\sqrt{1-t}\Big]\Big|_a^b + \frac{x^2}{\sqrt{1-x}}\ln|B^*|,$$

with B^* as in 3.1.4(a0).

Source: De Klerk

3.1.4(a3) For $a < x < b \leq 1$

$$\fint_a^b \frac{t^3}{(t-x)\sqrt{1-t}}\,dt =$$
$$\Big[-\tfrac{2}{15}(3t^2+4t+8)\sqrt{1-t} - \tfrac{2}{3}x(t+2)\sqrt{1-t} - 2x^2\sqrt{1-t}\Big]\Big|_a^b + \frac{x^3}{\sqrt{1-x}}\ln|B^*|,$$

with B^* as in 3.1.4(a0).

Source: De Klerk

3.1.4(an) For $a < x < b \leq 1$ and n an integer, one has

$$\fint_a^b \frac{t^n}{(t-x)\sqrt{1-t}}\,dt = \sum_{i=0}^{n-1} x^i \int_a^b \frac{t^{n-1-i}}{\sqrt{1-t}}\,dt + \frac{x^n}{\sqrt{1-x}}\ln|B^*|,$$

with B^* as in 3.1.4(a0).

Source: De Klerk

3.1.5(a0) For $-1 \leq a < x < b \leq 1$

$$\fint_a^b \frac{dt}{(t-x)\sqrt{1-t^2}} = \frac{-1}{\sqrt{1-x^2}}\ln|D^*|,$$

with

$$D^* = \frac{\sqrt{1-x^2}\sqrt{1-b^2} + (1-xb)}{\sqrt{1-x^2}\sqrt{1-a^2} + (1-xa)} \cdot \frac{a-x}{b-x}.$$

Source: De Klerk

Special case: For $-1 < x < 1$

$$\fint_{-1}^1 \frac{dt}{(t-x)\sqrt{1-t^2}} = 0.$$

Source: Kaya & Erdogan (1987) (19)

3.1.5(a1) For $-1 \le a < x < b \le 1$

$$\fint_a^b \frac{t}{(t-x)\sqrt{1-t^2}}\,dt = \sin^{-1} t|_a^b - \frac{x}{\sqrt{1-x^2}}\ln|D^*|,$$

with D^* as in 3.1.5(a0).

Source: De Klerk

Special case: For $-1 < x < 1$

$$\fint_{-1}^1 \frac{t}{(t-x)\sqrt{1-t^2}}\,dt = \pi.$$

Source: De Klerk

3.1.5(a2) For $-1 \le a < x < b \le 1$

$$\fint_a^b \frac{t^2}{(t-x)\sqrt{1-t^2}}\,dt = \left[-\sqrt{1-t^2} + x\sin^{-1} t\right]\Big|_a^b - \frac{x^2}{\sqrt{1-x^2}}\ln|D^*|,$$

with D^* as in 3.1.5(a0).

Source: De Klerk

Special case: For $-1 < x < 1$

$$\fint_{-1}^1 \frac{t^2}{(t-x)\sqrt{1-t^2}}\,dt = \pi x.$$

Source: De Klerk

3.1.5(a3) For $-1 \le a < x < b \le 1$

$$\fint_a^b \frac{t^3}{(t-x)\sqrt{1-t^2}}\,dt =$$

$$\left[-\frac{t}{2}\sqrt{1-t^2} + \tfrac{1}{2}\sin^{-1} t - x\sqrt{1-t^2} + x^2\sin^{-1} t\right]\Big|_a^b - \frac{x^3}{\sqrt{1-x^2}}\ln|D^*|,$$

with D^* as in 3.1.5(a0).

Source: De Klerk

Special case: For $-1 < x < 1$

$$\fint_{-1}^1 \frac{t^3}{(t-x)\sqrt{1-t^2}}\,dt = \pi(x^2 + \tfrac{1}{2}).$$

Source: De Klerk

3.1.5(an) For $-1 \le a < x < b \le 1$ and n an integer,

$$\fint_a^b \frac{t^n}{(t-x)\sqrt{1-t^2}}\,dt = \sum_{i=0}^{n-1} x^i \int_a^b \frac{t^{n-1-i}}{\sqrt{1-t^2}}\,dt - \frac{x^n}{\sqrt{1-x^2}}\ln|D^*|,$$

with D^* as in 3.1.5(a0).

Source: De Klerk

Special case:

$$\int_{-1}^{1} \frac{t^n}{(t-x)\sqrt{1-t^2}}\,dt = \begin{cases} 0, & n=0, \\ \pi \sum_{k=0}^{n-1} d_k x^k, & n \geq 1, \end{cases}$$

with,

$$d_k = \begin{cases} 0, & n-k \quad \text{even} \\ \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n-k}{2}\right)}{\Gamma\left(\frac{n-k+1}{2}\right)}, & n-k \quad \text{odd.} \end{cases}$$

Kaya & Erdogan (1987) (36, B8)

3.1.5(b)

$$\int_{-1}^{1} \frac{T_n(t)}{(t-x)\sqrt{1-t^2}}\,dt \;=\; \begin{cases} 0, & n=0 \\ \pi U_{n-1}(x), & n \geq 1, \end{cases}$$

where T_n and U_n are respectively the Chebyshev polynomials of the first and second kind.

Kaya & Erdogan (1987) (32)

3.2 Hypersingular integrals

3.2.1(a0) For $a < x < b$

$$\fint_a^b \frac{dt}{(t-x)^2} = -\frac{1}{b-x} - \frac{1}{x-a}.$$

Source: Special case of 3.3.1(a0); Kaya & Erdogan (1987) (9)

Special case: For $0 < x < 1$

$$\fint_0^1 \frac{dt}{(t-x)^2} = -\left(\frac{1}{1-x} + \frac{1}{x}\right).$$

Source: Hui & Shia (1999) (5)

Special case: For $x = 0$

$$\fint_0^1 \frac{dt}{t^2} = -1.$$

Source: Hui & Shia (1999) (5a)

3.2.1(a1) For $a < x < b$

$$\fint_a^b \frac{t}{(t-x)^2}\,dt = -\frac{b}{b-x} + \frac{a}{a-x} + \ln\frac{b-x}{x-a}.$$

Source: Special case of 3.3.1(a1)

3.2.1(a2) For $a < x < b$

$$\fint_a^b \frac{t^2}{(t-x)^2}\,dt = -\frac{b^2}{b-x} + \frac{a^2}{a-x} + 2(b-a) + 2x\ln\frac{b-x}{x-a}.$$

Source: Special case of 3.3.1(a2)

3.2.1(a3) For $a < x < b$

$$\begin{aligned}\fint_a^b \frac{t^3}{(t-x)^2}\,dt &= -\frac{b^3}{b-x} + \frac{a^3}{a-x} + \tfrac{3}{2}[(b-x)^2 - (a-x)^2] \\ &\quad +3.2x(b-a) + 3x^2\ln\frac{b-x}{x-a}.\end{aligned}$$

Source: Special case of 3.3.1(a3)

3.2.1(an) General cases: For $a < x < b$, and for (i) ν rational and (ii) $\nu = n$ an integer,

$$\fint_a^b \frac{t^\nu}{(t-x)^2}\,dt = \frac{1}{\nu-1}(b^{\nu-1} - a^{\nu-1}) + 2x\fint_a^b \frac{t^{\nu-1}}{(t-x)^2}\,dt - x^2\fint_a^b \frac{t^{\nu-2}}{(t-x)^2}\,dt,$$

Source: De Klerk

$$\fint_a^b \frac{t^n}{(t-x)^2}\,dt = nx^{n-1}\ln\frac{b-x}{x-a} + x^n\frac{a-b}{(b-x)(x-a)} + \sum_{k=1}^{n-1}\frac{kx^{k-1}}{n-k}(b^{n-k}-a^{n-k}),$$

Source: De Klerk

or, alternatively,

$$\fint_a^b \frac{t^n}{(t-x)^2}\,dt = nx^{n-1}\ln\frac{b-x}{x-a} + x^n\frac{a-b}{(b-x)(x-a)} + \sum_{k=2}^{n}\frac{\binom{n}{k}x^{n-k}[(b-x)^{k-1}-(a-x)^{k-1}]}{k-1}.$$

Source: De Klerk

Special cases: For $0 < x < 1$, and for (i) ν rational and (ii) $\nu = n$ an integer,

$$\fint_0^1 \frac{t^\nu}{(t-x)^2}\,dt = -\pi\nu x^{\nu-1}\cot\pi\nu - \frac{1}{1-x} - \nu\sum_{n=0}^{\infty}\frac{x^n}{n-\nu+1}.$$

Source: Kythe & Schäferkotter, p 259

$$\fint_0^1 \frac{t^n}{(t-x)^2}\,dt = \frac{x^{n-1}}{x-1} + nx^{n-1}\ln\frac{1-x}{x} + \sum_{j=2}^{n}\frac{(j-1)x^{j-2}}{n-j+1},$$

Source: De Klerk

or, alternatively,

$$\fint_0^1 \frac{t^n}{(t-x)^2}\,dt = \frac{x^{n-1}}{1-x} + nx^{n-1}\ln\frac{1-x}{x} + \sum_{j=2}^{n}\frac{\binom{n}{j}x^{n-j}\left[(1-x)^{j-1}-(-x)^{j-1}\right]}{j-1}.$$

Source: Kythe & Schäferkotter, p 259

3.2.1(b) For $-1 < x < 1$

$$\fint_{-1}^1 \frac{P_n(t)}{(t-x)^2}\,dt = -\frac{2(n+1)}{1-x^2}\Big[xQ_n(x) - Q_{n+1}(x)\Big], \quad n \geq 0,$$

where P_n and Q_n are respectively the Legendre polynomials of the first kind and Legendre functions of the second kind.

Source: Kaya & Erdogan (1987) (29)

3.2.1(c1) For $a < x < b$ and $\alpha = a - x, \quad \beta = b - x, \quad \alpha < 0 < \beta$,

$$\oint_a^b \frac{e^{kt}}{(t-x)^2}\,dt = e^{kx}\left[\Big(\frac{\beta-\alpha}{\alpha\beta}\Big) + k\ln\Big(-\frac{\beta}{\alpha}\Big) + \sum_{j=1}^{\infty}\frac{k^{j+1}(\beta^j-\alpha^j)}{j.(j+1)!}\right].$$

Special cases:

$$\oint_a^b \frac{e^{t}}{(t-x)^2}\,dt = e^{x}\left[\Big(\frac{\beta-\alpha}{\alpha\beta}\Big) + \ln\Big(-\frac{\beta}{\alpha}\Big) + \sum_{j=1}^{\infty}\frac{\beta^j-\alpha^j}{j.(j+1)!}\right].$$

$$\oint_a^b \frac{e^{-t}}{(t-x)^2}\,dt = e^{-x}\left[\Big(\frac{\beta-\alpha}{\alpha\beta}\Big) - \ln\Big(-\frac{\beta}{\alpha}\Big) + \sum_{j=1}^{\infty}\frac{(-1)^{j+1}(\beta^j-\alpha^j)}{j.(j+1)!}\right].$$

$$\oint_{-1}^1 \frac{e^{t}}{t^2}\,dt = -0.971\ 659\ 519.$$

Source: De Klerk

3.2.1(c2) For $a < x < b$ and $\alpha = a - x, \quad \beta = b - x, \quad \alpha < 0 < \beta$,

$$\oint_a^b \frac{e^{it}}{(t-x)^2}\,dt = e^{ix}\left[\Big(\frac{\beta-\alpha}{\alpha\beta}\Big) + i\,\Big(-\frac{\beta}{\alpha}\Big) + \sum_{j=1}^{\infty} i^{j+1}\frac{\beta^j-\alpha^j}{j.(j+1)!}\right]$$

Special case: $$\oint_{-1}^1 \frac{e^{it}}{t^2}\,dt = -2.972\ 770\ 753.$$

Source: De Klerk

3.2.1(c3) For $a < x < b$ and $\alpha = a - x, \quad \beta = b - x, \quad \alpha < 0 < \beta$,

$$\oint_a^b \frac{\sin t}{(t-x)^2}\,dt = \cos x\left[\ln\Big(-\frac{\beta}{\alpha}\Big) + \sum_{j=1}^{\infty}(-1)^j\frac{\beta^{2j}-\alpha^{2j}}{2j.(2j+1)!}\right] + \sin x\left[\Big(\frac{\beta-\alpha}{\alpha\beta}\Big) + \sum_{j=1}^{\infty}(-1)^j\frac{\beta^{2j-1}-\alpha^{2j-1}}{(2j-1).(2j)!}\right].$$

Special case: $$\oint_{-1}^1 \frac{\sin t}{t^2}\,dt = 0.0.$$

Source: De Klerk

3.2.1(c4) For $a < x < b$ and $\alpha = a - x, \quad \beta = b - x, \quad \alpha < 0 < \beta$,

$$\fint_a^b \frac{\cos t}{(t-x)^2}\,dt = \cos x\left[\left(\frac{\beta-\alpha}{\alpha\beta}\right) + \sum_{j=1}^{\infty}(-1)^j \frac{\beta^{2j-1}-\alpha^{2j-1}}{(2j-1).(2j)!}\right] - \sin x\left[\ln\left(-\frac{\beta}{\alpha}\right) + \sum_{j=1}^{\infty}(-1)^j \frac{\beta^{2j}-\alpha^{2j}}{2j.(2j+1)!}\right].$$

Special case: $$\fint_{-1}^{1} \frac{\cos t}{t^2}\,dt = -2.972\ 770\ 753.$$

Source: De Klerk

3.2.1(c5) For $a < x < b$, k a constant and $\alpha = a - x, \quad \beta = b - x, \quad \alpha < 0 < \beta$,

$$\fint_a^b \frac{\sin kt}{(t-x)^2}\,dt = k\cos kx\left[\ln\left(-\frac{\beta}{\alpha}\right) + \sum_{j=1}^{\infty}(-1)^j k^{2j}\frac{\beta^{2j}-\alpha^{2j}}{2j.(2j+1)!}\right] + k\sin kx\left[\frac{\beta-\alpha}{\alpha\beta k} + \sum_{j=1}^{\infty}(-1)^j k^{2j-1}\frac{\beta^{2j-1}-\alpha^{2j-1}}{(2j-1).(2j)!}\right].$$

Source: De Klerk

Special case: For $0 < x < 1$,

$$\fint_0^1 \frac{\sin \pi t}{(t-x)^2}\,dt = -\pi\sin\pi x\big[Si\big(\pi(1-x)\big) + Si(\pi x)\big] + \pi\cos\pi x\big[Ci(\pi x) - Ci\big(\pi(1-x)\big)\big],$$

$$Si(y) = \int_0^y \frac{\sin u}{u}\,du, \quad Ci(y) = \int_y^\infty \frac{\cos u}{u}\,du.$$

Source: Kythe & Schäferkotter, p 259

3.2.1(c6) For $a < x < b$, k a constant and $\alpha = a - x, \quad \beta = b - x, \quad \alpha < 0 < \beta$,

$$\fint_a^b \frac{\cos kt}{(t-x)^2}\,dt = k\cos kx\left[\frac{\beta-\alpha}{\alpha\beta k} + \sum_{j=1}^{\infty}(-1)^j k^{2j-1}\frac{\beta^{2j-1}-\alpha^{2j-1}}{(2j-1).(2j)!}\right] - k\sin kx\left[\ln\left(-\frac{\beta}{\alpha}\right) + \sum_{j=1}^{\infty}(-1)^j k^{2j}\frac{\beta^{2j}-\alpha^{2j}}{2j.(2j+1)!}\right].$$

Source: De Klerk

Special case: For $0 < x < 1$,

$$\fint_0^1 \frac{\cos \pi t}{(t-x)^2}\,dt = -\pi\sin\pi x\big[Ci(\pi x) - Ci\big(\pi(1-x)\big)\big] - \pi\cos\pi x\big[Si\big(\pi(1-x)\big) + Si(\pi x)\big] + \frac{1}{1-x} - \frac{1}{x},$$

$$Si(y) = \int_0^y \frac{\sin u}{u}\,du, \quad Ci(y) = \int_y^\infty \frac{\cos u}{u}\,du.$$

Source: De Klerk

3.2.1(d) Further cases

For $0 < x < 1$

$$\begin{aligned}
\oint_0^1 \frac{[t(1-t)]^{3/2}}{(t-x)^2}\,dt &= \frac{\pi}{2}\Big(\frac{3}{4} - 6x(1-x)\Big) \\
\oint_0^1 \frac{\frac{1}{2} + \frac{t-c}{2|t-c|}}{(t-x)^2}\,dt &= \frac{1}{x-1} + \frac{1}{c-x} \\
\oint_0^1 \frac{\frac{1}{2}\big(c + (1-2c)t + |t-c|\big)}{(t-x)^2}\,dt &= (c-1)\ln|x| - c\ln|1-x| + \ln|x-c| \\
\oint_0^1 \frac{\frac{1}{2} + \frac{1}{4\epsilon}\big(|t-c+\epsilon| + |t-c-\epsilon|\big)}{(t-x)^2}\,dt &= \frac{1}{x-1} + \frac{1}{2\epsilon}\ln\Big|\frac{x-c-\epsilon}{x-c+\epsilon}\Big|
\end{aligned}$$

$$\oint_0^1 \frac{\frac{3}{4}(t-c)^2 - \frac{1}{4}(t-c)|t-c| + \frac{1}{2}(t-c)(c^2+2c-1) + \frac{1}{2}c(c^2-1))}{(t-x)^2}\,dt$$

$$= (c+1) + \big(x + \frac{c^2}{2} - \frac{1}{2}\big)\ln|1-x| + (x-c)\ln|x-c| - \big(2x + \frac{c^2}{2} - c - \frac{1}{2}\big)\ln x$$

Source: Kythe & Schäferkotter (2005), p 259

3.2.2(a0) For $a < x < b \le 1$

$$\oint_a^b \frac{\sqrt{1-t}}{(t-x)^2}\,dt = -\frac{\ln|B^*|}{2\sqrt{1-x}} + \sqrt{1-x}\,\frac{1}{B^*}\,\frac{dB^*}{dx},$$

with

$$B^* = \frac{\sqrt{-a+1} + \sqrt{1-x}}{\sqrt{-a+1} - \sqrt{1-x}}\;\frac{\sqrt{-b+1} - \sqrt{1-x}}{\sqrt{-b+1} + \sqrt{1-x}}.$$

Source: De Klerk

Special case: For $-1 < x < 1$

$$\oint_{-1}^1 \frac{\sqrt{1-t}}{(t-x)^2}\,dt = -\frac{\ln|B|}{2\sqrt{1-x}} - \frac{\sqrt{2}}{1+x}, \qquad B = \frac{1 + \sqrt{\frac{1-x}{2}}}{1 - \sqrt{\frac{1-x}{2}}}.$$

Source: Kaya & Erdogan (1987) (22)

3.2.2(a1) For $a < x < b \le 1$

$$\oint_a^b \frac{t\sqrt{1-t}}{(t-x)^2}\,dt = 2\sqrt{1-t}\big|_a^b + \sqrt{1-x}\ln|B^*| - \frac{x}{2\sqrt{1-x}}\ln|B^*| + x\sqrt{1-x}\,\frac{1}{B^*}\,\frac{dB^*}{dx},$$

with B^* as in 3.2.2(a0).

Source: De Klerk

3.2.2(a2) For $a < x < b \le 1$

$$\begin{aligned}\int\!\!\!\!\!\!= _a^b \frac{t^2\sqrt{1-t}}{(t-x)^2}\,dt &= -\tfrac{2}{3}(1-t)^{3/2}|_a^b + 4x\sqrt{1-t}|_a^b \\ &+ 2x\sqrt{1-x}\ln|B^*| - \tfrac{1}{2}\frac{x^2}{\sqrt{1-x}}\ln|B^*| + x^2\sqrt{1-x}\,\frac{1}{B^*}\frac{dB^*}{dx},\end{aligned}$$

with B^* as in 3.2.2(a0).

Source: De Klerk

3.2.2(an) For $a < x < b \le 1$ and n an integer, one has

$$\begin{aligned}\int\!\!\!\!\!\!= _a^b \frac{t^n\sqrt{1-t}}{(t-x)^2}\,dt &= \sum_{i=0}^{n-1}\left(ix^{i-1}\int_a^b t^{n-1-i}\sqrt{1-t}\,dt\right) \\ &+nx^{n-1}\left[\sqrt{1-x}\ln|B^*| + 2\sqrt{-t+1}|_a^b\right] \\ &+x^n\left[-\tfrac{1}{2}\frac{\ln|B^*|}{\sqrt{1-x}} + \sqrt{1-x}\frac{1}{B^*}\frac{dB^*}{dx}\right],\end{aligned}$$

with B^* as in 3.2.2(a0) and

$$\int_a^b t^m\sqrt{1-t}\,dt = -\frac{2t^m(1-t)^{3/2}}{2m+3}\Big|_a^b + \frac{2m}{2m+3}\int_a^b t^{m-1}\sqrt{1-t}\,dt.$$

Source: De Klerk

3.2.3(a0) For $-1 \le a < x < b \le 1$

$$\int\!\!\!\!\!\!= _a^b \frac{\sqrt{1-t^2}}{(t-x)^2}\,dt = -\sin^{-1}t|_a^b + \frac{dC^*}{dx},$$

with

$$C^* = -\sqrt{1-x^2}\,\ln\left|\frac{2\sqrt{1-x^2}\sqrt{1-t^2}+2(1-xt)}{t-x}\right|\Big|_a^b.$$

Source: De Klerk

Special case: For $-1 < x < 1$

$$\int\!\!\!\!\!\!= _{-1}^1 \frac{\sqrt{1-t^2}}{(t-x)^2}\,dt = -\pi.$$

Source: Kaya & Erdogan (1987) (18)

3.2.3(a1) For $-1 \le a < x < b \le 1$ and n an integer, one has

$$\oint_a^b \frac{t\sqrt{1-t^2}}{(t-x)^2}\,dt = \left[\sqrt{1-t^2} - 2x\sin^{-1} t\right]\Big|_a^b + C^* + x\frac{dC^*}{dx},$$

with C^* as in 3.2.3(a0).

Source: De Klerk

Special case:

$$\oint_{-1}^1 \frac{t\sqrt{1-t^2}}{(t-x)^2}\,dt = -2\pi x.$$

Source: Kaya & Erdogan (35)

3.2.3(a2) For $-1 \le a < x < b \le 1$ and n an integer, one has

$$\oint_a^b \frac{t^2\sqrt{1-t^2}}{(t-x)^2}\,dt = \left[2x\sqrt{1-t^2} + (\tfrac{1}{2}t + x)\sqrt{1-t^2} + (\tfrac{1}{2} - 3x^2)\sin^{-1} t\right]\Big|_a^b + 2xC^* + x^2\frac{dC^*}{dx},$$

with C^* as in 3.2.3(a0).

Source: De Klerk

3.2.3(an) For $-1 \le a < x < b \le 1$

$$\begin{aligned}\oint_a^b \frac{t^n\sqrt{1-t^2}}{(t-x)^2}\,dt \quad &= \quad \sum_{i=1}^n \left(ix^{i-1} \int_a^b t^{n-1-i}\sqrt{1-t^2}\,dt \right) \\ &\quad + nx^{n-1}\left[\sqrt{1-t^2} - (n+1)x^n \sin^{-1} t\right]\Big|_a^b + nx^{n-1}C^* + x^n\frac{dC^*}{dx},\end{aligned}$$

with C^* as in 3.2.3(a0) and

Source: De Klerk

Special case: For $-1 < x < 1$

$$\oint_{-1}^1 \frac{t^n\sqrt{1-t^2}}{(t-x)^2}\,dt = \pi\sum_{k=0}^n c_k x^k, \quad n \ge 0,$$

with, For $k \le n$,

$$c_k = \begin{cases} 0, & n-k \quad \text{odd} \\ \frac{k+1}{2\sqrt{\pi}}\frac{\Gamma\left(\frac{n-k-1}{2}\right)}{\Gamma\left(\frac{n-k+2}{2}\right)}, & n-k \quad \text{even} \end{cases}$$

and with $\Gamma(-\frac{1}{2}) = -2\Gamma(\frac{1}{2}) = -2\sqrt{\pi}$.

Source: Kaya & Erdogan (1987) (37, B9)

3.2.3(b1) For $-1 < x < 1$

$$\oint_{-1}^1 \frac{U_n(t)\sqrt{1-t^2}}{(t-x)^2}\,dt = -\pi(n+1)U_n(x), \quad n \ge 0,$$

where U_n are the Chebyshev polynomials of the second kind.

Source: Kaya & Erdogan (1987) (31)

3.2.3(b2) For $-1 < x < 1$

$$\oint_{-1}^{1} \frac{T_n(t)}{(t-x)^2\sqrt{1-t^2}}\, dt = -\pi \frac{nT_n(x) - xU_{n-1}(x)}{x^2-1}, \quad n > 1,$$

Source: De Klerk

or

$$\oint_{-1}^{1} \frac{T_n(t)}{(t-x)^2\,\sqrt{1-t^2}}\, dt = \begin{cases} 0, & n = 0, 1 \\ \frac{\pi}{1-x^2}\Big[-\frac{n-1}{2}U_n(x) + \frac{n+1}{2}U_{n-2}(x)\Big], & n \geq 2, \end{cases}$$

where T_n and U_n are respectively the Chebyshev polynomials of the first and second kind.

Source: Kaya & Erdogan (1987) (33)

3.2.3(c1) For $-1 \leq a < x < b \leq 1$

$$\oint_{a}^{b} \frac{\sqrt{1-t^2}e^t}{(t-x)^2}\, dt = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{d}{dx}\left[\sum_{i=0}^{j-1} x^i \int_a^b t^{j-1-i}\sqrt{1-t^2}\; dt + x^j \left[\sqrt{1-t^2} - x\sin^{-1} t\right]\Big|_a^b + x^j C^* \right],$$

with

$$C^* = -\sqrt{1-x^2}\, \ln\left| \frac{2\sqrt{1-x^2}\sqrt{1-t^2} + 2(1-xt)}{t-x} \right|\Bigg|_a^b,$$

and $\sum_{i=0}^{q\,(<0)} \cdots = 0$.

Source: De Klerk

Special case: For $a = -1$, $b = 1$ and $x = 0$,

$$\oint_{-1}^{1} \frac{\sqrt{1-t^2}\, e^t}{t^2}\, dt = -2.339\ 556\ 253\ 339.$$

Source: Ashour & Ahmed (2007), p 1682

3.2.3(c2) For $-1 \leq a < x < b \leq 1$

$$\begin{aligned} \oint_{a}^{b} \frac{\sqrt{1-t^2}\sin t}{(t-x)^2}\, dt &= \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} \frac{d}{dx}\left[\sum_{i=0}^{2j} x^i \int_a^b t^{2j-i}\sqrt{1-t^2}\; dt \right. \\ &+ \left. x^{2j+1}\left[\sqrt{1-t^2} - x\sin^{-1} t\right]\Big|_a^b + x^{2j+1}C^* \right], \end{aligned}$$

with C^* as in 3.2.3(c1).

Source: De Klerk

Special case: For $a = -1$, $b = 1$ and $x = 0$,

$$\fint_{-1}^{1} \frac{\sqrt{1-t^2}\,\sin t}{t^2}\,dt = 0.$$

Source: De Klerk

3.2.3(c3) For $-1 \le a < x < b \le 1$

$$\begin{aligned}\fint_a^b \frac{\sqrt{1-t^2}\cos t}{(t-x)^2}\,dt &= \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!}\frac{d}{dx}\left[\sum_{i=0}^{2j-1} x^i \int_a^b t^{2j-1-i}\sqrt{1-t^2}\,dt\right. \\ &+ \left. x^{2j}\left[\sqrt{1-t^2} - x\sin^{-1} t\right]\Big|_a^b + x^{2j}C^*\right],\end{aligned}$$

with C^* as in 3.2.3(c1).

Source: De Klerk

Special case: For $a = -1$, $b = 1$ and $x = 0$,

$$\fint_{-1}^{1} \frac{\sqrt{1-t^2}\,\cos t}{t^2}\,dt = -3.910\ 898\ 042\ 871.$$

Source: Shand, Y. (2006), p 57, 58

3.2.4(a0) For $a < x < b \le 1$

$$\fint_a^b \frac{dt}{(t-x)^2\sqrt{1-t}} = \frac{1}{1-x}\left[\frac{\ln|B^*|}{2\sqrt{1-x}} + \sqrt{1-x}\,\frac{1}{B^*}\,\frac{dB^*}{dx}\right],$$

with

$$B^* = \frac{\sqrt{-a+1}+\sqrt{1-x}}{\sqrt{-a+1}-\sqrt{1-x}}\ \frac{\sqrt{-b+1}-\sqrt{1-x}}{\sqrt{-b+1}+\sqrt{1-x}}.$$

Source: De Klerk

Special case: For $-1 < x < 1$

$$\fint_{-1}^{1} \frac{dt}{(t-x)^2\sqrt{1-t}} = \frac{1}{1-x}\left[\frac{\ln|B|}{2\sqrt{1-x}} - \frac{\sqrt{2}}{1+x}\right], \qquad B = \frac{1+\sqrt{\frac{1-x}{2}}}{1-\sqrt{\frac{1-x}{2}}}.$$

Source: Kaya & Erdogan (1987) (24)

3.2.4(a1) For $a < x < b \le 1$

$$\fint_a^b \frac{t}{(t-x)^2\sqrt{1-t}}\,dt = \frac{1}{\sqrt{1-x}}\left[\ln|B^*| + \frac{x\ln|B^*|}{2(1-x)} + \frac{x}{B^*}\,\frac{dB^*}{dx}\right],$$

with B^* as in 3.2.4(a0).

Source: De Klerk

3.2.4(a2) For $a < x < b \leq 1$

$$\fint_a^b \frac{t^2}{(t-x)^2\sqrt{1-t}}\,dt = -2\sqrt{1-t}\Big|_a^b + \frac{1}{\sqrt{1-x}}\left[2x\ln|B^*| + \frac{x^2\ln|B^*|}{2(1-x)} + \frac{x^2}{B^*}\frac{dB^*}{dx}\right],$$

with B^* as in 3.2.4(a0).

Source: De Klerk

3.2.4(a3) For $a < x < b \leq 1$

$$\begin{aligned}\fint_a^b \frac{t^3}{(t-x)^2\sqrt{1-t}}\,dt &= -\left[\tfrac{2}{3}(t+2)\sqrt{1-t} + 4x\sqrt{1-t}\right]\Big|_a^b \\ &+ \frac{1}{\sqrt{1-x}}\left[3x^2\ln|B^*| + \frac{x^3\ln|B^*|}{2(1-x)} + \frac{x^3}{B^*}\frac{dB^*}{dx}\right],\end{aligned}$$

with B^* as in 3.2.4(a0).

Source: De Klerk

3.2.4(an) For $a < x < b \leq 1$

$$\begin{aligned}\fint_a^b \frac{t^n}{(t-x)^2\sqrt{1-t}}\,dt &= \sum_{i=1}^{n-1} i x^{i-1}\int_a^b \frac{t^{n-1-i}}{\sqrt{1-t}}\,dt \\ &+ \frac{1}{\sqrt{1-x}}\left[n x^{n-1}\ln|B^*| + \frac{x^n\ln|B^*|}{2(1-x)} + \frac{x^n}{B^*}\frac{dB^*}{dx}\right],\end{aligned}$$

with B^* as in 3.2.4(a0).

Source: De Klerk

3.2.5(a0) For $-1 \leq a < x < b \leq 1$

$$\fint_a^b \frac{dt}{(t-x)^2\sqrt{1-t^2}} = \frac{-1}{\sqrt{1-x^2}}\left[\frac{x\ln|D^*|}{1-x^2} + \frac{1}{D^*}\frac{dD^*}{dx}\right],$$

with

$$D^* = \frac{\sqrt{1-x^2}\sqrt{1-b^2} + (1-xb)}{\sqrt{1-x^2}\sqrt{1-a^2} + (1-xa)} \cdot \frac{a-x}{b-x}.$$

Source: De Klerk

Special case: For $-1 < x < 1$

$$\fint_{-1}^1 \frac{dt}{(t-x)^2\sqrt{1-t^2}} = 0.$$

Source: Kaya & Erdogan (1987) (20)

3.2.5(a1) For $-1 \le a < x < b \le 1$

$$\fint_a^b \frac{t}{(t-x)^2\sqrt{1-t^2}}\,dt = \frac{-1}{\sqrt{1-x^2}}\left[\ln|D^*| + \frac{x^2\ln|D^*|}{1-x^2} + \frac{x}{D^*}\,\frac{dD^*}{dx}\right],$$

with D^* as in 3.2.5(a0).

Source: De Klerk

Special case: For $-1 < x < 1$

$$\fint_{-1}^1 \frac{t}{(t-x)^2\sqrt{1-t^2}}\,dt = 0.$$

Source: De Klerk

3.2.5(a2) For $-1 \le a < x < b \le 1$

$$\fint_a^b \frac{t^2}{(t-x)^2\sqrt{1-t^2}}\,dt = \sin^{-1} t\big|_a^b - \frac{1}{\sqrt{1-x^2}}\left[2x\ln|D^*| + \frac{x^3\ln|D^*|}{1-x^2} + \frac{x^2}{D^*}\,\frac{dD^*}{dx}\right],$$

with D^* as in 3.2.5(a0).

Source: De Klerk

Special case: For $-1 < x < 1$

$$\fint_{-1}^1 \frac{t^2}{(t-x)^2\sqrt{1-t^2}}\,dt = \pi.$$

Source: De Klerk

3.2.5(a3) For $-1 \le a < x < b \le 1$

$$\begin{aligned}\fint_a^b \frac{t^3}{(t-x)^2\sqrt{1-t^2}}\,dt &= \left[-\sqrt{1-t^2} + 2x\sin^{-1} t\right]\Big|_a^b \\ &- \frac{1}{\sqrt{1-x^2}}\left[3x^2\ln|D^*| + \frac{x^4\ln|D^*|}{1-x^2} + \frac{x^3}{D^*}\,\frac{dD^*}{dx}\right],\end{aligned}$$

with D^* as in 3.2.5(a0).

Source: De Klerk

Special case: For $-1 < x < 1$

$$\fint_{-1}^1 \frac{t^3}{(t-x)^2\sqrt{1-t^2}}\,dt = 2\pi x.$$

Source: De Klerk

3.2.5(an) For $-1 \le a < x < b \le 1$

$$\begin{aligned} \fint_a^b \frac{t^n}{(t-x)^2\sqrt{1-t^2}}\,dt &= \sum_{i=1}^{n-1} i x^{i-1} \int_a^b \frac{t^{n-1-i}}{\sqrt{1-t^2}}\,dt \\ &- \frac{1}{\sqrt{1-x^2}}\left[n x^{n-1}\ln|D^*| + \frac{x^{n+1}\ln|D^*|}{1-x^2} + \frac{x^n}{D^*}\frac{dD^*}{dx}\right], \end{aligned}$$

with D^* as in 3.2.5(a0).

Source: De Klerk

Special case: For $-1 < x < 1$

$$\fint_{-1}^1 \frac{t^n}{(t-x)^2\sqrt{1-t^2}}\,dt = \begin{cases} 0, & n = 0, 1 \\ \pi \sum_{k=0}^{n-2} e_k x^k, & n \ge 2, \end{cases}$$

with

$$e_k = \begin{cases} 0, & n-k \quad \text{odd} \\ \frac{k+1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n-k-1}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)}, & n-k \quad \text{even.} \end{cases}$$

Source: Kaya & Erdogan (1987) (37, B9)

3.2.5(c1) For $-1 \le a < x < b \le 1$

$$\fint_a^b \frac{e^t}{(t-x)^2\sqrt{1-t^2}}\,dt = \sum_{j=0}^{\infty} \frac{1}{j!}\frac{d}{dx}\left[\sum_{i=0}^{j-1} x^i \int_a^b \frac{t^{j-1-i}}{\sqrt{1-t^2}}\,dt - \frac{x^j}{\sqrt{1-x^2}}\ln|D^*|\right],$$

with D^* as in 3.2.5(a0).

Source: De Klerk

3.2.6 Further cases:

3.2.6(a) For $-1 < x < 1$

$$\fint_{-1}^1 \frac{1/(t^2+\alpha^2)}{(t-x)^2\sqrt{1-t^2}}\,dt = \frac{-\pi(\alpha^2 - x^2)}{\alpha(\alpha^2+1)^{1/2}(\alpha^2+x^2)}, \qquad \alpha^2 > 1.$$

Source: Paget, D.F. (1981), p 452; Tsamasphyros, G. & Dimou, G. (1990), p 23.

3.2.6(b) For $-1 < x < 1$

$$\fint_{-1}^1 \frac{dt}{(t-x)^2\sqrt{\alpha^2 - t^2}} = \frac{x}{(\alpha^2 - x^2)^{3/2}} \ln\left[\frac{(\alpha^2-x^2)^{1/2} - x(\alpha^2-1)^{1/2}}{(\alpha^2-x^2)^{1/2} + x(\alpha^2-1)^{1/2}}\right] - \frac{2(\alpha^2-1)^{1/2}}{(\alpha^2-x^2)(1-x^2)},$$

$\alpha^2 > 1$.

Source: Paget, D.F. (1981), p 451

3.2.6(c) For $0 < x < 1$

$$\int\!\!\!\!\!\!= _0^1 \frac{t^6}{(t-x)^2}\,dt = \frac{6}{5} + \frac{3}{2}x + 2x^2 + 3x^3 + 6x^4 + \frac{1}{x-1} + 6x^5 \ln\frac{1-x}{x}.$$

Source: Wu & Sun (2008), pp 161-163

3.2.6(d) For $x = 0$

$$\int\!\!\!\!\!\!= _{-1}^1 \frac{t^4 + |t|^{4+\alpha}}{t^2}\,dt = \frac{12+2\alpha}{9+3\alpha}, \qquad 0 < \alpha \le 1.$$

Source: Wu & Sun (2008), pp 161-163

3.2.6(e) For $x = 0$

$$\int\!\!\!\!\!\!= _{-1}^1 \frac{t^4 + |t|^{3+\alpha}}{t^2}\,dt = \frac{10+2\alpha}{6+3\alpha}, \qquad 0 < \alpha \le 1.$$

Source: Wu & Sun (2008), pp 161-163

3.3 Generalization and a variety of cases

3.3.1(a0) For $a < x < b$

$$\int_a^b \frac{dt}{(t-x)^{p+1}} = \begin{cases} \ln \frac{b-x}{x-a}, & p=0; \\ -\frac{1}{p}\left[\frac{1}{(b-x)^p} - \frac{1}{(a-x)^p}\right], & p=1,2,\ldots \end{cases}$$

Source: Kaya & Erdogan (1987) (8); Monegato (1994) (2.8)

Special case: For $-1 < x < 1$

$$\int_{-1}^1 \frac{dt}{(t-x)^{p+1}} = \frac{1}{p}\Big[\frac{(-1)^p}{(1+x)^p} - \frac{1}{(1-x)^p}\Big], \quad p > 0.$$

Special case: For $x = 0$

$$\int_{-1}^1 \frac{dt}{t^{p+1}} = \frac{1}{p}\Big[(-1)^p - 1\Big], \quad p > 0.$$

3.3.1(a1) For $a < x < b$

$$\int_a^b \frac{t}{(t-x)^{p+1}}\, dt = \begin{cases} (b-a) + x \ln \frac{b-x}{x-a}, & p=0; \\ \frac{-b}{b-x} + \frac{a}{a-x} + \ln \frac{b-x}{x-a}, & p=1; \\ -\frac{1}{p}\left[\frac{b}{(b-x)^p} - \frac{a}{(a-x)^p}\right] + \frac{1}{p}\left(\frac{-1}{p-1}\right)\left[\frac{1}{(b-x)^{p-1}} - \frac{1}{(a-x)^{p-1}}\right], & p=2,3,\ldots \end{cases}$$

Source: De Klerk

3.3.1(a2) For $a < x < b$

$$\int_a^b \frac{t^2}{(t-x)^{p+1}}\, dt = \begin{cases} \frac{1}{2}[(b-x)^2 - (a-x)^2] + 2x(b-a) + x^2 \ln \frac{b-x}{x-a}, & p=0; \\ \frac{-b^2}{b-x} + \frac{a^2}{a-x} + 2(b-a) + 2x \ln \frac{b-x}{x-a}, & p=1; \\ -\frac{1}{2}\left[\frac{b^2}{(b-x)^2} - \frac{a^2}{(a-x)^2}\right] + \left(\frac{-b}{b-x} + \frac{a}{a-x}\right) + \ln \frac{b-x}{x-a}, & p=2; \\ -\frac{1}{p}\left[\frac{b^2}{(b-x)^p} - \frac{a^2}{(a-x)^p}\right] + \frac{2}{p}\left(\frac{-1}{p-1}\right)\left[\frac{b}{(b-x)^{p-1}} - \frac{a}{(a-x)^{p-1}}\right] & \\ \quad + \frac{2}{p}\left(\frac{1}{p-1}\right)\left(\frac{-1}{p-2}\right)\left[\frac{1}{(b-x)^{p-2}} - \frac{1}{(a-x)^{p-2}}\right], & p=3,4,\ldots. \end{cases}$$

Source: De Klerk

3.3.1(a3) For $a < x < b$

$$-\!\!\!\!\!\!\int_a^b \frac{t^3}{(t-x)^{p+1}}\,dt = \begin{cases} \frac{1}{3}[(b-x)^3-(a-x)^3]+\frac{3}{2}x[(b-x)^2-(a-x)^2] \\ \quad +3x^2(b-a)+x^3\ln\frac{b-x}{x-a}, & p=0; \\ -\left(\frac{b^3}{b-x}-\frac{a^3}{a-x}\right)+\frac{3}{2}[(b-x)^2-(a-x)^2] \\ \quad +3.2x(b-a)+3x^2\ln\frac{b-x}{x-a}, & p=1; \\ -\frac{1}{2}\left[\frac{b^3}{(b-x)^2}-\frac{a^3}{(a-x)^2}\right]+\frac{3}{2}\left[\frac{-b^2}{b-x}+\frac{a^2}{a-x}\right]+3(b-a) \\ \quad +3x\ln\frac{b-x}{x-a}, & p=2; \\ -\frac{1}{3}\left[\frac{b^3}{(b-x)^3}-\frac{a^3}{(a-x)^3}\right]-\frac{1}{2}\left[\frac{b^2}{(b-x)^2}-\frac{a^2}{(a-x)^2}\right] \\ \quad -\left[\frac{b}{b-x}-\frac{a}{a-x}\right]+\ln\frac{b-x}{x-a}, & p=3; \\ -\frac{1}{p}\left[\frac{b^3}{(b-x)^p}-\frac{a^3}{(a-x)^p}\right]+\frac{3}{p}\left(\frac{-1}{p-1}\right)\left[\frac{b^2}{(b-x)^{p-1}}-\frac{a^2}{(a-x)^{p-1}}\right] \\ \quad +\frac{3}{p}\left(\frac{2}{p-1}\right)\left(\frac{-1}{p-2}\right)\left[\frac{b}{(b-x)^{p-2}}-\frac{a}{(a-x)^{p-2}}\right] \\ \quad +\frac{3}{p}\left(\frac{2}{p-1}\right)\left(\frac{1}{p-2}\right)\left(\frac{-1}{p-3}\right)\left[\frac{b}{(b-x)^{p-3}}-\frac{a}{(a-x)^{p-3}}\right], & p=4,5,\ldots. \end{cases}$$

Source: De Klerk

Special case: For $0 < x < 1$

$$-\!\!\!\!\!\!\int_0^1 \frac{t^3}{(t-x)^{p+1}}\,dt = \begin{cases} \frac{3}{2}+3x+\frac{1}{x-1}+3x^2\ln\frac{1-x}{x}, & p=1, \\ 1+\frac{x}{2}-\frac{x^3-6x^2+6x}{2(x-1)^2}+3x\ln\frac{1-x}{x}, & p=2. \end{cases}$$

Source: Wu & Sun (2005), p360

3.3.1(b) For $-1 < x < 1$

$$-\!\!\!\!\!\!\int_{-1}^1 \frac{P_n(t)}{(t-x)^{p+1}}\,dt = \frac{1}{p!}\frac{d^p}{dx^p}-\!\!\!\!\!\!\int_{-1}^1 \frac{P_n(t)}{t-x}\,dt,$$

or, alternatively,

$$-\!\!\!\!\!\!\int_{-1}^1 \frac{P_n(t)}{(t-x)^{p+1}}\,dt = \frac{2}{p!}\frac{d^{p-1}}{dx^{p-1}}\left(\frac{nxQ_n(x)-nQ_{n-1}(x)}{1-x^2}\right),$$

where P_n and Q_n are respectively the Legendre polynomials of the first kind and Legendre functions of the second kind.

Special case:

$$-\!\!\!\!\!\!\int_{-1}^1 \frac{P_n(t)}{(t-x)^3}\,dt = \frac{2nx}{(1-x^2)^2}\Big[xQ_n(x)-Q_{n-1}(x)\Big]+\frac{n}{1-x^2}Q_n(x)(1+n), \quad n \geq 0.$$

Source: De Klerk

3.3.1(c1)

For $a < x < b$ and $\alpha = a - x, \quad \beta = b - x, \quad \alpha < 0 < \beta$, it follows

$$\ne\!\!\!\!\int_a^b \frac{e^{kt}}{(t-x)^{p+1}}\,dt =$$
$$\frac{e^{kx}}{p!}\left[k^p \ln\left(-\frac{\beta}{\alpha}\right) + \sum_{j=1}^{p} \frac{k^{p-j}}{j}\left(\prod_{l=0}^{j-1}(p-l)\right)\left(\frac{\beta^j-\alpha^j}{(\alpha\beta)^j}\right) + \sum_{j=1}^{\infty} \frac{k^{j+p}p!}{j(j+p)!}(\beta^j-\alpha^j)\right].$$

Source: De Klerk

3.3.1(c2)

For $a < x < b$ and $\alpha = a - x, \quad \beta = b - x, \quad \alpha < 0 < \beta$, it follows

$$\ne\!\!\!\!\int_a^b \frac{e^{it}}{(t-x)^{p+1}}\,dt =$$
$$\frac{e^{ix}}{p!}\left[i^p \ln\left(-\frac{\beta}{\alpha}\right) + \sum_{j=1}^{p} \frac{i^{p-j}}{j}\left(\prod_{l=0}^{j-1}(p-l)\right)\left(\frac{\beta^j-\alpha^j}{(\alpha\beta)^j}\right) + \sum_{j=1}^{\infty} \frac{p!\,i^{p+j}}{j(j+p)!}(\beta^j-\alpha^j)\right].$$

Source: De Klerk

3.3.1(c3 and c4)

For $a < x < b$ and $\alpha = a - x, \quad \beta = b - x, \quad \alpha < 0 < \beta$,

$$\begin{aligned}
\ne\!\!\!\!\int_a^b \frac{\cos kt + i\sin kt}{(t-x)^{p+1}}\,dt &= \frac{\cos kx}{p.p!}\left\{\prod_{l=0}^{p-1}(p-l)\right\}\frac{\beta^p-\alpha^p}{(\alpha\beta)^p} \\
&+ \sum_{j=1}^{p-1} i^j \left[\frac{\sin kx}{p!}\,\frac{k^{j-1}}{p+1-j}\left\{\prod_{l=0}^{p-j}(p-l)\right\}\frac{\beta^{p+1-j}-\alpha^{p+1-j}}{(\alpha\beta)^{p+1-j}}\right. \\
&\qquad \left. + \frac{\cos kx}{p!}\,\frac{k^j}{p-j}\left\{\prod_{l=0}^{p-1-j}(p-l)\right\}\frac{\beta^{p-j}-\alpha^{p-j}}{(\alpha\beta)^{p-j}}\right] \\
&+ i^p\left[\frac{\cos kx}{p!}k^p \ln\left(\frac{-\beta}{\alpha}\right) + \frac{\sin kx}{p!}k^{p-1}\left(\frac{\beta-\alpha}{\alpha\beta}\right)\right] \\
&+ i^{p+1}\left[\frac{\sin kx}{p!}k^p \ln\left(\frac{-\beta}{\alpha}\right) + \cos kx\,\frac{k^{p+1}(\beta-\alpha)}{1.(1+p)!}\right] \\
&+ \sum_{j=1}^{\infty} i^{p+1+j}\left[\sin kx\,\frac{k^{p+j}(\beta^j-\alpha^j)}{j.(j+p)!} + \cos kx\,\frac{k^{p+1+j}(\beta^{j+1}-\alpha^{j+1})}{(j+1).(j+1+p)!}\right],
\end{aligned}$$

with

$$\sum_{j=1}^{q\,(<1)} \cdots = 0, \qquad \prod_{j=1}^{q\,(<1)} \cdots = 0.$$

Source: De Klerk

Special cases:

For $0 < x < 1$

$$\begin{aligned}\fint_0^1 \frac{\sin \pi t}{(t-x)^3}\, dt &= \cos \pi x \left[\frac{\pi}{x(x-1)} + \sum_{j=0}^{\infty} (-1)^{j+1} \frac{\pi^{2j+3}}{(2j+1).(2j+3)!} \big((1-x)^{2j+1} - (-x)^{2j+1}\big) \right] \\ &+ \sin \pi x \left[\frac{(1-x)^2 - x^2}{2x^2(x-1)^2} - \frac{\pi^2}{2} \ln\left(\frac{1-x}{x}\right) \right. \\ &\left. + \sum_{j=1}^{\infty} (-1)^{j+1} \frac{\pi^{2j+2}}{(2j).(2j+2)!} \big((1-x)^{2j} - (-x)^{2j}\big) \right].\end{aligned}$$

Source: De Klerk

For $0 < x < 1$

$$\begin{aligned}\fint_0^1 \frac{\cos \pi t}{(t-x)^3}\, dt &= \cos \pi x \left[\frac{(1-x)^2 - x^2}{2x^2(x-1)^2} - \frac{\pi^2}{2} \ln\left(\frac{1-x}{x}\right) \right. \\ &\left. + \sum_{j=1}^{\infty} (-1)^{j+1} \frac{\pi^{2j+2}}{(2j).(2j+2)!} \big((1-x)^{2j} - (-x)^{2j}\big) \right] \\ &+ \sin \pi x \left[\frac{\pi}{x(1-x)} + \sum_{j=1}^{\infty} (-1)^{j+1} \frac{\pi^{2j+1}}{(2j-1).(2j+1)!} \big((1-x)^{2j-1} - (-x)^{2j-1}\big) \right]\end{aligned}$$

Source: De Klerk

3.3.1(d) Further cases:

For $p = 1, 2,$

$$\fint_{-1}^1 \frac{t^2 + \big(2 + \operatorname{sgn}(t)\big)|t|^{p+1+1/2}}{t^{p+1}}\, dt = \frac{24 - 10p}{3}, \qquad x = 0.$$

Source: Wu & Sun (2005), pp 360-361

For $p = 1, 2,$

$$\fint_{-1}^1 \frac{t^2 + \big(2 + \operatorname{sgn}(t)\big)|t|^{p+1/2}}{t^{p+1}}\, dt = 16 - 6p, \qquad x = 0.$$

Source: Wu & Sun (2005), pp 360-361

3.3.2(a) For $a < x < b \le 1$ and $p = 1, 2, \ldots,$

$$\fint_a^b \frac{f(t)\sqrt{1-t}}{(t-x)^{p+1}}\, dt = \frac{1}{p!} \frac{d^p}{dx^p} \int_a^b \frac{f(t)\sqrt{1-t}}{(t-x)}\, dt.$$

Source: De Klerk

3.3.2(a0) For $-1 < x < 1$ and $p = 1, 2, \ldots,$

$$\begin{aligned}\fint_{-1}^{1} \frac{\sqrt{1-t}}{(t-x)^{p+1}}\,dt &= -\frac{2\sqrt{2}(-1)^{p+1}}{p(1-x)(1+x)^p} \\ &\quad + \frac{2p-3}{2p(1-x)} \fint_{-1}^{1} \frac{\sqrt{1-t}}{(t-x)^p}\,dt.\end{aligned}$$

Source: Kaya & Erdogan (1987) (25)

3.3.2(an) For $-1 < x < 1$ and $p = 0, 1, \ldots,$

$$\fint_{-1}^{1} \frac{t^n\sqrt{1-t}}{(t-x)^{p+1}}\,dt = \sum_{m=1}^{p+1} \binom{n}{p+1-m} x^{n-p-1+m} \fint_{-1}^{1} \frac{\sqrt{1-t}}{(t-x)^m}\,dt + \sum_{k=0}^{n-p-1} A_k^{p+1} x^{n-p-1-k}$$

and $n \geq p$, and

$$A_k^{p+1} = 4\sqrt{2}\binom{n-k-1}{p} \sum_{i=p+1}^{k} (-1)^i \binom{k}{i} \frac{2^i}{2i+3}.$$

Source: Kaya & Erdogan (1987) (38, B5)

3.3.3(a) For $-1 \leq a < x < b \leq 1$ and $p = 1, 2, \ldots,$

$$\fint_{a}^{b} \frac{f(t)\sqrt{1-t^2}}{(t-x)^{p+1}}\,dt = \frac{1}{p!}\frac{d^p}{dx^p} \fint_{a}^{b} \frac{f(t)\sqrt{1-t^2}}{(t-x)}\,dt.$$

Source: De Klerk

3.3.3(a0) For $-1 < x < 1$ and $p = 1, 2, \ldots,$

$$\fint_{-1}^{1} \frac{\sqrt{1-t^2}}{(t-x)^{p+1}}\,dt = \begin{cases} -\pi, & p = 1; \\ 0, & p = 2, 3, \ldots. \end{cases}$$

Source: De Klerk

3.3.3(an) For $-1 < x < 1$ and $p = 1, 2, \ldots,$

$$\fint_{-1}^{1} \frac{t^n\sqrt{1-t^2}}{(t-x)^{p+1}}\,dt = \begin{cases} 0, & p > n+1 \\ \frac{\pi}{p!}\sum_{k=p}^{n+1} k(k-1)\ldots(k-(p-1))\, b_k\, x^{k-p}, & p \leq n+1, \end{cases}$$

with

$$b_k = \begin{cases} 0, & n-k \quad \text{even} \\ \frac{1}{2\sqrt{\pi}} \frac{\Gamma\left(\frac{n-k}{2}\right)}{\Gamma\left(\frac{n-k+3}{2}\right)}, & n-k \quad \text{odd.} \end{cases}$$

Source: Kaya & Erdogan (1987) (34, B6); De Klerk

3.3.3(b1) For $-1 < x < 1$

$$\oint_{-1}^{1} \frac{U_n(t)\sqrt{1-t^2}}{(t-x)^{p+1}}\,dt = \frac{1}{p!}\frac{d^p}{dx^p}\int_{-1}^{1} \frac{U_n(t)\sqrt{1-t^2}}{t-x}\,dt,$$

or, alternatively,

$$\oint_{-1}^{1} \frac{U_n(t)\sqrt{1-t^2}}{(t-x)^{p+1}}\,dt = \frac{-\pi}{p!}\frac{d^{p-1}}{dx^{p-1}}\left[\frac{d}{dx}\big(T_{n+1}(x)\big)\right] = \frac{-\pi}{p!}\frac{d^{p-1}}{dx^{p-1}}\Big[(n+1)U_n(x)\Big],$$

$n \geq 0$, where T_n and U_n are respectively the Chebyshev polynomials of the first and second kind.

Special case:

$$\oint_{-1}^{1} \frac{U_n(t)\sqrt{1-t^2}}{(t-x)^3}\,dt = -\frac{\pi}{2}(n+1)\frac{dU_n(x)}{dx} = \frac{(n+1)T_{n+1} - xU_n}{x^2-1}, \quad n \geq 0.$$

Source: De Klerk

3.3.3(b2) For $-1 < x < 1$

$$\oint_{-1}^{1} \frac{T_n(t)}{(t-x)^{p+1}\sqrt{1-t^2}}\,dt = \frac{1}{p!}\frac{d^p}{dx^p}\int_{-1}^{1} \frac{T_n(t)}{(t-x)\sqrt{1-t^2}}\,dt,$$

or, alternatively,

$$\oint_{-1}^{1} \frac{T_n(t)}{(t-x)^{p+1}\sqrt{1-t^2}}\,dt = \frac{\pi}{p!}\frac{d^{p-1}}{dx^{p-1}}\left[\frac{nT_n(x) - xU_{n-1}(x)}{x^2-1}\right],$$

$n \geq 1$, where T_n and U_n are respectively the Chebyshev polynomials of the first and second kind.

Special case:

$$\begin{aligned} \oint_{-1}^{1} \frac{T_n(t)}{(t-x)^3\sqrt{1-t^2}}\,dt &= \frac{\pi}{2}\frac{d}{dx}\left[\frac{nT_n(x) - xU_{n-1}(x)}{x^2-1}\right] \\ &= \frac{\pi}{2(x^2-1)^2}\Big[-3nxT_n(x) + \big(n^2(x^2-1) + 2x^2 + 1\big)U_{n-1}(x)\Big], \quad n \geq 1. \end{aligned}$$

Source: De Klerk

3.3.3(c1) For $-1 \leq a < x < b \leq 1$

$$\begin{aligned}\fint_a^b \frac{\sqrt{1-t^2}e^t}{(t-x)^{p+1}}\,dt &= \frac{1}{p!}\frac{d^p}{dx^p}\left[\sum_{j=0}^{\infty}\frac{1}{j!}\left\{\sum_{i=0}^{j-1}x^i\int_a^b t^{j-1-i}\sqrt{1-t^2}\,dt\right.\right.\\ &\left.\left. +x^j\left[\sqrt{1-t^2}-x\sin^{-1}t\right]\Big|_a^b + x^j C^*\right\}\right],\end{aligned}$$

with

$$C^* = -\sqrt{1-x^2}\,\ln\left|\frac{2\sqrt{1-x^2}\sqrt{1-t^2}+2(1-xt)}{t-x}\right|\Bigg|_a^b$$

and $\sum_{i=0}^{q\,(<0)}\cdots = 0$.
Special case:

$$\begin{aligned}\fint_a^b \frac{\sqrt{1-t^2}e^t}{(t-x)^3}\,dt &= \frac{1}{2}\frac{d^2}{dx^2}\left[\sum_{j=0}^{\infty}\frac{1}{j!}\left\{\sum_{i=0}^{j-1}x^i\int_a^b t^{j-1-i}\sqrt{1-t^2}\,dt\right.\right.\\ &\left.\left. +x^j\left[\sqrt{1-t^2}-x\sin^{-1}t\right]\Big|_a^b + x^j C^*\right\}\right],\end{aligned}$$

with C^* as above.

Source: De Klerk

3.3.4(a) For $a < x < b \leq 1$ and $p = 1, 2, \ldots,$

$$\fint_a^b \frac{f(t)}{(t-x)^{p+1}\sqrt{1-t}}\,dt = \frac{1}{p!}\frac{d^p}{dx^p}\fint_a^b \frac{f(t)}{(t-x)\sqrt{1-t}}\,dt.$$

Source: De Klerk

3.3.4(a0) For $-1 < x < 1$ and $p = 1, 2, \ldots,$

$$\begin{aligned}\fint_{-1}^1 \frac{dt}{(t-x)^{p+1}\sqrt{1-t}} &= -\frac{\sqrt{2}(-1)^{p+1}}{p(1-x)(1+x)^p}\\ &+\frac{2p-1}{2p(1-x)}\fint_{-1}^1 \frac{dt}{(t-x)^p\sqrt{1-t}}, \quad x<1.\end{aligned}$$

Source: Kaya & Erdogan (1987) (26)

3.3.4(an) For $-1 < x < 1$ and $p = 0, 1, \dots,$

$$\fint_{-1}^{1} \frac{t^n}{(t-x)^{p+1}\sqrt{1-t}}\,dt = \sum_{m=1}^{p+1} \binom{n}{p+1-m} x^{n-p-1+m} \fint_{-1}^{1} \frac{dt}{(t-x)^m\sqrt{1-t}} + \sum_{k=0}^{n-p-1} B_k^{p+1} x^{n-p-1-k}$$

and $n \geq p+1$, and

$$B_k^{p+1} = 2\sqrt{2}\binom{n-k-1}{p} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{2^i}{2i+1}.$$

Source: Kaya & Erdogan (1987) (39, B10)

3.3.5(a) For $a < x < b \leq 1$ and $p = 1, 2, 3, \dots,$

$$\fint_{a}^{b} \frac{f(t)}{(t-x)^{p+1}\sqrt{1-t^2}}\,dt = \frac{1}{p!}\frac{d^p}{dx^p} \fint_{a}^{b} \frac{f(t)}{(t-x)\sqrt{1-t^2}}\,dt.$$

Source: De Klerk

3.3.5(a0) For $a < x < b \leq 1$ and $p = 1, 2, 3, \dots,$

$$\fint_{a}^{b} \frac{dt}{(t-x)^{p+1}\sqrt{1-t^2}} = \frac{1}{p!}\frac{d^p}{dx^p}\left(\frac{-1}{\sqrt{1-x^2}} \ln|D^*|\right),$$

with

$$D^* = \frac{\sqrt{1-x^2}\sqrt{1-b^2} + (1-xb)}{\sqrt{1-x^2}\sqrt{1-a^2} + (1-xa)} \cdot \frac{a-x}{b-x}.$$

Source: De Klerk

3.3.5(an) For $a < x < b \leq 1$ and $p = 1, 2, 3, \dots,$

$$\fint_{-1}^{1} \frac{t^n}{(t-x)^{p+1}\sqrt{1-t^2}}\,dt = \frac{1}{p!}\frac{d^p}{dx^p}\left[\sum_{i=0}^{n-1} x^i \int_a^b \frac{t^{n-1-i}}{\sqrt{1-t^2}}\,dt - \frac{x^n}{\sqrt{1-x^2}} \ln|D^*|\right],$$

with D^* as in 3.3.5(a0).

Source: De Klerk

3.3.6 Further cases:

3.3.6(a) For $-1 < t < 1$ and $x = 0$,

$$\fint_{-1}^{1} \frac{e^{1-t} - 1}{(t-x)^{3/2}}\,dt = -8.120\ 313\ 877\ 115.$$

Source: Ashour, SA & Ahmed, HM. (2007), p 1680.

3.3.6(b) For $-1 < t < 1$ and $x = 0$,

$$\neq\!\!\!\!\!\!\int_{-1}^{1} \frac{\sqrt{1-t^2}\, e^t}{t^2}\, dt = -2.339\ 556\ 253\ 339.$$

Source: Ashour, SA & Ahmed, HM. (2007), p 1682.

Chapter 4

Singular integrals: numerically calculated

Theme (1990+)	Technique
$\fint_a^b \frac{f(t)}{t-x}\,dt$ – **Gauss quadrature**	
Ioakimidis (1993)	4.1.2.1
Korsunsky (1998)	4.1.2.2
Hui & Shia (1999)	4.1.2.3
Kolm & Rokhlin (2001)	4.1.2.4
Kythe & Schäferkotter (2005)	4.1.2.5
$\fint_a^b \frac{f(t)}{(t-x)^2}\,dt$ – **Newton-Cotes quadrature**	
Sun & Wu (2005)	4.2.1.1
Wu & Sun (2005)	4.2.1.2
Wu & Sun (2008)	4.2.1.3
$\fint_a^b \frac{f(t)}{(t-x)^2}\,dt$ – **Gauss quadrature**	
Korsunsky (1998)	4.2.2.1
Hui & Shia (1999)	4.2.2.2
Kolm & Rokhlin (2001)	4.2.2.3
Kythe & Schäferkotter (2005)	4.2.2.4
Shen (2005)	4.2.2.5
$\fint_a^b \frac{f(t)}{(t-x)^{p+1}}\,dt$ – **Newton-Cotes quadrature**	
Wu & Sun (2005)	4.3.1.1
$\fint_a^b \frac{f(t)}{(t-x)^{p+1}}\,dt$ – **Gauss quadrature**	
Kythe & Schäferkotter (2005)	4.3.2.2

4.1 Cauchy principal value integrals

4.1.1 Newton-Cotes quadrature

4.1.2 Gauss quadrature

Technique 4.1.2.1
Author and date: NI Ioakimidis (1993)
Article: The Gauss-Laguerre quadrature rule for finite-part integrals
Journal: Communications in Numerical Methods in Engineering, 9, 439-450.
Equations in article: 1, 12, 16, 22, 32
Numerical technique: Gauss-Laguerre quadrature

$$\int_0^\infty \frac{e^{-t}\,f(t)}{t}\,dt \equiv \int_0^\infty \frac{e^{-t}}{t}[f(t)-f(0)]\,dt - \gamma f(0) \simeq \sum_{i=1}^{n} w_{in}[f(t_{in})-f(0)] - \gamma f(0).$$

Algorithm:

Either follow Procedure A or Procedure B:

Procedure A:

Step 1: Choose n, $n \geq 2$, the number of nodes in the Gauss quadrature.

Step 2: Calculate

$$c_n = \frac{\sum_{k=1}^{n}\frac{1}{k}+\gamma}{\sum_{k=1}^{n-1}\frac{1}{k}+\gamma},$$

with $\gamma = 0.577\ 215\ 664\ 9...$.

Step 3: Find the classical Laguerre polynomials L_{n-1}, L_n and L_{n+1} and calculate the polynomials M_n, M_n' and M_{n+1} where

$$M_n(t) = L_n(t) - c_n L_{n-1}(t),\ n \geq 2.$$

Consult: Laguerre Polynomial

Step 4: Calculate the zeros t_{in}, $i = 1, \ldots, n$ of M_n.

Step 5: Calculate the weights

$$w_{in} = \frac{c_n}{n(n+1)\,M_n'(t_{in})\,M_{n+1}(t_{in})}, \quad i = 1, 2, \ldots, n,$$

or, alternatively,

$$w_{in} = -\frac{c_{n-1}}{n(n-1)\,M_n'(t_{in})\,M_{n-1}(t_{in})}, \quad i = 1, 2, \ldots, n.$$

Step 6: Compute the approximated value of the integral using the quadrature formula

$$\int_0^\infty \frac{e^{-t}\,f(t)}{t}\,dt \simeq \sum_{i=1}^{n} w_{in}[f(t_{in})-f(0)] - \gamma f(0).$$

Procedure B:

Step 1: Choose n, $n \geq 2$, the number of nodes in the Gauss quadrature.

Step 2: Compute the approximated value of the integral using the quadrature formula

$$\int_0^\infty \frac{e^{-t} f(t)}{t} \, dt \simeq \sum_{i=1}^{n} w_{in}[f(t_{in}) - f(0)] - \gamma f(0),$$

with $\gamma = 0.577\ 215\ 664\ 9...$, and t_{in} and w_{in} as given in the appendix.

Appendix: For $n = 10$; otherwise, consult original article.

t_{in}	w_{in}
−0.033 537 662 5	−2.206 874 697 7
0.339 472 493 7	1.300 355 767 3
1.214 488 133 1	0.277 123 918 2
2.617 128 891 9	0.046 818 132 2
4.587 964 406 2	0.005 042 935 0
7.192 862 083 9	0.000 308 685 5
10.538 392 798 1	0.000 009 473 1
14.806 984 300 8	0.000 000 120 8
20.353 250 286 8	0.000 000 000 1
28.089 410 590 7	0.000 000 000 0

Comments:

Technique 4.1.2.2
Author and date: AM Korsunsky (1998)
Article: Gauss-Chebyshev quadrature formulae for strongly singular integrals
Journal: Quarterly of Applied Mathematics, 56, 461-472.
Equations in article: 24a, 27, 30a, 32, 33, 45, 46
Numerical technique: Gauss-Chebyshev quadrature

$$\frac{1}{\pi}\fint_{-1}^{1}\left[\frac{f(t)}{t-x_k}+K(t,x_k)\right]\sqrt{1-t^2}\,dt \simeq \sum_{i=1}^{n}\frac{1-t_{in}^2}{n+1}\left[\frac{f(t_{in})}{t_{in}-x_k}+K(t_{in},x_k)\right],$$

$K(t,x)$ smooth.
Algorithm:

Step 1: Choose n, the number of nodes in the Gauss-Chebyshev quadrature.

Step 2: Calculate the possible singular values x_k, the zeros of T_{n+1}, $k = 1,\dots,n+1$, namely

$$x_k = \cos\left[\tfrac{(2k-1)\pi}{2(n+1)}\right].$$

Step 3: Calculate the nodes t_{in}, the zeros of the Chebyshev polynomial of the second kind, U_n given by

$$t_{in} = \cos\frac{i\pi}{n+1}, \quad i = 1,2,\dots,n.$$

Step 4: Compute for $k = 1,\dots,n+1$, the approximated value of the integral using the quadrature formula

$$\frac{1}{\pi}\fint_{-1}^{1}\left[\frac{f(t)}{t-x_k}+K(t,x_k)\right]\sqrt{1-t^2}\,dt \simeq \sum_{i=1}^{n}\frac{1-t_{in}^2}{n+1}\left[\frac{f(t_{in})}{t_{in}-x_k}+K(t_{in},x_k)\right],$$

$K(t,x)$ smooth.

Comments:

Technique 4.1.2.3(a)
Author and date: CY Hui & D Shia (1999)
Article: Evaluations of hypersingular integrals using Gaussian quadrature
Journal: International Journal for Numerical Methods in Engineering, 44, 205-214.
Equations in article: 6, 8, 9, 10, 11
Numerical technique: Gauss-Legendre quadrature

$$\fint_{-1}^{1} \frac{f(t)}{t-x}\, dt \simeq -\frac{2f(x)Q_n(x)}{P_n(x)} + \sum_{i=1}^{n} \frac{w_{in} f(t_{in})}{t_{in}-x}, \quad x \neq t_{in}.$$

Algorithm:

Step 1: Choose n, the number of nodes in the Gauss-Legendre quadrature.

Step 2: Find the Legendre polynomials of the first kind, P_n and P_{n+1} and the Legendre function of the second kind Q_n.
Consult: Legendre polynomial and Legendre functions

Step 3: Calculate the zeros t_{in}, $i = 1, \ldots, n$ of P_n.
Consult: Legendre polynomial zeros

Step 4: Calculate the weights, w_{in},

$$w_{in} = -\frac{2}{(n+1)P_n'(t_{in})P_{n+1}(t_{in})}.$$

Step 5: Compute the approximated value of the integral using the quadrature formula

$$\fint_{-1}^{1} \frac{f(t)}{t-x}\, dt \simeq -\frac{2f(x)Q_n(x)}{P_n(x)} + \sum_{i=1}^{n} \frac{w_{in} f(t_{in})}{t_{in}-x}, \quad x \neq t_{in}.$$

Comments:

Technique 4.1.2.3(b)
Author and date: CY Hui & D Shia (1999)
Article: Evaluations of hypersingular integrals using Gaussian quadrature
Journal: International Journal of Numerical Methods in Engineering, 44, 205-214.
Equations in article: 6, 8, 9, 12, 13, 14, 15
Numerical technique: Gauss-Chebyshev quadrature

$$\int_{-1}^{1} \frac{\sqrt{1-t^2}f(t)}{t-x}\,dt \simeq -\frac{\pi f(x)}{U_{n-1}(x)}T_n(x) + \sum_{i=1}^{n-1} \frac{w_{in} f(t_{in})}{t_{in}-x}, \quad x \neq t_{in}.$$

Algorithm:

Step 1: Choose n, the number of nodes in the Gauss-Chebyshev quadrature.

Step 2: Find the Chebyshev polynomial of the first kind, T_n, and the Chebyshev polynomial of the second kind U_{n-1}.
Consult: Chebyshev polynomial 1 and Chebyshev polynomial 2

Step 3: Calculate the nodes, t_{in}, which are the zeros of U_{n-1} given by

$$t_{in} = \cos\frac{i\pi}{n}, \quad i = 1, 2, \ldots, n-1.$$

Step 4: Calculate the weights, w_{in},

$$w_{in} = \frac{\pi}{n}\sin^2\frac{i\pi}{n}, \quad i = 1, 2, \ldots, n-1.$$

Step 5: Compute the approximated value of the integral using the quadrature formula

$$\int_{-1}^{1} \frac{\sqrt{1-t^2}f(t)}{t-x}\,dt \simeq -\frac{\pi f(x)}{U_{n-1}(x)}T_n(x) + \sum_{i=1}^{n-1} \frac{w_{in} f(t_{in})}{t_{in}-x}, \quad x \neq t_{in}.$$

Comments:

Technique 4.1.2.4
Author and date: P Kolm & V Rokhlin (2001)
Article: Numerical quadratures for singular and hypersingular integrals
Journal: Computers and Mathematics with Applications, 41, 327-352.
Equations in article: 3, 51, 71, 75
Numerical technique: Gauss Legendre-quadrature

$$\int_{-1}^{1} \frac{f(t)}{t-x}\,dt \simeq \sum_{i=1}^{n} \Big[w_{in} \sum_{j=0}^{n-1} (2j+1) P_j(t_{in}) Q_j(x) \Big] f(t_{in}).$$

Algorithm:

Step 1: Choose n, the number of nodes in the Gauss-Legendre quadrature.

Step 2: Find the Legendre polynomials of the first kind, $P_0, \ldots, P_n$ and the Legendre functions of the second kind $Q_0, \ldots, Q_{n-1}$.
Consult: Legendre polynomial and Legendre functions

Step 3: Calculate the zeros t_{in}, $i = 1, \ldots, n$ of P_n.
Consult: Legendre polynomial zeros

Step 4: Calculate the weights, w_{in},

$$w_{in} = \int_{-1}^{1} \prod_{j=1; j\neq i}^{n} \Big(\frac{t - t_{jn}}{t_{in} - t_{jn}} \Big)^2 dt, \quad i = 1, \ldots, n.$$

Step 5: Compute the approximated value of the integral using the quadrature formula

$$\int_{-1}^{1} \frac{f(t)}{t-x}\,dt \simeq \sum_{i=1}^{n} \Big[- w_{in} \sum_{j=0}^{n-1} (2j+1) P_j(t_{in}) Q_j(x) \Big] f(t_{in}).$$

Comments:

Technique 4.1.2.5(a)
Author and date: PK Kythe & MR Schäferkotter (2005)
Chapter: Singular Integrals
Book: Handbook of Computational Methods for Integration.
Section in book: 5.5
Numerical technique: Gauss-Legendre quadrature

$$\fint_a^b \frac{w(t)f(t)}{t-x}\,dt \simeq \sum_{i=1}^{n} w_{in}\frac{f(t_{in})}{t_{in}-x} + f(x)\left[\fint_a^b \frac{w(t)}{t-x}\,dt - \sum_{i=1}^{n}\frac{w_{in}}{t_{in}-x}\right].$$

Algorithm:

Step 1: Choose n, the number of nodes in the Gauss-Legendre quadrature.

Step 2: Find the Legendre polynomials of the first kind, P_n and P_{n+1}.
Consult: Legendre polynomial

Step 3: Calculate the zeros t_{in}, $i = 1, \ldots, n$ of P_n.
Consult: Legendre polynomial zeros

Step 4: Calculate the weights, w_{in},

$$w_{in} = \frac{2(1-t_{in}^2)}{(n+1)^2[P_{n+1}(t_{in})]^2}, \quad i = 1, \ldots, n.$$

Step 5: Compute the approximated value of the integral using the Gauss-Legendre quadrature formula

$$\fint_{-1}^{1} \frac{f(t)}{t-x}\,dt \simeq \sum_{i=1}^{n} w_{in}\frac{f(t_{in})}{t_{in}-x} + f(x)\left[\ln\left(\frac{1-x}{1+x}\right) - \sum_{i=1}^{n}\frac{w_{in}}{t_{in}-x}\right].$$

Comments:

Technique 4.1.2.5(b)
Author and date: PK Kythe & MR Schäferkotter (2005)
Chapter: Singular Integrals
Book: Handbook of Computational Methods for Integration.
Section in book: 5.5
Numerical technique: Gauss-Chebyshev quadrature

$$\int_a^b \frac{w(t)f(t)}{t-x}\,dt \simeq \sum_{i=1}^{n} w_{in}\frac{f(t_{in})}{t_{in}-x} + f(x)\left[\int_a^b \frac{w(t)}{t-x}\,dt - \sum_{i=1}^{n}\frac{w_{in}}{t_{in}-x}\right].$$

Algorithm:

Step 1: Choose n, the number of nodes in the Gauss-Chebyshev quadrature.

Step 2: Calculate the zeros t_{in}, $i = 1, \ldots, n$ of T_n, the Chebyshev polynomial of the first kind, given by

$$t_{in} = \cos\frac{(2i-1)\pi}{2n}, \quad i = 1, 2, \ldots, n.$$

Step 3: Calculate the weights, w_{in}, given by

$$w_{in} = \frac{(1-t_{in}^2)\pi}{n}, \quad i = 1, \ldots, n.$$

Step 4: Compute the approximated value of the integral using the Gauss-Chebyshev quadrature formula

$$\int_{-1}^{1} \frac{\sqrt{1-t^2}\,f(t)}{t-x}\,dt \simeq \sum_{i=1}^{n} w_{in}\frac{f(t_{in})}{t_{in}-x} - f(x)\left[\pi x + \sum_{i=1}^{n}\frac{w_{in}}{t_{in}-x}\right],$$

Comments:

4.2 Hypersingular integrals

4.2.1 Newton-Cotes quadrature

Technique 4.2.1.1(a)
Author and date: W Sun & J Wu (2005)
Article: Newton-Cotes formulae for the numerical evaluation of certain hypersingular integrals
Journal: Computing, 75, 297-309.
Equations in article: 1, 3, 3a, 4
Numerical technique: Newton-Cotes quadrature

$$\neq\!\!\!\!\!\!\int_a^b \frac{f(t)}{(t-x)^2}\,dt \simeq \sum_{i=0}^{n}\Big[\frac{1-\delta_{i0}}{h_i}\ln\Big|\frac{t_i-x}{t_{i-1}-x}\Big| - \frac{1-\delta_{in}}{h_{i+1}}\ln\Big|\frac{t_{i+1}-x}{t_i-x}\Big| - \frac{\delta_{i0}}{x-t_0} + \frac{\delta_{in}}{x-t_n}\Big] f(t_i),$$

with δ_{ij} the Kronecker-delta.

Algorithm:

Step 1: Choose n, with $n-1$ the number of internal nodes in $[a,b]$.

Step 2: Let $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ a partition of the interval $[a,b]$ with $h_i = t_i - t_{i-1}$, $i = 1,\ldots,n$.

Step 3: Assure that $x \neq t_i$, $i = 1, 2, \ldots, n-1$.

Step 4: Compute the approximated value of the integral using the Newton-Cotes quadrature formula

$$\neq\!\!\!\!\!\!\int_a^b \frac{f(t)}{(t-x)^2}\,dt \simeq \sum_{i=0}^{n}\Big[\frac{1-\delta_{i0}}{h_i}\ln\Big|\frac{t_i-x}{t_{i-1}-x}\Big| - \frac{1-\delta_{in}}{h_{i+1}}\ln\Big|\frac{t_{i+1}-x}{t_i-x}\Big| - \frac{\delta_{i0}}{x-t_0} + \frac{\delta_{in}}{x-t_n}\Big] f(t_i),$$

with δ_{ij} the Kronecker-delta.

Comments:

Technique 4.2.1.1(b)
Author and date: W Sun & J Wu (2005)
Article: Newton-Cotes formulae for the numerical evaluation of certain hypersingular integrals
Journal: Computing, 75, 297-309.
Equations in article: 16, 16a, 22, 23, 24
Numerical technique: Newton-Cotes quadrature (with singular point equal to a node)

$$\begin{aligned}\oint_a^b \frac{f(t)}{(t-x)^2}\,dt \;&\simeq\; \sum_{i=0,\,i\neq k}^{n} \frac{\theta_i f(t_i)}{(i-k)^2 h} + f(t_k)\Big[\frac{b-a}{(a-t_k)(b-t_k)} - \sum_{i=0,\,i\neq k}^{n} \frac{\theta_i}{(i-k)^2 h}\Big] \\ &+\; \frac{f(t_{k+1})-f(t_{k-1})}{2h}\left[\ln\frac{b-t_k}{t_k-a} - \sum_{i=0,\,i\neq k}^{n} \frac{\theta_i}{i-k}\right] \\ &+\; \frac{f(t_{k+1})-2f(t_k)+f(t_{k-1})}{2h}.\end{aligned}$$

Algorithm:

Step 1: Choose n, with $n-1$ the number of internal nodes in $[a,b]$.

Step 2: Let $a=t_0<t_1<\ldots<t_{n-1}<t_n=b$ be a uniform partition of the interval $[a,b]$ with $h=h_i=t_i-t_{i-1}=(b-a)/n$, $i=1,\ldots,n$.

Step 3: Let $x=t_k$ where k is any specific choice from the values $1,\ldots,n-1$.

Step 4: Calculate $\theta_i = 1-(\delta_{i0}+\delta_{in})/2$, $i=0,\ldots,n$, with δ_{ij} the Kronecker-delta.

Step 5: Compute the approximated value of the integral using the Newton-Cotes quadrature formula

$$\begin{aligned}\oint_a^b \frac{f(t)}{(t-x)^2}\,dt \;&\simeq\; \sum_{i=0,\,i\neq k}^{n} \frac{\theta_i f(t_i)}{(i-k)^2 h} + f(t_k)\Big[\frac{b-a}{(a-t_k)(b-t_k)} - \sum_{i=0,\,i\neq k}^{n} \frac{\theta_i}{(i-k)^2 h}\Big] \\ &+\; \frac{f(t_{k+1})-f(t_{k-1})}{2h}\left[\ln\frac{b-t_k}{t_k-a} - \sum_{i=0,\,i\neq k}^{n} \frac{\theta_i}{i-k}\right] \\ &+\; \frac{f(t_{k+1})-2f(t_k)+f(t_{k-1})}{2h}.\end{aligned}$$

Comments:

Technique 4.2.1.1(c)
Author and date: W Sun & J Wu (2005)
Article: Newton-Cotes formulae for the numerical evaluation of certain hypersingular integrals
Journal: Computing, 75, 297-309.
Equations in article: 16, 28, 29
Numerical technique: Newton-Cotes quadrature (with singular point not very close to either one of the end points a or b)

$$\begin{aligned}\int_a^b \frac{f(t)}{(t-x)^2}\,dt \;&\simeq\; \sum_{i=0,\,i\neq k}^{n} \frac{\theta_i f(t_i)}{(i-k)^2 h} + f(t_k)\Big[\frac{b-a}{(a-t_k)(b-t_k)} - \sum_{i=0,\,i\neq k}^{n} \frac{\theta_i}{(i-k)^2 h}\Big] \\ &+\; \frac{f(t_{k+1}) - 2f(t_k) + f(t_{k-1})}{2h}.\end{aligned}$$

Algorithm:

Step 1: Choose n, with $n-1$ the number of internal nodes in $[a,b]$.

Step 2: Let $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ be a uniform partition of the interval $[a,b]$ with $h = h_i = t_i - t_{i-1} = (b-a)/n$, $i = 1, \ldots, n$.

Step 3: Ensure that the singular point x is not very close to either one of the end points a or b.

Step 4: Calculate $\theta_i = 1 - (\delta_{i0} + \delta_{in})/2$, $i = 0, \ldots, n$, with δ_{ij} the Kronecker-delta.

Step 5: Compute the approximated value of the integral using the Newton-Cotes quadrature formula

$$\begin{aligned}\int_a^b \frac{f(t)}{(t-x)^2}\,dt \;&\simeq\; \sum_{i=0,\,i\neq k}^{n} \frac{\theta_i f(t_i)}{(i-k)^2 h} + f(t_k)\Big[\frac{b-a}{(a-t_k)(b-t_k)} - \sum_{i=0,\,i\neq k}^{n} \frac{\theta_i}{(i-k)^2 h}\Big] \\ &+\; \frac{f(t_{k+1}) - 2f(t_k) + f(t_{k-1})}{2h}.\end{aligned}$$

Comments:

Technique 4.2.1.2
Author and date: J Wu & W Sun (2005)
Article: The superconvergence of the composite trapezoidal rule for the Hadamard finite-part integrals.
Journal: Numerische Mathematik, 102, 343-363.
Equations in article: 1.1, 1.4, 2.1, 2.3
Numerical technique: Newton-Cotes quadrature

$$\fint_a^b \frac{f(t)}{(t-x)^2}\,dt \simeq \sum_{i=0}^{n} w_i(x) f(t_i).$$

Algorithm:

Step 1: Let $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ be a uniform partition of the interval $[a, b]$ with $h = t_i - t_{i-1}$, $i = 1, \ldots, n$, and with $n-1$ the number of internal nodes, to be determined in step 2.

Step 2: Call the singular point x the node t_m. Choose both m and n integers and in such a way that

$$n = \Big(\frac{b-a}{x-a}\Big)(m + \tfrac{1}{2} \pm \tfrac{1}{3}).$$

This will ensure superconvergence; otherwise make a choice so that $x \simeq t_m$.

Step 3: Calculate the weights $w_i(x)$, $i = 0, 1, \ldots, n$,

$$w_i(x) = \frac{1-\delta_{i0}}{h} \ln\left|\frac{t_i - x}{t_{i-1} - x}\right| - \frac{1-\delta_{in}}{h} \ln\left|\frac{t_{i+1} - x}{t_i - x}\right| - \frac{\delta_{i0}}{x - t_0} + \frac{\delta_{in}}{x - t_n},$$

with δ_{ij} the Kronecker-delta.

Step 4: Compute the approximated value of the integral using the Newton-Cotes quadrature formula

$$\fint_a^b \frac{f(t)}{(t-x)^2}\,dt \simeq \sum_{i=0}^{n} w_i(x) f(t_i).$$

Comments:

Technique 4.2.1.3
Author and date: J Wu & W Sun (2008)
Article: The superconvergence of Newton-Cotes rules for the Hadamard finite-part integral on an interval.
Journal: Numerische Mathematik, 109, 143-165.
Equations in article: 2.2, 2.3
Numerical technique: Newton-Cotes quadrature

$$\fint_a^b \frac{f(t)}{(t-x)^2}\,dt \simeq \sum_{i=0}^{n-1}\sum_{j=0}^{k} w_{ij}^k(x) f(t_{ij}).$$

Algorithm:

Step 1: Choose $n-1$, the number of internal nodes, with $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ a uniform partition of the interval $[a, b]$ with $h = t_i - t_{i-1}$, $i = 1, \ldots, n$.

Step 2: In order to construct a piecewise Lagrange interpolation polynomial of degree k for each subinterval, choose k, and perform a further partition of each subinterval, $t_i = t_{i0} < t_{i1} < \ldots < t_{ik} = t_{i+1}$, $i = 0, \ldots, n-1$.

Step 3: Calculate for each subinterval the polynomials

$$l_{ki}(t) = \prod_{j=0}^{k}(t - t_{ij}), \quad i = 0, \ldots, n-1,$$

as well as the derivatives $l'_{ki}(t), \quad i = 0, \ldots, n-1$.

Step 4: Calculate, for $i = 0, \ldots, n-1$ and $j = 0, \ldots, k$ the weights $w_{ij}^{(k)}(x)$,

$$w_{ij}^{(k)}(x) = \frac{1}{l'_{ki}(t_{ij})} \fint_{t_{i0}\,(=t_i)}^{t_{ik}\,(=t_{i+1})} \frac{1}{(t-x)^2} \prod_{m=0;\, m\neq j}^{k} (t - t_{im})\,dt.$$

Step 5: Compute the approximated value of the integral using the Newton-Cotes quadrature formula

$$\fint_a^b \frac{f(t)}{(t-x)^2}\,dt \simeq \sum_{i=0}^{n-1}\sum_{j=0}^{k} w_{ij}^k(x) f(t_{ij}).$$

Step 6: Remark: If $x = t_{[n/2]} + (\tau + 1)h/2$, where τ is a zero of

$$S_k(t) = \psi'_k(t) + \sum_{i=1}^{\infty}[\psi'_k(2i + t) + \psi'_k(-2i + t)], \quad t \in (-1, 1),$$

and where ψ_k is a function of second kind, associated with a polynomial of equally-distributed zeros, then this choice will ensure superconvergence. Consult the article for technical details and published values of τ.

Comments:

4.2.2 Gauss quadrature

Technique 4.2.2.1
Author and date: AM Korsunsky (1998)
Article: Gauss-Chebyshev quadrature formulae for strongly singular integrals
Journal: Quarterly of Applied Mathematics, 56, 461-472.
Equations in article: 45, 46, 47
Numerical technique: Gauss-Chebyshev quadrature

$$\frac{1}{\pi}\,=\!\!\!\!\!\!\int_{-1}^{1}\frac{\sqrt{1-t^2}\,f(t)}{(t-x_k)^2}\,dt \simeq \sum_{i=1}^{n}\left[\frac{1-t_{in}^2}{n+1}\,\frac{1}{(t_{in}-x_k)^2}+\frac{1-t_{in}^2}{t_{in}-x_k}\,\frac{(-1)^{i+k}}{\sqrt{1-x_k^2}}\right]f(t_{in}).$$

Algorithm:

Step 1: Choose n, the number of nodes in the Gauss-Chebyshev quadrature.

Step 2: Calculate the values x_k, the zeros of $T_{n+1}(t)$, $k = 1, \dots, n+1$, namely,

$$x_k = \cos\left[\tfrac{(2k-1)\pi}{2(n+1)}\right].$$

Step 3: Calculate the nodes, t_{in}, the zeros of the Chebyshev polynomial of the second kind, $U_n(t)$ given by

$$t_{in} = \cos\frac{i\pi}{n+1}, \quad i = 1, 2, \dots, n.$$

Step 4: Compute the approximated value of the integral using the quadrature formula

$$\frac{1}{\pi}\,=\!\!\!\!\!\!\int_{-1}^{1}\frac{\sqrt{1-t^2}\,f(t)}{(t-x_k)^2}\,dt \simeq \sum_{i=1}^{n}\left[\frac{1-t_{in}^2}{n+1}\,\frac{1}{(t_{in}-x_k)^2}+\frac{1-t_{in}^2}{t_{in}-x_k}\,\frac{(-1)^{i+k}}{\sqrt{1-x_k^2}}\right]f(t_{in}).$$

Comments:

Technique 4.2.2.2(a)
Author and date: CY Hui & D Shia (1999)
Article: Evaluations of hypersingular integrals using Gaussian quadrature
Journal: International Journal of Numerical Methods in Engineering, 44, 205-214.
Equations in article: 6, 8, 9, 18, 19
Numerical technique: Gauss-Legendre quadrature

$$\int_{-1}^{1} \frac{f(t)}{(t-x)^2}\,dt \simeq -\frac{2f'(x)Q_n(x)}{P_n(x)} - \frac{2f(x)(1-x^2)^{-1}}{P_n(x)^2} + \sum_{i=1}^{n} \frac{w_{in} f(t_{in})}{(t_{in}-x)^2}, \quad x \neq t_{in}.$$

Algorithm:

Step 1: Choose n, the number of nodes in the Gauss-Legendre quadrature.

Step 2: Find the Legendre polynomials of the first kind, P_n and P_{n+1}, and the Legendre function of the second kind Q_n.
Consult: Legendre polynomial and Legendre functions

Step 3: Calculate the zeros t_{in}, $i = 1, \ldots, n$ of P_n.
Consult: Legendre polynomial zeros

Step 4: Calculate the weights, w_{in},

$$w_{in} = -\frac{2}{(n+1)P_n'(t_{in})P_{n+1}(t_{in})}, \; i = 1, \ldots, n.$$

Step 5: Compute the approximated value of the integral using the quadrature formula

$$\int_{-1}^{1} \frac{f(t)}{(t-x)^2}\,dt \simeq -\frac{2f'(x)Q_n(x)}{P_n(x)} - \frac{2f(x)(1-x^2)^{-1}}{P_n(x)^2} + \sum_{i=1}^{n} \frac{w_{in} f(t_{in})}{(t_{in}-x)^2}, \quad x \neq t_{in}.$$

Comments:

Technique 4.2.2.2(b)
Author and date: CY Hui & D Shia (1999)
Article: Evaluations of hypersingular integrals using Gaussian quadrature
Journal: International Journal of Numerical Methods in Engineering, 44, 205-214.
Equations in article: 6, 8, 9, 18, 20
Numerical technique: Gauss-Chebyshev quadrature

$$\displaystyle{\neq}\!\!\!\!\int_{-1}^{1} \frac{\sqrt{1-t^2}f(t)}{(t-x)^2}\,dt \simeq -\frac{\pi f'(x)T_n(x)}{U_{n-1}(x)} - \frac{\pi f(x)[U_{n-1}(t)T_n'(t) - T_n(t)U_{n-1}'(t)]|_{t=x}}{[U_{n-1}(x)]^2} + \sum_{i=1}^{n-1} \frac{w_{in} f(t_{in})}{(t_{in}-x)^2}), \quad x \neq t_{in}.$$

Algorithm:

Step 1: Choose n, the number of nodes in the Gauss-Chebyshev quadrature.

Step 2: Find the Chebyshev polynomial of the first kind, T_n, and the Chebyshev polynomial of the second kind U_{n-1}.
Consult: Chebyshev polynomial 1 and Chebyshev polynomial 2

Step 3: Calculate the nodes, t_{in}, which are the zeros of U_{n-1} given by

$$t_{in} = \cos\frac{i\pi}{n}, \quad i = 1, 2, \ldots, n-1.$$

Step 4: Calculate the weights, w_{in},

$$w_{in} = \frac{\pi}{n}\sin^2\frac{i\pi}{n}, \quad i = 1, 2, \ldots, n-1.$$

Step 5: Compute the approximated value of the integral using the quadrature formula

$$\displaystyle{\neq}\!\!\!\!\int_{-1}^{1} \frac{\sqrt{1-t^2}f(t)}{(t-x)^2}\,dt \simeq -\frac{\pi f'(x)T_n(x)}{U_{n-1}(x)} - \frac{\pi f(x)[U_{n-1}(t)T_n'(t) - T_n(t)U_{n-1}'(t)]|_{t=x}}{[U_{n-1}(x)]^2} + \sum_{i=1}^{n-1} \frac{w_{in} f(t_{in})}{(t_{in}-x)^2}), \quad x \neq t_{in}.$$

Comments:

Technique 4.2.2.3
Author and date: P Kolm & V Rokhlin (2001)
Article: Numerical quadratures for singular and hypersingular integrals
Journal: Computers and mathematics with Applications, 41, 327-352.
Equations in article: 4, 51, 73, 77
Numerical technique: Gauss-Legendre quadrature

$$\begin{aligned}\neq\!\!\!\!\!\!\int_{-1}^{1} \frac{f(t)}{(t-x)^2}\,dt \;&\simeq\; \sum_{i=1}^{n} w_{in}\Bigg[-\sum_{j=0}^{n-2}\sum_{k=j}^{[(n+j-3)/2]}(2j+1)(4k+3-2i)Q_j(x)P_{2k+1-i}(t_{in})\\ &+\;\sum_{j=0}^{n-1}\frac{2j+1}{2}P_j(t_{in})\left(\frac{1}{x-1}-\frac{(-1)^j}{x+1}\right)\Bigg]f(t_{in})\end{aligned}$$

where $[(n+j-3)/2]$ denotes the integer part of $(n+j-3)/2$.

Algorithm:

Step 1: Choose n, the number of nodes in the Gauss-Legendre quadrature.

Step 2: Find the Legendre polynomials of the first kind, $P_0,\ldots,P_n$ and the Legendre functions of the second kind $Q_0,\ldots,Q_{n-1}$.
Consult: Legendre polynomial and Legendre functions

Step 3: Calculate the zeros t_{in}, $i=1,\ldots,n$ of P_n.
Consult: Legendre polynomial zeros

Step 4: Calculate the weights, w_{in},

$$w_{in} = \int_{-1}^{1}\prod_{j=1;j\neq i}^{n}\left(\frac{t-t_{jn}}{t_{in}-t_{jn}}\right)^2 dt, \quad i=1,\ldots,n.$$

Step 5: Compute the approximated value of the integral using the quadrature formula

$$\begin{aligned}\neq\!\!\!\!\!\!\int_{-1}^{1} \frac{f(t)}{(t-x)^2}\,dt \;&\simeq\; \sum_{i=1}^{n} w_{in}\Bigg[-\sum_{j=0}^{n-2}\sum_{k=j}^{[(n+j-3)/2]}(2j+1)(4k+3-2i)Q_j(x)P_{2k+1-i}(t_{in})\\ &+\;\sum_{j=0}^{n-1}\frac{2j+1}{2}P_j(t_{in})\left(\frac{1}{x-1}-\frac{(-1)^j}{x+1}\right)\Bigg]f(t_{in}),\end{aligned}$$

where $[(n+j-3)/2]$ denotes the integer part of $(n+j-3)/2$.

Comments:

Technique 4.2.2.4(a)
Author and date: PK Kythe & MR Schäferkotter (2005)
Book: Handbook of Computational Methods for Integration.
Chapter: Singular Integrals
Section in book: 5.5
Numerical technique: Gauss quadrature (general)

$$\begin{aligned}\int_a^b \frac{w(t)f(t)}{(t-x)^2}\,dt \;&\simeq\; \sum_{i=1}^{n} w_{in}\frac{f(t_{in})}{(t_{in}-x)^2} + f(x)\left[\int_a^b \frac{w(t)}{(t-x)^2}\,dt - \sum_{i=1}^{n}\frac{w_{in}}{(t_{in}-x)^2}\right]\\ &+f'(x)\left[\int_a^b \frac{w(t)}{t-x}\,dt - \sum_{i=1}^{n}\frac{w_{in}}{t_{in}-x}\right].\end{aligned}$$

Algorithm:

Step 1: Choose n, the number of nodes in the Gauss-Legendre quadrature.

Step 2: Find the Legendre polynomials of the first kind, P_n and P_{n+1}.
Consult: Legendre Polynomial

Step 3: Calculate the zeros t_{in}, $i = 1,\dots,n$ of P_n.
Consult: Legendre polynomial zeros

Step 4: Calculate the weights, w_{in},

$$w_{in} = \frac{2(1-t_{in}^2)}{(n+1)^2[P_{n+1}(t_{in})]^2}, \quad i = 1,\dots,n.$$

Step 5: Compute the approximated value of the integral using the Gauss-Legendre quadrature formula

$$\begin{aligned}\int_{-1}^1 \frac{f(t)}{(t-x)^2}\,dt \;&\simeq\; \sum_{i=1}^{n} w_{in}\frac{f(t_{in})}{(t_{in}-x)^2} + f(x)\left[\frac{2}{x^2-1} - \sum_{i=1}^{n}\frac{w_{in}}{(t_{in}-x)^2}\right]\\ &+f'(x)\left[\ln\left(\frac{1-x}{1+x}\right) - \sum_{i=1}^{n}\frac{w_{in}}{t_{in}-x}\right].\end{aligned}$$

Comments:

Technique 4.2.2.4(b)
Author and date: PK Kythe & MR Schäferkotter (2005)
Book: Handbook of Computational Methods for Integration.
Chapter: Singular Integrals
Section in book: 5.5
Numerical technique: Gauss quadrature (general)

$$\begin{aligned}\int_a^b \frac{w(t)f(t)}{(t-x)^2}\,dt \;\simeq\; & \sum_{i=1}^{n} w_{in}\frac{f(t_{in})}{(t_{in}-x)^2} + f(x)\left[\int_a^b \frac{w(t)}{(t-x)^2}\,dt - \sum_{i=1}^{n}\frac{w_{in}}{(t_{in}-x)^2}\right] \\ & + f'(x)\left[\int_a^b \frac{w(t)}{t-x}\,dt - \sum_{i=1}^{n}\frac{w_{in}}{t_{in}-x}\right].\end{aligned}$$

Algorithm:

Step 1: Choose n, the number of nodes in the Gauss-Chebyshev quadrature.

Step 2: Calculate the zeros t_{in}, $i = 1,\dots,n$ of $T_n(t)$, the Chebyshev polynomial of the first kind, given by

$$t_{in} = \cos\frac{(2i-1)\pi}{2n}, \quad i = 1, 2, \dots, n.$$

Step 3: Calculate the weights, w_{in}, given by

$$w_{in} = \frac{(1-t_{in}^2)\pi}{n}, \quad i = 1, \dots, n.$$

Step 4: Compute the approximated value of the integral using the Gauss-Chebyshev quadrature formula

$$\begin{aligned}\int_a^b \frac{\sqrt{1-t^2}\; f(t)}{(t-x)^2}\,dt \;\simeq\; & \sum_{i=1}^{n} w_{in}\frac{f(t_{in})}{(t_{in}-x)^2} - f(x)\left[\pi + \sum_{i=1}^{n}\frac{w_{in}}{(t_{in}-x)^2}\right] \\ & - f'(x)\left[\pi x + \sum_{i=1}^{n}\frac{w_{in}}{t_{in}-x}\right].\end{aligned}$$

Comments:

Technique 4.2.2.5(a)
Author and date: Y Shen (2006)
Article: Numerical quadratures for Hadamard hypersingular integrals
Journal: Numerical Mathematics, 15, 50-59.
Equations in article: 3.1, 3.2, 3.3
Numerical technique: Gauss-Legendre quadrature

$$\displaystyle\not{\int}_{-1}^{1} \frac{f(t)}{(t-x)^2}\,dt \simeq -2Q_0'(x)f(x) + \sum_{i=1}^{n} \tilde{w}_{in}(x)\frac{f(t_{in})-f(x)}{t_{in}-x}.$$

Algorithm:

Step 1: Choose n, the number of nodes in the Gauss-Legendre quadrature.

Step 2: Find the Legendre polynomials of the first kind, $P_0, \ldots, P_n$ and the Legendre functions of the second kind, $Q_0, \ldots, Q_{n-1}$.
Consult: Legendre polynomial and Legendre functions

Step 3: Calculate the zeros t_{in}, $i = 1, \ldots, n$ of P_n.
Consult: Legendre polynomial zeros

Step 4: Calculate the weights, w_{in},

$$w_{in} = \frac{2}{(1-t_{in}^2)[P_n'(t_{in})]^2}, \quad i = 1, \ldots, n.$$

Step 5: Calculate

$$\tilde{w}_{in}(x) = -w_{in}\sum_{j=0}^{n-1}(2j+1)P_j(t_{in})Q_j(x), \quad i = 1, \ldots, n,$$

or

$$\tilde{w}_{in}(x) = -w_{in}\sum_{j=0}^{n}(2j-1)P_{j-1}(t_{in})Q_{j-1}(x), \quad i = 1, \ldots, n.$$

Step 6: Compute the approximated value of the integral using the Gauss-Legendre quadrature formula

$$\displaystyle\not{\int}_{-1}^{1} \frac{f(t)}{(t-x)^2}\,dt \simeq -2Q_0'(x)f(x) + \sum_{i=1}^{n} \tilde{w}_{in}(x)\frac{f(t_{in})-f(x)}{t_{in}-x}.$$

Comments:

Technique 4.2.2.5(b)
Author and date: Y Shen (2006)
Article: Numerical quadratures for Hadamard hypersingular integrals
Journal: Numerical Mathematics, 15, 50-59.
Equations in article: 3.7, 3.8, 3.9, 3.10
Numerical technique: Gauss-Chebyshev quadrature (1)

$$\neq\!\!\!\!\!\!\int_{-1}^{1} \frac{f(t)\sqrt{1-t^2}}{(t-x)^2}\,dt \simeq -\pi f(x) + \sum_{i=1}^{n} \tilde{w}_{in}(x)\frac{f(t_{in})-f(x)}{t_{in}-x}.$$

Algorithm:

Step 1: Choose n, the number of nodes in the Gauss-Chebyshev quadrature.

Step 2: Find the Chebyshev polynomials of the first kind, $T_0, \ldots, T_n$ and the Chebyshev polynomials of the second kind, $U_0, \ldots, U_{n-1}$.
Consult: Chebyshev polynomial 1 and Chebyshev polynomial 2

Step 3: Calculate the zeros t_{in}, $i = 1, \ldots, n$ of T_n,

$$t_{in} = \cos\frac{i\pi}{n+1}.$$

Step 4: Calculate the weights, w_{in},

$$w_{in} = \frac{\pi}{n+1}\sin^2\frac{i\pi}{n+1}, \quad i = 1, \ldots, n.$$

Step 5: Calculate

$$\tilde{w}_{in}(x) = -2w_{in}\sum_{j=0}^{n-1} U_j(t_{in})T_{j+1}(x), \quad i = 1, \ldots, n.$$

Step 6: Compute the approximated value of the integral using the Gauss-Chebyshev quadrature formula

$$\neq\!\!\!\!\!\!\int_{-1}^{1} \frac{f(t)\sqrt{1-t^2}}{(t-x)^2}\,dt \simeq -\pi f(x) + \sum_{i=1}^{n} \tilde{w}_{in}(x)\frac{f(t_{in})-f(x)}{t_{in}-x}.$$

Comments:

Technique 4.2.2.5(c)
Author and date: Y Shen (2006)
Article: Numerical quadratures for Hadamard hypersingular integrals
Journal: Numerical Mathematics, 15, 50-59.
Equations in article: 3.16, 3.17, 3.18, 3.19
Numerical technique: Gauss-Chebyshev quadrature (2)

$$\oint_{-1}^{1} \frac{f(t)}{(t-x)^2} \frac{1}{\sqrt{1-t^2}}\, dt \simeq \sum_{i=1}^{n} \tilde{w}_{in}(x) \frac{f(t_{in}) - f(x)}{t_{in} - x}.$$

Algorithm:

Step 1: Choose n, the number of nodes in the Gauss-Chebyshev quadrature.

Step 2: Find the Chebyshev polynomials of the first kind, $T_0, \ldots, T_{n-1}$ and the Chebyshev polynomials of the second kind, $U_0, \ldots, U_{n-2}$.
Consult: Chebyshev polynomial 1 and Chebyshev polynomial 2

Step 3: Calculate the nodes t_{in}, $i = 1, \ldots, n$,

$$t_{in} = \cos \frac{(2i-1)\pi}{2n}.$$

Step 4: Calculate the weights, w_{in},

$$w_{1n} = w_{2n} = \ldots = w_{nn} = \frac{\pi}{n}.$$

Step 5: Calculate

$$\tilde{w}_{in}(x) = 2w_{in} \sum_{j=1}^{n-1} T_j(t_{in}) U_{j-1}(x), \quad i = 1, \ldots, n.$$

Step 6: Compute the approximated value of the integral using the Gauss-Chebyshev quadrature formula

$$\oint_{-1}^{1} \frac{f(t)}{(t-x)^2} \frac{1}{\sqrt{1-t^2}}\, dt \simeq \sum_{i=1}^{n} \tilde{w}_{in}(x) \frac{f(t_{in}) - f(x)}{t_{in} - x}.$$

Comments:

4.3 Generalizations

4.3.1 Newton-Cotes quadrature

Technique 4.3.1.1
Author and date: J Wu & W Sun (2005)
Article: The superconvergence of the composite trapezoidal rule for the Hadamard finite-part integrals.
Journal: Numerische Mathematik, 102, 343-363.
Equations in article: 1.1, 1.4, 2.2, 2.3
Numerical technique: Newton-Cotes quadrature

$$\oint_a^b \frac{f(t)}{(t-x)^3}\,dt \simeq \sum_{i=0}^{n} w_i(x) f(t_i).$$

Algorithm:

Step 1: Let $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ be a uniform partition of the interval $[a, b]$ with $h = t_i - t_{i-1}$, $i = 1, \ldots, n$, and with $n-1$ the number of internal nodes, to be determined in step 2.

Step 2: Call the singular point x the node t_m. Choose both m and n integers and in such a way that

$$n = \Big(\frac{b-a}{x-a}\Big)(m + \tfrac{1}{2}).$$

This will ensure superconvergence; otherwise make a choice so that $x \simeq t_{mn}$.

Step 3: Calculate the weights $w_i(x)$, $i = 0, 1, \ldots, n$,

$$w_i(x) = \frac{1}{2}\Big[\frac{\delta_{i0}}{(t_0-x)^2} - \frac{\delta_{in}}{(t_n-x)^2} + \frac{1-\delta_{i0}}{(t_{i-1}-x)(t_i-x)} - \frac{1-\delta_{in}}{(t_i-x)(t_{i+1}-x)}\Big],$$

with δ_{ij} the Kronecker-delta.

Step 4: Compute the approximated value of the integral using the Newton-Cotes quadrature formula

$$\oint_a^b \frac{f(t)}{(t-x)^3}\,dt \simeq \sum_{i=0}^{n} w_i(x) f(t_i).$$

Comments:

4.3.2 Gauss quadrature

Technique 4.3.2.1(a)
Author and date: PK Kythe & MR Schäferkotter (2005)
Book: Handbook of Computational Methods for Integration.
Chapter: Singular Integrals
Section in book: 5.5
Numerical technique: Gauss-Legendre quadrature

$$\begin{aligned}\int_{-1}^{1} \frac{f(t)}{(t-x)^{p+1}}\,dt &\simeq \sum_{i=1}^{n} w_{in}\frac{f(t_{in})}{(t_{in}-x)^{p+1}} \\ &+ \sum_{j=0}^{p}\frac{f^{(j)}(x)}{j!}\left[\begin{Bmatrix}\frac{1}{p-j}\left[\frac{(-1)^{p-j}}{(1+x)^{p-j}}-\frac{1}{(1-x)^{p-j}}\right], & j<p \\ \ln\left(\frac{1-x}{1+x}\right), & j=p\end{Bmatrix} - \sum_{i=1}^{n}\frac{w_{in}}{(t_{in}-x)^{p+1-j}}\right].\end{aligned}$$

Algorithm:

Step 1: Choose n, the number of nodes in the Gauss-Legendre quadrature.

Step 2: Find the Legendre polynomials of the first kind, P_n and P_{n+1}.
Consult: Legendre Polynomial

Step 3: Calculate the zeros t_{in}, $i = 1, \ldots, n$ of P_n.
Consult: Legendre Polynomial zeros

Step 4: Calculate the weights, w_{in},

$$w_{in} = \frac{2(1-t_{in}^2)}{(n+1)^2[P_{n+1}(t_{in})]^2}, \quad i = 1, \ldots, n.$$

Step 5: Compute the approximated value of the integral using the Gauss-Legendre quadrature formula

$$\begin{aligned}\int_{-1}^{1} \frac{f(t)}{(t-x)^{p+1}}\,dt &\simeq \sum_{i=1}^{n} w_{in}\frac{f(t_{in})}{(t_{in}-x)^{p+1}} \\ &+ \sum_{j=0}^{p}\frac{f^{(j)}(x)}{j!}\left[\begin{Bmatrix}\frac{1}{p-j}\left[\frac{(-1)^{p-j}}{(1+x)^{p-j}}-\frac{1}{(1-x)^{p-j}}\right], & j<p \\ \ln\left(\frac{1-x}{1+x}\right), & j=p\end{Bmatrix} - \sum_{i=1}^{n}\frac{w_{in}}{(t_{in}-x)^{p+1-j}}\right].\end{aligned}$$

Comments:

Technique 4.3.2.1(b)
Author and date: PK Kythe & MR Schäferkotter (2005)
Book: Handbook of Computational Methods for Integration.
Chapter: Singular Integrals
Section in book: 5.5
Numerical technique: Gauss-Chebyshev quadrature

$$\begin{aligned}\displaystyle ⨍_{-1}^{1} \frac{\sqrt{1-t^2}f(t)}{(t-x)^{p+1}}\,dt \;&\simeq\; \sum_{i=1}^{n} w_{in}\frac{f(t_{in})}{(t_{in}-x)^{p+1}} \\ &+\; \sum_{j=0}^{p}\frac{f^{(j)}(x)}{j!}\left[\left\{\begin{array}{cc} 0, & j\le p-2 \\ -\pi, & j=p-1 \\ -\pi x, & j=p \end{array}\right\} - \sum_{i=1}^{n}\frac{w_{in}}{(t_{in}-x)^{p+1-j}}\right].\end{aligned}$$

Algorithm:

Step 1: Choose n, the number of nodes in the Gauss-Chebyshev quadrature.

Step 2: Calculate the zeros t_{in}, $i = 1, \dots, n$ of $T_n(t)$, the Chebyshev polynomial of the first kind, given by

$$t_{in} = \cos\frac{(2i-1)\pi}{2n}, \quad i = 1, 2, \dots, n.$$

Step 3: Calculate the weights, w_{in}, given by

$$w_{in} = \frac{(1-t_{in}^2)\pi}{n}, \quad i = 1, \dots, n.$$

Step 4: Compute the approximated value of the integral using the Gauss-Chebyshev quadrature formula

$$\begin{aligned}\displaystyle ⨍_{-1}^{1} \frac{\sqrt{1-t^2}f(t)}{(t-x)^{p+1}}\,dt \;&\simeq\; \sum_{i=1}^{n} w_{in}\frac{f(t_{in})}{(t_{in}-x)^{p+1}} \\ &+\; \sum_{j=0}^{p}\frac{f^{(j)}(x)}{j!}\left[\left\{\begin{array}{cc} 0, & j\le p-2 \\ -\pi, & j=p-1 \\ -\pi x, & j=p \end{array}\right\} - \sum_{i=1}^{n}\frac{w_{in}}{(t_{in}-x)^{p+1-j}}\right].\end{aligned}$$

Comments:

Chapter 5

Bibliography

Ashour, S.A. & Ahmed, H.M. 2007. A Gauss quadrature rule for hypersingular integrals. *Applied Mathematics and Computation*, 186, 1671-1682.

Davis, P.J. & Rabinowitz, P. 2007(1984). Methods of numerical integration. Dover.

Hui, C-Y. & Shia, D. 1999. Evaluations of hypersingular integrals using Gaussian quadrature. *International Journal for Numerical methods in engineering*, 44, 205-214.

Ioakimidis, N.I. 1986. On the Gaussian quadrature rule for finite-part integrals with a first-order singularity. *Communications in Applied Numerical methods*, 2, 123-132.

Ioakimidis, N.I. & Pitta, M.S. 1988. Remarks on the Gaussian quadrature rule for finite-part integrals with a second-order singularity. *Computer methods in Applied Mechanics and engineering*, 69, 325-343.

Ioakimidis, N.I. 1993. The Gauss-Laguerre quadrature rule for finite-part integrals. *Communications in Numerical methods in engineering*, 9, 439-450.

Kaya, A.C.& Erdogan, F. 1987. On the solution of integral equations with strongly singular kernels. *Quarterly of Applied Mathematics*, 45, 105-122.

Kolm, P. & Rokhlin, V. 2001. Numerical quadratures for singular and hypersingular integrals. *Computers and Mathematics with Applications*, 41, 327-352.

Korsunsky, A.M. 1997. Gauss-Jacobi quadratures for hypersingular integrals. Proceedings of the 1st UK Conference on boundary integral methods, 164-177. Editors L. Elliott, D.B. Ingham, D. Lesnic.

Korsunsky, A.M. 1998. Gauss-Chebyshev quadrature formulae for strongly singular integrals. *Quarterly of Applied Mathematics*, 56, 461-472.

Kythe, P.K. & Schäferkotter, M.R. 2005. Handbook of Computational methods for Integration. Chapman & Hall.

Linz, P. 1985. On the approximate computation of certain strongly singular integrals. *Computing*, 35, 345-353.

Monegato, G. 1987. On the weights of certain quadratures for the numerical evaluation of Cauchy principal value integrals and their derivatives. *Numerische Mathematik*, 50, 273-281.

Monegato, G. 1994. Numerical evaluation of hypersingular integrals. *Journal of Computational and Applied Mathematics*, 50, 9-31.

Paget, D.F. 1981. The numerical evaluation of Hadamard finite-part integrals. *Numerische Mathematik*, 36, 447-453.

Ramm, A.G. & of der Sluis, A. 1990. Calculating singular integrals as an ill-posed problem. *Numerische Mathematik*, 57, 139-145.

Shand, Y. 2006. Numerical quadratures for Hadamard hypersingular integrals. *Numerical mathematics*, 15, 50-59.

Sun, W. & Wu, J. 2005. Newton-Cotes formulae for the numerical evaluation of certain hypersingular integrals. *Computing*, 75, 297-309.

Tsamasphyros, G. & Dimou, G. 1990. Gauss quadrature rules for finite-part integrals. *International Journal for Numerical methods in engineering*, 30, 13-26.

Wu, J. & Sun, W. 2005. The superconvergence of the composite trapezoidal rule for Hadamard finite-part integrals. *Numerische Mathematik*, 102, 343-363.

Wu, J. & Sun, W. 2008. The superconvergence of Newton-Cotes rules for the Hadamard finite-part integral on an interval. *Numerische Mathematik*, 109, 143-165.