Fourier transform and distributions

with applications to the Schrödinger operator

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Lecture Notes

2nd Edition

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Contents

1 Introduction

This text assumes that the reader is familiar with the following concepts:

1) Metric spaces and their completeness.

2) Lebesgue integral in a bounded domain $\Omega \subset \mathbb{R}^n$ and in $\mathbb{R}^n$.

3) Banach spaces (L^p, $1 \le p \le \infty$, C^k) and Hilbert spaces (L^2): If $1 \le p < \infty$ then we set

$$L^p(\Omega) := \{f : \Omega \to \mathbb{C} \, \text{measurable} : \|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p dx\right)^{1/p} < \infty\}$$

while

$$L^\infty(\Omega) := \{f : \Omega \to \mathbb{C} \, \text{measurable} : \|f\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| < \infty\}.$$

Moreover

$$C^k(\overline{\Omega}) \quad := \quad \{f : \overline{\Omega} \to \mathbb{C} : \|f\|_{C^k(\overline{\Omega})} := \max_{x\in\overline{\Omega}} \sum_{|\alpha|\le k} |\partial^\alpha f(x)| < \infty\},$$

where $\overline{\Omega}$ is the closure of Ω. We say that $f \in C^\infty(\Omega)$ if $f \in C^k(\overline{\Omega_1})$ for all $k \in \mathbb{N}$ and for all bounded subsets $\Omega_1 \subset \Omega$. The space $C^\infty(\Omega)$ is not a normed space. The inner product in $L^2(\Omega)$ is denoted by

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x)\overline{g(x)}dx.$$

Also in $L^2(\Omega)$, the duality pairing is given by

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx.$$

4) Hölder's inequality: Let $1 \le p \le \infty, u \in L^p$ and $v \in L^{p'}$ with

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Then $uv \in L^1$ and

$$\int_\Omega |u(x)v(x)|dx \le \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}} \left(\int_\Omega |v(x)|^{p'} dx\right)^{\frac{1}{p'}},$$

where the Hölder conjugate exponent p' of p is obtained via

$$p' = \frac{p}{p-1}$$

with the understanding that $p' = \infty$ if $p = 1$ and $p' = 1$ if $p = \infty$.

5) Lebesgue's theorem about dominated convergence:

Let $A \subset \mathbb{R}^n$ be measurable and let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions converging to $f(x)$ point-wise in A. If there exists function $g \in L^1(A)$ such that $|f_k(x)| \leq g(x)$ in A, then $f \in L^1(A)$ and

$$\lim_{k\to\infty} \int_A f_k(x)dx = \int_A f(x)dx.$$

6) Fubini's theorem about the interchange of the order of integration:

$$\int_{X\times Y} |f(x,y)|dxdy = \int_X dx \left(\int_Y |f(x,y)|dy \right) = \int_Y dy \left(\int_X |f(x,y)|dx \right),$$

if one of the three integrals exists.

2 Fourier transform in Schwartz space

Consider the Euclidean space $\mathbb{R}^n, n \geq 1$ with $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and with $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ and scalar product $(x, y) = \sum_{j=1}^n x_j y_j$. The open ball of radius $\delta > 0$ centered at $x \in \mathbb{R}^n$ is denoted by

$$U_\delta(x) := \{y \in \mathbb{R}^n : |x - y| < \delta\}.$$

Recall the Cauchy-Bunjakovsky inequality

$$|(x, y)| \leq |x||y|.$$

Following L. Schwartz we call an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n), \alpha_j \in \mathbb{N} \cup \{0\} \equiv \mathbb{N}_0$ an n-dimensional multi-index. Denote

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!$$

and

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad 0^0 = 1, \quad 0! = 1.$$

Moreover, multi-indices α and β can be ordered according to

$$\alpha \leq \beta$$

if and only if $\alpha_j \leq \beta_j$ for all $j = 1, 2, \ldots, n$. Let us also introduce a shorthand notation

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \quad \partial_j = \frac{\partial}{\partial x_j}.$$

Definition. The Schwartz space $S(\mathbb{R}^n)$ of rapidly decaying functions is defined as

$$S(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : |f|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^n} \left|x^\alpha \partial^\beta f(x)\right| < \infty \text{ for any } \alpha, \beta \in \mathbb{N}_0^n\}.$$

The following properties of $S = S(\mathbb{R}^n)$ are readily verified.

1) S is a linear space.

2) $\partial^\alpha : S \to S$ for any $\alpha \geq 0$.

3) $x^\beta \cdot : S \to S$ for any $\beta \geq 0$.

4) If $f \in S(\mathbb{R}^n)$ then $|f(x)| \leq c_m(1 + |x|)^{-m}$ for any $m \in \mathbb{N}$. The converse is not true (see part 3) of Example 2.1).

5) It follows from part 4) that $S(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for any $1 \leq p \leq \infty$.

Example 2.1. 1) $f(x) = e^{-a|x|^2} \in S$ for any $a > 0$.

2) $f(x) = e^{-a(1+|x|^2)^a} \in S$ for any $a > 0$.

3) $f(x) = e^{-|x|} \notin S$.

4) $C_0^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n)$, where $C_0^\infty(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \operatorname{supp} f \text{ is compact in } \mathbb{R}^n\}$ and $\operatorname{supp} f = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}$.

The space $S(\mathbb{R}^n)$ is not a normed space because $|f|_{\alpha,\beta}$ is only a seminorm for $\alpha \geq 0$ and $\beta > 0$ i.e. the condition

$$|f|_{\alpha,\beta} = 0 \quad \text{if and only if} \quad f = 0$$

fails to hold for e.g. constant function f. But the space (S, ρ) is a metric space if the metric ρ is defined by

$$\rho(f, g) = \sum_{\alpha,\beta \geq 0} 2^{-|\alpha|-|\beta|} \cdot \frac{|f-g|_{\alpha,\beta}}{1+|f-g|_{\alpha,\beta}}.$$

Exercise 1. Prove that ρ is a metric, that is,

1) $\rho(f, g) \geq 0$ and $\rho(f, g) = 0$ if and only if $f = g$.

2) $\rho(f, g) = \rho(g, f)$.

3) $\rho(g, h) \leq \rho(g, f) + \rho(f, h)$.

4) (In addition) $|\rho(f, h) - \rho(g, h)| \leq \rho(f, g)$.

Theorem 1 (Completeness)**.** *The space (S, ρ) is a complete metric space i.e. every Cauchy sequence converges.*

Proof. Let $\{f_k\}_{k=1}^\infty$, $f_k \in S$, be a Cauchy sequence, that is, for any $\varepsilon > 0$ there exists $n_0(\varepsilon) \in \mathbb{N}$ such that

$$\rho(f_k, f_m) < \varepsilon, \quad k, m \geq n_0(\varepsilon).$$

It follows that

$$\sup_{x \in K} \left|\partial^\beta (f_k - f_m)\right| < \varepsilon$$

for any $\beta \geq 0$ and for any compact set K in $\mathbb{R}^n$. It means that $\{f_k\}_{k=1}^\infty$ is a Cauchy sequence in the Banach space $C^{|\beta|}(K)$. Hence there exists a function $f \in C^{|\beta|}(K)$ such that

$$\lim_{k\to\infty} f_k \overset{C^{|\beta|}(K)}{=} f.$$

That's why we may conclude that our function $f \in C^\infty(\mathbb{R}^n)$. It only remains to prove that $f \in S$. It is clear that

$$\begin{aligned} \sup_{x\in K} |x^\alpha \partial^\beta f| &\leq \sup_{x\in K} |x^\alpha \partial^\beta (f_k - f)| + \sup_{x\in K} |x^\alpha \partial^\beta f_k| \\ &\leq C_\alpha(K) \sup_{x\in K} |\partial^\beta (f_k - f)| + \sup_{x\in K} |x^\alpha \partial^\beta f_k|. \end{aligned}$$

Taking $k \to \infty$ we obtain

$$\sup_{x\in K} |x^\alpha \partial^\beta f| \quad \leq \quad \limsup_{k\to\infty} |f_k\,|_{\alpha,\beta} < \infty.$$

The last inequality is valid since $\{f_k\}_{k=1}^\infty$ is a Cauchy sequence, so that $|f_k|_{\alpha,\beta}$ is bounded. The last inequality doesn't depend on K either. That's why we may conclude that $|f|_{\alpha,\beta} < \infty$ or $f \in S$. □

Definition. We say that $f_k \xrightarrow{S} f$ as $k \to \infty$ if and only if

$$|f_k - f\,|_{\alpha,\beta} \to 0, \quad k \to \infty$$

for all $\alpha, \beta \geq 0$.

Exercise 2. Prove that $\overline{C_0^\infty(\mathbb{R}^n)} = S$, that is, for any $f \in S$ there exists $\{f_k\}_{k=1}^\infty$, $f_k \in C_0^\infty(\mathbb{R}^n)$, such that $f_k \xrightarrow{S} f, k \to \infty$.

Now we are in the position to define the Fourier transform in $S(\mathbb{R}^n)$.

Definition. The Fourier transform $Ff(\xi)$ or $\widehat{f}(\xi)$ of the function $f(x) \in S$ is defined by

$$Ff(\xi) \equiv \widehat{f}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i(x,\xi)} f(x)dx, \quad \xi \in \mathbb{R}^n.$$

Remark. This integral is well-defined since

$$\left|\widehat{f}(\xi)\right| \leq c_m (2\pi)^{-n/2} \int_{\mathbb{R}^n} (1 + |x|)^{-m} dx < \infty,$$

for $m > n$.

Next we prove the following properties of the Fourier transform:

1) F is a linear continuous map from S into S.

2) $\xi^\alpha \partial_\xi^\beta \widehat{f}(\xi) = (-i)^{|\alpha|+|\beta|} \widehat{\partial_x^\alpha (x^\beta f(x))}$.

Indeed, we have

$$\partial_\xi^\beta \widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} (-ix)^\beta e^{-i(x,\xi)} f(x)dx$$

and hence

$$\left\|\partial_\xi^\beta \widehat{f}(\xi)\right\|_{L^\infty(\mathbb{R}^n)} \leq c_m (2\pi)^{-n/2} \int_{\mathbb{R}^n} \frac{|x|^{|\beta|}}{(1 + |x|)^m} dx < \infty$$

if we choose $m > n + |\beta|$. At the same time we obtained the formula

$$\partial_\xi^\beta \widehat{f}(\xi) = \widehat{(-ix)^\beta f(x)}. \tag{2.1}$$

Further, integration by parts gives us

$$\xi^\alpha \widehat{f}(\xi) = (-i)^{|\alpha|} (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i(x,\xi)} \partial_x^\alpha f(x)dx.$$

That's why we have the estimate

$$\left\|\xi^\alpha \widehat{f}\right\|_{L^\infty(\mathbb{R}^n)} \le c\int_{\mathbb{R}^n} |\partial_x^\alpha f(x)|\,dx < \infty$$

since $\partial_x^\alpha f(x) \in S$ for any $\alpha \ge 0$, if $f(x) \in S$. And also we have the formula

$$\xi^\alpha \widehat{f} = \widehat{(-i)^{|\alpha|}\partial_x^\alpha f}. \tag{2.2}$$

If we combine these two last estimates we may conclude that $F : S \to S$ and F is a continuous map since F maps every bounded set from S to a bounded set from S again.

The formulas (2.1) and (2.2) show us that it is more convenient to use the following notation:

$$D_j = -i\partial_j = -i\frac{\partial}{\partial x_j}, \quad D^\alpha = D_1^{\alpha_1}\cdots D_n^{\alpha_n}.$$

For this new derivative the formulas (2.1) and (2.2) can be rewritten as

$$D_\xi^\alpha \widehat{f} = (-1)^{|\alpha|}\widehat{x^\alpha f}, \quad \xi^\alpha \widehat{f} = \widehat{D^\alpha f}.$$

Example 2.2. It is true that

$$F(e^{-\frac{1}{2}|x|^2})(\xi) = e^{-\frac{1}{2}|\xi|^2}.$$

Proof. The definition gives us directly

$$\begin{aligned} F(e^{-\frac{1}{2}|x|^2})(\xi) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i(x,\xi)-\frac{1}{2}|x|^2}dx \\ &= (2\pi)^{-n/2}e^{-\frac{1}{2}|\xi|^2}\int_{\mathbb{R}^n} e^{-\frac{1}{2}(|x|^2+2i(x,\xi)-|\xi|^2)}dx \\ &= (2\pi)^{-n/2}e^{-\frac{1}{2}|\xi|^2}\prod_{j=1}^n\int_{-\infty}^{\infty} e^{-\frac{1}{2}(t+i\xi_j)^2}dt. \end{aligned}$$

In order to calculate the last integral we consider the function $f(z) = e^{-\frac{z^2}{2}}$ of the complex variable z and the domain D_R depicted in Figure 1. We consider the positive direction of going around the boundary ∂D_R. It is clear that $f(z)$ is a holomorphic function in this domain and due to Cauchy theorem we have

$$\oint_{\partial D_R} e^{-\frac{z^2}{2}}dz = 0.$$

But

$$\oint_{\partial D_R} e^{-\frac{z^2}{2}}dz = \int_{-R}^R e^{-\frac{t^2}{2}}dt + i\int_0^{\xi_j} e^{-\frac{1}{2}(R+i\tau)^2}d\tau + \int_R^{-R} e^{-\frac{1}{2}(t+i\xi_j)^2}dt + i\int_{\xi_j}^0 e^{-\frac{1}{2}(-R+i\tau)^2}d\tau.$$

If $R \to \infty$ then

$$\int_0^{\xi_j} e^{-\frac{1}{2}(\pm R+i\tau)^2}d\tau \to 0.$$

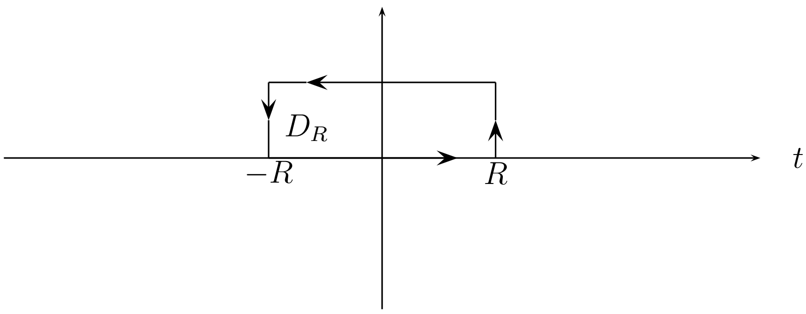

Figure 1: Domain D_R.

Hence

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(t+i\xi_j)^2} dt = \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt, \quad j = 1, \dots, n.$$

Using Fubini's theorem and polar coordinates we can evaluate the last integral as

$$\begin{aligned}\left(\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt\right)^2 &= \int_{\mathbb{R}^2} e^{-\frac{1}{2}(t^2+s^2)} dt ds = \int_0^{2\pi} d\theta \int_0^{\infty} e^{-\frac{r^2}{2}} r dr \\ &= 2\pi \int_0^{\infty} e^{-m} dm = 2\pi.\end{aligned}$$

Thus

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(t+i\xi_j)^2} dt = \sqrt{2\pi}$$

and

$$F(e^{-\frac{|x|^2}{2}})(\xi) = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}|\xi|^2} \prod_{j=1}^{n} \sqrt{2\pi} = e^{-\frac{1}{2}|\xi|^2}.$$

□

Exercise 3. Let $P(D)$ be a differential operator

$$P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha$$

with constant coefficients. Prove that $\widehat{P(D)u} = P(\xi)\widehat{u}$.

Definition. Adopt the following notation for translation and dilation of a function

$$(\tau_h f)(x) := f(x-h), \quad (\sigma_\lambda f)(x) := f(\lambda x), \quad \lambda \neq 0.$$

Exercise 4. Let $f \in S(\mathbb{R}^n), h \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}, \lambda \neq 0$. Prove that

1) $\widehat{\sigma_\lambda f}(\xi) = |\lambda|^{-n} \sigma_{\frac{1}{\lambda}} \widehat{f}(\xi)$ and $\sigma_\lambda \widehat{f}(\xi) = |\lambda|^{-n} \widehat{\sigma_{\frac{1}{\lambda}} f}(\xi)$

2) $\widehat{\tau_h f}(\xi) = e^{-i(h,\xi)}\widehat{f}(\xi)$ and $\tau_h\widehat{f}(\xi) = \widehat{e^{i(h,\cdot)}f}(\xi)$.

Exercise 5. Let A be a real-valued $n\times n$-matrix such that A^{-1} exists. Denote $f_A(x) := f(A^{-1}x)$. Prove that

$$\widehat{f_A}(\xi) = (\widehat{f})_A(\xi)$$

if and only if A is an orthogonal matrix (a rotation), that is, $A^T = A^{-1}$.

Let us now consider f and g from $S(\mathbb{R}^n)$. Then

$$\begin{aligned}(Ff,g)_{L^2} &= \int_{\mathbb{R}^n}\widehat{f}(\xi)\overline{g(\xi)}d\xi = (2\pi)^{-n/2}\int_{\mathbb{R}^n}\overline{g(\xi)}\left(\int_{\mathbb{R}^n}e^{-i(x,\xi)}f(x)dx\right)d\xi \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n}f(x)\left(\int_{\mathbb{R}^n}\overline{e^{i(x,\xi)}g(\xi)}d\xi\right)dx = (f,F^*g)_{L^2},\end{aligned}$$

where $F^*g(x) := Fg(-x)$.

Remark. Here F^* is the adjoint operator (in the sense of L^2) which maps S into S since $F: S\to S$. The inverse Fourier transform F^{-1} is defined as: $F^{-1} := F^*$.

In order to justify this definition we will prove the following theorem.

Theorem 2 (Fourier inversion formula)**.** *Let f be a function from $S(\mathbb{R}^n)$. Then*

$$F^*Ff = f.$$

To this end we will prove first the following (somewhat technical) lemma.

Lemma 1. *Let $f_0(x)$ be a function from $L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} f_0(x)dx = 1$ and let $f(x)$ be a function from $L^\infty(\mathbb{R}^n)$ which is continuous at $\{0\}$. Then*

$$\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n}\varepsilon^{-n}f_0\left(\frac{x}{\varepsilon}\right)f(x)dx = f(0).$$

Proof. Since

$$\int_{\mathbb{R}^n}\varepsilon^{-n}f_0\left(\frac{x}{\varepsilon}\right)f(x)dx - f(0) = \int_{\mathbb{R}^n}\varepsilon^{-n}f_0\left(\frac{x}{\varepsilon}\right)(f(x)-f(0))dx,$$

then we may assume without loss of generality that $f(0) = 0$. Since f is continuous at $\{0\}$ then for any $\eta > 0$ there exists $\delta > 0$ such that

$$|f(x)| < \frac{\eta}{\|f_0\|_{L^1}},$$

whenever $|x| < \delta$. Note that

$$\left|\int_{\mathbb{R}^n}f_0dx\right| \le \|f_0\|_{L^1}.$$

That's why we may conclude that

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\varepsilon^{-n}f_0\left(\frac{x}{\varepsilon}\right)f(x)dx\right| &\le \frac{\eta}{\|f_0\|_{L^1}}\cdot\varepsilon^{-n}\int_{|x|<\delta}\left|f_0\left(\frac{x}{\varepsilon}\right)\right|dx\\ &+ \|f\|_{L^\infty}\varepsilon^{-n}\int_{|x|>\delta}\left|f_0\left(\frac{x}{\varepsilon}\right)\right|dx\\ &\le \frac{\eta}{\|f_0\|_{L^1}}\cdot\|f_0\|_{L^1}+\|f\|_{L^\infty}\int_{|y|>\frac{\delta}{\varepsilon}}|f_0(y)|\,dy=\eta+\|f\|_{L^\infty}I_\varepsilon.\end{aligned}$$

But $I_\varepsilon \to 0$ as $\varepsilon \downarrow 0$. This proves the lemma. □

Proof of theorem 2. Let us consider $v(x) = e^{-\frac{|x|^2}{2}}$. We know from Example 2.2 that $\int_{\mathbb{R}^n} v(x)dx = (2\pi)^{n/2}$ and $Fv = v$. If we apply Lemma 1 with $f_0 = (2\pi)^{-n/2}v(x)$ and $f \in S(\mathbb{R}^n)$ then

$$\begin{aligned}(2\pi)^{n/2}f(0) &= \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n}\varepsilon^{-n}v\left(\frac{x}{\varepsilon}\right)f(x)dx=\lim_{\varepsilon\downarrow 0}\langle f,\varepsilon^{-n}\sigma_{\frac{1}{\varepsilon}}v\rangle_{L^2}=\lim_{\varepsilon\downarrow 0}\langle f,\varepsilon^{-n}\sigma_{\frac{1}{\varepsilon}}Fv\rangle_{L^2}\\ &\overset{\text{Ex.4}}{=} \lim_{\varepsilon\downarrow 0}\langle f,F(\sigma_\varepsilon v)\rangle_{L^2}=\lim_{\varepsilon\downarrow 0}\langle Ff,\sigma_\varepsilon v\rangle_{L^2}=\langle Ff,1\rangle=\int_{\mathbb{R}^n}Ff(\xi)e^{i(0,\xi)}d\xi\end{aligned}$$

after using the Lebesgue theorem of dominated convergence in the last step. This proves that

$$f(0) = (2\pi)^{-n/2}\int_{\mathbb{R}^n}Ff(\xi)e^{i(0,\xi)}d\xi = (F^*Ff)(0).$$

The proof is now finished by

$$\begin{aligned}f(x) &= (\tau_{-x}f)(0)=(F^*F(\tau_{-x}f))(0)=(2\pi)^{-n/2}\int_{\mathbb{R}^n}F(\tau_{-x}f)(\xi)e^{i(0,\xi)}d\xi\\ &\overset{\text{Ex.4}}{=}(2\pi)^{-n/2}\int_{\mathbb{R}^n}e^{i(x,\xi)}Ff(\xi)d\xi=F^*Ff(x).\end{aligned}$$

□

Corollary. *The Fourier transform is an isometry (in the sense of L^2).*

Proof. The fact that Fourier transform retains norm of $f \in S$ follows from the following *Parseval equality*

$$\|Ff\|_{L^2}^2=(Ff,Ff)_{L^2}=(f,F^*Ff)_{L^2}=(f,f)_{L^2}=\|f\|_{L^2}^2.$$

□

Note that

$$(Ff,g)_{L^2}=(f,F^*g)_{L^2}$$

means that

$$\int_{\mathbb{R}^n}\widehat{f}(\xi)\overline{g(\xi)}d\xi=\int_{\mathbb{R}^n}f(x)\overline{F^*g(x)}dx=\int_{\mathbb{R}^n}f(x)F(\overline{g})(x)dx.$$

It implies that

$$\int_{\mathbb{R}^n}\widehat{f}(\xi)g(\xi)d\xi=\int_{\mathbb{R}^n}f(x)\widehat{g}(x)dx$$

or

$$\langle Ff,g\rangle_{L^2}=\langle f,Fg\rangle_{L^2}.$$

3 Fourier transform in $L^p(\mathbb{R}^n), 1 \le p \le 2$

Let us start with a preliminary proposition.

Proposition. *Let X be a linear normed space and $E \subset X$ a subspace of X such that $\overline{E} = X$, that is, the closure $\overline{E}$ of E in the sense of the norm in X is equal to X. Let Y be a Banach space. If $T : E \to Y$ is continuous linear map, i.e. there exists $M > 0$ such that*

$$\|Tu\|_Y \le M \|u\|_X, \quad u \in E,$$

then there exists a unique linear continuous map $T_{ex} : X \to Y$ such that $T_{ex}|_E = T$ and

$$\|T_{ex}u\|_Y \le M \|u\|_X, \quad u \in X.$$

Exercise 6. Prove the previous proposition.

Lemma 1. *Let $1 \le p < \infty$. Then*

$$\overline{C_0^\infty(\mathbb{R}^n)} \stackrel{L^p}{=} L^p(\mathbb{R}^n),$$

that is, $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ in the sense of L^p-norm.

Proof. We will use the fact that the set of finite linear combinations of characteristic functions of bounded measurable sets in $\mathbb{R}^n$ is dense in $L^p(\mathbb{R}^n), 1 \le p < \infty$. This is a well-known fact from functional analysis.

Let now $A \subset \mathbb{R}^n$ be a bounded measurable set and let $\varepsilon > 0$. Then there exists a closed set F and open set Q such that $F \subset A \subset Q$ and $\mu(Q \setminus F) < \varepsilon^p$ (or only $\mu(Q) < \varepsilon^p$ if there is no closed set $F \subset A$). Here μ is the Lebesgue measure in $\mathbb{R}^n$. Let now φ be a function from $C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi \subset Q, \varphi|_F \equiv 1$ and $0 \le \varphi \le 1$. Then

$$\|\varphi - \chi_A\|_{L^p(\mathbb{R}^n)}^p = \int_{\mathbb{R}^n} |\varphi(x) - \chi_A(x)|^p \, dx \le \int_{Q \setminus F} 1 dx = \mu(Q \setminus F) < \varepsilon^p$$

or

$$\|\varphi - \chi_A\|_{L^p(\mathbb{R}^n)} < \varepsilon,$$

where χ_A denotes the characteristic function of A i.e.

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

Thus, we may conclude that $\overline{C_0^\infty(\mathbb{R}^n)} = L^p(\mathbb{R}^n)$ for $1 \le p < \infty$. □

Remark. Lemma 1 does not hold for $p = \infty$. Indeed, for a function $f \equiv c_0 \ne 0$ and for any function $\varphi \in C_0^\infty(\mathbb{R}^n)$ we have that

$$\|f - \varphi\|_{L^\infty(\mathbb{R}^n)} \ge |c_0| > 0.$$

Hence we cannot approximate any function from $L^\infty(\mathbb{R}^n)$ by functions from $C_0^\infty(\mathbb{R}^n)$. It means that

$$\overline{C_0^\infty(\mathbb{R}^n)}^{L^\infty} \neq L^\infty(\mathbb{R}^n).$$

But the following result holds:

Exercise 7. Prove that $\overline{S(\mathbb{R}^n)} \overset{L^\infty}{=} \dot{C}(\mathbb{R}^n)$, where

$$\dot{C}(\mathbb{R}^n) := \{f \in C(\mathbb{R}^n) : \lim_{|x|\to\infty} f(x) = 0\}.$$

Now we are in the position to extend F from $S \subset L^1$ to L^1.

Theorem 1 (Riemann-Lebesgue lemma). *Let $F : S \to S$ be the Fourier transform in Schwartz space $S(\mathbb{R}^n)$. Then there exists a unique extension F_{ex} as a map from $L^1(\mathbb{R}^n)$ to $\dot{C}(\mathbb{R}^n)$ with norm $\|F_{ex}\|_{L^1\to L^\infty} = (2\pi)^{-n/2}$.*

Proof. We know that $\|Ff\|_{L^\infty} \le (2\pi)^{-n/2} \|f\|_{L^1}$ for $f \in S$. Now we apply the preliminary proposition to $E = S, X = L^1$ and $Y = L^\infty$. Since $\overline{S} \overset{L^1}{=} L^1$ (it follows from $C_0^\infty \subset S$ and $\overline{C_0^\infty} \overset{L^1}{=} L^1$) for any $f \in L^1(\mathbb{R}^n)$ there exists $\{f_k\} \subset S$ such that $\|f_k - f\|_{L^1} \to 0$ as $k \to \infty$. In that case we can define

$$F_{ex} f \overset{L^\infty}{:=} \lim_{k\to\infty} F f_k.$$

Since $\overline{S} \overset{L^\infty}{=} \dot{C}$ (see Exercise 7) then $F_{ex} f \in \dot{C}$ and $\|F_{ex}\|_{L^1\to L^\infty} \le (2\pi)^{-n/2}$. On the other hand

$$\|Ff\|_{L^\infty} \ge |\widehat{f}(0)| = (2\pi)^{-n/2} \|f\|_{L^1}$$

for $f \in L^1$ and $f \ge 0$. Hence $\|F_{ex}\|_{L^1\to L^\infty} = (2\pi)^{-n/2}$. □

Alternative proof. If $f \in L^1(\mathbb{R}^n)$ then we can define Fourier transform $Ff(\xi)$ directly by

$$Ff(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i(x,\xi)} f(x)dx,$$

since

$$\left|\int_{\mathbb{R}^n} e^{-i(x,\xi)} f(x)dx\right| \le \int_{\mathbb{R}^n} |f(x)|dx = \|f\|_{L^1}.$$

Also we have

$$\begin{aligned}(2\pi)^{n/2} \left\|\widehat{f}(\xi+h) - \widehat{f}(\xi)\right\|_{L^\infty(\mathbb{R}^n)} &= \sup_{\xi\in\mathbb{R}^n} \left|\int_{\mathbb{R}^n} e^{-i(\xi,x)}(e^{-i(h,x)} - 1) f(x)dx\right| \\ &\le 2\int_{|x|>\frac{\varepsilon}{|h|}} |f(x)|dx + \varepsilon \int_{|x||h|\le\varepsilon} |f(x)|dx := I_1 + I_2.\end{aligned}$$

Here we have used the fact that $|e^{iy} - 1| \le |y|$ for $y \in \mathbb{R}$ with $|y| \le 1$. It is easily seen that $I_1 \to 0$ for $|h| \to 0$ and $I_2 \to 0$ for $\varepsilon \to 0$, since $f \in L^1(\mathbb{R}^n)$.

It means that the Fourier transform $\widehat{f}(\xi)$ is continuous (even uniformly continuous) on $\mathbb{R}^n$. Moreover, we have

$$2\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i(x,\xi)} \left(f(x) - f\left(x + \frac{\xi\pi}{|\xi|^2} \right) \right) dx.$$

This equality follows from

$$\widehat{f}(\xi) = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\pi} e^{-i(x,\xi)} f(x) dx = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\left(\xi, \frac{\xi}{|\xi|^2}\right)\pi} e^{-i(x,\xi)} f(x) dx$$

and Exercise 4. Thus,

$$2|\widehat{f}(\xi)| \le (2\pi)^{-n/2} \left\| f(x) - f\left(x + \frac{\xi\pi}{|\xi|^2} \right) \right\|_{L^1(\mathbb{R}^n)} \to 0$$

for $|\xi| \to \infty$. □

Theorem 2 (Plancherel)**.** *Let $F : S \to S$ be the Fourier transform in S with $\|Ff\|_{L^2} = \|f\|_{L^2}$. Then there exists a unique extension F_{ex} of F to L^2-space, such that $F_{ex} : L^2 \overset{onto}{\to} L^2$ and $\|F_{ex}\|_{L^2 \to L^2} = 1$. Also the Parseval equality remains valid.*

Proof. We know that $\overline{S} \overset{L^2}{=} L^2$ since $\overline{C_0^\infty} \overset{L^2}{=} L^2$. Thus for any $f \in L^2(\mathbb{R}^n)$ there exists $\{f_k\}_{k=1}^\infty \subset S(\mathbb{R}^n)$ such that $\|f_k - f\|_{L^2(\mathbb{R}^n)} \to 0$ as $k \to \infty$. By Parseval equality in S we get

$$\|Ff_k - Ff_l\|_{L^2} = \|f_k - f_l\|_{L^2} \to 0, \quad k, l \to \infty.$$

Thus $\{Ff_k\}_{k=1}^\infty$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$ and, therefore, $Ff_k \overset{L^2}{\to} g$, where $g \in L^2$. That's why we may put $F_{ex}f := g$. Also we have the Parseval equality

$$\|F_{ex}f\|_{L^2} = \lim_{k\to\infty} \|Ff_k\|_{L^2} = \lim_{k\to\infty} \|f_k\|_{L^2} = \|f\|_{L^2}$$

which proves the statement about the operator norm. □

Remark. In L^2-space we also have the Fourier inversion formula $F_{ex}^* F_{ex} f = f$ or $F_{ex}^{-1} F_{ex} f = f$.

Exercise 8. Prove that if $f \in L^2(\mathbb{R}^n)$ then

1) $F_{ex}f(\xi) \overset{L^2}{=} \lim_{R\to+\infty} Ff_R(\xi)$, where $f_R(x) = \chi_{\{x:|x|\le R\}}(x) f(x)$

2) $F_{ex}f(\xi) \overset{L^2}{=} \lim_{\varepsilon\to+0} F(e^{-\varepsilon|x|} f)$.

Exercise 9. Let us assume that $f \in L^1(\mathbb{R}^n)$ and $Ff(\xi) \in L^1(\mathbb{R}^n)$. Prove that

$$f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x,\xi)} Ff(\xi) d\xi = F^{-1}Ff(x).$$

It means that the Fourier inversion formula is valid.

Exercise 10. Let f_1 and f_2 belong to $L^2(\mathbb{R}^n)$. Prove that

$$(f_1, f_2)_{L^2} = (Ff_1, Ff_2)_{L^2}.$$

Theorem 3 (Riesz - Torin interpolation theorem)**.** *Let T be a linear continuous map from $L^{p_1}(\mathbb{R}^n)$ to $L^{q_1}(\mathbb{R}^n)$ with norm estimate M_1 and from $L^{p_2}(\mathbb{R}^n)$ to $L^{q_2}(\mathbb{R}^n)$ with norm estimate M_2. Then T is a linear continuous map from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with p and q such that*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}, \quad 0 \le \theta \le 1,$$

with norm estimate $M_1^\theta M_2^{1-\theta}$.

Proof. Let F and G be two functions with the properties:

1) $F, G \ge 0$,

2) $\|F\|_{L^1} = \|G\|_{L^1} = 1$.

Let us consider now the function $\Phi(z)$ of complex variable $z \in \mathbb{C}$ given by

$$\Phi(z) := M_1^{-z} M_2^{-1+z} \int_{\mathbb{R}^n} T(f_0 F^{\frac{z}{p_1} + \frac{1-z}{p_2}})(x) g_0 G^{\frac{z}{q_1'} + \frac{1-z}{q_2'}}(x) dx,$$

where $\frac{1}{q_1} + \frac{1}{q_1'} = 1, \frac{1}{q_2} + \frac{1}{q_2'} = 1, |f_0| \le 1$ and $|g_0| \le 1$. The two functions f_0 and g_0 will be selected later. We assume also that $0 \le \mathrm{Re}\,(z) \le 1$.

Our aim is to prove the inequality

$$|\langle Tf, g\rangle_{L^2}| \le M_1^\theta M_2^{1-\theta} \|f\|_{L^p} \|g\|_{L^{q'}},$$

where

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

Since T is a linear continuous map and $F^{\frac{z}{p_1} + \frac{1-z}{p_2}}, G^{\frac{z}{q_1'} + \frac{1-z}{q_2'}}$ are holomorphic functions with respect to z (consider $a^z = e^{z \ln a}, a > 0$) we may conclude that $\Phi(z)$ is a holomorphic function also.

1) Let us assume now that $\mathrm{Re}\,(z) = 0$, i.e. $z = iy$. Then we have $\Phi(iy) = M_1^{-iy} M_2^{-1+iy} \langle T(f_0 F^{\frac{iy}{p_1} + \frac{1-iy}{p_2}}), g_0 G^{\frac{iy}{q_1'} + \frac{1-iy}{q_2'}} \rangle_{L^2}$. Since $|a^{ix}| = 1$ for $a, x \in \mathbb{R}, a > 0$, it follows from Hölder inequality and the assumptions on T that

$$\begin{aligned} |\Phi(iy)| &\le M_2^{-1} M_2 \left\| f_0 F^{\frac{iy}{p_1} + \frac{1-iy}{p_2}} \right\|_{L^{p_2}} \left\| g_0 G^{\frac{iy}{q_1'} + \frac{1-iy}{q_2'}} \right\|_{L^{q_2'}} \\ &= \left\| |f_0| F^{\frac{1}{p_2}} \right\|_{L^{p_2}} \left\| |g_0| G^{\frac{1}{q_2'}} \right\|_{L^{q_2'}} \le \|F\|_{L^1}^{\frac{1}{p_2}} \|G\|_{L^1}^{\frac{1}{q_2'}} = 1. \end{aligned}$$

2) Let us assume now that $\operatorname{Re}(z) = 1$, i.e., $z = 1 + iy$. Then we have similarly that

$$\begin{aligned} |\Phi(1+iy)| &\le M_1^{-1} M_1 \left\| f_0 F^{\frac{1+iy}{p_1} + \frac{-iy}{p_2}} \right\|_{L^{p_1}} \left\| g_0 G^{\frac{1+iy}{q_1'} + \frac{-iy}{q_2'}} \right\|_{L^{q_1'}} \\ &= \left\| |f_0| F^{\frac{1}{p_1}} \right\|_{L^{p_1}} \left\| |g_0| G^{\frac{1}{q_1'}} \right\|_{L^{q_1'}} \le \|F\|_{L^1}^{\frac{1}{p_1}} \|G\|_{L^1}^{\frac{1}{q_1'}} = 1. \end{aligned}$$

If we apply now the *Phragmen-Lindelev's theorem* for the domain $0 < \operatorname{Re}(z) < 1$ we obtain that $|\Phi(z)| \le 1$ for any z such that $0 < \operatorname{Re}(z) < 1$. Then $|\Phi(\theta)| \le 1$ also for $0 < \theta < 1$. But this is equivalent to the estimate

$$|\langle T(f_0 F^{\frac{1}{p}}), g_0 G^{\frac{1}{q'}} \rangle_{L^2}| \le M_1^{\theta} M_2^{1-\theta}, \tag{3.1}$$

where $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$, $\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$ and $\frac{1}{q} + \frac{1}{q'} = 1$. In order to finish the proof of this theorem let us choose (for arbitrary functions $f \in L^p$ and $g \in L^{q'}$ with p and q' as above) the functions F, G, f_0 and g_0 as follows:

$$F = |f_1|^p, \quad G = |g_1|^{q'}, \quad f_0 = \operatorname{sgn} f_1, \quad g_0 = \operatorname{sgn} g_1,$$

where $f_1 = \frac{f}{\|f\|_{L^p}}$, $g_1 = \frac{g}{\|g\|_{L^{q'}}}$ and

$$\operatorname{sgn} f = \begin{cases} 1, & f > 0 \\ 0, & f = 0 \\ -1, & f < 0. \end{cases}$$

In that case $f_1 = f_0 F^{\frac{1}{p}}$ and $g_1 = g_0 G^{\frac{1}{q'}}$. Applying the estimate (3.1) we obtain

$$\left| \left\langle T\left(\frac{f}{\|f\|_{L^p}} \right), \frac{g}{\|g\|_{L^{q'}}} \right\rangle_{L^2} \right| \le M_1^{\theta} M_2^{1-\theta},$$

which is equivalent to

$$|\langle Tf, g \rangle_{L^2}| \le M_1^{\theta} M_2^{1-\theta} \|f\|_{L^p} \|g\|_{L^{q'}}.$$

It implies the desired final estimate

$$\|Tf\|_{L^q} \le M_1^{\theta} M_2^{1-\theta} \|f\|_{L^p}.$$

□

Theorem 4 (Hausdorff-Young)**.** *Let $F : S \to S$ be the Fourier transform in Schwartz space. Then there exists a unique extension F_{ex} as a linear continuous map*

$$F_{ex} : L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n),$$

where $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. What is more, we have the norm estimate

$$\|F_{ex}\|_{L^p \to L^{p'}} \le (2\pi)^{-n\left(\frac{1}{p} - \frac{1}{2}\right)}.$$

It is called the Hausdorff-Young inequality.

Proof. We know from Theorems 1 and 2 that there exists a unique extension F_{ex} of the Fourier transform from S to S for spaces:

1) $F_{ex}: \quad L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ with norm estimate $M_1 = (2\pi)^{-\frac{n}{2}}$

2) $F_{ex}: \quad L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with norm estimate $M_2 = 1$.

Applying now Theorem 3 we obtain that $F_{ex} : L^p \to L^q$, where

$$\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{2} = \frac{1}{2} + \frac{\theta}{2}, \quad \frac{1}{q} = \frac{\theta}{\infty} + \frac{1-\theta}{2} = \frac{1}{2} - \frac{\theta}{2}.$$

It follows that

$$\frac{1}{p} + \frac{1}{q} = 1$$

i.e. $q = p'$ and $\theta = \frac{2}{p} - 1$. For θ to satisfy the condition $0 \le \theta \le 1$ we get $1 \le p \le 2$. We may also conclude that

$$\|F_{ex}\|_{L^p \to L^{p'}} \le ((2\pi)^{-\frac{n}{2}})^\theta 1^{1-\theta} = (2\pi)^{-n\left(\frac{1}{p}-\frac{1}{2}\right)}.$$

□

Remark. In order to obtain F_{ex} in $L^p(\mathbb{R}^n), 1 \le p \le 2$, constructively we can apply the following procedure. Let us assume that $f \in L^p(\mathbb{R}^n), 1 \le p \le 2$, and $\{f_k\}_{k=1}^\infty \subset S(\mathbb{R}^n)$ such that

$$\|f_k - f\|_{L^p(\mathbb{R}^n)} \to 0, \quad k \to \infty.$$

It follows from Hausdorff-Young inequality that

$$\|Ff_k - Ff_l\|_{L^{p'}(\mathbb{R}^n)} \le C_n \|f_k - f_l\|_{L^p(\mathbb{R}^n)}.$$

It means that $\{Ff_k\}_{k=1}^\infty$ is a Cauchy sequence in $L^{p'}(\mathbb{R}^n)$. That's why we can define

$$F_{ex}f \overset{L^{p'}}{:=} \lim_{k\to\infty} Ff_k.$$

And we also have the Hausdorff-Young inequality

$$\|F_{ex}f\|_{L^{p'}} = \lim_{k\to\infty} \|Ff_k\|_{L^{p'}} \le \lim_{k\to\infty} C_n \|f_k\|_{L^p} = C_n \|f\|_{L^p}.$$

Example 3.1 (Fourier transform on the line). Let $f_2(x) = \frac{1}{(x-i\varepsilon)^2}$, where $\varepsilon > 0$ is fixed. It is clear that $f_2 \in L^1(\mathbb{R})$ and

$$\widehat{f_2}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-ix\xi}dx}{(x-i\varepsilon)^2}.$$

In order to calculate this integral we consider the function $F(z) := \frac{e^{-iz\xi}}{(z-i\varepsilon)^2}$ of complex variable $z \in \mathbb{C}$. It is easily seen that $z = i\varepsilon, \varepsilon > 0$ is a pole of order 2. We consider the cases $\xi > 0$ and $\xi < 0$ separately, see Figure 2.

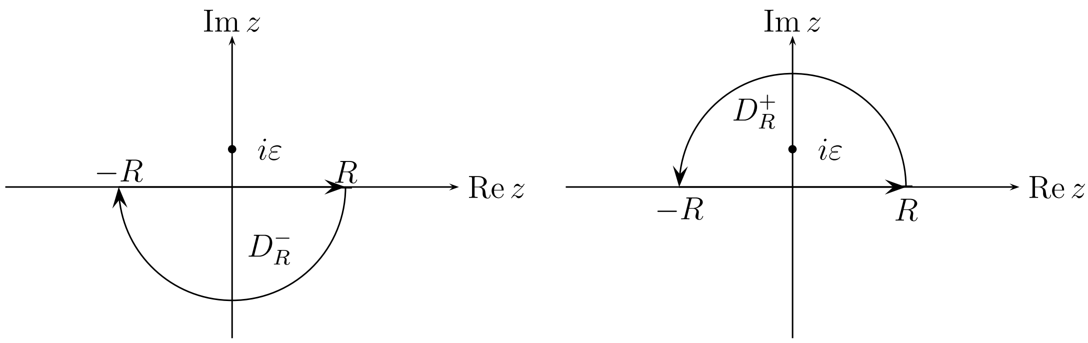

Figure 2: Domains D_R^- and D_R^+ of integration.

1) $\xi > 0$. By Cauchy theorem we have

$$\oint_{\partial D_R^-} F(z)dz = 0 = \int_{-R}^{R} F(z)dz + \int\limits_{\substack{|z|=R \\ \operatorname{Im} z<0}} F(z)dz = I_1 + I_2.$$

It follows that

$$I_1 \to \int_{-\infty}^{+\infty} \frac{e^{-ix\xi}dx}{(x-i\varepsilon)^2}, \quad R \to \infty,$$

and

$$I_2 \to 0, \quad R \to \infty$$

due to Jordan's lemma, since $\xi \operatorname{Im} z < 0$. That's why we may conclude that

$$\int_{-\infty}^{+\infty} \frac{e^{-ix\xi}dx}{(x-i\varepsilon)^2} = 0$$

for $\xi > 0$.

2) $\xi < 0$. In this case again $\xi \operatorname{Im} z < 0$. So we may apply Jordan's lemma again and obtain

$$\oint_{\partial D_R^+} F(z)dz = \int_{-R}^{R} \frac{e^{-ix\xi}dx}{(x-i\varepsilon)^2} + \int\limits_{\substack{|z|=R \\ \operatorname{Im} z>0}} F(z)dz = 2\pi i \operatorname*{Res}_{z=i\varepsilon} F(z).$$

Hence

$$\int_{-\infty}^{\infty} \frac{e^{-ix\xi}dx}{(x-i\varepsilon)^2} = 2\pi i((z-i\varepsilon)^2 F(z))' \mid_{z=i\varepsilon} = 2\pi\xi e^{\varepsilon\xi}.$$

If we combine these two cases we obtain

$$\widehat{\frac{1}{(x-i\varepsilon)^2}}(\xi) = \sqrt{2\pi}\xi H(-\xi)e^{\varepsilon\xi},$$

where

$$H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

is the Heaviside function. Similarly we obtain

$$\widehat{\frac{1}{(x+i\varepsilon)^2}}(\xi) = -\sqrt{2\pi}\xi H(\xi)e^{-\varepsilon\xi}.$$

Example 3.2. Let $f_1(x) = \frac{1}{x-i\varepsilon}$, where $\varepsilon > 0$ is fixed. It is clear that $f_1 \notin L^1(\mathbb{R})$, but $f_1 \in L^p(\mathbb{R}), 1 < p \leq 2$. Analogous to Example 3.1 we obtain:

$$\widehat{\frac{1}{x+i\varepsilon}}(\xi) = \begin{cases} -i\sqrt{2\pi}H(\xi)e^{-\varepsilon\xi}, & \xi \neq 0 \\ -i\sqrt{\frac{\pi}{2}}, & \xi = 0 \end{cases}$$

and

$$\widehat{\frac{1}{x-i\varepsilon}}(\xi) = \begin{cases} i\sqrt{2\pi}H(-\xi)e^{\varepsilon\xi}, & \xi \neq 0 \\ i\sqrt{\frac{\pi}{2}}, & \xi = 0. \end{cases}$$

Exercise 11. Find the Fourier transforms of the following functions on the line.

a) $f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$

b) $f(x) = e^{-|x|}$ and $f(x) = \frac{1}{1+x^2}$

c) $f_3(x) = \frac{1}{(x \pm i\varepsilon)^3}, \varepsilon > 0.$

Exercise 12. Define the Laplace transform by

$$L(p) := \int_0^\infty f(x)e^{-px}dx,$$

where $|f(x)| \leq Me^{ax}, x > 0, f(x) = 0, x < 0$ and $p = p_1 + ip_2, p_1 > a$. Prove that

a) $L(p) = \sqrt{2\pi}\widehat{f(x)e^{-p_1x}}(p_2)$.

b) Apply the Fourier inversion formula to prove the Mellin formula

$$f(x) = \frac{1}{2\pi i}\int_{p_1-i\infty}^{p_1+i\infty} L(p)e^{px}dp, \quad p_1 > a.$$

4 Tempered distributions

In this chapter we will consider two types of distributions: Schwartz distributions and tempered distributions. To that end we consider the space $D := C_0^\infty(\mathbb{R}^n)$ of test functions. It is clear that D is a linear space and $D \subset S$. A notion of convergence is given in

Definition. A sequence $\{\varphi_k\}_{k=1}^\infty$ is a *null-sequence* in D if and only if

1) there exists a compact set $K \subset \mathbb{R}^n$, such that $\operatorname{supp} \varphi_k \subset K$ for any k and

2) for any $\alpha \geq 0$ we have

$$\sup_{x \in K} |D^\alpha \varphi_k(x)| \to 0, \quad k \to \infty.$$

We denote this fact by $\varphi_k \xrightarrow{D} 0$. As usual, $\varphi_k \xrightarrow{D} \varphi \in D$ means that $\varphi_k - \varphi \xrightarrow{D} 0$.

Now we are in the position to define the *Schwartz distribution space.*

Definition. Functional $T : D \to \mathbb{C}$ is a Schwartz distribution if it is linear and continuous, that is,

1) $T(\alpha_1\varphi_1 + \alpha_2\varphi_2) = \alpha_1 T(\varphi_1) + \alpha_2 T(\varphi_2)$ for any $\varphi_1, \varphi_2 \in D$ and $\alpha_1, \alpha_2 \in \mathbb{C}$

2) for any null-sequence φ_k in D it holds that $T(\varphi_k) \to 0$ in $\mathbb{C}$ as $k \to \infty$.

The linear space of Schwartz distributions is denoted by D'. The action of T on φ is denoted by $T(\varphi) = \langle T, \varphi \rangle$.

Example 4.1. Every locally integrable function f, that is, $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, defines a Schwartz distribution by the formula

$$\langle T_f, \varphi \rangle := \int_{\mathbb{R}^n} f(x)\varphi(x)dx.$$

It is clear that T_f is a linear map. It remains to prove only that T_f is continuous map on D. Let $\{\varphi_k\}_{k=1}^\infty$ be a null-sequence in D. Then

$$|\langle T_f, \varphi_k \rangle| \leq \sup_{x \in K} |\varphi_k(x)| \int_K |f(x)| dx \to 0, \quad k \to \infty$$

by the definition of null-sequence.

Example 4.2. If $\langle T, \varphi \rangle := \varphi(0)$, then $T \in D'$. Indeed, T is linear and if $\varphi_k \xrightarrow{D} 0$ then $\langle T, \varphi_k \rangle = \varphi_k(0) \to 0$ for $k \to \infty$. This distribution is called the δ-function and is denoted by δ i.e.

$$\langle \delta, \varphi \rangle = \varphi(0), \quad \varphi \in D.$$

Remark. A distribution T is *regular* if it can be written in the form

$$\langle T, \varphi \rangle = \int_{\mathbb{R}^n} f(x)\varphi(x)dx$$

for some locally integrable function f. All other distributions are *singular*.

Exercise 13. Prove that δ is a singular distribution.

Definition. The functional T defined by

$$\langle T, \varphi \rangle := \lim_{\varepsilon \to +0} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x} dx \equiv \text{p.v.} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x} dx$$

on $D(\mathbb{R})$ is called the *principal value* of $\frac{1}{x}$. We denote it by $T = \text{p.v.}\frac{1}{x}$.

Remark. Note that $\frac{1}{x} \notin L^1_{\text{loc}}(\mathbb{R})$ but we have the following

Exercise 14. Prove that

$$\langle \text{p.v.}\frac{1}{x}, \varphi \rangle = \int_0^{\infty} \frac{\varphi(x) - \varphi(-x)}{x} dx = \text{p.v.} \int_{-\infty}^{\infty} \frac{\varphi(x) - \varphi(0)}{x} dx.$$

Example 4.3. Let σ be a hypersurface of dimension $n-1$ in $\mathbb{R}^n$ and let $d\sigma$ stand for an element of surface area on σ. Consider the functional

$$\langle T, \varphi \rangle = \int_{\sigma} a(x)\varphi(x)d\sigma$$

on D, where $a(x)$ is a locally integrable function over σ. We can interpret T in terms of surface source. Indeed,

$$\langle \int_{\sigma} a(\xi)\delta(x-\xi)d\sigma_{\xi}, \varphi \rangle := \int_{\sigma} a(\xi) \langle \delta(x-\xi), \varphi(x) \rangle d\sigma_{\xi} = \int_{\sigma} a(\xi)\varphi(\xi)d\sigma_{\xi}.$$

It is easy to see that T is a singular distribution. This distribution is known as the *simple layer*.

Definition. If $T \in D'$ and $g \in C^{\infty}(\mathbb{R}^n)$ then we may define the product gT by

$$\langle gT, \varphi \rangle := \langle T, g\varphi \rangle, \quad \varphi \in D.$$

This product is well-defined because $g\varphi \in D$.

If f is a locally integrable function whose derivative $\frac{\partial f}{\partial x_j}$ is also locally integrable, then

$$\left\langle \frac{\partial f}{\partial x_j}, \varphi \right\rangle = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j} \varphi(x) dx = -\int_{\mathbb{R}^n} f \frac{\partial \varphi}{\partial x_j} dx = -\left\langle f, \frac{\partial \varphi}{\partial x_j} \right\rangle, \quad \varphi \in D$$

by integration by parts. This property is used to define the derivative of any distribution.

Definition. Let T be a distribution from D'. For any multi-index α we define the derivative $\partial^\alpha T$ by

$$\langle \partial^\alpha T, \varphi\rangle := \langle T, (-1)^{|\alpha|}\partial^\alpha \varphi\rangle, \quad \varphi \in D.$$

It is easily seen that $\partial^\alpha T \in D'$.

Example 4.4. Consider the Heaviside function $H(x)$. Since $H \in L^1_{\text{loc}}(\mathbb{R})$ then

$$\langle H', \varphi\rangle = -\langle H, \varphi'\rangle = -\int_0^\infty \varphi'(x)dx = \varphi(0) = \langle \delta, \varphi\rangle.$$

Hence $H' = \delta$.

Example 4.5. Let us prove that in the sense of Schwartz distributions $(\log|x|)' = \text{p.v.}\frac{1}{x}$. Indeed,

$$\begin{aligned}
\langle (\log|x|)', \varphi\rangle &= -\langle \log|x|, \varphi'\rangle = -\int_{-\infty}^{\infty} \log(|x|)\varphi'(x)dx \\
&= -\int_0^\infty \log(x)\varphi'(x)dx - \int_{-\infty}^0 \log(-x)\varphi'(x)dx \\
&= -\int_0^\infty \log(x)(\varphi'(x) + \varphi'(-x))dx = -\int_0^\infty \log(x)(\varphi(x) - \varphi(-x))'dx \\
&= -\log(x)\left[\varphi(x) - \varphi(-x)\right]_0^\infty + \int_0^\infty \frac{\varphi(x) - \varphi(-x)}{x}dx = \langle \text{p.v.}\frac{1}{x}, \varphi\rangle
\end{aligned}$$

by integration by parts and Exercise 14.

Exercise 15. Prove that

$$\left(\text{p.v.}\frac{1}{x}\right)' = -\text{p.v.}\frac{1}{x^2}.$$

The following characterization of D' is given without a proof: $T \in D'$ if and only if for any compact $K \subset \mathbb{R}^n$ there exists $n_0(K) \in \mathbb{N}_0$ such that

$$|\langle T, \varphi\rangle| \le C_0 \sum_{|\alpha| \le n_0} \sup_{x\in K} |D^\alpha \varphi|$$

for any $\varphi \in D$ with $\operatorname{supp}\varphi \subset K$.

Definition. Functional $T : S \to \mathbb{C}$ is a *tempered distribution* if

1) T is linear i.e. $\langle T, \alpha\varphi + \beta\psi\rangle = \alpha\langle T, \varphi\rangle + \beta\langle T, \psi\rangle$ for all $\alpha, \beta \in \mathbb{C}$ and $\varphi, \psi \in S$

2) T is a continuous on S, i.e. there exists $n_0 \in \mathbb{N}_0$ and constant $c_0 > 0$ such that

$$|\langle T, \varphi\rangle| \le c_0 \sum_{|\alpha|, |\beta| \le n_0} |\varphi|_{\alpha,\beta}$$

for any $\varphi \in S$.

The space of tempered distributions is denoted by S'. In addition, for $T_k, T \in S'$ the convergence $T_k \xrightarrow{S'} T$ means that $\langle T_k, \varphi\rangle \xrightarrow{\mathbb{C}} \langle T, \varphi\rangle$ for all $\varphi \in S$.

Remark. Since $D \subset S$ the space of tempered distributions is more narrow than the space of Schwartz distributions, $S' \subset D'$. Later we will consider even more narrow distribution space $\mathcal{E}'$ which consists of continuous linear functionals on the (widest test function) space $\mathcal{E} := C^\infty(\mathbb{R}^n)$. In short, $D \subset S \subset \mathcal{E}$ implies that

$$\mathcal{E}' \subset S' \subset D'.$$

It turns out that members of $\mathcal{E}'$ have compact support and that's why they are called *distributions with compact support.* But more on that later.

Example 4.6. Let us consider $\mathbb{R}^1$.

1) It is clear that $f(x) = e^{|x|^2}$ is a Schwartz distribution, but not a tempered distribution, because part 2) of the previous definition is not satisfied.

2) If $f(x) = \sum_{k=0}^{m} a_k x^k$ is a polynomial then $f(x) \in S'$ since

$$\begin{aligned} |\langle T_f, \varphi\rangle| &= \left| \int_{\mathbb{R}} \sum_{k=0}^{m} a_k x^k \varphi(x) dx \right| \\ &\le \sum_{k=0}^{m} |a_k| \int_{\mathbb{R}} (1+|x|)^{-1-\delta}(1+|x|)^{1+\delta}|x|^k|\varphi(x)|dx \\ &\le C \sum_{k=0}^{m} |a_k||\varphi|_{0,k+1+\delta} \int_{\mathbb{R}} (1+|x|)^{-1-\delta} dx, \end{aligned}$$

so the condition 2) is satisfied e.g. for $\delta = 1, n_0 = m+2$. This polynomial is a regular distribution since $\langle T_f, \varphi\rangle = \int_{\mathbb{R}} f(x)\varphi(x)dx$ is well-defined.

Definition. Let T be a distribution from D'. Then the support of T is defined by

$$\operatorname{supp} T := \mathbb{R}^n \setminus A,$$

where $A = \{x \in \mathbb{R}^n : \langle T, \varphi\rangle = 0 \, \text{for any} \, \varphi \in C^\infty \, \text{with} \operatorname{supp} \varphi \subset U_\delta(x)\}$.

Exercise 16. Prove that

1) if f is continuous then
$$\operatorname{supp} T_f = \operatorname{supp} f$$

2) $\operatorname{supp}(\partial^\alpha T) \subset \operatorname{supp} T$

3) $\operatorname{supp} \delta = \{0\}$.

Example 4.7. 1) The weighted Lebesgue spaces are defined as

$$L^p_\sigma(\mathbb{R}^n) := \{f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^p_\sigma} := \Big(\int_{\mathbb{R}^n} (1+|x|)^{\sigma p}|f(x)|^p dx\Big)^{\frac{1}{p}} < \infty\}$$

for $1 \le p < \infty$ and

$$L^\infty_\sigma(\mathbb{R}^n) := \{f \in L^\infty_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^\infty_\sigma} := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}(1+|x|)^\sigma |f(x)| < \infty\}.$$

If $f \in L^1_{-\delta}(\mathbb{R}^n)$ for some $\delta > 0$ then $T_f \in S'$. In fact,

$$|\langle T_f, \varphi\rangle| = \left|\int_{\mathbb{R}^n} f\varphi dx\right| \le \|f\|_{L^1_{-\delta}} \|\varphi\|_{L^\infty_\delta}.$$

It means that $\int_{\mathbb{R}} f\varphi dx$ is well-defined in this case and

$$\langle T_f, \varphi\rangle := \int_{\mathbb{R}^n} f\varphi dx.$$

2) If $f \in L^p, 1 \le p \le \infty$, then $f \in S'$. Indeed,

$$L^p(\mathbb{R}^n) \subset L^1_{-\delta}(\mathbb{R}^n) \quad \text{for} \quad \delta > \frac{n}{p'},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. This follows from Hölder's inequality

$$\int_{\mathbb{R}} (1+|x|)^{-\delta}|f(x)|dx \le \Big(\int_{\mathbb{R}} (1+|x|)^{-\delta p'} dx\Big)^{\frac{1}{p'}} \|f\|_{L^p}.$$

3) Let $T \in S'$, and $\varphi_0(x) \in C_0^\infty(\mathbb{R}^n)$ with $\varphi_0(0) = 1$. The product $\varphi_0\left(\frac{x}{k}\right)T$ is well-defined in S' by

$$\left\langle \varphi_0\left(\frac{x}{k}\right)T, \varphi\right\rangle := \left\langle T, \varphi_0\left(\frac{x}{k}\right)\varphi\right\rangle.$$

If we consider the sequence $T_k := \varphi_0\left(\frac{x}{k}\right)T$ then

1) $\langle T_k, \varphi\rangle \equiv \langle T, \varphi_0(\frac{x}{k})\varphi\rangle \overset{k\to\infty}{\longrightarrow} \langle T, \varphi\rangle$ (since $\varphi_0(\frac{x}{k})\varphi \overset{S}{\to} \varphi$) so that $T_k \overset{S'}{\to} T$.
2) T_k has compact support as a tempered distribution. This fact follows from the compactness of $\varphi_k = \varphi_0(\frac{x}{k})$.

Now we are ready to prove more serious and more useful fact.

Theorem 1. *Let $T \in S'$. Then there exists $T_k \subset S$ such that*

$$\langle T_k, \varphi\rangle = \int_{\mathbb{R}^n} T_k(x)\varphi(x)dx \to \langle T, \varphi\rangle, \quad k \to \infty,$$

where $\varphi \in S$. In short, $\overline{S} \overset{S'}{=} S'$.

Proof. Let $j(x)$ be a function from $D \equiv C_0^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} j(x)dx = 1$ and $j(-x) = j(x)$. Let $j_k(x) := k^n j(kx)$. By Lemma 1 of Chapter 2 we have

$$\lim_{k\to\infty} \langle j_k, \varphi\rangle = \lim_{k\to\infty} \int_{\mathbb{R}^n} j_k(x)\varphi(x)dx = \varphi(0)$$

for any $\varphi \in S$. That is, $j_k(x) \xrightarrow{S'} \delta(x)$.

The *convolution* of two integrable functions g and φ is defined by

$$(g * \varphi)(x) := \int_{\mathbb{R}^n} g(x-y)\varphi(y)dy.$$

If h and g are integrable functions and $\varphi \in S$ then it follows from Fubini's theorem that

$$\begin{aligned}\langle h * g, \varphi\rangle &= \int_{\mathbb{R}^n} \varphi(x)dx \int_{\mathbb{R}^n} h(x-y)g(y)dy = \int_{\mathbb{R}^n} g(y)dy \int_{\mathbb{R}^n} h(x-y)\varphi(x)dx \\ &= \int_{\mathbb{R}^n} g(y)dy \int_{\mathbb{R}^n} Rh(y-x)\varphi(x)dx = \langle g, Rh * \varphi\rangle,\end{aligned}$$

where $Rh(z) := h(-z)$ is the reflection of h.

Let now $\varphi_0(x) \in D$ with $\varphi_0(0) = 1$. For any $T \in S'$ let us put $T_k := j_k * \widetilde{T_k}$, where $\widetilde{T_k} = \varphi_0\left(\frac{x}{k}\right)T$. From above considerations we know that $\langle j_k * \widetilde{T_k}, \varphi\rangle = \langle \widetilde{T_k}, Rj_k * \varphi\rangle$.

Let us prove that this T_k meets the requirements of this theorem. First of all,

$$\begin{aligned}\langle T_k, \varphi\rangle &\equiv \langle j_k * \widetilde{T_k}, \varphi\rangle = \langle \widetilde{T_k}, Rj_k * \varphi\rangle = \langle \varphi_0\left(\frac{x}{k}\right)T, j_k * \varphi\rangle \\ &= \langle T, \varphi_0\left(\frac{x}{k}\right)(j_k * \varphi)\rangle \to \langle T, \varphi\rangle, \quad k \to \infty,\end{aligned}$$

because

a) $\varphi_0\left(\frac{x}{k}\right) \to 1$ pointwise for $k \to \infty$, since $\varphi_0(0) = 1$ and $\varphi_0(\frac{x}{k})\varphi \xrightarrow{S} \varphi$

b) $j_k * \varphi \xrightarrow{S} \varphi$ for $k \to \infty$ by Lemma 1 of Chapter 2:

$$\int_{\mathbb{R}^n} j_k(x-y)\varphi(y)dy = \int_{\mathbb{R}^n} j_k(z)\varphi(x-z)dz \to \varphi(x).$$

Finally $j_k(x) \in C_0^\infty(\mathbb{R}^n)$ implies that $T_k \in C_0^\infty(\mathbb{R}^n) \subset S$ also. □

Definition. Let us assume that $L : S \to S$ is a linear continuous map. The adjoint map $L' : S' \to S'$ is defined by

$$\langle L'T, \varphi\rangle := \langle T, L\varphi\rangle, \quad T \in S'.$$

Clearly, L' is also a linear continuous map.

Corollary. *Any linear continuous map (operator) $L : S \to S$ admits a linear continuous extension $\widetilde{L} : S' \to S'$.*

Proof. If $T \in S'$ then by Theorem 1 there exists $T_k \subset S$ such that $T_k \xrightarrow{S'} T$. Then

$$\langle LT_k, \varphi\rangle = \langle T_k, L'\varphi\rangle \to \langle T, L'\varphi\rangle := \langle \widetilde{L}T, \varphi\rangle, \quad k \to \infty.$$

□

Now we are in the position to formulate

Theorem 2 (Properties of tempered distributions)**.** *The following linear continuous operators from S into S admit unique linear continuous extensions as maps from S' into S':*

1) $\langle uT, \varphi\rangle := \langle T, u\varphi\rangle, \quad u \in S;$

2) $\langle \partial^\alpha T, \varphi\rangle := \langle T, (-1)^{|\alpha|}\partial^\alpha \varphi\rangle;$

3) $\langle \tau_h T, \varphi\rangle := \langle T, \tau_{-h}\varphi\rangle;$

4) $\langle \sigma_\lambda T, \varphi\rangle := \langle T, |\lambda|^{-n}\sigma_{\frac{1}{\lambda}}\varphi\rangle, \quad \lambda \neq 0;$

5) $\langle FT, \varphi\rangle := \langle T, F\varphi\rangle.$

Proof. See the previous definition, Theorem 1 and its corollary. □

Remark. Since $\langle F^{-1}FT, \varphi\rangle = \langle FT, F^{-1}\varphi\rangle = \langle T, FF^{-1}\varphi\rangle = \langle T, \varphi\rangle$ we have that $F^{-1}F = FF^{-1} = I$ in S'.

Example 4.8. 1) Since

$$\begin{aligned}\langle F1, \varphi\rangle \equiv \langle 1, F\varphi\rangle &= \int_{\mathbb{R}^n} (F\varphi)(\xi)d\xi = (2\pi)^{\frac{n}{2}}(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{i(0,\xi)}F\varphi d\xi \\ &= (2\pi)^{\frac{n}{2}}F^{-1}F\varphi(0) = (2\pi)^{\frac{n}{2}}\varphi(0) = (2\pi)^{\frac{n}{2}}\langle \delta, \varphi\rangle\end{aligned}$$

for any $\varphi \in S$ we have that

$$\widehat{1} = (2\pi)^{\frac{n}{2}}\delta$$

in S'.

2) $\widehat{\delta} = (2\pi)^{-\frac{n}{2}} \cdot 1$, since

$$\langle \widehat{\delta}, \varphi\rangle = \langle \delta, F\varphi\rangle = F\varphi(0) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-i(0,x)}\varphi(x)dx = (2\pi)^{-\frac{n}{2}}\langle 1, \varphi\rangle, \quad \varphi \in S.$$

Moreover, $F^{-1}\delta = (2\pi)^{-\frac{n}{2}} \cdot 1$ in S'.

3) $\widehat{e^{-a\frac{x^2}{2}}} = a^{-\frac{n}{2}}e^{-\frac{\xi^2}{2a}}, \operatorname{Re} a \geq 0, a \neq 0$. Indeed, for $a > 0$ we know that

$$F(e^{-a\frac{x^2}{2}}) = F(e^{-\frac{(\sqrt{a}x)^2}{2}}) = a^{-\frac{n}{2}}e^{-\frac{\xi^2}{2a}}.$$

If a is such that $\operatorname{Re} a \geq 0, a \neq 0$, then we can use analytic continuation of these formulas.

4) Consider $(1-\Delta)u = f$, where $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ is the Laplacian in $\mathbb{R}^n$ and u and $f \in S'$. This equation can be solved in S' using the Fourier transform. Indeed, we get

$$(1+|\xi|^2)\widehat{u} = \widehat{f}$$

or

$$\widehat{u} = (1+|\xi|^2)^{-1}\widehat{f}$$

or

$$u = F^{-1}((1+|\xi|^2)^{-1}Ff).$$

If $f \in S$ then $Ff \in S$ and $(1+|\xi|^2)^{-1}Ff \in S$ also and then $u \in S$ exists. If $f \in S'$ then by Theorem 1 there exists $f_k \in S$ such that $f_k \xrightarrow{S'} f$. That's why we may conclude that

$$u \stackrel{S'}{=} \lim_{k\to\infty} u_k,$$

where $u_k = F^{-1}((1+|\xi|^2)^{-1}Ff_k)$.

Exercise 17. Let $P(D)$ be an elliptic partial differential operator

$$P(D) = \sum_{|\alpha|\le m} a_\alpha D^\alpha$$

with constant coefficients and $P(\xi) \neq 0$ for $\xi \neq 0$. Prove that if $u \in S'$ and $Pu = 0$ then u is a polynomial.

Corollary. *If $\Delta u = 0$ in S' and $|u| \le$ const then $u \equiv$const.*

Exercise 18. Prove that

1) $F(\text{p.v.}\frac{1}{x}) = -i\sqrt{\frac{\pi}{2}}\operatorname{sgn}\xi$

2) $F(\text{p.v.}\frac{1}{x^2}) = -\sqrt{\frac{\pi}{2}}|\xi|$.

Definition. Introduce the tempered distributions

$$\frac{1}{x \pm i0} := \lim_{\varepsilon\downarrow 0}\frac{1}{x \pm i\varepsilon}$$

i.e.

$$\left\langle \frac{1}{x\pm i0}, \varphi\right\rangle = \lim_{\varepsilon\downarrow 0}\left\langle \frac{1}{x\pm i\varepsilon}, \varphi\right\rangle, \quad \varphi \in S.$$

In a similar fashion,

$$\frac{1}{(x\pm i0)^2} := \lim_{\varepsilon\downarrow 0}\frac{1}{(x\pm i\varepsilon)^2}$$

in S'.

Example 4.9. We know from Example 3.2 that

$$\widehat{\frac{1}{x+i\varepsilon}}(\xi) = \begin{cases} -i\sqrt{2\pi}H(\xi)e^{-\varepsilon\xi}, & \xi \neq 0 \\ -i\sqrt{\frac{\pi}{2}}, & \xi = 0 \end{cases}$$

and

$$\widehat{\frac{1}{x-i\varepsilon}}(\xi) = \begin{cases} i\sqrt{2\pi}H(-\xi)e^{\varepsilon\xi}, & \xi \neq 0 \\ i\sqrt{\frac{\pi}{2}}, & \xi = 0. \end{cases}$$

Hence

$$\widehat{\frac{1}{x+i0}} = \lim_{\varepsilon\downarrow 0} \widehat{\frac{1}{x+i\varepsilon}} = \begin{cases} -i\sqrt{2\pi}H(\xi), & \xi \neq 0 \\ -i\sqrt{\frac{\pi}{2}}, & \xi = 0 \end{cases}$$

and

$$\widehat{\frac{1}{x-i0}} = \lim_{\varepsilon\downarrow 0} \widehat{\frac{1}{x-i\varepsilon}} = \begin{cases} i\sqrt{2\pi}H(-\xi), & \xi \neq 0 \\ i\sqrt{\frac{\pi}{2}}, & \xi = 0. \end{cases}$$

It follows from Exercise 18 that

$$\widehat{\frac{1}{x+i0}} + \widehat{\frac{1}{x-i0}} = -i\sqrt{2\pi}\operatorname{sgn}\xi = 2\left(-i\sqrt{\frac{\pi}{2}}\operatorname{sgn}\xi\right) = 2\widehat{\text{p.v.}\frac{1}{x}}$$

and thus

$$\frac{1}{x+i0} + \frac{1}{x-i0} = 2\text{p.v.}\frac{1}{x}.$$

In a similar fashion,

$$\widehat{\frac{1}{x-i0}} - \widehat{\frac{1}{x+i0}} = i\sqrt{2\pi}\cdot 1 = i\sqrt{2\pi}\sqrt{2\pi}\widehat{\delta} = 2\pi i\widehat{\delta}$$

and so

$$\frac{1}{x-i0} - \frac{1}{x+i0} = 2\pi i\delta.$$

Add and subtract to get finally

$$\frac{1}{x+i0} = \text{p.v.}\frac{1}{x} - i\pi\delta \qquad \text{and} \qquad \frac{1}{x-i0} = \text{p.v.}\frac{1}{x} + i\pi\delta.$$

Exercise 19. Prove that

1)

$$\widehat{\frac{1}{(x+i0)^2}} = -\sqrt{2\pi}\xi H(\xi) \quad \text{and} \quad \widehat{\frac{1}{(x-i0)^2}} = \sqrt{2\pi}\xi H(-\xi)$$

2)

$$\frac{1}{(x+i0)^2} + \frac{1}{(x-i0)^2} = 2\text{p.v.}\frac{1}{x^2} \quad \text{and} \quad \frac{1}{(x-i0)^2} - \frac{1}{(x+i0)^2} = -2\pi i\delta'$$

3)

$$\frac{1}{(x+i0)^2} = \text{p.v.}\frac{1}{x^2} + \pi i \delta' \quad \text{and} \quad \frac{1}{(x-i0)^2} = \text{p.v.}\frac{1}{x^2} - \pi i \delta'$$

4)

$$\widehat{\log|x|} = -\sqrt{\frac{\pi}{2}}\text{p.v.}\frac{1}{|\xi|}$$

5)

$$\widehat{x^\beta} = (2\pi)^{n/2} i^{|\beta|} \partial^\beta \delta.$$

Exercise 20. Prove that

1)

$$\widehat{H}(\xi) = -\frac{i}{\sqrt{2\pi}} \cdot \frac{1}{\xi - i0}$$

2)

$$\widehat{\text{sgn}}(\xi) = -\frac{i}{\sqrt{\frac{\pi}{2}}}\text{p.v.}\frac{1}{\xi}.$$

Example 4.10. Since

$$\begin{aligned}\langle \widehat{\partial^\alpha \delta}, \varphi \rangle &= \langle \partial^\alpha \delta, \widehat{\varphi} \rangle = (-1)^{|\alpha|} \langle \delta, \partial^\alpha \widehat{\varphi} \rangle = \langle \delta, i^{|\alpha|} \widehat{\xi^\alpha \varphi} \rangle \\ &= \langle \widehat{\delta}, (i\xi)^\alpha \varphi \rangle = \langle (2\pi)^{-\frac{n}{2}}, (i\xi)^\alpha \varphi \rangle = \langle (2\pi)^{-\frac{n}{2}} (i\xi)^\alpha, \varphi \rangle\end{aligned}$$

we get

$$\widehat{\partial^\alpha \delta} = (2\pi)^{-\frac{n}{2}} (i\xi)^\alpha.$$

In particular, in dimension one,

$$\widehat{\delta^{(k)}} = \frac{1}{\sqrt{2\pi}} i^k \xi^k, \quad \widehat{x^k} = \sqrt{2\pi} i^k \delta^{(k)}.$$

Consider the Cauchy-Riemann operators

$$\bar{\partial} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and

$$\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

in $\mathbb{R}^2$.

Let us prove the following facts about these operators:

1)

$$\partial \cdot \bar{\partial} = \bar{\partial} \cdot \partial = \frac{1}{4}\Delta,$$

2)

$$\frac{1}{\pi}\bar{\partial}\left(\frac{1}{z}\right) = \delta \quad \text{in } \mathbb{R}^2$$

The last fact means that

$$\frac{1}{\pi}\frac{1}{x+iy}$$

is the fundamental solution of $\bar{\partial}$. Taking the Fourier transform of 2) gives us

$$\widehat{\bar{\partial}\frac{1}{z}}(\xi) = \pi\widehat{\delta}(\xi)$$

which is equivalent to

$$\frac{1}{2}(i\xi_1 - \xi_2)\cdot\widehat{\frac{1}{z}}(\xi) = \pi\cdot(2\pi)^{-1}\cdot 1 = \frac{1}{2}$$

or

$$\widehat{\frac{1}{z}}(\xi) = \frac{1}{i\xi_1 - \xi_2} = -i\frac{1}{\xi_1 + i\xi_2}, \quad \xi \neq 0.$$

Let us check that this indeed holds true. We have, by Example 3.2,

$$\begin{aligned}
\widehat{\frac{1}{z}}(\xi) &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{e^{-i(\xi_1 x+\xi_2 y)}}{x+iy}dxdy = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\xi_2 y}dy\int_{-\infty}^{\infty}\frac{e^{-i\xi_1 x}}{x+iy}dx \\
&= \frac{1}{2\pi}\int_0^{\infty}e^{-i\xi_2 y}dy\int_{-\infty}^{\infty}\frac{e^{-i\xi_1 x}}{x+iy}dx + \frac{1}{2\pi}\int_{-\infty}^{0}e^{-i\xi_2 y}dy\int_{-\infty}^{\infty}\frac{e^{-i\xi_1 x}}{x+iy}dx \\
&= \frac{1}{2\pi}\int_0^{\infty}e^{-i\xi_2 y}\sqrt{2\pi}(-i\sqrt{2\pi}H(\xi_1)e^{-y\xi_1})dy \\
&+ \frac{1}{2\pi}\int_{-\infty}^{0}e^{-i\xi_2 y}\sqrt{2\pi}(i\sqrt{2\pi}H(-\xi_1)e^{-y\xi_1})dy \\
&= -i\left(H(\xi_1)\int_0^{\infty}e^{-y(\xi_1+i\xi_2)}dy - H(-\xi_1)\int_{-\infty}^{0}e^{-y(\xi_1+i\xi_2)}dy\right).
\end{aligned}$$

For $\xi_1 > 0$ we have

$$-i\int_0^{\infty}e^{-y(\xi_1+i\xi_2)}dy = i\left.\frac{e^{-y(\xi_1+i\xi_2)}}{\xi_1+i\xi_2}\right|_0^{\infty} = -i\frac{1}{\xi_1+i\xi_2}.$$

For $\xi_1 < 0$ we have

$$i\int_{-\infty}^{0}e^{-y(\xi_1+i\xi_2)}dy = -i\left.\frac{e^{-y(\xi_1+i\xi_2)}}{\xi_1+i\xi_2}\right|_{-\infty}^{0} = -i\frac{1}{\xi_1+i\xi_2}.$$

Hence

$$\widehat{\frac{1}{z}}(\xi) = -\frac{i}{\xi_1+i\xi_2}$$

which proves 2). Part 1) is established with a simple calculation:

$$\partial\cdot\bar{\partial} = \frac{1}{4}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{1}{4}\left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right) = \frac{1}{4}\Delta = \bar{\partial}\cdot\partial.$$

5 Convolutions in S and S'

Consider first the direct product of distributions. Let $T_1, T_2, \dots, T_n$ be one-dimensional tempered distributions, $T_j \in S'(\mathbb{R}), j = 1, 2, \dots, n$. The product $T_1(x_1) \cdots T_n(x_n)$ can be formally defined by

$$\begin{aligned}\langle T_1(x_1) \cdots T_n(x_n), \varphi(x_1, \dots, x_n)\rangle &= \langle T_1(x_1) \cdots T_{n-1}(x_{n-1}), \varphi_1(x_1, \dots, x_{n-1})\rangle \\ &= \langle T_1(x_1) \cdots T_{n-2}(x_{n-2}), \varphi_2(x_1, \dots, x_{n-2})\rangle \\ &= \cdots = \langle T_1(x_1), \varphi_{n-1}(x_1)\rangle,\end{aligned}$$

where

$$\begin{aligned}\varphi_1(x_1, \dots, x_{n-1}) &:= \langle T_n(x_n), \varphi(x_1, \dots, x_n)\rangle \in S(\mathbb{R}^{n-1}) \\ \varphi_j(x_1, \dots, x_{n-j}) &:= \langle T_{n-j+1}, \varphi_{j-1}(x_1, \dots, x_{n-j+1})\rangle \in S(\mathbb{R}^{n-j}).\end{aligned}$$

In this sense it is clear that

$$\delta(x_1, \dots, x_n) = \delta(x_1) \cdots \delta(x_n).$$

But the product $T_1(x)T_2(x)$, where x are the same, in general case does not exist, that is, it is impossible to define such product. We remedy this by recalling

Definition. The convolution $\varphi * \psi$ of the functions $\varphi \in S$ and $\psi \in S$ is defined as

$$(\varphi * \psi)(x) := \int_{\mathbb{R}^n} \varphi(x - y)\psi(y)dy.$$

We can observe the following immediately.

1) The convolution is commutative for any $n \geq 1$. If $n \geq 2$, then

$$(\varphi * \psi)(x) = \int_{\mathbb{R}^n} \varphi(x - y)\psi(y)dy = \int_{\mathbb{R}^n} \varphi(z)\psi(x - z)dz = (\psi * \varphi)(x).$$

If $n = 1$ then

$$\begin{aligned}(\varphi * \psi)(x) &= \int_{-\infty}^{\infty} \varphi(x - y)\psi(y)dy = -\int_{\infty}^{-\infty} \varphi(z)\psi(x - z)dz \\ &= \int_{-\infty}^{\infty} \psi(x - z)\varphi(z)dz = (\psi * \varphi)(x).\end{aligned}$$

2) It is also clear that the convolution is well-defined for φ and ψ from S, and moreover for any $\alpha \geq 0$,

$$\begin{aligned}\partial_x^\alpha(\varphi * \psi)(x) &= (\partial^\alpha\varphi * \psi)(x) = \int_{\mathbb{R}^n} \partial_x^\alpha\varphi(x - y)\psi(y)dy \\ &= \int_{\mathbb{R}^n} (-1)^{|\alpha|}\partial_y^\alpha\varphi(x - y)\psi(y)dy = (-1)^{2|\alpha|} \int_{\mathbb{R}^n} \varphi(x - y)\partial_y^\alpha\psi(y)dy \\ &= (\varphi * \partial^\alpha\psi)(x),\end{aligned}$$

where we integrated by parts and used the fact that $\partial_{x_j}\varphi(x-y) = -\partial_{y_j}\varphi(x-y)$.

We would like to prove that for φ and ψ from S it follows that $\varphi * \psi$ from S also. In fact,

a) $\varphi * \psi \in C^\infty(\mathbb{R}^n)$ since $\partial^\alpha(\varphi * \psi) = \varphi * \partial^\alpha \psi$ and $\partial^\alpha : S \to S$.

b) $\varphi * \psi$ decreases at the infinity faster than any inverse power:

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} \varphi(x-y)\psi(y)dy \right| &\le c_1 \int_{|y|\le \frac{|x|}{2}} \frac{1}{|x-y|^m} |\psi(y)|dy + c_2 \int_{|y|>\frac{|x|}{2}} |\psi(y)|dy \\
&\le \frac{c_1'}{|x|^m} \int_{|y|\le \frac{|x|}{2}} |\psi(y)|dy + c_2 \int_{|y|>\frac{|x|}{2}} |y|^{-m}|y|^m|\psi(y)|dy \\
&\le \frac{c_1''}{|x|^m} + \frac{c_2''}{|x|^m} = c|x|^{-m}, \quad m \in \mathbb{N}.
\end{aligned}$$

Next we collect some important inequalities involving the convolution.

1) Hölder's inequality implies that

$$\|\varphi * \psi\|_{L^\infty(\mathbb{R}^n)} \le \|\varphi\|_{L^p(\mathbb{R}^n)} \cdot \|\psi\|_{L^{p'}(\mathbb{R}^n)}, \tag{5.1}$$

where $\frac{1}{p} + \frac{1}{p'} = 1, 1 \le p \le \infty$. It means that the convolution is well-defined even for $\varphi \in L^p(\mathbb{R}^n)$ and $\psi \in L^{p'}(\mathbb{R}^n)$. In particular,

$$\|\varphi * \psi\|_{L^\infty(\mathbb{R}^n)} \le \|\varphi\|_{L^1(\mathbb{R}^n)} \cdot \|\psi\|_{L^\infty(\mathbb{R}^n)}. \tag{5.2}$$

2) It follows from Fubini's theorem that

$$\begin{aligned}
\|\varphi * \psi\|_{L^1} &\le \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} |\varphi(x-y)||\psi(y)|dy \\
&= \int_{\mathbb{R}^n} |\psi(y)|dy \int_{\mathbb{R}^n} |\varphi(x-y)|dx = \|\varphi\|_{L^1} \|\psi\|_{L^1}.
\end{aligned} \tag{5.3}$$

3) Interpolating (5.2) and (5.3) leads us to

$$\|\varphi * \psi\|_{L^p} \le \|\varphi\|_{L^1} \cdot \|\psi\|_{L^p}. \tag{5.4}$$

4) Interpolating (5.1) and (5.3) leads us to

$$\|\varphi * \psi\|_{L^s} \le \|\varphi\|_{L^r} \cdot \|\psi\|_{L^q}, \tag{5.5}$$

where

$$\frac{1}{s} = \frac{\theta}{\infty} + \frac{1-\theta}{1}, \quad \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{1}, \quad \frac{1}{q} = \frac{\theta}{p'} + \frac{1-\theta}{1}.$$

It follows that

$$\theta = 1 - \frac{1}{s}.$$

Thus

$$\frac{1}{r}=\frac{1}{p}\left(1-\frac{1}{s}\right)+\frac{1}{s}=\frac{1}{p}+\frac{1}{s}\left(1-\frac{1}{p}\right)=\frac{1}{p}+\frac{1}{s}\cdot\frac{1}{p'}=1-\frac{1}{p'}+\frac{1}{s}\cdot\frac{1}{p'}.$$

But

$$\frac{1}{p'}=\frac{1}{q}\cdot\frac{1}{\theta}-\frac{1-\theta}{\theta}=\left(\frac{1}{q}-1\right)\frac{1}{\theta}+1=\left(\frac{1}{q}-1\right)\cdot\left(\frac{1}{1-\frac{1}{s}}\right)+1.$$

Finally we have

$$\frac{1}{r}=1-\frac{1}{p'}\left(1-\frac{1}{s}\right)=1-\left(1-\frac{1}{s}\right)\left[\left(\frac{1}{q}-1\right)\frac{1}{1-\frac{1}{s}}+1\right]=1-\frac{1}{q}+1-1+\frac{1}{s}$$

or

$$1+\frac{1}{s}=\frac{1}{r}+\frac{1}{q}.$$

Now we are in the position to consider the Fourier transform of a convolution.

1) Let $\varphi,\psi\in S$. Then $\varphi*\psi\in S$ and $F(\varphi*\psi)\in S$. Moreover,

$$\begin{aligned}F(\varphi*\psi) &= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{-i(x,\xi)}dx\int_{\mathbb{R}^n}\varphi(x-y)\psi(y)dy\\ &= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\psi(y)dy\int_{\mathbb{R}^n}e^{-i(x,\xi)}\varphi(x-y)dx\\ &= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\psi(y)e^{-i(y,\xi)}dy\int_{\mathbb{R}^n}\varphi(z)e^{-i(z,\xi)}dz=(2\pi)^{\frac{n}{2}}F\varphi\cdot F\psi,\end{aligned}$$

i.e.

$$\widehat{\varphi*\psi}=(2\pi)^{\frac{n}{2}}\widehat{\varphi}\cdot\widehat{\psi}.$$

Similarly,

$$F^{-1}(\varphi*\psi)=(2\pi)^{\frac{n}{2}}F^{-1}\varphi\cdot F^{-1}\psi.$$

Hence

$$\varphi*\psi=(2\pi)^{\frac{n}{2}}F(\underbrace{F^{-1}\varphi}_{\varphi_1}\cdot\underbrace{F^{-1}\psi}_{\psi_1})$$

which implies that

$$F\varphi_1*F\psi_1=(2\pi)^{\frac{n}{2}}F(\varphi_1\cdot\psi_1)$$

or

$$\widehat{\varphi\cdot\psi}=(2\pi)^{-\frac{n}{2}}\widehat{\varphi}*\widehat{\psi}.$$

2) Let us assume that $\varphi\in L^1$ and $\psi\in L^p, 1\le p\le 2$. Then (5.4) implies that $\varphi*\psi\in L^p, 1\le p\le 2$. Further, $F(\varphi*\psi)$ belongs to $L^{p'}$ by Hausdorff-Young inequality. Thus,

$$\widehat{\varphi*\psi}=(2\pi)^{\frac{n}{2}}\widehat{\varphi}\cdot\widehat{\psi}\in L^{p'}.$$

Lemma 1. *Let $\varphi(x)$ be a function from $L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi(x)dx = 1$ and let $\psi(x)$ be a function from $L^2(\mathbb{R}^n)$. Let us set $\varphi_\varepsilon(x) := \varepsilon^{-n}\varphi\left(\frac{x}{\varepsilon}\right), \varepsilon > 0$. Then*

$$\lim_{\varepsilon\to+0} \varphi_\varepsilon * \psi \overset{L^2(\mathbb{R}^n)}{=} \psi.$$

Proof. By (5.4) we have that $\varphi_\varepsilon * \psi \in L^2(\mathbb{R}^n)$. Then

$$\widehat{\varphi_\varepsilon * \psi} = (2\pi)^{\frac{n}{2}}\widehat{\varphi_\varepsilon} \cdot \widehat{\psi}$$

in L^2. But

$$\widehat{\varphi_\varepsilon} = \varepsilon^{-n}\widehat{\varphi\left(\frac{x}{\varepsilon}\right)} = \varepsilon^{-n}\widehat{\sigma_{\frac{1}{\varepsilon}}\varphi}(\xi) = \varepsilon^{-n}\left(\frac{1}{\varepsilon}\right)^{-n}\widehat{\varphi}(\varepsilon\xi) = \widehat{\varphi}(\varepsilon\xi) \overset{L^\infty}{\to} \widehat{\varphi}(0), \quad \varepsilon \to +0.$$

Note also that

$$\widehat{\varphi}(0) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-i(0,x)}\varphi(x)dx = (2\pi)^{-\frac{n}{2}}.$$

Hence

$$\widehat{\varphi_\varepsilon * \psi} = (2\pi)^{\frac{n}{2}}\widehat{\varphi}(\varepsilon\xi) \cdot \widehat{\psi}(\xi) \overset{L^2}{\to} \widehat{\psi}(\xi), \quad \varepsilon \to +0.$$

By Fourier inversion formula it follows that

$$\varphi_\varepsilon * \psi \overset{L^2}{\to} \psi, \quad \varepsilon \to +0.$$

□

Theorem 1. *For any fixed function φ from $S(\mathbb{R}^n)$ the map $\varphi * T$ has, as a linear continuous map from S to S (with respect to T), a unique linear continuous extension as a map from S' to S' (with respect to T) as follows:*

$$\langle \varphi * T, \psi\rangle := \langle T, R\varphi * \psi\rangle,$$

where $R\varphi(x) := \varphi(-x)$. Moreover, this extension has the properties

1) $\widehat{\varphi * T} = (2\pi)^{\frac{n}{2}}\widehat{\varphi} \cdot \widehat{T}$
2) $\partial^\alpha(\varphi * T) = \partial^\alpha\varphi * T = \varphi * \partial^\alpha T.$

Proof. Let us assume that φ, ψ and T belong to the Schwartz space $S(\mathbb{R}^n)$. Then we have checked already the properties 1) and 2) above. But we can easily check that for such functions the definition is also true. In fact,

$$\begin{aligned}\langle \varphi * T, \psi\rangle &= \int_{\mathbb{R}^n}(\varphi * T)(x)\psi(x)dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\varphi(x-y)T(y)dy\psi(x)dx \\ &= \int_{\mathbb{R}^n} T(y)\int_{\mathbb{R}^n}\varphi(x-y)\psi(x)dxdy \\ &= \int_{\mathbb{R}^n} T(y)dy\int_{\mathbb{R}^n} R\varphi(y-x)\psi(x)dx = \langle T, R\varphi * \psi\rangle.\end{aligned}$$

For the case $T \in S'$ the statement of this theorem follows from the fact that $\overline{S} \overset{S'}{=} S'$ (see Theorem 1 from Chapter 4). □

Corollary. *Since $\varphi * T = T * \varphi$ for φ and T from S then we may define $T * \varphi$ as follows (for $T \in S'$)*

$$\langle T * \varphi, \psi \rangle := \langle T, R\varphi * \psi \rangle.$$

Example 5.1. 1) It is true that $\delta * \varphi = \varphi$. Indeed,

$$\begin{aligned} \langle \delta * \varphi, \psi \rangle &= \langle \delta, R\varphi * \psi \rangle = (R\varphi * \psi)(0) \\ &= \int_{\mathbb{R}^n} \varphi(y - x)\psi(y)dy \,|_{x=0} = \int_{\mathbb{R}^n} \varphi(y)\psi(y)dy = \langle \varphi, \psi \rangle. \end{aligned}$$

Alternatively, we note that

$$\widehat{\delta * \varphi} = (2\pi)^{\frac{n}{2}} \widehat{\delta} \cdot \widehat{\varphi} = 1 \cdot \widehat{\varphi} = \widehat{\varphi}$$

is equivalent to

$$\delta * \varphi = \varphi$$

in S'.

2) Property 2) of Theorem 1 and part 1) of this example imply that

$$\partial^\alpha(\delta * \varphi) = \delta * \partial^\alpha \varphi = \partial^\alpha \varphi.$$

3) Let us consider again the equation $(1 - \Delta)u = f$ for u and $f \in L^2$ (or even from S'). Then $(1 + |\xi|^2)\widehat{u} = \widehat{f}$ is still valid in L^2 and $\widehat{u} = (1 + |\xi|^2)^{-1}\widehat{f}$ or

$$u(x) = F^{-1}\left(\frac{1}{1 + |\xi|^2}\widehat{f}\right) = (2\pi)^{-\frac{n}{2}} F^{-1}\left(\frac{1}{1 + |\xi|^2}\right) * f = \int_{\mathbb{R}^n} K(x - y)f(y)dy,$$

where

$$K(x - y) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{i(x-y,\xi)}}{1 + |\xi|^2} d\xi.$$

This is the inverse Fourier transform of locally integrable function. This function K is the *free space Green's function* of the operator $1 - \Delta$ in $\mathbb{R}^n$. We calculate this integral precisely later.

Lemma 2. *Let $j(x)$ be a function from $L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} j(x)dx = 1$. Set $j_\varepsilon(x) = \varepsilon^{-n} j\left(\frac{x}{\varepsilon}\right), \varepsilon > 0$. Then*

$$\|j_\varepsilon * f - f\|_{L^p} \to 0, \quad \varepsilon \to +0$$

for any function $f \in L^p(\mathbb{R}^n), 1 \le p < \infty$. In the case $p = \infty$ we can state only the fact

$$\int_{\mathbb{R}^n} (j_\varepsilon * f)\overline{g}dx \to \int_{\mathbb{R}^n} f \cdot \overline{g}dx, \quad \varepsilon \to +0$$

for any $g \in L^1(\mathbb{R}^n)$.

Exercise 21. Prove Lemma 2 and find a counterexample as to why the first part fails for $p = \infty$.

Remark. If $j \in C_0^\infty(\mathbb{R}^n)$ or $S(\mathbb{R}^n)$ then $j_\varepsilon * f \in C_0^\infty(\mathbb{R}^n)$ or $S(\mathbb{R}^n)$ also for any $f \in L^p(\mathbb{R}^n), 1 \le p < \infty$.

6 Sobolev spaces

Lemma 1. *For any function $f \in L^2(\mathbb{R}^n)$ the following statements are equivalent:*

1) $\frac{\partial f}{\partial x_j}(x) \in L^2(\mathbb{R}^n)$,

2) $\xi_j \widehat{f}(\xi) \in L^2(\mathbb{R}^n)$,

3) $\lim\limits_{t\to 0} \frac{\Delta_j^t f(x)}{t}$ *exists in* $L^2(\mathbb{R}^n)$. *Here* $\Delta_j^t f(x) := f(x+te_j) - f(x)$ *with* $t \in \mathbb{R}$ *and* $e_j = (0,\dots,1,0,\dots,0)$.

4) *There exists* $\{f_k\}_{k=1}^\infty, f_k \in S$, *such that* $f_k \xrightarrow{L^2} f$ *and* $\frac{\partial f_k}{\partial x_j}$ *has a limit in* $L^2(\mathbb{R}^n)$,

Proof. 1) $\Leftrightarrow$ 2)**:** Since

$$\widehat{D_j f} = \xi_j \widehat{f}$$

we have

$$\left\|\xi_j \widehat{f}\right\|_{L^2} = \|D_j f\|_{L^2}$$

by the Parseval equality.

2) $\Rightarrow$ 3)**:** Let $\xi_j \widehat{f}$ be a function from $L^2(\mathbb{R}^n)$. Then the equality

$$\widehat{\frac{1}{t}\Delta_j^t f}(\xi) = \frac{1}{t}(e^{it\xi_j} - 1)\widehat{f}(\xi) = \frac{e^{it\xi_j} - 1}{t\xi_j} \cdot \xi_j \widehat{f}(\xi)$$

holds. But

$$\frac{e^{it\xi_j} - 1}{t\xi_j} \to i$$

pointwise as $t \to 0$. Hence

$$\widehat{\frac{1}{t}\Delta_j^t f} \xrightarrow{L^2} i\xi_j \widehat{f}, \quad t \to 0$$

i.e. (again due to Parseval equality)

$$\frac{1}{t}\Delta_j^t f \xrightarrow{L^2} \frac{\partial f}{\partial x_j}, \quad t \to 0.$$

The same arguments lead us to the statement that 3) $\Rightarrow$ 1).

4) $\Rightarrow$ 1)**:** Let f_k be a sequence from S such that $f_k \xrightarrow{L^2} f$. Then $f_k \xrightarrow{S'} f$ and $\frac{\partial f_k}{\partial x_j} \xrightarrow{S'} \frac{\partial f}{\partial x_j}$ also. By the condition 4) we have that the limit $\lim\limits_{k\to\infty} \frac{\partial f_k}{\partial x_j} \stackrel{L^2}{=} g$ exists. That's why we may conclude that $\frac{\partial f_k}{\partial x_j} \xrightarrow{S'} g$. It means that $g = \frac{\partial f}{\partial x_j}$ in S'.

2) $\Rightarrow$ 4)**:** Write $\widehat{f}(\xi)$ as the sum of two functions $\widehat{f}(\xi) = g(\xi) + h(\xi)$, where

$$g(\xi) = \widehat{f}(\xi)\chi_{\{|\xi_j|<1\}}, \quad h(\xi) = \widehat{f}(\xi)\chi_{\{|\xi_j|>1\}}.$$

Let $\{g_k\}$ be a sequence in S such that, $g_k \overset{L^2}{\to} g$ and $\operatorname{supp} g_k \subset \{|\xi_j| < 2\}$. Let $\{h_k\}$ be a sequence in S such that, $h_k \overset{L^2}{\to} \xi_j h$ and $\operatorname{supp} h_k \subset \{|\xi_j| > \frac{1}{2}\}$. If we define the sequence $f_k(x) = F^{-1}\left(g_k + \frac{h_k}{\xi_j}\right)(x)$, then

$$\widehat{f_k}(\xi) = g_k + \frac{h_k}{\xi_j} \overset{L^2}{\to} g(\xi) + h(\xi) = \widehat{f}(\xi).$$

But

$$\widehat{\frac{\partial f_k}{\partial x_j}} = i\xi_j g_k + ih_k \overset{L^2}{\to} i\xi_j(g+h) = i\xi_j\widehat{f}.$$

It means that (by Fourier inversion formula or Parseval equality)

$$\frac{\partial f_k}{\partial x_j} \overset{L^2}{\to} F^{-1}(i\xi_j\widehat{f}) = \frac{\partial f}{\partial x_j}.$$

□

We have also the following generalization of Lemma 1 to multi-index α.

Lemma 2. *Let f be a function from $L^2(\mathbb{R}^n)$ and let $s \in \mathbb{N}$. Then the following statements are equivalent:*

1) $D^\alpha f \in L^2(\mathbb{R}^n), |\alpha| \le s$;
2) $\xi^\alpha \widehat{f} \in L^2(\mathbb{R}^n), |\alpha| \le s$;
3) $\lim_{h\to 0} \frac{\Delta_h^\alpha f}{h^\alpha}$ *exists in* $L^2(\mathbb{R}^n)$, $|\alpha| \le s$. *Here* $\Delta_h^\alpha f := (\Delta_{h_1}^{\alpha_1} \cdots \Delta_{h_n}^{\alpha_n})f$ *and* $h \in \mathbb{R}^n$ *with* $h_j \neq 0$ *for all* $j = 1, 2, \ldots, n$.
4) *There exists* $f_k \in S$ *such that,* $f_k \overset{L^2}{\to} f$ *and* $D^\alpha f_k$ *has a limit in* $L^2(\mathbb{R}^n)$ *for* $|\alpha| \le s$.

Proof. Follows from Lemma 1 by induction on $|\alpha|$. □

Definition. Let $s > 0$ be an integer. Then

$$H^s(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) : \sum_{|\alpha|\le s} \|D^\alpha f\|_{L^2} < \infty\}$$

is the (L^2-based) Sobolev space of order s with the norm

$$\|f\|_{H^s(\mathbb{R}^n)} := \left(\sum_{|\alpha|\le s} \|D^\alpha f\|^2_{L^2(\mathbb{R}^n)}\right)^{1/2}.$$

Remark. It is easy to check that $H^s(\mathbb{R}^n), s \in \mathbb{N}$, can be characterized by

$$H^s(\mathbb{R}^n)=\{f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi < \infty\}.$$

Proof. It follows from Parseval equality that

$$\begin{aligned}\sum_{|\alpha|\le s} \|D^\alpha f\|_{L^2}^2 &= \sum_{|\alpha|\le s} \left\|\widehat{D^\alpha f}\right\|_{L^2}^2 = \sum_{|\alpha|\le s} \left\|\xi^\alpha \widehat{f}\right\|_{L^2}^2 \\ &= \sum_{|\alpha|\le s} \int_{\mathbb{R}^n} |\xi^\alpha|^2 |\widehat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} \sum_{|\alpha|\le s} |\xi^\alpha|^2 |\widehat{f}(\xi)|^2 d\xi.\end{aligned}$$

But it is easily seen that there are positive constants c_1 and c_2 such that

$$c_1(1+|\xi|^2)^s \le \sum_{|\alpha|\le s} |\xi^\alpha|^2 \le c_2(1+|\xi|^2)^s$$

or

$$\sum_{|\alpha|\le s} |\xi|^{2\alpha} \asymp (1+|\xi|^2)^s.$$

Therefore we may conclude, that

$$\sum_{|\alpha|\le s} \|D^\alpha f\|_{L^2} < \infty \quad \Leftrightarrow \quad \sum_{|\alpha|\le s} \|D^\alpha f\|_{L^2}^2 < \infty \quad \Leftrightarrow \quad \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi < \infty.$$

□

This property for an integer s justifies the following definition.

Definition. Let s be a real number. Then

$$H^s(\mathbb{R}^n) := \{f \in S' : (1+|\xi|^2)^{\frac{s}{2}} \widehat{f} \in L^2(\mathbb{R}^n)\}$$

with the norm

$$\|f\|_{H^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}}.$$

Definition. Let $s > 0$ be an integer and $1 \le p \le \infty$. Then

$$W_p^s(\mathbb{R}^n) := \{f \in L^p(\mathbb{R}^n) : \sum_{|\alpha|\le s} \|D^\alpha f\|_{L^p(\mathbb{R}^n)} < \infty\}$$

is called the Sobolev space with norm

$$\|f\|_{W_p^s(\mathbb{R}^n)} := \left(\sum_{|\alpha|\le s} \|D^\alpha f\|_{L^p(\mathbb{R}^n)}^p\right)^{1/p}.$$

Exercise 22. Let $s > 0$ be an even integer and $1 \le p \le \infty$. Prove that

$$|f|_{W_p^s(\mathbb{R}^n)} := \Big(\int_{\mathbb{R}^n} |F^{-1}((1+|\xi|^2)^{\frac{s}{2}}\widehat{f})|^p dx\Big)^{\frac{1}{p}}$$

is an equivalent norm in $W_p^s(\mathbb{R}^n)$.

Definition. Let $s > 0$ be a real number and $1 \le p \le \infty$. Then

$$W_p^s(\mathbb{R}^n) := \{f \in S' : \Big(\int_{\mathbb{R}^n} |F^{-1}((1+|\xi|^2)^{\frac{s}{2}}\widehat{f})|^p dx\Big)^{\frac{1}{p}} < \infty\}$$

with the norm

$$\|f\|_{W_p^s(\mathbb{R}^n)} := \Big(\int_{\mathbb{R}^n} |F^{-1}((1+|\xi|^2)^{\frac{s}{2}}\widehat{f})|^p dx\Big)^{\frac{1}{p}}.$$

Exercise 23. Let $s \in \mathbb{R}$. Prove that

$$f \in H^s(\mathbb{R}^n)$$

if and only if

$$\widehat{f} \in L_s^2(\mathbb{R}^n).$$

Proposition. *Let us assume that $0 < s < 1$. Then*

$$\int_{\mathbb{R}^n} (1+|\xi|^{2s})|\widehat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |f(x)|^2 dx + A_s \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}} dxdy, \tag{6.1}$$

where A_s is a positive constant depending on s and n.

Remark. Since $1+|\xi|^{2s} \asymp (1+|\xi|^2)^s, 0 < s < 1$, then the right hand side of (6.1) is an equivalent norm in $H^s(\mathbb{R}^n)$.

Proof. Denote by I the double integral appearing in the right hand side of (6.1). Then

$$\begin{aligned} I &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(y+z)-f(y)|^2 |z|^{-n-2s} dydz \\ &= \int_{\mathbb{R}^n} |z|^{-n-2s} dz \int_{\mathbb{R}^n} |e^{i(z,\xi)}-1|^2 |\widehat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 d\xi \int_{\mathbb{R}^n} \frac{|e^{i(z,\xi)}-1|^2}{|z|^{n+2s}} dz \end{aligned}$$

by Parseval equality and Exercise 4. We claim that

$$\int_{\mathbb{R}^n} \frac{|e^{i(z,\xi)}-1|^2}{|z|^{n+2s}} dz = |\xi|^{2s} A_s^{-1}.$$

Indeed, if we consider the Householder reflection matrix

$$A := I - \frac{2vv^T}{|v|^2}, \quad v = \xi - |\xi|e_1, \quad \xi \in \mathbb{R}^n$$

then $A^T = A^{-1} = A$ and $A\xi = |\xi|e_1 = (|\xi|, 0, \dots, 0)$. It follows that

$$|\xi|^{-2s}\int_{\mathbb{R}^n}\frac{|e^{i(z,\xi)}-1|^2}{|z|^{n+2s}}dz = |\xi|^{-2s}\int_{\mathbb{R}^n}\frac{|e^{i(Az,A\xi)}-1|^2}{|z|^{n+2s}}dz = |\xi|^{-2s}\int_{\mathbb{R}^n}\frac{|e^{i(y,A\xi)}-1|^2}{|y|^{n+2s}}dy$$
$$= |\xi|^{-2s}\int_{\mathbb{R}^n}\frac{|e^{iy_1|\xi|}-1|^2}{|y|^{n+2s}}dy = \int_{\mathbb{R}^n}\frac{|e^{iz_1}-1|^2}{|z|^{n+2s}}dz := A_s^{-1}.$$

Therefore

$$\int_{\mathbb{R}^n}|f(x)|^2dx + A_s\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f(x)-f(y)|^2|x-y|^{-n-2s}dxdy = \int_{\mathbb{R}^n}|\widehat{f}(\xi)|^2d\xi + \int_{\mathbb{R}^n}|\xi|^{2s}|\widehat{f}(\xi)|^2d\xi.$$

□

Remark. Note that A_s exists only for $0 < s < 1$.

Exercise 24. Prove that

$$\|f\|^2_{H^{k+s}(\mathbb{R}^n)} \asymp \int_{\mathbb{R}^n}(1+|\xi|^{2k+2s})|\widehat{f}(\xi)|^2d\xi \asymp \left\|\widehat{f}\right\|^2_{L^2} + \sum_{|\alpha|=k}\int_{\mathbb{R}^n}|\xi|^{2s}|\widehat{D^\alpha f}|^2d\xi$$
$$= \|f\|^2_{L^2} + A_s\sum_{|\alpha|=k}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|D^\alpha f(x) - D^\alpha f(y)|^2\cdot|x-y|^{-n-2s}dxdy.$$

Example 6.1. $1 \notin H^s(\mathbb{R}^n)$ for any s. Indeed, assume that $1 \in H^{s_0}(\mathbb{R}^n)$ for some s_0 (it is clear that $s_0 > 0$). It means that $(1+|\xi|^2)^{\frac{s_0}{2}}\widehat{1} \in L^2(\mathbb{R}^n)$. It follows from this fact that $\widehat{1} \in L^2_{\text{loc}}(\mathbb{R}^n)$ and, further, $\widehat{1} \in L^1_{\text{loc}}(\mathbb{R}^n)$. But $\widehat{1} = (2\pi)^{\frac{n}{2}}\delta$, and we know that δ is not a regular distribution.

Next we list some properties of $H^s(\mathbb{R}^n)$.

1) Since $f \in H^s(\mathbb{R}^n)$ if and only if $\widehat{f}(\xi) \in L^2_s(\mathbb{R}^n)$ and $L^2_s(\mathbb{R}^n)$ is a separable Hilbert space with the scalar product

$$(f_1, g_1)_{L^2_s(\mathbb{R}^n)} = \int_{\mathbb{R}^n}(1+|\xi|^2)^s f_1\cdot\overline{g_1}d\xi$$

then $H^s(\mathbb{R}^n)$ is also a separable Hilbert space and the scalar product can be defined by

$$(f, g)_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n}(1+|\xi|^2)^s\widehat{f}\cdot\overline{\widehat{g}}d\xi.$$

Let us define $H^s(\mathbb{R}^n)$ for negative s.

Definition. Let us set for any positive real number s that

$$H^{-s}(\mathbb{R}^n) := (H^s(\mathbb{R}^n))^*,$$

where $(H^s(\mathbb{R}^n))^*$ denotes the adjoint space of $H^s(\mathbb{R}^n)$ in the sense of Hilbert space $L^2(\mathbb{R}^n)$ with the norm defined by

$$\|f\|_{H^{-s}(\mathbb{R}^n)} := \sup_{0\neq g\in H^s(\mathbb{R}^n)}\frac{|(f,g)_{L^2(\mathbb{R}^n)}|}{\|g\|_{H^s(\mathbb{R}^n)}}.$$

2) For $-\infty < s < t < \infty$ it follows that $S \subset H^t(\mathbb{R}^n) \subset H^s(\mathbb{R}^n) \subset S'$.

Example 6.2. $\delta \in H^s(\mathbb{R}^n)$ if and only if $s < -\frac{n}{2}$. Indeed, if we denote

$$\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}$$

then $\delta \in H^s(\mathbb{R}^n)$ is equivalent to $(2\pi)^{-\frac{n}{2}} \langle \xi \rangle^s \in L^2(\mathbb{R}^n)$ which in turn is equivalent to $s < -\frac{n}{2}$.

3) Let φ be a function from $H^s(\mathbb{R}^n)$ and ψ a function from $H^{-s}(\mathbb{R}^n)$. Then $\widehat{\varphi} \in L^2_s(\mathbb{R}^n)$ and $\widehat{\psi} \in L^2_{-s}(\mathbb{R}^n)$, so that $\widehat{\varphi} \cdot \widehat{\psi} \in L^1(\mathbb{R}^n)$ by Hölder inequality. That's why we may define (momentarily, and with slight abuse of notation)

$$\langle \varphi, \psi \rangle_{L^2(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \widehat{\varphi} \cdot \widehat{\psi} d\xi$$

and get

$$|\langle \varphi, \psi \rangle_{L^2(\mathbb{R}^n)}| \le \|\varphi\|_{H^s(\mathbb{R}^n)} \cdot \|\psi\|_{H^{-s}(\mathbb{R}^n)}.$$

For example, if φ is a function from $H^{\frac{n}{2}+1+\varepsilon}(\mathbb{R}^n), \varepsilon > 0$ and $\psi = \frac{\partial \delta}{\partial x_j}$, then

$$\left\langle \frac{\partial \delta}{\partial x_j}, \varphi \right\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \widehat{\frac{\partial \delta}{\partial x_j}} \cdot \widehat{\varphi} d\xi = i(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \xi_j \widehat{\varphi}(\xi) d\xi$$

is well-defined, since $\widehat{\varphi} \in L^2_{\frac{n}{2}+1+\varepsilon}(\mathbb{R}^n)$ and $\xi_j \in L^2_{-\frac{n}{2}-1-\varepsilon}(\mathbb{R}^n)$.

4) Let

$$P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha$$

be a differential operator with constant coefficients. Then $P(D) : H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$ for any real s.

Proof.

$$\begin{aligned}
\|P(D)f\|^2_{H^{s-m}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (1 + |\xi|^2)^{s-m} |\widehat{P(D)f}|^2 d\xi \\
&= \int_{\mathbb{R}^n} (1 + |\xi|^2)^{s-m} |P(\xi)|^2 \cdot |\widehat{f}(\xi)|^2 d\xi \\
&\le c \int_{\mathbb{R}^n} (1 + |\xi|^2)^{s-m} (1 + |\xi|^2)^m |\widehat{f}(\xi)|^2 d\xi = c \|f\|^2_{H^s(\mathbb{R}^n)}.
\end{aligned}$$

□

There is a generalization of this result. Let $P(x, D)$ be a differential operator

$$P(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$$

with variable coefficients such that $|a_\alpha(x)| \le c_0$ for all $x \in \mathbb{R}^n$ and $|\alpha| \le m$. Then

$$P(x,D): H^m(\mathbb{R}^n) \to L^2(\mathbb{R}^n).$$

Indeed,

$$\begin{aligned} \|P(x,D)f\|_{L^2} &\le c_0 \sum_{|\alpha|\le m} \|D^\alpha f\|_{L^2} = c_0 \sum_{|\alpha|\le m} \left\|\xi^\alpha \widehat{f}\right\|_{L^2} \\ &\le c_0' \left\|(1+|\xi|^2)^{\frac{m}{2}} \widehat{f}\right\|_{L^2} = c_0' \|f\|_{H^m}. \end{aligned}$$

5)

Lemma 3. *Let φ be a function from S and f a function from $H^s(\mathbb{R}^n), s \in \mathbb{R}$. Then $\varphi \cdot f \in H^s(\mathbb{R}^n)$ and*

$$\|\varphi f\|_{H^s} \le c \left\|(1+|\xi|^2)^{\frac{|s|}{2}} \widehat{\varphi}\right\|_{L^1} \cdot \|f\|_{H^s}.$$

Proof. We know that

$$\widehat{\varphi \cdot f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \widehat{\varphi}(\xi-\eta)\widehat{f}(\eta)d\eta.$$

Hence

$$\langle\xi\rangle^s \widehat{\varphi \cdot f} = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \frac{\langle\xi\rangle^s}{\langle\eta\rangle^s} \widehat{\varphi}(\xi-\eta)\langle\eta\rangle^s \widehat{f}(\eta)d\eta.$$

Let us prove that

$$\langle\xi\rangle^s \le 2^{\frac{|s|}{2}} \langle\eta\rangle^s \cdot \langle\xi-\eta\rangle^{|s|}$$

for any $s \in \mathbb{R}$. Indeed,

$$\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}} \le (1+|\eta|^2)^{\frac{1}{2}} + |\xi-\eta| = \langle\eta\rangle + |\xi-\eta| \le \langle\eta\rangle(1+|\xi-\eta|).$$

Since $1+|\xi-\eta| \le \sqrt{2}\langle\xi-\eta\rangle$ we have

$$\langle\xi\rangle^s \le 2^{\frac{s}{2}} \langle\eta\rangle^s \cdot \langle\xi-\eta\rangle^s$$

for $s \ge 0$. Moreover, for $s < 0$ we have

$$\frac{\langle\xi\rangle^s}{\langle\eta\rangle^s} = \frac{\langle\eta\rangle^{|s|}}{\langle\xi\rangle^{|s|}} \le 2^{\frac{|s|}{2}} \langle\eta-\xi\rangle^{|s|}.$$

It now follows from (5.4) that

$$\|\varphi f\|_{H^s} = \left\|\langle\xi\rangle^s \widehat{\varphi f}\right\|_{L^2} \le c \left\||\langle\xi\rangle^{|s|}\widehat{\varphi}| * |\langle\eta\rangle^s \widehat{f}|\right\|_{L^2} \le c \left\|\langle\xi\rangle^{|s|}\widehat{\varphi}\right\|_{L^1} \cdot \left\|\langle\eta\rangle^s \widehat{f}\right\|_{L^2}$$

for any $s \in \mathbb{R}$. □

6) Let us consider now distributions with compact support in more detail than in Chapter 4.

Definition. Denote $\mathcal{E} = C^\infty(\mathbb{R}^n)$. We say that $T \in \mathcal{E}'$ if T is a linear functional on $\mathcal{E}$ which is also continuous i.e. $\varphi_k \to 0$ in $\mathcal{E}$ implies that $\langle T, \varphi_k\rangle \to 0$ in $\mathbb{C}$. Here $\varphi_k \to 0$ in $\mathcal{E}$ means that

$$\sup_K |\partial^\alpha \varphi_k| \to 0, \quad k \to \infty$$

for any compact subset $K \subset \mathbb{R}^n$ and for any multi-index α.

It can be proved that $T \in \mathcal{E}'$ if and only if there exist $c_0 > 0, R_0 > 0$ and $n_0 \in \mathbb{N}_0$ such that

$$|\langle T, \varphi\rangle| \le c_0 \sum_{|\alpha|\le n_0} \sup_{|x|\le R_0} |D^\alpha \varphi(x)|$$

for any $\varphi \in C^\infty(\mathbb{R}^n)$. Moreover, members of $\mathcal{E}'$ have compact support.

Assume that $T \in \mathcal{E}'$. Since $\varphi(x) = e^{-i(x,\xi)} \in C^\infty(\mathbb{R}^n)$ then $\langle T, e^{-i(x,\xi)}\rangle$ is well-defined and there exists $c_0 > 0, R_0 > 0$ and $n_0 \in \mathbb{N}_0$ such that

$$|\langle T, e^{-i(x,\xi)}\rangle| \le c_0 \sum_{|\alpha|\le n_0} \sup_{|x|\le R_0} |D_x^\alpha e^{-i(x,\xi)}| \le c_0 \sum_{|\alpha|\le n_0} |\xi^\alpha| \asymp (1+|\xi|^2)^{\frac{n_0}{2}}.$$

If we now set

$$\widehat{T}(\xi) := (2\pi)^{-n/2}\langle T, e^{-i(x,\xi)}\rangle$$

then $\widehat{T}$ is a usual function of ξ. The same is true for

$$\partial^\alpha \widehat{T}(\xi) = (2\pi)^{-n/2}(-1)^{|\alpha|}\langle T, \partial^\alpha e^{-i(x,\xi)}\rangle$$

and hence $\widehat{T} \in C^\infty(\mathbb{R}^n)$. On the other hand $|\langle T, e^{-i(x,\xi)}\rangle| \le c_0\langle\xi\rangle^{n_0}$ implies that $|\widehat{T}(\xi)| \le c_0'\langle\xi\rangle^{n_0}$ and hence $\widehat{T} \in L^2_s(\mathbb{R}^n)$ for $s < -n_0 - \frac{n}{2}$. So, by Exercise 23, we may conclude that any $T \in \mathcal{E}'$ belongs to $H^s(\mathbb{R}^n)$ for $s < -n_0 - \frac{n}{2}$.

7)

Lemma 4. *The closure of $C_0^\infty(\mathbb{R}^n)$ in the norm of $H^s(\mathbb{R}^n)$ is $H^s(\mathbb{R}^n)$ for any $s \in \mathbb{R}$. In short, $\overline{C_0^\infty(\mathbb{R}^n)} \overset{H^s}{=} H^s(\mathbb{R}^n)$.*

Proof. Let f be an arbitrary function from $H^s(\mathbb{R}^n)$ and let f_R be a new function such that

$$\widehat{f_R}(\xi) = \chi_R(\xi)\widehat{f}(\xi) = \begin{cases} \widehat{f}(\xi), & |\xi| < R, \\ 0, & |\xi| > R. \end{cases}$$

Then $f_R(x) = F^{-1}(\chi_R \widehat{f})(x) = (2\pi)^{-\frac{n}{2}}(F^{-1}\chi_R * f)(x)$. It follows from above considerations that $F^{-1}\chi_R \in C^\infty(\mathbb{R}^n)$ as an inverse Fourier transform of a compactly supported function (but $\notin C_0^\infty(\mathbb{R}^n)$) and

$$\|f - f_R\|^2_{H^s} = \int_{\mathbb{R}^n} |\widehat{f}(\xi) - \widehat{f_R}(\xi)|^2\langle\xi\rangle^{2s}d\xi = \int_{|\xi|>R} |\widehat{f}(\xi)|^2\langle\xi\rangle^{2s}d\xi \to 0, \quad R \to \infty$$

since $f \in H^s(\mathbb{R}^n)$. This was the first step.

The second step is as follows. Let $j(\xi) \in C_0^\infty(|\xi| < 1)$ with $\int_{\mathbb{R}^n} j(\xi)d\xi = 1$. Let us set $j_k(\xi) := k^n j(k\xi)$. We remember from Lemma 2 of Chapter 5 that $j_k * g \xrightarrow{L^p} g, 1 \le p < \infty$. Define the sequence $v_k := F^{-1}(j_k * \widehat{f_R})$. Since $\widehat{v_k} = j_k * \widehat{f_R}$ then $\operatorname{supp} \widehat{v_k} \subset U_{R+1}(0)$ and so $\widehat{v_k} \in C_0^\infty(\mathbb{R}^n)$. Hence $v_k \in S$. That's why $v_k \in H^s(\mathbb{R}^n)$ and

$$\begin{aligned} \|v_k - f_R\|^2_{H^s(\mathbb{R}^n)} &= \int_{|\xi|<R+1} \langle\xi\rangle^{2s} |j_k * \widehat{f_R} - \widehat{f_R}|^2 d\xi \\ &\le C_R \int_{|\xi|<R+1} |j_k * \widehat{f_R} - \widehat{f_R}|^2 d\xi \to 0, \quad k \to \infty. \end{aligned}$$

Since $v_k \notin C_0^\infty(\mathbb{R}^n)$ we take a function $\kappa \in C_0^\infty(\mathbb{R}^n)$ with $\kappa(0) = 1$. Then

$$\kappa\left(\frac{x}{A}\right) v_k \xrightarrow{S} v_k, \quad A \to \infty.$$

This fact implies that $\kappa\left(\frac{x}{A}\right) v_k \xrightarrow{H^s} v_k$ as $A \to \infty$. Setting $f_k(x) := \kappa\left(\frac{x}{A}\right) v_k(x) \in C_0^\infty(\mathbb{R}^n)$ we get finally

$$\|f - f_k\|_{H^s} \le \|f - f_R\|_{H^s} + \|f_R - v_k\|_{H^s} + \left\|v_k - \kappa\left(\frac{x}{A}\right) v_k\right\|_{H^s} \to 0$$

if A, k and R are large enough. □

Now we are in the position to formulate the main result concerning $H^s(\mathbb{R}^n)$.

Theorem 1 (Sobolev embedding theorem)**.** *Let f be a function from $H^s(\mathbb{R}^n)$ for $s > k + \frac{n}{2}$, where $k \in \mathbb{N}_0$. Then $D^\alpha f \in \dot{C}(\mathbb{R}^n)$ for all α such that $|\alpha| \le k$. In short,*

$$H^s \subset \dot{C}^k(\mathbb{R}^n), \quad s > k + \frac{n}{2}.$$

Proof. Let $f \in H^s(\mathbb{R}^n) \subset S'$. Then

$$D^\alpha f = F^{-1}F(D^\alpha f) = F^{-1}(\xi^\alpha \widehat{f}(\xi)).$$

What is more,

$$\begin{aligned} \int_{\mathbb{R}^n} |\xi^\alpha \widehat{f}(\xi)| d\xi &\le c \int_{\mathbb{R}^n} |\xi|^{|\alpha|} |\widehat{f}(\xi)| d\xi = c \int_{\mathbb{R}^n} \frac{|\xi|^{|\alpha|}}{\langle\xi\rangle^s} \langle\xi\rangle^s |\widehat{f}(\xi)| d\xi \\ &\le c \left(\int_{\mathbb{R}^n} \frac{|\xi|^{2|\alpha|}}{\langle\xi\rangle^{2s}} d\xi\right)^{1/2} \left(\int_{\mathbb{R}^n} \langle\xi\rangle^{2s} |\widehat{f}(\xi)|^2 d\xi\right)^{1/2} \le c' \|f\|_{H^s(\mathbb{R}^n)} \end{aligned}$$

if and only if $2s - 2|\alpha| > n$ or $s > |\alpha| + n/2$.

It means that for such s and α the function $D^\alpha f$ is a Fourier transform of some function from $L^1(\mathbb{R}^n)$. Due to Riemann-Lebesgue lemma we have that $D^\alpha f$ from $\dot{C}(\mathbb{R}^n)$. □

Lemma 5. *$L^2_s(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ if and only if $q = 2$ and $s \geq 0$ or $1 \leq q < 2$ and $s > n\left(\frac{1}{q} - \frac{1}{2}\right)$.*

Exercise 25. Prove Lemma 5.

Lemma 6 (Hörmander)**.** *1) $F : H^s(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ for $1 \leq q < 2$ and $s > n\left(\frac{1}{q} - \frac{1}{2}\right)$.*

2) $F : L^p(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n)$ for $2 < p \leq \infty$ and $s > n\left(\frac{1}{2} - \frac{1}{p}\right)$.

3) $F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

Proof. 1) See Lemma 5.

2) Let f be a function from $L^p(\mathbb{R}^n)$ for $2 < p \leq \infty$. Then $f \in S'$ (tempered distribution) and $|\langle \widehat{f}, \varphi\rangle_{L^2(\mathbb{R}^n)}| = |\langle f, \widehat{\varphi}\rangle_{L^2(\mathbb{R}^n)}| \leq \|f\|_p \cdot \|\widehat{\varphi}\|_{p'}$, where $1 \leq p' < 2$. But if $\varphi \in H^s(\mathbb{R}^n)$ for $s > n\left(\frac{1}{p'} - \frac{1}{2}\right)$ then $\|\widehat{\varphi}\|_{L^{p'}} \leq c\|\varphi\|_{H^s}$. So

$$|\langle \widehat{f}, \varphi\rangle_{L^2(\mathbb{R}^n)}| \leq c\|f\|_p \cdot \|\varphi\|_{H^s}.$$

That's why (by duality)

$$\left\|\widehat{f}\right\|_{H^{-s}} \leq c\|f\|_{L^p}$$

for $s > n\left(\frac{1}{p'} - \frac{1}{2}\right) = n\left(\frac{1}{2} - \frac{1}{p}\right)$.

3) This is simply the Parseval equality $\left\|\widehat{f}\right\|_{L^2} = \|f\|_{L^2}$.

□

Exercise 26. Prove that

1) $\chi_{[0,1]} \in H^s(\mathbb{R})$ if and only if $s < 1/2$.

2) $\chi_{[0,1]\times[0,1]} \in H^s(\mathbb{R}^2)$ if and only if $s < 1/2$.

3) $K(x) := F^{-1}\left(\frac{1}{1+|\xi|^2}\right) \in H^s(\mathbb{R})$ if and only if $s < 2 - n/2$.

4) Let $f(x) = \chi(x)\log\log|x|^{-1}$ in $\mathbb{R}^2$, where $\chi(x) \in C_0^\infty(|x| < 1/3)$. Prove that $f \in H^1(\mathbb{R}^2)$ but $f \notin L^\infty(\mathbb{R}^2)$.

Remark. This counterexample shows us that Sobolev embedding theorem is sharp.

Lemma 7. *Let us assume that φ and f from $H^s(\mathbb{R}^n)$ for $s > \frac{n}{2}$. Then $F(\varphi f) \in L^1(\mathbb{R}^n)$.*

Proof. Since $f, \varphi \in H^s(\mathbb{R}^n)$ then $\widehat{f}, \widehat{\varphi} \in L^2_s(\mathbb{R}^n)$ for $s > \frac{n}{2}$. But this implies (see Lemma 5) that $\widehat{f}$ and $\widehat{\varphi} \in L^1(\mathbb{R}^n)$ and

$$F(\varphi f) = (2\pi)^{-\frac{n}{2}}\widehat{\varphi} * \widehat{f}$$

also belongs to $L^1(\mathbb{R}^n)$. □

Remark. It is possible to prove that if $\varphi, f \in H^s(\mathbb{R}^n)$ for $s > \frac{n}{2}$ then $\varphi f \in H^s(\mathbb{R}^n)$ with the same s.

Exercise 27. Prove that $W^1_p(\mathbb{R}^n) \cdot W^1_p(\mathbb{R}^n) \subset W^1_p(\mathbb{R}^n)$ if $p > n$.

7 Homogeneous distributions

We start this chapter with the Fourier transform of a radially symmetric function.

Lemma 1. *Let $f(x)$ be a radially symmetric function in $\mathbb{R}^n$ i.e. $f(x) = f_1(|x|)$. Let us assume also that $f(x) \in L^1(\mathbb{R}^n)$. Then the Fourier transform $\widehat{f}(\xi)$ is also radial and*

$$\widehat{f}(\xi) = |\xi|^{1-\frac{n}{2}} \int_0^\infty f_1(r) r^{\frac{n}{2}} J_{\frac{n-2}{2}}(r|\xi|) dr,$$

where $J_\nu(\cdot)$ is the Bessel function of order ν.

Proof. Let us take the Fourier transform

$$\widehat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i(x,\xi)} f_1(|x|) dx = (2\pi)^{-\frac{n}{2}} \int_0^\infty f_1(r) r^{n-1} dr \int_{\mathbb{S}^{n-1}} e^{-i|\xi| r(\varphi,\theta)} d\theta,$$

where $x = r\theta, \xi = |\xi|\varphi$ and $\theta, \varphi \in \mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$. It is known that

$$\int_{\mathbb{S}^{n-1}} e^{-i|\xi| r(\varphi,\theta)} d\theta = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_0^\pi e^{-i|\xi| r \cos\psi} (\sin\psi)^{n-2} d\psi,$$

where Γ is the gamma function. This fact implies that $\widehat{f}(\xi)$ is a radial function, since the last integral depends only on $|\xi|$. A property of Bessel functions is that

$$\int_0^\pi e^{-i|\xi| r \cos\psi} (\sin\psi)^{n-2} d\psi = 2^{\frac{n}{2}-1} \sqrt{\pi}\, \Gamma\left(\frac{n-1}{2}\right) \frac{J_{\frac{n-2}{2}}(r|\xi|)}{(r|\xi|)^{\frac{n-2}{2}}}. \tag{7.1}$$

Collecting these things we obtain

$$\widehat{f}(\xi) = |\xi|^{\frac{2-n}{2}} \int_0^\infty f_1(r) r^{\frac{n}{2}} J_{\frac{n-2}{2}}(r|\xi|) dr.$$

□

Remark. If we put variable $u = \cos\psi$ in the integral appearing in (7.1), then we obtain

$$I = \int_0^\pi e^{-i|\xi| r \cos\psi} (\sin\psi)^{n-2} d\psi = \int_{-1}^1 e^{-i|\xi| r u} (\sqrt{1-u^2})^{n-3} du.$$

In particular, if $n = 3$ then (7.1) implies that

$$I = \int_{-1}^1 e^{-i|\xi| r u} du = 2\frac{\sin(|\xi| r)}{|\xi| r} = \sqrt{2\pi} \frac{J_{\frac{1}{2}}(r|\xi|)}{(r|\xi|)^{\frac{1}{2}}}$$

i.e.

$$J_{\frac{1}{2}}(r|\xi|) = \sqrt{\frac{2}{\pi}} \frac{\sin(|\xi| r)}{(|\xi| r)^{\frac{1}{2}}}.$$

If $n = 2$ then

$$I = \int_{-1}^{1} \frac{e^{-i|\xi|ru}}{\sqrt{1-u^2}} du = \pi J_0(r|\xi|)$$

i.e.

$$J_0(r|\xi|) = \frac{1}{\pi} \int_{-1}^{1} \frac{e^{-i|\xi|ru}}{\sqrt{1-u^2}} du.$$

Remark. For further considerations we state the small and large argument asymptotics of J_ν for $\nu > -1$ as

$$J_\nu(|x|) \cong \begin{cases} c_\nu |x|^\nu, & |x| \to +0 \\ c'_\nu \frac{1}{\sqrt{|x|}} \cos(A_\nu |x| + B_\nu), & |x| \to +\infty. \end{cases}$$

Exercise 28. Prove that $\widehat{f}(A\xi) = \widehat{f}(\xi)$ if A is a linear transformation in $\mathbb{R}^n$ with $A' = A^{-1}$ and f is radially symmetric.

Let us return again to the distribution (cf. Example 5.1)

$$K_1(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} F^{-1}\left(\frac{1}{1+|\xi|^2}\right)(x).$$

Let us assume now that $n = 1, 2, 3, 4$. Then the last integral can be understood in the classical sense. It follows from Lemma 1 that

$$K_1(x) = \widetilde{K_1}(|x|) = (2\pi)^{-\frac{n}{2}} |x|^{1-\frac{n}{2}} \int_0^\infty \frac{r^{\frac{n}{2}} J_{\frac{n-2}{2}}(r|x|) dr}{1+r^2} = (2\pi)^{-\frac{n}{2}} |x|^{2-n} \int_0^\infty \frac{\rho^{\frac{n}{2}} J_{\frac{n-2}{2}}(\rho) d\rho}{\rho^2 + |x|^2}.$$

It is not too difficult to prove that for $|x| < 1$ we have

$$|K_1(x)| \le c \begin{cases} 1, & n = 1, \\ \log \frac{1}{|x|}, & n = 2, \\ |x|^{2-n}, & n = 3, 4. \end{cases}$$

Exercise 29. Prove this fact.

Remark. A little later we will prove estimates for $K_1(x)$ for any dimension and for all $x \in \mathbb{R}^n$.

There is one more important example. If we have the equation $(-1 - \Delta)u = f$ in $L^2(\mathbb{R}^n)$ (or even in S), then formally $u = (2\pi)^{-\frac{n}{2}} F^{-1}\left(\frac{1}{|\xi|^2 - 1}\right) * f = K_{-1} * f$, where

$$K_{-1}(|x|) = (2\pi)^{-\frac{n}{2}} |x|^{2-n} \int_0^\infty \frac{\rho^{\frac{n}{2}} J_{\frac{n-2}{2}}(\rho) d\rho}{\rho^2 - |x|^2}.$$

But there is a problem with the convergence of this integral near $\rho = |x|$. That's why this integral must be regularized as

$$\lim_{\varepsilon \downarrow 0} \int_0^\infty \frac{\rho^{\frac{n}{2}} J_{\frac{n-2}{2}}(\rho) d\rho}{\rho^2 - |x|^2 - i\varepsilon}.$$

Recall that

1) $\sigma_\lambda f(x) := f(\lambda x), \quad \lambda \neq 0$ and

2) $\langle \sigma_\lambda T, \varphi \rangle := \lambda^{-n} \langle T, \sigma_{\frac{1}{\lambda}} \varphi \rangle, \quad \lambda > 0.$

Definition. A tempered distribution T is said to be a homogeneous distribution of degree $m \in \mathbb{C}$ if

$$\sigma_\lambda T = \lambda^m T$$

for any $\lambda > 0$. In other words,

$$\langle \sigma_\lambda T, \varphi \rangle = \lambda^m \langle T, \varphi \rangle$$

or

$$\langle T, \varphi \rangle = \lambda^{-n-m} \langle T, \sigma_{\frac{1}{\lambda}} \varphi \rangle$$

for $\varphi \in S$. The space of all such distributions is denoted by $H_m(\mathbb{R}^n)$.

Lemma 2. $F : H_m(\mathbb{R}^n) \to H_{-m-n}(\mathbb{R}^n)$.

Proof. Let $T \in H_m(\mathbb{R}^n)$ and $\varphi \in S$. Then

$$\begin{aligned} \langle \sigma_\lambda \widehat{T}, \varphi \rangle &= \lambda^{-n} \langle \widehat{T}, \sigma_{\frac{1}{\lambda}} \varphi \rangle = \lambda^{-n} \langle T, \widehat{\sigma_{\frac{1}{\lambda}} \varphi} \rangle = \lambda^{-n} \langle T, \lambda^n \sigma_\lambda \widehat{\varphi} \rangle \\ &= \langle T, \sigma_\lambda \widehat{\varphi} \rangle = \lambda^{-n} \langle \sigma_{\frac{1}{\lambda}} T, \widehat{\varphi} \rangle = \lambda^{-n} \lambda^{-m} \langle T, \widehat{\varphi} \rangle = \lambda^{-n-m} \langle \widehat{T}, \varphi \rangle. \end{aligned}$$

□

Definition. $H_m^*(\mathbb{R}^n) := \{T \in H_m(\mathbb{R}^n) : T \in C^\infty(\mathbb{R}^n \setminus \{0\})\}$.

Exercise 30. Prove that

1) if $T \in H_m^*$ then $D^\alpha T \in H_{m-|\alpha|}^*$ and $x^\alpha T \in H_{m+|\alpha|}^*$

2) $F : H_m^* \to H_{-m-n}^*$.

Exercise 31. Let $\rho(x)$ be a function from $C^\infty(\mathbb{R}^n)$ with $|D^\alpha \rho(x)| \leq c \langle x \rangle^{m-|\alpha|}$ for all $\alpha \geq 0$ and $m \in \mathbb{R}$. Prove that $\widehat{\rho}(\xi) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and $(1 - \varphi)\widehat{\rho} \in S$, where $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\varphi \equiv 1$ in $U_\delta(0)$.

Example 7.1. 1) $\delta \in H_{-n}^*(\mathbb{R}^n)$. Indeed,

$$\langle \sigma_\lambda \delta, \varphi \rangle = \lambda^{-n} \langle \delta, \sigma_{\frac{1}{\lambda}} \varphi \rangle = \lambda^{-n} \sigma_{\frac{1}{\lambda}} \varphi(0) = \lambda^{-n} \varphi(0) = \lambda^{-n} \langle \delta, \varphi \rangle.$$

But $\operatorname{supp} \delta = \{0\}$. It means that $\delta \in C^\infty(\mathbb{R}^n \setminus \{0\})$. Alternatively one could note that

$$\widehat{\delta} = (2\pi)^{-\frac{n}{2}} \cdot 1 \in H_0^*(\mathbb{R}^n)$$

and use Exercise 30 to conclude that

$$\delta = F^{-1}((2\pi)^{-\frac{n}{2}} \cdot 1) \in H_{-n}^*(\mathbb{R}^n).$$

2) Let us assume that $\omega \in C^\infty(\mathbb{S}^{n-1})$ and $m > -n$. Set $T_m(x) := |x|^m \omega\left(\frac{x}{|x|}\right), x \in \mathbb{R}^n \setminus \{0\}$. Then $T_m(x) \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $T_m \in H^*_m(\mathbb{R}^n)$. Indeed,

$$\langle \sigma_\lambda T_m, \varphi \rangle = \int_{\mathbb{R}^n} \sigma_\lambda T_m(x)\varphi(x)dx = \int_{\mathbb{R}^n} |\lambda x|^m \omega\left(\frac{\lambda x}{|\lambda x|}\right)\varphi(x)dx = \lambda^m \langle T_m, \varphi\rangle.$$

Since $|x|^m$ and $\omega\left(\frac{x}{|x|}\right)$ are from $C^\infty(\mathbb{R}^n \setminus \{0\})$ then $T_m \in H^*_m(\mathbb{R}^n)$. Moreover, $D^\alpha T_m \in H^*_{m-|\alpha|}(\mathbb{R}^n)$ and $x^\alpha T_m \in H^*_{m+|\alpha|}(\mathbb{R}^n)$ by Exercise 30.

3) Let now $m = -n$ in part 2) and in addition assume that $\int_{\mathbb{S}^{n-1}} \omega(\theta)d\theta = 0$. Note that $T_{-n}(x) \notin L^1_{\text{loc}}(\mathbb{R}^n)$. But we can define T_{-n} as a distribution from S' by

$$\langle \text{p.v.}T_{-n}, \varphi \rangle := \int_{\mathbb{R}^n} T_{-n}(x)[\varphi(x) - \varphi(0)\psi(|x|)]dx,$$

where $\varphi \in S(\mathbb{R}^n)$ and $\psi \in S(\mathbb{R})$ with $\psi(0) = 1$. We assume that ψ is fixed. But it is clear that this definition does not depend on ψ, because $\int_{\mathbb{S}^{n-1}} \omega(\theta)d\theta = 0$.

Exercise 32. Prove that,

$$\langle \text{p.v.}T_{-n}, \varphi \rangle = \lim_{\varepsilon \to +0} \int_{|x| \geq \varepsilon} T_{-n}(x)\varphi(x)dx,$$

where $T_{-n} = |x|^{-n}\omega\left(\frac{x}{|x|}\right), \int_{\mathbb{S}^{n-1}} \omega(\theta)d\theta = 0$.

Let us prove that:

1) $\text{p.v.}T_{-n} \in H^*_{-n}$ and

2) $\widehat{\text{p.v.}T_{-n}} \in H^*_0(\mathbb{R}^n)$ and moreover it is bounded.

Proof. Part 1) is clear. Part 2) follows from

$$\begin{aligned}
|\langle \widehat{\text{p.v.}T_{-n}}, \varphi \rangle| &= |\langle \text{p.v.}T_{-n}, \widehat{\varphi} \rangle| = \left|\int_{\mathbb{R}^n} T_{-n}(x)[\widehat{\varphi}(x) - \widehat{\varphi}(0)\psi(|x|)]dx\right| \\
&\leq (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} |\varphi(\xi)|d\xi \left|\int_{\mathbb{R}^n} T_{-n}(x)[e^{-i(x,\xi)} - \psi(|x|)]dx\right| \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} |\varphi(\xi)|d\xi \left|\int_{\mathbb{S}^{n-1}} \omega(\theta)d\theta \int_0^\infty \frac{1}{r}[e^{-ir(\theta,\xi)} - \psi(r)]dr\right| \\
&\leq c\,\|\varphi\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

Hence $\widehat{\text{p.v.}T_{-n}} \in L^\infty(\mathbb{R}^n)$ by duality. □

Part 2) implies that

3) $\text{p.v.}T_{-n}* : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

Indeed, if $f \in L^2(\mathbb{R}^n)$ then

$$F(\text{p.v.}T_{-n} * f) = (2\pi)^{\frac{n}{2}} \widehat{\text{p.v.}T_{-n}} \cdot \widehat{f}$$

which implies that

$$\|\text{p.v.}T_{-n} * f\|_{L^2(\mathbb{R}^n)} \le (2\pi)^{\frac{n}{2}} \left\|\widehat{\text{p.v.}T_{-n}}\right\|_{L^\infty} \cdot \|f\|_{L^2}.$$

Remark. Actually, it follows from the Calderón-Zigmund theory that

$$\text{p.v.}T_{-n}* : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \quad 1 < p < \infty.$$

Now we want to consider more difficult case than previous one. Denote

$$\langle \text{p.v.}\frac{1}{|x|^n}, \varphi\rangle := \int_{\mathbb{R}^n} |x|^{-n}[\varphi(x) - \varphi(0)\psi(|x|)]dx, \tag{7.2}$$

where $\varphi \in S$ and $\psi \in S$ with $\psi(0) = 1$. But now we don't have the condition $\int_{\mathbb{S}^{n-1}} \omega(\theta)d\theta = 0$ as above. This is the reason why (7.2) must depend on the function $\psi(|x|)$. We will try to choose an appropriate function ψ. Applying the operator σ_λ we get

$$\begin{aligned}
\langle \sigma_\lambda\left(\text{p.v.}\frac{1}{|x|^n}\right), \varphi\rangle &= \langle \text{p.v.}\frac{1}{|x|^n}, \lambda^{-n}\sigma_{\frac{1}{\lambda}}\varphi\rangle = \int_{\mathbb{R}^n} |x|^{-n}\lambda^{-n}\left[\varphi\left(\frac{x}{\lambda}\right) - \varphi(0)\psi(|x|)\right]dx \\
&= \lambda^{-n}\int_{\mathbb{R}^n} |y|^{-n}[\varphi(y) - \varphi(0)\psi(\lambda|y|)]dy \\
&= \lambda^{-n}\int_{\mathbb{R}^n} |y|^{-n}[\varphi(y) - \varphi(0)\psi(|y|)]dy \\
&- \lambda^{-n}\int_{\mathbb{R}^n} |y|^{-n}\varphi(0)[\psi(\lambda|y|) - \psi(|y|)]dy \\
&= \langle \lambda^{-n}\text{p.v.}\frac{1}{|x|^n}, \varphi\rangle + \text{Rest},
\end{aligned}$$

where

$$\begin{aligned}
\text{Rest} &= -\lambda^{-n}\varphi(0)\int_{\mathbb{R}^n} |y|^{-n}[\psi(\lambda|y|) - \psi(|y|)]dy \\
&= -\lambda^{-n}\langle\delta, \varphi\rangle\int_0^\infty \frac{\psi(\lambda r) - \psi(r)}{r}dr\int_{\mathbb{S}^{n-1}} d\theta \\
&= -\omega_n\lambda^{-n}\langle\delta, \varphi\rangle\int_0^\infty \frac{\psi(\lambda r) - \psi(r)}{r}dr,
\end{aligned}$$

where $\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ is the area of the unit sphere $\mathbb{S}^{n-1}$. Let us denote the last integral by $G(\lambda), \lambda > 0$. Then

$$G'(\lambda) = \int_0^\infty \psi'(\lambda r)dr = \frac{1}{\lambda}\int_0^\infty \psi'(t)dt = -\frac{1}{\lambda}\psi(0) = -\frac{1}{\lambda}.$$

Also we have that $G(1) = 0$. That's why we may conclude that $G(\lambda) = -\ln\lambda$. It implies that

$$\text{Rest} = \omega_n \lambda^{-n} \ln\lambda \langle \delta, \varphi \rangle$$

and so

$$\sigma_\lambda \left(\text{p.v.} \frac{1}{|x|^n} \right) = \lambda^{-n} \text{p.v.} \frac{1}{|x|^n} + \omega_n \lambda^{-n} \ln\lambda \cdot \delta(x).$$

Taking the Fourier transform we get

$$F\left(\sigma_\lambda \left(\text{p.v.} \frac{1}{|x|^n} \right) \right) = \lambda^{-n} F(\text{p.v.} \frac{1}{|x|^n}) + (2\pi)^{-\frac{n}{2}} \omega_n \lambda^{-n} \ln\lambda$$

or

$$\lambda^{-n} \sigma_{\frac{1}{\lambda}} F\left(\text{p.v.} \frac{1}{|x|^n} \right) = \lambda^{-n} F(\text{p.v.} \frac{1}{|x|^n}) + (2\pi)^{-\frac{n}{2}} \omega_n \lambda^{-n} \ln\lambda$$

or

$$F\left(\text{p.v.} \frac{1}{|x|^n} \right) \left(\frac{\xi}{\lambda} \right) = F\left(\text{p.v.} \frac{1}{|x|^n} \right)(\xi) + (2\pi)^{-\frac{n}{2}} \omega_n \ln\lambda.$$

Let us put now $\lambda = |\xi|$. Then

$$F\left(\text{p.v.} \frac{1}{|x|^n} \right)(\xi) = -(2\pi)^{-\frac{n}{2}} \omega_n \ln|\xi| + F\left(\text{p.v.} \frac{1}{|x|^n} \right) \left(\frac{\xi}{|\xi|} \right).$$

Since $\text{p.v.} \frac{1}{|x|^n}$ for such ψ is a homogeneous distribution and radial then $F\left(\text{p.v.} \frac{1}{|x|^n} \right)$ is also a homogeneous distribution and radial. That's why $F\left(\text{p.v.} \frac{1}{|x|^n} \right) \left(\frac{\xi}{|\xi|} \right)$ depends only on $\left| \frac{\xi}{|\xi|} \right| = 1$. So this term is a constant that depends on the choice of ψ. We will choose our function $\psi(|x|)$ so that this constant is zero. Then, finally

$$F\left(\text{p.v.} \frac{1}{|x|^n} \right)(\xi) = -(2\pi)^{-\frac{n}{2}} \omega_n \ln|\xi|.$$

Now let us consider $T_{-m} = |x|^{-m}, 0 < m < n$. It is clear that $|x|^{-m} \in L^1_{\text{loc}}(\mathbb{R}^n)$. That's why the situation is more simple. We have

$$\langle \widehat{|x|^{-m}}, \varphi \rangle = \langle |x|^{-m}, \widehat{\varphi} \rangle = \int_{\mathbb{R}^n} |x|^{-m} \widehat{\varphi}(x) dx.$$

Lemma 1 implies that

$$\widehat{|x|^{-m}} = |\xi|^{1-\frac{n}{2}} \int_0^\infty \frac{r^{\frac{n}{2}} J_{\frac{n-2}{2}}(r|\xi|)}{r^m} dr = |\xi|^{-n+m} \int_0^\infty \rho^{\frac{n}{2}-m} J_{\frac{n-2}{2}}(\rho) d\rho.$$

Last integral converges if $\frac{n-1}{2} < m < n$. That's why we may write that

$$\widehat{|x|^{-m}} = C_{n,m} |\xi|^{m-n}, \quad \frac{n-1}{2} < m < n.$$

In fact, this is true even for m such that $0 < \mathrm{Re}\,(m) < n$. It follows by analytic continuation on m. In order to calculate the constant $C_{n,m}$ let us apply this distribution to $\varphi = e^{-\frac{|x|^2}{2}}$. Since $\widehat{\varphi} = \varphi$ we get

$$\langle |x|^{-m}, e^{-\frac{|x|^2}{2}} \rangle = \langle C_{n,m}|\xi|^{m-n}, e^{-\frac{|\xi|^2}{2}} \rangle.$$

The left hand side is

$$\begin{aligned}\int_{\mathbb{R}^n} |x|^{-m} e^{-\frac{|x|^2}{2}} dx &= \omega_n \int_0^\infty r^{n-m-1} e^{-\frac{r^2}{2}} dr \\ &= 2^{\frac{n-m-2}{2}} \omega_n \int_0^\infty t^{\frac{n-m}{2}-1} e^{-t} dt = 2^{\frac{n-m-2}{2}} \omega_n \Gamma\left(\frac{n-m}{2}\right).\end{aligned}$$

Using this the right hand side becomes

$$C_{n,m} \langle |\xi|^{m-n}, e^{-\frac{|\xi|^2}{2}} \rangle = C_{n,m} 2^{\frac{n-(n-m)-2}{2}} \omega_n \Gamma\left(\frac{m}{2}\right).$$

That's why

$$C_{n,m} 2^{\frac{m-2}{2}} \omega_n \Gamma\left(\frac{m}{2}\right) = 2^{\frac{n-m-2}{2}} \omega_n \Gamma\left(\frac{n-m}{2}\right)$$

which gives us

$$C_{n,m} = 2^{\frac{n}{2}-m} \frac{\Gamma\left(\frac{n-m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}.$$

Finally, we have

$$\widehat{|x|^{-m}} = 2^{\frac{n}{2}-m} \frac{\Gamma\left(\frac{n-m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \cdot |\xi|^{m-n}. \tag{7.3}$$

Definition. The *Hilbert transform* Hf of $f \in S$ is defined by

$$Hf := \frac{1}{\pi}\left(\text{p.v.}\frac{1}{x} * f\right)$$

i.e.

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{|x-t| \geq \varepsilon} \frac{f(t)dt}{x-t}, \quad x \in \mathbb{R}.$$

Exercise 33. Prove that

1) $\|Hf\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$.

2) Hilbert transform has an extension to functions from $L^2(\mathbb{R})$.

3) $H^2 = -I$ i.e. $H^{-1} = -H$.

4) $(Hf_1, Hf_2)_{L^2} = (f_1, f_2)_{L^2}$ for $f_1 \in L^p$ and $f_2 \in L^{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1, 1 < p < \infty$.

5) $H: L^p(\mathbb{R}) \to L^p(\mathbb{R}), 1 < p < \infty$ i.e.

$$\left\| \frac{1}{\pi} \int_{|x-t|\geq\varepsilon} \frac{f(t)dt}{x-t} \right\|_{L^p} \leq c \|f\|_{L^p}$$

for any $\varepsilon > 0$ where c does not depend on ε.

An n-dimensional analogue of the Hilbert transform is developed via

Definition. The functions

$$R_j(x) := \frac{x_j}{|x|^{n+1}}, \quad x \neq 0, \quad j = 1, 2, \ldots, n$$

are called the *Riesz kernels*.

Remark. We can rewrite $R_j(x)$ in the form $R_j(x) = |x|^{-n}\omega_j(x)$, where $\omega_j(x) = \frac{x_j}{|x|}$. That's why we may conclude that

1) $\int_{\mathbb{S}^{n-1}} \omega_j(\theta)d\theta = 0$
2) $R_j(\lambda x) = \lambda^{-n} R_j(x), \quad \lambda > 0.$

These properties imply that we may define *Riesz transform* by

$$R_j * f = \text{p.v.} R_j * f,$$

because in our previous notation $R_j(x) = T_{-n} \in H^*_{-n}(\mathbb{R}^n)$ is a homogeneous distribution. Let us calculate the Fourier transform of the Riesz kernels. Due to homogeneity it suffices to consider $|\xi| = 1$. We have,

$$\begin{aligned} \widehat{R_j}(\xi) &= \widehat{\text{p.v.} R_j}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \frac{e^{-i(x,\xi)} x_j}{|x|^{n+1}} dx \\ &= \lim_{\varepsilon\to+0, \mu\to+\infty} (2\pi)^{-\frac{n}{2}} \int_{\varepsilon<|x|<\mu} \frac{e^{-i(x,\xi)} x_j}{|x|^{n+1}} dx. \end{aligned}$$

Split

$$\int_{\varepsilon<|x|<\mu} \frac{e^{-i(x,\xi)} x_j}{|x|^{n+1}} dx = \int_{\varepsilon<|x|<1} \frac{e^{-i(x,\xi)} x_j}{|x|^{n+1}} dx + \int_{1<|x|<\mu} \frac{e^{-i(x,\xi)} x_j}{|x|^{n+1}} dx := I_1 + I_2.$$

For I_1 we will use integration by parts:

$$\begin{aligned} I_1 &= \frac{1}{1-n} \int_{\varepsilon<|x|<1} e^{-i(x,\xi)} \frac{\partial}{\partial x_j}(|x|^{1-n}) dx \\ &= c_n i \xi_j \int_{\varepsilon<|x|<1} \frac{e^{-i(x,\xi)}}{|x|^{n-1}} dx + \int_{|x|=1} \frac{e^{-i(x,\xi)} x_j}{|x|^n} d\sigma - \int_{|x|=\varepsilon} \frac{e^{-i(x,\xi)} x_j}{|x|^n} d\sigma \\ &\overset{\varepsilon\downarrow 0}{\to} c_n i \xi_j \int_{|x|<1} \frac{e^{-i(x,\xi)}}{|x|^{n-1}} dx + \int_{|x|=1} e^{-i(x,\xi)} x_j d\sigma - 0. \end{aligned}$$

But

$$\begin{aligned}\int_{|x|=1} x_j e^{-i(x,\xi)} d\sigma &= i\frac{\partial}{\partial \xi_j}\int_{|x|=1} e^{-i(x,\xi)} d\sigma = i\frac{\partial}{\partial \xi_j}\int_{|x|=1} \cos(|\xi|\cdot x_1) d\sigma \\ &= -\frac{i\xi_j}{|\xi|}\int_{|x|=1} x_1 \cdot \sin(|\xi|\cdot x_1) d\sigma = -i\xi_j \cdot C_1, \quad |\xi| = 1,\end{aligned}$$

where we have used the fact that a rotation maps ξ to $(|\xi|, 0, \dots, 0)$. Similarly we may conclude that

$$\int_{|x|<1} \frac{e^{-i(x,\xi)}}{|x|^{n-1}} dx = \int_{|x|<1} \cos(|\xi| x_1)|x|^{1-n} dx = C_2, \quad |\xi| = 1.$$

If we collect all of these things we obtain:

$$I_1 \overset{\varepsilon\downarrow 0}{\to} C_n i\xi_j, \quad |\xi| = 1.$$

For I_2 we will use the following technique

$$\begin{aligned} I_2 \overset{\mu\to+\infty}{\to} & \int_{|x|>1} \frac{e^{-i(x,\xi)} x_j}{|x|^{n+1}} dx = i\frac{\partial}{\partial \xi_j}\int_{|x|>1} \frac{e^{-i(x,\xi)}}{|x|^{n+1}} dx \\ = & \; i\frac{\partial}{\partial \xi_j}\int_{|x|>1} |x|^{-n-1}\cos(|\xi|\cdot x_1) dx \\ = & -\frac{i\xi_j}{|\xi|}\int_{|x|>1} \frac{x_1 \sin(|\xi| x_1)}{|x|^{n+1}} dx \\ = & -i\xi_j \cdot \mathrm{const}, \quad |\xi| = 1.\end{aligned}$$

Exercise 34. Prove the convergence of the last integral.

Collecting these integrals we obtain that

$$\widehat{\mathrm{p.v.}R_j} = i\xi_j \cdot C_n$$

for $|\xi| = 1$. But we know from Exercise 30 that $\widehat{\mathrm{p.v.}R_j} \in H_0^*(\mathbb{R}^n)$. That's why we may conclude that $\widehat{\mathrm{p.v.}R_j}(\xi) = iC_n\frac{\xi_j}{|\xi|}$. Further, we have

$$\widehat{R_j * f} = (2\pi)^{\frac{n}{2}}\widehat{R_j}\cdot\widehat{f} = iC_n'\frac{\xi_j}{|\xi|}\widehat{f}$$

or

$$R_j * f = iC_n' F^{-1}\left(\frac{\xi_j}{|\xi|}\widehat{f}\right).$$

Corollary.

$$\sum_{j=1}^n R_j * R_j * = -\widetilde{C_n}\delta * .$$

Proof.

$$\sum_{j=1}^{n} F(R_j * R_j * f) = \sum_{j=1}^{n} iC'_n \frac{\xi_j}{|\xi|} \cdot iC'_n \frac{\xi_j}{|\xi|} \widehat{f}(\xi) = -(C'_n)^2 \widehat{f}(\xi).$$

□

Remark. By Parseval equality we have

$$\|R_j * f\|_{L^2} = \left\|\widehat{R_j * f}\right\|_{L^2} = C \left\| \frac{\xi_j}{|\xi|} \widehat{f} \right\|_{L^2} \leq C \left\|\widehat{f}\right\|_{L^2} = C \|f\|_{L^2}$$

i.e.

$$R_j * : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

and it follows from Calderón-Zigmund theory that

$$R_j * : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \quad 1 < p < \infty.$$

Let us now introduce the *Riesz potential* by

$$I^{-1} f := F^{-1} \left(\frac{1}{|\xi|} \widehat{f}(\xi) \right) = (2\pi)^{-\frac{n}{2}} F^{-1} \left(\frac{1}{|\xi|} \right) * f = I_1 * f,$$

where, by (7.3),

$$I_1(x) = c_n \frac{1}{|x|^{n-1}}.$$

That's why

$$I^{-1} f(x) = c_n \int_{\mathbb{R}^n} \frac{f(y)dy}{|x - y|^{n-1}}.$$

It is straightforward to check that $\frac{\partial}{\partial x_j} I_1 = c'_n R_j$ and hence

$$\frac{\partial}{\partial x_j} I^{-1} f = c'_n R_j * f.$$

We would like to prove that

$$I^{-1} : L^s_\sigma(\mathbb{R}^n) \to W^1_s(\mathbb{R}^n)$$

for some s and σ. Since $R_j *$ is a bounded map from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ we may conclude that

$$\frac{\partial}{\partial x_j} I^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n). \tag{7.4}$$

Now let us assume for simplicity that $n \geq 3$. Let us try to prove that

$$I^{-1} : L^2_\sigma(\mathbb{R}^n) \to L^2(\mathbb{R}^n). \tag{7.5}$$

Indeed, for $f \in L^2(\mathbb{R}^n)$

$$I^{-1} f \in L^2(\mathbb{R}^n)$$

if and only if

$$\frac{1}{|\xi|}\widehat{f} \in L^2(\mathbb{R}^n).$$

Let us assume now that $\sigma > 1$.

Lemma 5 in Chapter 6 implies that $L^2_\sigma(\mathbb{R}^n) \subset L^r(\mathbb{R}^n)$ for any $1 \le r < 2$ and $\sigma > n\left(\frac{1}{r} - \frac{1}{2}\right)$. But for $\sigma > 1$ we may find appropriate r such that $r < \frac{2n}{n+2}$. That's why we may conclude that for function $f \in L^2_\sigma(\mathbb{R}^n)$ with $\sigma > 1$ it follows from Hausdorff-Young inequality that $\widehat{f} \in L^{r'}(\mathbb{R}^n)$ for some $r' > \frac{2n}{n-2}$ or $|\widehat{f}|^2 \in L^{\frac{r'}{2}}(\mathbb{R}^n)$. This fact implies that for $|\xi| < 1$ we have

$$|\xi|^{-1}\widehat{f}(\xi) \in L^2_{\text{loc}}.$$

Indeed,

$$\int_{|\xi|<1} |\xi|^{-2}|\widehat{f}(\xi)|^2 d\xi \le \left(\int_{|\xi|<1} |\widehat{f}(\xi)|^{r'} d\xi\right)^{2/r'} \left(\int_{|\xi|<1} |\xi|^{-2\left(\frac{r'}{2}\right)'} d\xi\right)^{\frac{1}{\left(\frac{r'}{2}\right)'}} < \infty$$

since $\frac{r'}{2} > \frac{n}{n-2}$ and $\left(\frac{r'}{2}\right)' < \frac{n}{2}$. For $|\xi| > 1$ the function $|\xi|^{-1}\widehat{f}(\xi)$ belongs to $L^2(\mathbb{R}^n)$. This fact follows from the inequality $|\xi|^{-1}|\widehat{f}(\xi)| < |\widehat{f}(\xi)|$ and from the positivity of σ (see Lemma 5 in Chapter 6). This proves (7.5) for $\sigma > 1$.

If we collect (7.4) and (7.5) then we will obtain that

$$I^{-1} : L^2_\sigma(\mathbb{R}^n) \to W^1_2(\mathbb{R}^n), \quad \sigma > 1.$$

Let us consider now $L^\infty_\sigma(\mathbb{R}^n)$ for $\sigma > 1$. If $f \in L^\infty_\sigma(\mathbb{R}^n)$ then $|f(x)| \le C(1+|x|)^{-\sigma}$ and thus

$$|I^{-1}f(x)| \le C\int_{\mathbb{R}^n} \frac{(1+|y|)^{-\sigma}dy}{|x-y|^{n-1}} < \infty.$$

It means that

$$I^{-1} : L^\infty_\sigma(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n).$$

Interpolating this with (7.5) we can get the following result:

$$I^{-1} : L^s_\sigma(\mathbb{R}^n) \to L^s(\mathbb{R}^n), \quad 2 \le s \le \infty, \quad \sigma > 1.$$

If we recall the fact that $R_j* : L^s(\mathbb{R}^n) \to L^s(\mathbb{R}^n)$ for any $1 < s < \infty$ then we obtain

$$I^{-1} : L^s_\sigma(\mathbb{R}^n) \to W^1_s(\mathbb{R}^n), \qquad 2 \le s < \infty, \quad \sigma > 1.$$

8 Fundamental solutions of elliptic partial differential operators

Let us consider a linear partial differential operator of order m in the form

$$L(x,D)=\sum_{|\alpha|\le m}a_\alpha(x)D^\alpha,\quad x\in\mathbb{R}^n,$$

where $\alpha=(\alpha_1,\dots,\alpha_n)$ is a multi-index, $D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$ and $D_j=\frac{1}{i}\frac{\partial}{\partial x_j}$.

In this chapter Ω is a bounded domain in $\mathbb{R}^n$ or $\Omega=\mathbb{R}^n$.

Definition. A fundamental solution for L in Ω is a distribution E in x, which satisfies

$$L_xE(x|y)=\delta(x-y)$$

in $D'(\Omega)$ with parameter $y\in\Omega$, i.e., $\langle L_xE,\varphi\rangle=\varphi(y)$ for $\varphi\in C_0^\infty(\Omega)$.

We understand that $\langle LE,\varphi\rangle$ is defined in distributional form

$$\langle LE,\varphi\rangle=\langle E,L'\varphi\rangle,$$

where L' is the formal adjoint operator for L given by

$$L'f=\sum_{|\alpha|\le m}(-1)^{|\alpha|}D^\alpha(a_\alpha(x)f(x)).$$

In that case $L'\varphi$ must be in $D(\Omega)$ for φ from $D(\Omega)$. This will be the case, for example, for $a_\alpha(x)\in C^\infty(\Omega)$.

Any two fundamental solutions for L with the same parameter y differ by a solution of the homogeneous equation $Lu=0$. Unless boundary conditions are imposed, the homogeneous equation will have many solutions and the fundamental solution will not be uniquely determined. In most problems there are grounds of symmetry or causality for selecting the particular fundamental solution for the appropriate physical behavior.

We also observe that if L has constant coefficients, we can find the fundamental solution in the form $E(x|y)=E(x-y|0):=E(x-y)$. This fact follows from the properties of the Fourier transform:

$$\widehat{L_xE(x-y)}=\sum_{|\alpha|\le m}a_\alpha\xi^\alpha\widehat{E(x-y)}=\sum_{|\alpha|\le m}a_\alpha\xi^\alpha e^{-i(\xi,y)}\widehat{E(x)}=e^{-i(\xi,y)}\widehat{\delta(x)}=\widehat{\delta(x-y)}$$

i.e.

$$L_xE(x-y)=\delta(x-y).$$

Exercise 35. Let L be a differential operator with constant coefficients. Prove that $u=q*E=E*q$ solves the inhomogeneous equation

$$Lu=q$$

in D'.

Remark. In many cases the fundamental solution is a function. That's why we can write u as an integral

$$u(x) = \int_\Omega E(x-y)q(y)dy.$$

Remark. In order for convolution product $E * q$ (or $q * E$) to be well-defined we have to assume that, for example, q vanishes outside a finite sphere.

Remark. If L does not have constant coefficients, we can no longer appeal to convolution products; instead one can often show that

$$u(x) = \int_\Omega E(x|y)q(y)dy.$$

Definition. Denote by $a_0(x,\xi)$ the main (or principal) symbol

$$a_0(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha, \quad \xi \in \mathbb{R}^n$$

of $L(x,\xi)$. Assume that $a_\alpha(x)$ are "smooth". Operator $L(x,D)$ is said to be elliptic in Ω if for any $x \in \Omega$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ it follows that

$$a_0(x,\xi) \neq 0.$$

Exercise 36. Let $a_\alpha(x)$ be real for $|\alpha| = m$. Prove that the previous definition is equivalent to

1) m is even,

2) $a_0(x,\xi) \geq C_K|\xi|^m, C_K > 0$, for any compact set $K \subset \Omega$ and for all $\xi \in \mathbb{R}^n$ and $x \in K$.

Let us consider the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u, & t > 0, x \in \mathbb{R}^n \\ u(x,0) = f(x), & x \in \mathbb{R}^n \end{cases}$$

in $S'(\mathbb{R}^n)$. Take the Fourier transform with respect to x to obtain

$$\begin{cases} \frac{\partial}{\partial t}\widehat{u}(\xi,t) = -\xi^2\widehat{u}(\xi,t), & t > 0 \\ \widehat{u}(\xi,0) = \widehat{f}(\xi). \end{cases}$$

This initial value problem for an ordinary differential equation has the solution

$$\widehat{u}(\xi,t) = e^{-t|\xi|^2}\widehat{f}(\xi).$$

Hence

$$u(x,t) = F^{-1}(e^{-t|\xi|^2}\widehat{f}(\xi)) = (2\pi)^{-\frac{n}{2}}F^{-1}(e^{-t|\xi|^2}) * f = P(\cdot,t) * f,$$

where

$$P(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-t|\xi|^2} e^{i(x,\xi)} d\xi = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}.$$

This formula implies that

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y)dy.$$

Definition. The function $P(x,t)$ is the fundamental solution of the heat equation and satisfies

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta\right) P(x,t) = 0, \quad t > 0 \\ \lim_{t\downarrow 0} P(x,t) \stackrel{S'}{=} \delta(x). \end{cases}$$

We can generalize this situation as follows. Let us consider an elliptic differential operator

$$L(D) = \sum_{|\alpha|\le m} a_\alpha D^\alpha$$

with constant coefficients. Assume that $L(\xi) = \sum_{|\alpha|\le m} a_\alpha \xi^\alpha > 0$ for all $\xi \in \mathbb{R}^n \backslash \{0\}$. If we consider $P_L(x,t)$ as a solution of

$$\begin{cases} \left(\frac{\partial}{\partial t} + L(D)\right) P_L(x,t) = 0, \quad t > 0 \\ \lim_{t\downarrow 0} P_L(x,t) \stackrel{S'}{=} \delta(x) \end{cases}$$

then $P_L(x,t)$ is the fundamental solution of $\frac{\partial}{\partial t} + L(D)$ and can be calculated by

$$P_L(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-tL(\xi)} e^{i(x,\xi)} d\xi.$$

Lemma 1. *Let $P_L(x,t)$ be as above. Then the function*

$$F(x,\lambda) \stackrel{S'}{:=} \lim_{\varepsilon\downarrow 0} \int_\varepsilon^\infty e^{-\lambda t} P_L(x,t)dt \tag{8.1}$$

is a fundamental solution of the operator $L(D) + \lambda I, \lambda > 0$.

Proof. By definitions of F and P_L we have

$$\langle F(x,\lambda), \varphi\rangle = \lim_{\varepsilon\downarrow 0} \langle \int_\varepsilon^\infty e^{-\lambda t} P_L(x,t)dt, \varphi\rangle = \lim_{\varepsilon\downarrow 0} \int_\varepsilon^\infty e^{-\lambda t} \langle P_L, \varphi\rangle dt.$$

Therefore

$$\begin{aligned} \langle (L(D)+\lambda)F, \varphi\rangle &= \lim_{\varepsilon\downarrow 0} \int_\varepsilon^\infty e^{-\lambda t} \langle (L(D)+\lambda)P_L, \varphi\rangle dt \\ &= \lim_{\varepsilon\downarrow 0} \int_\varepsilon^\infty e^{-\lambda t} \langle L(D)P_L, \varphi\rangle dt + \lambda \int_0^\infty e^{-\lambda t} \langle P_L, \varphi\rangle dt \\ &= \lim_{\varepsilon\downarrow 0} \int_\varepsilon^\infty e^{-\lambda t} \langle -\frac{\partial}{\partial t} P_L, \varphi\rangle dt + \lambda \langle F, \varphi\rangle \\ &= \lim_{\varepsilon\downarrow 0} \left[-e^{-\lambda t} \langle P_L, \varphi\rangle |_\varepsilon^\infty - \lambda \int_\varepsilon^\infty e^{-\lambda t} \langle P_L, \varphi\rangle dt \right] + \lambda \langle F, \varphi\rangle \\ &= \lim_{\varepsilon\downarrow 0} e^{-\lambda\varepsilon} \langle P_L(\cdot,\varepsilon), \varphi\rangle = \langle \delta, \varphi\rangle. \end{aligned}$$

□

Exercise 37. Let us define a fundamental solution $\Gamma(x,t)$ of $\frac{\partial}{\partial t}+L(D)$ as a solution of

$$\begin{cases}(\frac{\partial}{\partial t}+L)\Gamma(x,t)=\delta(x)\delta(t)\\ \Gamma(x,0)=0.\end{cases}$$

Prove that

$$F(x,\lambda):=\int_0^\infty e^{-\lambda t}\Gamma(x,t)dt$$

is a fundamental solution of the operator $L(D)+\lambda I, \lambda>0$.

Example 8.1. Assume that $L(D)=\sum_{j=1}^n\left(\frac{1}{i}\frac{\partial}{\partial x_j}\right)^2=-\Delta$. Then $L(\xi)=|\xi|^2$ and the fundamental solution $F(x,\lambda)$ of the operator $L(D)+\lambda=-\Delta+\lambda$ has the form

$$\begin{aligned} F(x,\lambda) &= \int_0^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\lambda t}\cdot e^{-\frac{x^2}{4t}}dt=\frac{1}{(4\pi)^{\frac{n}{2}}}\int_0^\infty e^{-\lambda t-\frac{x^2}{4t}}\cdot t^{-\frac{n}{2}}dt\\ &= \frac{1}{(4\pi)^{\frac{n}{2}}}\lambda^{\frac{n}{2}-1}\int_0^\infty e^{-\tau-\frac{(\sqrt{\lambda}|x|)^2}{4\tau}}\cdot\tau^{-\frac{n}{2}}d\tau=\frac{1}{(4\pi)^{\frac{n}{2}}}\lambda^{\frac{n}{2}-1}\int_0^\infty e^{-\tau-\frac{r^2}{4\tau}}\cdot\tau^{-\frac{n}{2}}d\tau, \end{aligned}$$

where $r=\sqrt{\lambda}|x|$. From our previous considerations we know that

$$F(x,\lambda)=(2\pi)^{-n/2}F^{-1}\left(\frac{1}{|\xi|^2+\lambda}\right)(x),$$

where F^{-1} is the inverse Fourier transform. The function

$$K_\nu(r)=\frac{1}{2}\left(\frac{r}{2}\right)^\nu\int_0^\infty e^{-t-\frac{r^2}{4t}}\cdot t^{-1-\nu}dt$$

is called the Macdonald function of order ν. So, we have

$$F(x,\lambda)=(2\pi)^{-\frac{n}{2}}\left(\frac{|x|}{\sqrt{\lambda}}\right)^{1-\frac{n}{2}}K_{\frac{n}{2}-1}(\sqrt{\lambda}|x|).$$

It is known that

$$K_\nu(r)=\frac{\pi i}{2}e^{i\pi\frac{\nu}{2}}H_\nu^{(1)}(ir),\quad r>0,$$

where $H_\nu^{(1)}$ is the Hankel function of first kind and order ν.

Next we want to obtain estimates for $F(x,\lambda)$ for $x\in\mathbb{R}^n,\lambda>0$ and $n\geq 1$. Let us consider the integral $\int_0^\infty e^{-\tau-\frac{r^2}{4\tau}}\tau^{-\frac{n}{2}}d\tau$ in two parts $I_1+I_2=\int_0^1+\int_1^\infty$.

1) If $0<r<1$ then

$$I_1=\int_0^1 e^{-y-\frac{r^2}{4y}}y^{-\frac{n}{2}}dy\leq\int_0^1 e^{-\frac{r^2}{4y}}y^{-\frac{n}{2}}dy=c_nr^{2-n}\int_{\frac{r^2}{4}}^\infty e^{-z}z^{\frac{n}{2}-2}dz=c_nr^{2-n}I_1'.$$

$n=1$: Since $I_1' \sim cr^{-1}, r \to +0$ then $|I_1| \le c_n$
$n=2$: Since $I_1' \sim c\ln\frac{1}{r}, r \to +0$ then $|I_1| \le c_n \ln\frac{1}{r}$
$n \ge 3$: Since $I_1' \sim c, r \to +0$ then $|I_1| \le c_n r^{2-n}$.

For I_2 we can simply argue that

$$I_2 = \int_1^\infty e^{-y-\frac{r^2}{4y}} y^{-\frac{n}{2}} dy \le e^{-\frac{r^2}{4}} \int_1^\infty e^{-y} dy \le e^{-\frac{r^2}{4}} \le 1, \quad r \to +0.$$

2) If $r > 1$ then

$$I_1 \le \int_0^1 e^{-\frac{r^2}{4y}} y^{-\frac{n}{2}} dy = c_n r^{2-n} \int_{\frac{r^2}{4}}^\infty e^{-z} z^{\frac{n}{2}-2} dz \le c_n \begin{cases} r^{-2} e^{-\frac{r^2}{4}}, & n = 1,2,3,4 \\ r^{2-n} e^{-\delta r^2}, & n \ge 5, \end{cases}$$

where $0 < \delta < \frac{1}{4}$. The last inequality follows from the fact that $z^{\frac{n}{2}-2} \le c_\varepsilon e^{\varepsilon z}$ for $\frac{n}{2} - 2 > 0$ and for any $\varepsilon > 0$ $(z > 1)$.
Since

$$I_2 \le \int_1^\infty e^{-y-\frac{r^2}{4y}} dy$$

we perform the change of variable $z := y + \frac{r^2}{4y}$. Then $z \ge r$ and $z \to +\infty$. Thus

$$\begin{aligned} \int_1^\infty e^{-y-\frac{r^2}{4y}} dy &= c \int_r^\infty e^{-z} \left(1 + \frac{z}{\sqrt{z^2 - r^2}}\right) dz \\ &= c \int_r^\infty e^{-z} dz + c \int_r^\infty e^{-z} \frac{z dz}{\sqrt{z^2 - r^2}} \\ &= c e^{-r} + c \left(e^{-z} \sqrt{z^2 - r^2} \Big|_r^\infty + \int_r^\infty e^{-z} \sqrt{z^2 - r^2} dz \right) \\ &= c \left(e^{-r} + \int_r^\infty e^{-z} \sqrt{z^2 - r^2} dz \right) \le c e^{-\delta r} \end{aligned}$$

for any $0 < \delta < 1$.

If we collect all these estimates we obtain that

1) If $\sqrt{\lambda}|x| < 1$ then

$$|F(x,\lambda)| \le c_n \lambda^{\frac{n}{2}-1} \begin{cases} 1, & n = 1 \\ \log\frac{1}{\sqrt{\lambda}|x|}, & n = 2 \\ (\sqrt{\lambda}|x|)^{2-n}, & n \ge 3 \end{cases} \le c_n' \lambda^{\frac{n}{2}-1} e^{-\delta\sqrt{\lambda}|x|} \begin{cases} 1, & n = 1 \\ \log\frac{1}{\sqrt{\lambda}|x|}, & n = 2 \\ (\sqrt{\lambda}|x|)^{2-n}, & n \ge 3. \end{cases}$$

2) If $\sqrt{\lambda}|x| > 1$ then

$$|F(x,\lambda)| \le c_n e^{-\delta\sqrt{\lambda}|x|}, \quad n \ge 1.$$

We will rewrite these estimates in more appropriate form for all $\lambda > 0$ and $x \in \mathbb{R}^n$ as

$$|F(x,\lambda)| \le c_n e^{-\delta\sqrt{\lambda}|x|} \begin{cases} \frac{1}{\sqrt{\lambda}}, & n = 1 \\ 1 + |\log \frac{1}{\sqrt{\lambda}|x|}|, & n = 2 \\ |x|^{2-n}, & n \ge 3. \end{cases}$$

Remark. It is not too difficult to observe that $F(x,\lambda)$ is positive.

Example 8.2. Recall from Chapter 7 that the solution of the equation $(-1-\Delta)u = f$ can be written in the form

$$u(x) = K_{-1} * f = F^{-1}\left(\frac{1}{|\xi|^2 - 1}\right) * f,$$

where

$$K_{-1}(|x|) = c_n |x|^{2-n} \lim_{\varepsilon \downarrow 0} \int_0^\infty \frac{\rho^{\frac{n}{2}} J_{\frac{n-2}{2}}(\rho) d\rho}{\rho^2 - |x|^2 - i\varepsilon}.$$

Actually K_{-1} is a fundamental solution of the operator $-1-\Delta$. Let us consider more general operator $-\Delta - \lambda$ for $\lambda > 0$ or even for $\lambda \in \mathbb{C}$. The operator $-\Delta - \lambda$ is called the Helmholtz operator. Its fundamental solution $E_n(x,\lambda)$ satisfies

$$-\Delta E_n - \lambda E_n = \delta(x).$$

We define $\sqrt{\lambda}$ with nonnegative imaginary part i.e. $\sqrt{\lambda} = \alpha + i\beta$, where $\beta \ge 0$ and $\beta = 0$ if and only if $\lambda \in [0, +\infty)$. We require that E_n is radially symmetric. Then, for $x \neq 0$, E_n must solve the equation

$$(r^{n-1}u')' + \lambda r^{n-1} u = 0.$$

This equation can be reduced to one of Bessel type by making the substitution $u = wr^{1-\frac{n}{2}}$. A straightforward calculation shows that

$$(rw')' - \left(1 - \frac{n}{2}\right)^2 \frac{w}{r} + \lambda r w = 0$$

or

$$w'' + \frac{w'}{r} + \left(\lambda - \left(1 - \frac{n}{2}\right)^2 \frac{1}{r^2}\right) w = 0$$

or

$$v''(r\sqrt{\lambda}) + \frac{v'(r\sqrt{\lambda})}{r\sqrt{\lambda}} + \left(1 - \left(1 - \frac{n}{2}\right)^2 \frac{1}{\lambda r^2}\right) v(r\sqrt{\lambda}) = 0, \quad w(r) = v(r\sqrt{\lambda}).$$

This is the Bessel equation of order $\frac{n}{2} - 1$. Its two linearly independent solutions are the Bessel functions $J_{\frac{n}{2}-1}$ and $Y_{\frac{n}{2}-1}$ of the first and second kind, respectively. Therefore the general solution is of the form

$$w(r) = c_0' J_{\frac{n}{2}-1}(\sqrt{\lambda} r) + c_1' Y_{\frac{n}{2}-1}(\sqrt{\lambda} r).$$

For us it is convenient to write it in terms of Hankel functions of first and second kind as

$$w(r) = c_0 H^{(1)}_{\frac{n}{2}-1}(\sqrt{\lambda} r) + c_1 H^{(2)}_{\frac{n}{2}-1}(\sqrt{\lambda} r),$$

where

$$H^{(1)}_\nu(z) = J_\nu(z) + iY_\nu(z), \quad H^{(2)}_\nu(z) = J_\nu(z) - iY_\nu(z).$$

The corresponding general solution u is

$$u(r) = r^{1-\frac{n}{2}} \left[c_0 H^{(1)}_{\frac{n}{2}-1}(\sqrt{\lambda} r) + c_1 H^{(2)}_{\frac{n}{2}-1}(\sqrt{\lambda} r) \right].$$

If $\lambda \notin [0, +\infty)$ then $\sqrt{\lambda}$ has positive imaginary part and the solution $H^{(2)}_{\frac{n}{2}-1}(\sqrt{\lambda} r)$ is exponentially large at $z = +\infty$, whereas $H^{(1)}_{\frac{n}{2}-1}(\sqrt{\lambda} r)$ is exponentially small. Hence we take

$$E_n(x, \lambda) = c_0 r^{1-\frac{n}{2}} H^{(1)}_{\frac{n}{2}-1}(\sqrt{\lambda} r).$$

Exercise 38. Prove that

$$\lim_{\varepsilon \downarrow 0} \int_{|x|=\varepsilon} \frac{\partial E_n}{\partial r} d\sigma(x) = 1$$

or

$$\lim_{r \to 0} r^{n-1} \omega_n \frac{\partial E_n}{\partial r} = 1,$$

where $\omega_n = |\mathbb{S}^{n-1}|$ is the area (measure) of the unit sphere $\mathbb{S}^{n-1}$.

For small values of r, we have the asymptotic expansions

$$H^{(1)}_{\frac{n-2}{2}}(r) \sim -\frac{i 2^{\frac{n-2}{2}} \Gamma(\frac{n-2}{2})}{\pi} r^{-\frac{n-2}{2}}, \quad n \neq 2$$

and

$$H^{(1)}_0(r) \sim \frac{2i}{\pi} \log r.$$

It can be proved using Exercise 38 that

$$c_0 = \frac{i}{4} \left(\frac{\sqrt{\lambda}}{2\pi} \right)^{\frac{n-2}{2}}.$$

Thus for $n \geq 2$ and $\lambda \notin [0, +\infty)$ we obtain

$$E_n(x, \lambda) = \frac{i}{4} \left(\frac{\sqrt{\lambda}}{2\pi |x|} \right)^{\frac{n-2}{2}} H^{(1)}_{\frac{n-2}{2}}(\sqrt{\lambda} |x|). \tag{8.2}$$

A direct calculation shows that for $n = 1$ we have

$$E_1(x, \lambda) = \frac{i}{2\sqrt{\lambda}} e^{i\sqrt{\lambda}|x|}$$

for all $\lambda \neq 0$. The formula (8.2) is valid also for $\lambda \in (0, +\infty)$. This fact follows from the definition:

$$\begin{aligned} E_n(x,\lambda) &= \lim_{\varepsilon \downarrow 0} E_n(x, \lambda + i\varepsilon) = \frac{i}{4} \lim_{\varepsilon \downarrow 0} \left(\frac{\sqrt{\lambda + i\varepsilon}}{2\pi |x|} \right)^{\frac{n-2}{2}} H^{(1)}_{\frac{n-2}{2}}(\sqrt{\lambda + i\varepsilon}|x|) \\ &= \frac{i}{4} \left(\frac{\sqrt{\lambda}}{2\pi |x|} \right)^{\frac{n-2}{2}} H^{(1)}_{\frac{n-2}{2}}(\sqrt{\lambda}|x|). \end{aligned}$$

Remark. We may conclude that

$$(2\pi)^{-n/2} F^{-1} \left(\frac{1}{|\xi|^2 - \lambda - i0} \right) = \frac{i}{4} \left(\frac{\sqrt{\lambda}}{2\pi |x|} \right)^{\frac{n-2}{2}} H^{(1)}_{\frac{n-2}{2}}(\sqrt{\lambda}|x|),$$

for $\lambda > 0$. A direct calculation shows that

$$E_n(x,0) = \begin{cases} -\frac{|x|}{2}, & n = 1 \\ \frac{1}{2\pi} \log \frac{1}{|x|}, & n = 2 \\ \frac{|x|^{2-n}}{(n-2)\omega_n}, & n \geq 3. \end{cases}$$

9 Schrödinger operator

There are many inverse scattering problems which are connected with the reconstruction of the quantum mechanical potential in the Schrödinger operator (Hamiltonian) $H = -\Delta + q(x)$. This operator is defined in $\mathbb{R}^n$. Here and throughout we assume that q is real-valued.

First of all we have to define H as a self-adjoint operator in $L^2(\mathbb{R}^n)$. Our basic assumption is that the potential $q(x)$ belongs to $L^p(\mathbb{R}^n)$ for $\frac{n}{2} < p \le \infty$ and has the following special behavior at infinity:

$$|q(x)| \le c|x|^{-\mu}, \quad |x| > R \tag{9.1}$$

with some $\mu \ge 0$ and $R > 0$ large enough. This parameter μ will be specified later, depending on the situation. We would like to construct the self-adjoint extension of this operator by Friedrichs method, because formally our operator is defined now only for smooth functions, say for functions from $C_0^\infty(\mathbb{R}^n)$. In order to construct such extension let us consider the Hilbert space H_1 which is defined as follows

$$H_1 = \{f \in L^2(\mathbb{R}^n) : \nabla f(x) \in L^2(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n} |q(x)||f(x)|^2 dx < \infty\}.$$

The inner product in H_1 is defined by

$$(f, g)_{H_1} = (\nabla f, \nabla g)_{L^2} + \int_{\mathbb{R}^n} q(x) f(x) \overline{g(x)} dx + \mu_0 (f, g)_{L^2},$$

with $\mu_0 > 0$ large enough and fixed.

Lemma 1. *Assume that $f \in W_2^1(\mathbb{R}^n)$ and $q \in L^p(\mathbb{R}^n)$ for $\frac{n}{2} < p \le \infty, n \ge 2$. Then for any $0 < \varepsilon < 1$ there exists $c_\varepsilon > 0$ such that*

$$|(qf, f)_{L^2}| \le \varepsilon \|\nabla f\|^2_{L^2(\mathbb{R}^n)} + c_\varepsilon \|f\|^2_{L^2(\mathbb{R}^n)}.$$

Proof. If $p = \infty$ then

$$\begin{aligned} |(qf, f)_{L^2}| &\le \int_{\mathbb{R}^n} |q(x)||f(x)|^2 dx \le \|q\|_{L^\infty(\mathbb{R}^n)} \|f\|^2_{L^2(\mathbb{R}^n)} \\ &\le \varepsilon \|\nabla f\|^2_{L^2(\mathbb{R}^n)} + \|q\|_{L^\infty(\mathbb{R}^n)} \|f\|^2_{L^2(\mathbb{R}^n)}. \end{aligned}$$

If $\frac{n}{2} < p < \infty$ then we estimate

$$\begin{aligned} |(qf, f)_{L^2}| &\le \int_{|q(x)|<A} |q(x)||f(x)|^2 dx + \int_{|q(x)|>A} |q(x)||f(x)|^2 dx \\ &\le \int_{|q(x)|>A} |q(x)||f(x)|^2 dx + A \|f\|^2_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Let us consider the integral appearing in the last estimate. For $n \geq 3$ it follows from Hölder inequality that

$$\begin{aligned}
\int_{|q(x)|>A} |q(x)||f(x)|^2 dx &\leq \left(\int_{|q(x)|>A} |q(x)|^{\frac{n}{2}} dx\right)^{\frac{2}{n}} \left(\int_{|q(x)|>A} |f(x)|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}} \\
&\leq A^{\left(\frac{n}{2}-p\right)\frac{2}{n}} \left(\int_{|q(x)|>A} |q(x)|^p dx\right)^{\frac{2}{n}} c_1 \|f\|^2_{W_2^1(\mathbb{R}^n)} \\
&\leq c_1 A^{1-\frac{2p}{n}} \|q\|^{\frac{2p}{n}}_{L^p(\mathbb{R}^n)} \|f\|^2_{W_2^1(\mathbb{R}^n)}.
\end{aligned}$$

For getting the last inequality we used the fact that $\frac{n}{2} < p < \infty$ and a well-known embedding: $W_2^1(\mathbb{R}^n) \subset L^{\frac{2n}{n-2}}(\mathbb{R}^n), n \geq 3$, with the norm estimate

$$\|f\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq \sqrt{c_1} \|f\|_{W_2^1(\mathbb{R}^n)}.$$

Collecting these estimates we obtain

$$\begin{aligned}
|(qf, f)_{L^2}| &\leq c_1 A^{1-\frac{2p}{n}} \|q\|^{\frac{2p}{n}}_{L^p(\mathbb{R}^n)} \|f\|^2_{W_2^1(\mathbb{R}^n)} + A \|f\|^2_{L^2(\mathbb{R}^n)} \\
&= c_1 A^{1-\frac{2p}{n}} \|q\|^{\frac{2p}{n}}_{L^p(\mathbb{R}^n)} \|\nabla f\|^2_{L^2(\mathbb{R}^n)} + \left(A + c_1 A^{1-\frac{2p}{n}} \|q\|^{\frac{2p}{n}}_{L^p(\mathbb{R}^n)}\right) \|f\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

The claim follows now from the last inequality since $A^{1-\frac{2p}{n}}$ can be chosen sufficiently small for $\frac{n}{2} < p < \infty$. □

Exercise 39. Prove Lemma 1 for $n = 2$.

Exercise 40. Let us assume that $q(x)$ satisfies the conditions

1) $|q| \leq c_1 |x|^{-\gamma_1}, \quad |x| < 1,$

and

2) $|q| \leq c_2 |x|^{-\gamma_2}, \quad |x| > 1.$

Find the conditions on γ_1 and γ_2 which ensure the statement of Lemma 1.

Remark. Lemma 1 holds for any potential $q \in L^p(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ for $p > \frac{n}{2}, n \geq 2$.

Using Lemma 1 we obtain

$$\begin{aligned}
\|f\|^2_{H_1} &= \|\nabla f\|^2_{L^2(\mathbb{R}^n)} + \mu_0 \|f\|^2_{L^2(\mathbb{R}^n)} + (qf, f)_{L^2} \\
&\geq \|\nabla f\|^2_{L^2(\mathbb{R}^n)} + \mu_0 \|f\|^2_{L^2(\mathbb{R}^n)} - \varepsilon \|\nabla f\|^2_{L^2(\mathbb{R}^n)} - c_\varepsilon \|f\|^2_{L^2(\mathbb{R}^n)} \\
&= (1-\varepsilon) \|\nabla f\|^2_{L^2(\mathbb{R}^n)} + (\mu_0 - c_\varepsilon) \|f\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

We choose here $0 < \varepsilon < 1$ and $\mu_0 > c_\varepsilon$. On the other hand

$$\|f\|^2_{H_1} \leq (1+\varepsilon) \|\nabla f\|^2_{L^2(\mathbb{R}^n)} + (\mu_0 + c_\varepsilon) \|f\|^2_{L^2(\mathbb{R}^n)}.$$

These two inequalities mean that the new Hilbert space H_1 is equivalent to the space $W_2^1(\mathbb{R}^n)$ up to equivalent norms. Thus we may conclude that for any $f \in H_1$ our operator is well defined by

$$(f, (H + \mu_0)f)_{L^2(\mathbb{R}^n)} = \|f\|_{H_1}^2 .$$

Moreover, since $H + \mu_0$ is positive then

$$\|f\|_{H_1}^2 = \left\|(H + \mu_0)^{\frac{1}{2}} f\right\|_{L^2(\mathbb{R}^n)}^2$$

and the following statements hold:

1) Domain of $(H + \mu_0)^{\frac{1}{2}}$ is $W_2^1(\mathbb{R}^n)$

2) $D(H + \mu_0) \equiv D(H) \subset W_2^1(\mathbb{R}^n)$

3) $D(H) = \{f \in W_2^1(\mathbb{R}^n) : Hf \in L^2(\mathbb{R}^n)\}$.

Remark.

$$(H+\mu_0)f = (H+\mu_0)^{\frac{1}{2}}(H+\mu_0)^{\frac{1}{2}} f \Leftrightarrow D(H) = \{f \in W_2^1(\mathbb{R}^n) : g := (H+\mu_0)^{\frac{1}{2}} f \in W_2^1(\mathbb{R}^n)\}.$$

Remark. Let us consider this extension procedure from another point of view. The inequality

$$(f, (H + \mu_0)f)_{L^2} \geq (1 - \varepsilon) \|\nabla f\|_{L^2(\mathbb{R}^n)}^2 + (\mu_0 - c_\varepsilon) \|f\|_{L^2(\mathbb{R}^n)}^2$$

allows us to conclude that

a) $(f, (H + \mu_0)f)_{L^2} \geq c' \|f\|_{L^2(\mathbb{R}^n)}^2$ and

b) $(f, (H + \mu_0)f)_{L^2} \geq c'' \|f\|_{W_2^1(\mathbb{R}^n)}^2$

for any $f \in C_0^\infty(\mathbb{R}^n)$. It means that there exists $(H + \mu_0)^{-1}$ which is also defined for $g \in C_0^\infty(\mathbb{R}^n)$ and satisfies the inequality

a) $\|(H + \mu_0)^{-1} g\|_{L^2(\mathbb{R}^n)} \leq \frac{1}{c'} \|g\|_{L^2(\mathbb{R}^n)}$ or even

b) $\|(H + \mu_0)^{-1} g\|_{W_2^1(\mathbb{R}^n)} \leq \frac{1}{c''} \|g\|_{W_2^{-1}(\mathbb{R}^n)}$, where $W_2^{-1}(\mathbb{R}^n)$ is the conjugate (adjoint) space of $W_2^1(\mathbb{R}^n)$.

Since $(H + \mu_0)^{-1}$ is bounded operator and $\overline{C_0^\infty(\mathbb{R}^n)} \overset{L^2}{=} L^2(\mathbb{R}^n)$ and $\overline{C_0^\infty(\mathbb{R}^n)} \overset{W_2^{-1}}{=} W_2^{-1}(\mathbb{R}^n)$ then we can extend $(H + \mu_0)^{-1}$ as the bounded operator onto $L^2(\mathbb{R}^n)$ in the first case and onto $W_2^{-1}(\mathbb{R}^n)$ in the second case. The extension for the differential operator is $H + \mu_0 = ((H + \mu_0)^{-1})^{-1}$ and $D(H + \mu_0) = R((H + \mu_0)^{-1})$ in both cases. It is also clear that $H + \mu_0$ and $(H + \mu_0)^{-1}$ are self-adjoint operators.

Lemma 2. *Let us assume that $q \in L^p(\mathbb{R}^n)$ for $2 \le p \le \infty$ if $n = 2, 3$ and $q \in L^p(\mathbb{R}^n)$ for $\frac{n}{2} < p \le \infty$ if $n \ge 4$. Then*

$$W_2^2(\mathbb{R}^n) \subset D(H).$$

Proof. Since $H = -\Delta + q$ and $D(H) = \{f \in W_2^1(\mathbb{R}^n) : Hf \in L^2(\mathbb{R}^n)\}$ then for required embedding it is enough to show that for $f \in W_2^2(\mathbb{R}^n)$ it follows that $qf \in L^2(\mathbb{R}^n)$.

If $p = \infty$ then

$$\int_{\mathbb{R}^n} |qf|^2 dx \le \|q\|^2_{L^\infty(\mathbb{R}^n)} \|f\|^2_{L^2(\mathbb{R}^n)} < \infty$$

for any $f \in W_2^2(\mathbb{R}^n), n \ge 2$.

For finite p let us consider first the case $n = 2, 3$. Since $W_2^2(\mathbb{R}^n) \subset C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ (Sobolev embedding) then

$$\begin{aligned} \int_{\mathbb{R}^n} |qf|^2 dx &= \int_{|q|<A} |qf|^2 dx + \int_{|q|>A} |qf|^2 dx \\ &\le A^2 \int_{|q|<A} |f|^2 dx + \|f\|^2_{L^\infty(\mathbb{R}^n)} \int_{|q|>A} |q|^p |q|^{2-p} dx \\ &\le A^2 \|f\|^2_{L^2(\mathbb{R}^n)} + C \|f\|^2_{W_2^2(\mathbb{R}^n)} A^{2-p} \|q\|^p_{L^p(\mathbb{R}^n)} < \infty. \end{aligned}$$

In the case $n \ge 4$ we have the embeddings:

- $n = 4 :\quad f \in W_2^2(\mathbb{R}^4) \subset L^p(\mathbb{R}^4), \quad p < \infty.$
- $n \ge 5 :\quad f \in W_2^2(\mathbb{R}^n) \subset L^{\frac{2n}{n-4}}(\mathbb{R}^n).$

That's why applying the Hölder inequality we obtain

- $n \ge 5 :$

$$\begin{aligned} \int_{\mathbb{R}^n} |qf|^2 dx &= \int_{|q|<A} |qf|^2 dx + \int_{|q|>A} |qf|^2 dx \\ &\le A^2 \|f\|^2_{L^2(\mathbb{R}^n)} + \left(\int_{|q|>A} |q|^{\frac{n}{2}} dx \right)^{\frac{4}{n}} \left(\int_{|q|>A} |f|^{\frac{2n}{n-4}} dx \right)^{\frac{n-4}{n}} \\ &\le A^2 \|f\|^2_{L^2(\mathbb{R}^n)} + C A^{(\frac{n}{2}-p)\frac{4}{n}} \left(\int_{|q|>A} |q|^p dx \right)^{\frac{4}{n}} \|f\|^2_{W_2^2(\mathbb{R}^n)} < \infty. \end{aligned}$$

- $n = 4 :$

$$\int_{\mathbb{R}^4} |qf|^2 dx \le \left(\int_{\mathbb{R}^4} |q|^p dx \right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^4} |f|^{p'} dx \right)^{\frac{2}{p'}} < \infty$$

for $2 < p < \infty$ and $p' < \infty$.

□

Exercise 41. Prove this lemma for $q \in L^p(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$, $\frac{n}{2} < p \leq \infty$ if $n \geq 4$ and for $q \in L^2(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ if $n = 2, 3$.

Remark. For $n \geq 5$ we may consider $q \in L^{\frac{n}{2}}(\mathbb{R}^n)$.

Lemma 3. *Let us assume that $q \in L^n(\mathbb{R}^n), n \geq 3$. Then*

$$D(H) = W_2^2(\mathbb{R}^n).$$

Proof. The embedding $W_2^2(\mathbb{R}^n) \subset D(H)$ was proved in Lemma 2. Let us now assume that $f \in D(H)$ i.e. $f \in W_2^1(\mathbb{R}^n)$ and $Hf \in L^2(\mathbb{R}^n)$. Note that for $g := Hf \in L^2$ we have the following representation

$$\begin{aligned} -f &= (-\Delta+1)^{-1}(q-1)f - (-\Delta+1)^{-1}g \\ &= (-\Delta+1)^{-1}(qf) - \underbrace{(-\Delta+1)^{-1}g}_{\in W_2^2} - \underbrace{(-\Delta+1)^{-1}f}_{\in W_2^2}. \end{aligned}$$

That's why it is enough to show that $qf \in L^2(\mathbb{R}^n)$. We use the same arguments as in Lemma 1 and Lemma 2. So it suffices to show that for any $f \in W_2^1(\mathbb{R}^n)$ it follows that $qf \in L^2(\mathbb{R}^n)$. From the embedding $W_2^1(\mathbb{R}^n) \subset L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ for $n \geq 3$ we have by Hölder inequality

$$\begin{aligned} \int_{\mathbb{R}^n} |q(x)|^2 |f(x)|^2 dx &= \int_{|q|<A} |q(x)|^2 |f(x)|^2 dx + \int_{|q|>A} |q(x)|^2 |f(x)|^2 dx \\ &\leq A^2 \|f\|^2_{L^2(\mathbb{R}^n)} + \left(\int_{|q|>A} |q|^n dx \right)^{\frac{2}{n}} \left(\int_{|q|>A} |f|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} < \infty. \end{aligned}$$

Thus lemma is proved. □

Exercise 42. Describe the domain of H for the case $\frac{n}{2} < p < n, n \geq 3$. Hint: Prove that $D(H) \subset W_2^2(\mathbb{R}^n) + W_s^2(\mathbb{R}^n)$ with some $s = s(p)$.

Let us present some mathematical background material concerning self-adjoint operators in Hilbert spaces. If $A : \mathcal{H} \to \mathcal{H}$ is a linear operator in a Hilbert space $\mathcal{H}$ with $\overline{D(A)} = \mathcal{H}$ then the set

$$\rho(A) := \{\lambda \in \mathbb{C} : (A - \lambda I)^{-1} \,\text{exists as a bounded operator from}\, \mathcal{H} \,\text{to}\, D(A)\}$$

is called the *resolvent set* of A and its complement

$$\sigma(A) := \mathbb{C} \setminus \rho(A)$$

is called the *spectrum* of A. The operator-valued function

$$R_\lambda := (A - \lambda I)^{-1}, \quad \lambda \in \rho(A),$$

is called the *resolvent* of A.

If $A = A^*$ (or even if A is closed) then the resolvent set is open and the spectrum is closed. Moreover, $\sigma(A) \neq \emptyset$ and $\sigma(A) \subset \mathbb{R}$ in this case.

Definition. If $A = A^*$ then

a) the *point spectrum* $\sigma_p(A)$ of A is the set of all eigenvalues of A, i.e.,

$$\sigma_p(A) = \{\lambda \in \sigma(A) : \operatorname{Ker}(A - \lambda I) \neq 0\}.$$

It means that there exists a non-trivial $f \in D(A)$ such that $Af = \lambda f$. The linear subspace

$$\{f \in D(A) : Af = \lambda f\}$$

is called the *eigenspace* of A corresponding to λ.

b) the complement $\sigma(A) \setminus \sigma_p(A)$ is the *continuous spectrum* $\sigma_c(A)$ of A.

c) the *discrete spectrum* $\sigma_d(A)$ of A is defined as

$$\sigma_d(A) := \{\lambda \in \sigma_p(A) : \dim \operatorname{Ker}(A - \lambda I) < \infty\}$$

and λ must also be isolated in $\sigma(A)$.

d) the set $\sigma_{\text{ess}}(A) := \sigma(A) \setminus \sigma_d(A)$ is called the *essential spectrum* of A.

Remark. $\lambda \in \sigma_c(A)$ means that $(A - \lambda I)^{-1}$ does exist but it is not bounded. It is equivalent to $\overline{R(A - \lambda I)} \neq \mathcal{H}$, i.e., there exists $f \in \mathcal{H}$ such that $f \notin R(A - \lambda I)$.

Theorem 1 (Spectral theorem of J. Neumann)**.** *Let us assume that $A : \mathcal{H} \to \mathcal{H}$ and $\overline{D(A)} = \mathcal{H}$. Then $A = A^*$ if and only if there exists a spectral family $\{E_\lambda\}_{\lambda=-\infty}^{\infty}$ i.e. an orthogonal projection E_λ (E_λ is a bounded and self-adjoint on $\mathcal{H}$ with $E_\lambda^2 = E_\lambda$) satisfying the conditions*

1) $E_\lambda \leq E_\mu$ for $\lambda \leq \mu$ (that is $E_\lambda \mathcal{H} \subset E_\mu \mathcal{H}$)

2) $E_{\lambda+0} = E_\lambda$ in norm

3) $E_{-\infty} = 0$ and $E_\infty = I$

4) $E_\lambda A = A E_\lambda$ on $D(A)$.

The domain $D(A)$ can be defined (or described) as:

$$D(A) = \{f \in \mathcal{H} : \int_{-\infty}^{\infty} \lambda^2 d_\lambda \|E_\lambda f\|^2 < \infty\}.$$

Moreover, if $f \in D(A)$ then

$$Af \stackrel{L^2}{=} \int_{-\infty}^{\infty} \lambda dE_\lambda f, \quad \|Af\|_{L^2}^2 = \int_{-\infty}^{\infty} \lambda^2 d \|E_\lambda f\|^2 .$$

Remark. The spectral family $\{E_\lambda\}$ is uniquely defined.

Remark. If $F(\cdot)$ is an arbitrary complex-valued function then the operator $F(A)$ can be defined by

$$F(A) = \int_{-\infty}^{\infty} F(\lambda) dE_\lambda$$

with domain

$$\{f \in \mathcal{H} : \int_{-\infty}^{\infty} |F(\lambda)|^2 d \left\|E_\lambda f\right\|^2 < \infty\}.$$

Exercise 43. Prove that for any $f, g \in \mathcal{H}$ the real-valued function $V(\lambda) := (E_\lambda f, g)$ is a function of bounded variation.

Let us return to our Schrödinger operator $H = -\Delta + q$ with $q \in L^p(\mathbb{R}^n)$, $\frac{n}{2} < p \leq \infty, n \geq 2$. We proved that $H = H^*$ and for $f \in D(H)$ it follows that

$$(Hf, f)_{L^2} \geq -c_0 \left\|f\right\|^2_{L^2(\mathbb{R}^n)}, \tag{9.2}$$

where

$$D(H) = \{f \in W^1_2(\mathbb{R}^n) : Hf \in L^2(\mathbb{R}^n)\}.$$

That's why we may conclude that

$$H = \int_{-c_0}^{\infty} \lambda dE_\lambda \quad \Leftrightarrow \quad (Hf, f)_{L^2} = \int_{-c_0}^{\infty} \lambda d(E_\lambda f, f).$$

Our next problem is to investigate the spectrum of this Schrödinger operator.

Exercise 44. Let $A : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be such that $Af(t) := tf(t), t \in \mathbb{R}$ for $f \in D(A)$. Define $D(A)$ and formulate the spectral theorem in this case.

Exercise 45. Let A be a self-adjoint operator in Hilbert space $\mathcal{H}$. Assume that $z \in \mathbb{C} \setminus \mathbb{R}$, that is, $\operatorname{Im} z \neq 0$. Prove that the resolvent $R_z = (A - zI)^{-1}$ is a bounded operator in $\mathcal{H}$ with the norm estimate $\|R_z\|_{\mathcal{H}\to\mathcal{H}} \leq \frac{1}{|\operatorname{Im} z|}$.

Exercise 46. Let A be a self-adjoint operator in Hilbert space $\mathcal{H}$ with the spectral family $\{E_\lambda\}_{\lambda=-\infty}^{\infty}$ and let $f(t) = \frac{t-i}{t+i}, t \in \mathbb{R}$. Define the *Cayley transform* by $U_A := \int_{-\infty}^{\infty} \frac{\lambda - i}{\lambda + i} dE_\lambda$. Prove that

1) $\|U_A u\| = \|u\|$ for any $u \in \mathcal{H}$.

2) $A = i(I + U_A)(I - U_A)^{-1}$.

We will need also the following facts about the spectrum of a self-adjoint operator in $\mathcal{H}$.

1) A real number λ belongs to $\sigma(A)$ if and only if there is a sequence $\{f_m\} \subset D(A)$ such that $\|f_m\| = 1$ and $\|(A - \lambda) f_m\| \to 0$ as $m \to \infty$.

2) The essential spectrum $\sigma_{\text{ess}}(A)$ is the union of

   a) $\sigma_c(A)$;

b) the limiting points of $\sigma_p(A)$;

c) eigenvalues of infinite multiplicity.

3) A real number λ belongs to $\sigma_{\text{ess}}(A)$ if and only if there is a sequence $\{f_m\} \subset D(A)$ such that

a) $\|f_m\| = 1$

b) $f_m \to 0$ weakly

c) $(A - \lambda)f_m \to 0$ in norm.

4) If $\lambda_0 \in \sigma(A)$ is not an isolated point of $\sigma(A)$ then $\lambda_0 \in \sigma_{\text{ess}}(A)$. In other words, if $\lambda_0 \in \sigma(A)$ and $\lambda_0 \notin \sigma_{\text{ess}}(A)$ then λ_0 is isolated.

5) If $\lambda \in \sigma(A) \setminus \sigma_{\text{ess}}(A)$ then λ is an eigenvalue of A of finite multiplicity.

6) If A is a self-adjoint and K a compact operator then $\sigma_{\text{ess}}(A + K) = \sigma_{\text{ess}}(A)$.

Let us consider now Laplacian $H_0 = -\Delta$ in $\mathbb{R}^n, n \geq 1$. Since $(-\Delta f, f)_{L^2} = \|\nabla f\|^2_{L^2(\mathbb{R}^n)} \geq 0$ for any $f \in W_2^1(\mathbb{R}^n)$, then H_0 is a non-negative operator. Moreover, $H_0 = H_0^*$ with domain $D(H_0) = W_2^2(\mathbb{R}^n)$ and this operator has the spectral representation

$$H_0 f = \int_0^\infty \lambda dE_\lambda f.$$

It follows that $\sigma(H_0) \subset [0, +\infty)$, but actually $\sigma(H_0) = [0, +\infty)$ and even $\sigma(H_0) = \sigma_c(H_0) = \sigma_{\text{ess}}(H_0) = [0, +\infty)$. In order to understand this fact it is enough to observe that for any $\lambda \in [0, +\infty)$ the homogenous equation $(H_0 - \lambda)u = 0$ has a solution of the form $u(x, \overrightarrow{k}) = e^{i(\overrightarrow{k}, x)}$, where $(\overrightarrow{k}, \overrightarrow{k}) = \lambda$ and $\overrightarrow{k} \in \mathbb{R}^n$. These solutions $u(x, \overrightarrow{k})$ are called *generalized eigenfunctions*, but $u(x, \overrightarrow{k}) \notin L^2(\mathbb{R}^n)$. These solutions are bounded and correspond to the continuous spectrum of H_0. That's why $u(x, \overrightarrow{k})$ are not eigenfunctions, but generalized eigenfunctions. If we consider the solutions of the equation $(H_0 - \lambda)u = 0$ for $\lambda < 0$, then these solutions will be exponentially increasing at the infinity. It implies that $\lambda < 0$ does not belong to $\sigma(H_0)$.

For the spectral representation of H_0 we have two forms:

1) The *Neumann spectral representation*

$$-\Delta f = \int_0^\infty \lambda dE_\lambda f, \quad f \in W_2^2(\mathbb{R}^n),$$

2) and *Scattering theory* representation

$$-\Delta f = F^{-1}(|\xi|^2 \widehat{f}) = (2\pi)^{-n} \int_{\mathbb{R}^n} |\xi|^2 e^{i(\xi,x)} d\xi \int_{\mathbb{R}^n} e^{-i(\xi,y)} f(y) dy.$$

Exercise 47. Determine the connection between these two representations.

There are some important remarks about the resolvent $(-\Delta - z)^{-1}$ for $z \notin [0, +\infty)$. A consequence of the spectral theorem is that

$$(-\Delta - z)^{-1} = \int_0^\infty (\lambda - z)^{-1} dE_\lambda, \quad z \in \mathbb{C} \setminus [0, +\infty),$$

and for such z the operator $(-\Delta - z)^{-1}$ is bounded operator in $L^2(\mathbb{R}^n)$. Moreover, with respect to $z \notin [0, +\infty)$ the operator $(-\Delta - z)^{-1}$ as a operator-valued function is a *holomorphic* function. This fact follows immediately from

$$((-\Delta - z)^{-1})'_z = \int_0^\infty (\lambda - z)^{-2} dE_\lambda = (-\Delta - z)^{-2}.$$

The last integral converges as well as the previous one (even better). Now we are in the position to formulate a theorem about the spectrum of $H = -\Delta + q$.

Theorem 2. *Assume that $q \in L^p(\mathbb{R}^n)$, $\frac{n}{2} < p \le \infty, n \ge 2$ and $q(x) \to 0$ as $|x| \to +\infty$. Then*

1) $\sigma_c(H) \supset (0, +\infty)$;

2) $\sigma_p(H) \subset [-c_0, 0]$ is of finite multiplicity with only accumulation point at $\{0\}$ with c_0 from (9.2).

In order to prove this theorem we will prove two lemmas.

Lemma 4. *Assume that the potential $q(x)$ satisfies the assumptions of Theorem 2. Assume in addition that $q(x) \in L^2(\mathbb{R}^n)$ for $n = 2, 3$. Then*

$$(-\Delta - z)^{-1} \circ q : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is a compact operator for $z \notin [0, +\infty)$.

Proof. Due to our assumptions on the potential $q(x)$ it can be represented as the sum $q(x) = q_1(x) + q_2(x)$, where $q_1 \in L^p(|x| < R)$ with the same p and $q_2 \to 0$ as $|x| \to \infty$. We may assume (without loss of generality) that q_2 is supported in $\{x \in \mathbb{R}^n : |x| > R\}$ and that it is a continuous function. Let us consider first the case when $n = 2, 3$. If $f \in L^2(\mathbb{R}^n)$ then $q_1 f \in L^1(|x| < R)$ and $(-\Delta - z)^{-1}(q_1 f) \in W_1^2(\mathbb{R}^n)$ (by Fourier transform). By the embedding theorem for Sobolev spaces we have that

$$(-\Delta - z)^{-1}(q_1 f) \in W_1^2(\mathbb{R}^n) \subset W_2^{2-\frac{n}{2}}(\mathbb{R}^n), \quad n = 2, 3$$

with the norm estimate

$$\begin{aligned}
\left\|(-\Delta - z)^{-1}(q_1 f)\right\|_{L^2(\mathbb{R}^n)} &\le \left\|(-\Delta - z)^{-1}(q_1 f)\right\|_{W_2^{2-\frac{n}{2}}} \\
&\le c\left\|(-\Delta - z)^{-1}(q_1 f)\right\|_{W_1^2} \le c\left\|q_1 f\right\|_{L^1(\mathbb{R}^n)} \\
&\le c\left\|q_1\right\|_{L^2(|x|<R)} \left\|f\right\|_{L^2(|x|<R)}
\end{aligned}$$

or

$$\left\|(-\Delta - z)^{-1} \circ q_1\right\|_{L^2(|x|<R)\to L^2(\mathbb{R}^n)} \le c \left\|q_1\right\|_{L^2},$$

where c may depend only on z.

In the case $n \ge 4$ and $q \in L^p(|x| < R), p > \frac{n}{2}$, we may obtain by Hölder inequality that

$$q_1 f \in L^s(|x| < R), \quad s > \frac{2n}{n+4},$$

for $f \in L^2(\mathbb{R}^n)$ and, therefore, $(-\Delta - z)^{-1}(q_1 f) \in W_s^2(\mathbb{R}^n)$. Again by embedding theorem for Sobolev spaces we have

$$(-\Delta - z)^{-1}(q_1 f) \in W_2^{2-n\left(\frac{1}{s}-\frac{1}{2}\right)}(\mathbb{R}^n)$$

for some $s > \frac{2n}{n+4}$, with the norm estimate

$$\left\|(-\Delta - z)^{-1} \circ q_1\right\|_{L^2(|x|<R)\to L^2(\mathbb{R}^n)} \le c \left\|q_1\right\|_{L^p(|x|<R)}.$$

In order to prove that $(-\Delta - z)^{-1} \circ q_1$ is a compact operator we approximate it as follows:

$$A := (-\Delta - z)^{-1} \circ q_1, \quad A_j := \varphi_j(x) A,$$

where $\varphi_j(x) \in C_0^\infty(\mathbb{R}^n), |\varphi_j(x)| \le C$ and

$$\|A - A_j\|_{L^2 \to L^2} \to 0, \quad j \to \infty.$$

The reason is that $(-\Delta - z)^{-1} \circ q_1$ is actually an integral operator with a kernel $K_z(x-y)$ which tends to 0 when $|x| \to \infty$ uniformly with respect to $|y| < R$ (note that q_1 is supported in $|y| < R$). That's why we can approximate this kernel K_z by the functions $\varphi_j \in C_0^\infty(\mathbb{R}^n)$. But A_j is a compact operator for each $j = 1, 2, \dots$ because the embedding

$$W_2^\alpha(|x| < R) \subset L^2(|x| < R)$$

is compact for positive α. It implies that also A is compact operator.

Next we consider q_2. Since for $f(x) \in L^2(\mathbb{R}^n)$ we know that $(-\Delta - z)^{-1} f \in W_2^2(\mathbb{R}^n)$ then we may conclude that $q_2(-\Delta - z)^{-1} f \in L^2(|x| > R)$. Actually

$$q_2 \cdot : W_2^2(\mathbb{R}^n) \to L^2(|x| > R)$$

is compact embedding. In order to establish this fact let us consider again $\varphi_j(x) \in C_0^\infty(\mathbb{R}^n), |\varphi_j(x)| \le c$ and $\varphi_j \to q_2$ as $j \to \infty$. We can state this because $\overline{C_0^\infty} \stackrel{L^\infty}{=} \dot{C}$. That's why we required such behavior of $q(x)$ at the infinity ($q \to 0$ as $|x| \to +\infty$). If we denote $A := q_2(-\Delta - z)^{-1}$ and $A_j := \varphi_j(-\Delta - z)^{-1}$ then we obtain

$$\|A - A_j\|_{L^2 \to L^2} \le \sup_x |\varphi_j - q_2| \left\|(-\Delta - z)^{-1}\right\|_{L^2 \to L^2} \le c \sup_x |\varphi_j - q_2| \to 0, \ j \to +\infty. \tag{9.3}$$

But we know that $W^2_{2,\text{comp}} \subset L^2_{\text{comp}}$ is compact embedding. This implies (together with (9.3)) that A is compact operator. Thus lemma is proved, because

$$(-\Delta - z)^{-1} \circ q_2 = (q_2(-\Delta - \overline{z})^{-1})^*.$$

□

Lemma 5. *Let Q be an open and connected set in $\mathbb{C}$. Let $A(z)$ be compact, operator valued and holomorphic function in Q and in $L^2(\mathbb{R}^n)$. If $(I + A(z_0))^{-1}$ exists for some $z_0 \in Q$ then $(I + A(z))^{-1}$ exists in all of Q except for finitely many points from Q with only possible accumulation points on ∂Q.*

Proof. We will prove this lemma for the concrete operator $A(z) := (-\Delta - z)^{-1}q(x)$. Lemma 4 shows us that $A(z)$ is compact operator for $z \notin [0, +\infty)$. The remarks about $R_z = (-\Delta - z)^{-1}$ show us that $A(z)$ is a holomorphic function in $\mathbb{C} \setminus [0, +\infty)$. Also we can prove that $(I + (-\Delta - z)^{-1}q)^{-1}$ exists for any $z \in \mathbb{C} \setminus \mathbb{R}$ or for real $z < -c_0$, where $-\Delta + q \geq -c_0$. Indeed, if $z \in \mathbb{C}$ with $\operatorname{Im} z \neq 0$ then $(I + (-\Delta - z)^{-1}q)u = 0$ or $(-\Delta - z)u = -qu$ or $(\Delta u, u) + z(u, u) = (qu, u)$. It implies for $z, \operatorname{Im} z \neq 0$, that $(u, u) = 0$ if and only if $u = 0$. In the real case $z < -c_0$ we have that equality $(I + (-\Delta - z)^{-1}q)u = 0$ implies

$$((-\Delta + q)u, u) - z(u, u) = 0.$$

It follows that

$$(-c_0 - z) \|u\|^2_{L^2} \leq 0$$

or $u = 0$. These remarks show us that in $\mathbb{C} \setminus [0, +\infty)$ our operator $I + (-\Delta - z)^{-1}q$ may be non-invertible only on $[-c_0, 0)$.

Let us consider an open and connected set Q in $\mathbb{C} \setminus [0, +\infty)$ such that $[-c_0, 0) \subset Q$, see Figure 3. It is easily seen that there exists $z_0 \in Q$ such that $(I + (-\Delta - z_0)^{-1}q)^{-1}$ exists also. It is not so difficult to show that there exists $\delta > 0$ such that $(I + (-\Delta - z)^{-1}q)^{-1}$ exists in $U_\delta(z_0)$. Indeed, let us choose $\delta > 0$ such that

$$\|A(z) - A(z_0)\|_{L^2 \to L^2} < \frac{1}{\|(I + A(z_0))^{-1}\|_{L^2 \to L^2}} \tag{9.4}$$

for all z such that $|z - z_0| < \delta$. Then

$$(I + A(z))^{-1} = (I + A(z_0))^{-1}(I + B)^{-1},$$

where $B := (A(z) - A(z_0))(I + A(z_0))^{-1}$. But $\|B\| < 1$ due to (9.4) and then

$$(I + B)^{-1} = I - B + B^2 + \cdots + (-1)^n B^n + \cdots$$

exists in the strong topology from L^2 to L^2. That's why we may conclude that $I + A(z)$ may be non-invertible only for finitely many points in Q. This fact follows from the holomorphicity of $A(z)$ with respect to z by analogy with the theorem about zeros of the

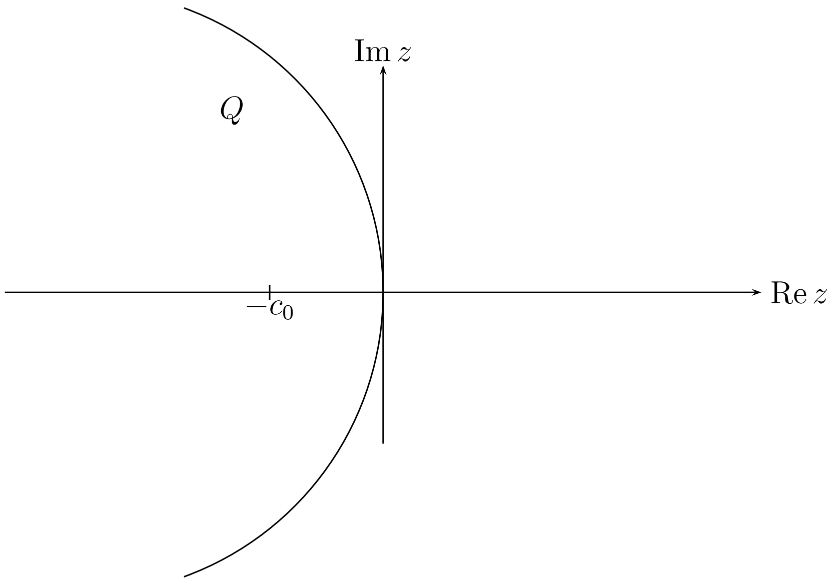

Figure 3: The set Q.

holomorphic function in complex analysis. Moreover, since $A(z)$ is compact operator then by Fredholm's alternative $\operatorname{Ker}(I + A(z))$ has finite dimension. That's why we may conclude that $(I + (-\Delta - z)^{-1}q)^{-1}$ does not exist only in the finite numbers of the points (at most) on $[-c_0, -\varepsilon]$ for any $\varepsilon > 0$ and these points are finite multiplicity. This finishes the proof. $\square$

Let us return to the proof of Theorem 2.

Proof of Theorem 2. Let μ be a positive number and $\mu + c_0 > 0$ $(H \geq -c_0 I)$. Let us consider for such μ the second resolvent equation

$$(H + \mu)^{-1} = (H_0 + \mu)^{-1} - (H + \mu)^{-1} \circ q \circ (H_0 + \mu)^{-1}, \tag{9.5}$$

where $H_0 = -\Delta$ and $H = -\Delta + q(x)$. It follows from Lemma 4 that $q \circ (H_0 + \mu)^{-1}$ is compact operator in $L^2(\mathbb{R}^n)$. It means that $(H + \mu)^{-1}$ is a compact perturbation of $(H_0 + \mu)^{-1}$. Hence, by fact 6) above we have

$$\sigma_{\text{ess}}((H + \mu)^{-1}) = \sigma_{\text{ess}}((H_0 + \mu)^{-1}).$$

But $\sigma_{\text{ess}}((H_0 + \mu)^{-1}) = [0, \frac{1}{\mu}] = \sigma_c((H_0 + \mu)^{-1})$. That's why we may conclude that

$$\sigma_{\text{ess}}(H + \mu) = [\mu, +\infty].$$

Outside of this set we have only points of the discrete spectrum with one possible accumulation point at μ. This statement is a simple corollary from Lemma 5. Moreover, these points of discrete spectrum are situated on $[\mu - c_0, \mu)$ and they are of finite multiplicity. Hence the discrete spectrum $\sigma_d(H)$ of H belongs to $[-c_0, 0)$ with only one possible accumulation point at $\{0\}$. And $(0, +\infty)$ is continuous part of $\sigma(H)$. There is only one problem. As a matter of fact, Weyl's theorem (fact 6)) states that the operators H and H_0 don't have the same spectrum but the same essential spectrum. That's why on $(0, +\infty)$ there can be eigenvalues of infinite multiplicity (see the definition of σ_{ess}). In order to eliminate such possibility and to prove that $0 \in \sigma_c(H)$ and $\sigma_d(H)$ is finite let us assume additionally that our potential $q(x)$ has a special behavior at the infinity:

$$|q(x)| \leq c|x|^{-\mu}, \quad |x| \to +\infty,$$

where $\mu > 2$. In that case we can prove that on the interval $[-c_0, 0)$ the operator H has at most finitely many points of discrete spectrum. And we prove also that $0 \in \sigma_c(H)$.

Assume on the contrary that H has infinitely many points of discrete spectrum or one of them has an infinite multiplicity. It means that in $D(H)$ there exists the infinite dimensional space of the functions $\{u\}$ which satisfy the equation

$$(-\Delta + q)u = \lambda u, \qquad -c_0 \leq \lambda \leq 0.$$

It follows that

$$\int_{\mathbb{R}^n} (|\nabla u(x)|^2 + q^+(x)|u(x)|^2)dx \leq \int_{\mathbb{R}^n} q^-(x)|u(x)|^2 dx,$$

where q^+ and q^- are the positive and negative parts of the potential $q(x)$, respectively. Let us consider an infinite sequence of functions $\{u(x)\}$ which are orthogonal in the metric $\int_{\mathbb{R}^n} q^-(x)|u(x)|^2 dx$. That's why this sequence is uniformly bounded in the metric $\int_{\mathbb{R}^n}(|\nabla u|^2 + |q||u|^2)dx$ and hence, in the metric $\int_{\mathbb{R}^n}(|\nabla u|^2 + |u|^2)dx$. But for every eigenfunction $u(x)$ of the operator H with eigenvalue $\lambda \in [-c_0, 0]$ the following inequality holds (see [1]):

$$|u(x)| \leq c|\lambda| \int_{|x-y|\leq 1} |u(y)|dy,$$

where c does not depend on x. It follows from this inequality that

a) $\lambda = 0$ is not an eigenvalue.

b) this orthogonal sequence is uniformly bounded in every fixed ball.

Lemma 6. *Denote by U the set of functions $u(x) \in D(H)$ which are uniformly bounded in every fixed ball in $\mathbb{R}^n$. Then U is a precompact set in the metric*

$$\int_{\mathbb{R}^n} |q||u|^2 dx$$

if it is a bounded set in the metric

$$\int_{\mathbb{R}^n} (|\nabla u|^2 + |u|^2)dx.$$

Proof. Let $\{u_k(x)\}_{k=1}^{\infty} \subset U$ be an arbitrary sequence which is bounded in the second metric. Then for $u(x) := u_k(x) - u_m(x)$ we have for r large enough that

$$\begin{aligned}\int_{\mathbb{R}^n} |q(x)||u(x)|^2 dx &\leq c\int_{|x|>r} \frac{|u(x)|^2}{|x|^\mu} dx + \int_{|x|\leq r, |q(x)|\leq A} |q(x)||u(x)|^2 dx \\ &+ \int_{|x|\leq r, |q(x)|>A} |q(x)||u(x)|^2 dx := I_0 + I_1 + I_2.\end{aligned}$$

For $n \geq 3$ (for $n = 2$ we need some changes) and $\mu > 2$ we get

$$I_0 \leq cr^{2-\mu} \int_{|x|>r} |x|^{-2}|u(x)|^2 dx \leq cr^{2-\mu} \int_{\mathbb{R}^n} |\nabla u(x)|^2 dx, \quad u \in W_2^1(\mathbb{R}^n).$$

Due to the uniform boundedness of U in every ball we may conclude that

$$I_2 \leq c \int_{|x|\leq r, |q(x)|>A} |q(x)| dx \to 0$$

as $A \to +\infty$ uniformly on U with fixed r. Since the embedding $W_2^1 \subset L^2$ for every ball is compact the boundedness of the sequence in the second metric implies the precompactness in L^2 for every ball. That's why we have

$$I_1 \leq A \int_{|x|\leq r} |u(x)|^2 dx \to 0, \quad m, k \to \infty$$

with r and A fixed. After limiting processes, these inequalities for I_0, I_1 and I_2 show that

$$\int_{\mathbb{R}^n} |q(x)||u(x)|^2 dx \to 0, \quad m, k \to \infty.$$

Thus lemma is proved. □

Let us return to the proof of 1). By Lemma 6 we obtain that our sequence (which is orthogonal in the metric $\int q^-|u|^2 dx$) is a Cauchy sequence in the first metric. But this fact contradicts its orthogonality. Thus 1) is proved.

2) Let us discuss (briefly) the situation with a positive eigenvalue at the continuous spectrum. If we consider the homogeneous equation

$$[I + (-\Delta - k^2 - i0)^{-1} q] f = 0, \quad k^2 > 0,$$

in the space $\dot{C}(\mathbb{R}^n)$ then by Green's formula one can show (see, [2] or [3]) that the solution $f(x)$ of this equation behaves at the infinity as $o(|x|^{-\frac{n-1}{2}})$. That's why may conclude (T. Kato) that $f(x) \equiv 0$ outside some ball in $\mathbb{R}^n$. By the unique continuation principle for the Schrödinger operator (see, for example, [4]) it follows that $f \equiv 0$ in the whole $\mathbb{R}^n$. □

Let us consider now the spectral representation of the Schrödinger operator $H = -\Delta + q(x)$, with $q(x)$ as in Theorem 2 with the behavior $\mathcal{O}(|x|^{-\mu}), \mu > 2$ at the infinity. For any $f \in D(H)$,

$$Hf(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} k^2 u(x, \overrightarrow{k}) d\overrightarrow{k} \int_{\mathbb{R}^n} f(y) \overline{u(y, \overrightarrow{k})} dy + \sum_{j=1}^{M} \lambda_j f_j u_j(x),$$

where $u(x, \overrightarrow{k})$ are the solutions of the equation $Hu = k^2 u$, $u_j(x)$ are the ortonormal eigenfunctions corresponding to the negative eigenvalues λ_j, taking into account multiplicity of λ_j and $f_j = (f, u_j)_{L^2(\mathbb{R}^n)}$. The functions $u(x, \overrightarrow{k})$ are called *generalized eigenfunctions*. In the case when $q \equiv 0$ the generalized eigenfunctions have the form $u(x, \overrightarrow{k}) = e^{i(x, \overrightarrow{k})}$. This fact follows by Fourier transform. Indeed,

$$(-\Delta - k^2)u = 0$$

if and only if

$$(|\xi|^2 - k^2)\widehat{u} = 0$$

or

$$\widehat{u} = \sum_{\alpha} c_\alpha \delta^{(\alpha)}(\xi - \overrightarrow{k}),$$

since

$$|\xi|^2 = k^2$$

if and only if

$$\xi - \overrightarrow{k} = 0.$$

Hence

$$\begin{aligned} u(x, \overrightarrow{k}) &= \sum_{\alpha} c_\alpha F^{-1}(\delta^{(\alpha)}(\xi - \overrightarrow{k}))(x) \\ &= \sum_{\alpha} c_\alpha e^{i(x, \overrightarrow{k})} F^{-1}(\delta^{(\alpha)}(\xi))(x) = \sum_{\alpha} c'_\alpha e^{i(x, \overrightarrow{k})} x^\alpha. \end{aligned}$$

But $u(x, \overrightarrow{k})$ must be bounded. That's why $u(x, \overrightarrow{k}) = c'_0 e^{i(x, \overrightarrow{k})}$. We choose $c'_0 = 1$. If we have the Schrödinger operator $H = -\Delta + q$ with $q \neq 0$, then it is natural to look for the solutions of $Hu = k^2 u$ of the form $u(x, \overrightarrow{k}) = e^{i(x, \overrightarrow{k})} + u_{\text{sc}}(x, \overrightarrow{k})$. Due to this representation we have

$$(-\Delta - k^2)(e^{i(x, \overrightarrow{k})} + u_{\text{sc}}) = -qu$$

or

$$(-\Delta - k^2)u_{\text{sc}} = -qu.$$

In order to find u_{sc} let us recall that from Chapter 8 we know the fundamental solution of the operator $-\Delta - k^2$. That's why

$$u(x, k) = e^{i(x, \overrightarrow{k})} - \int_{\mathbb{R}^n} G_k^+(|x - y|)q(y)u(y)dy,$$

where

$$G_k^+(|x|) = \frac{i}{4}\left(\frac{|k|}{2\pi|x|}\right)^{\frac{n-2}{2}} H^{(1)}_{\frac{n-2}{2}}(|k||x|)$$

is the fundamental solution for the operator $-\Delta - k^2$. This equation is called the *Lippmann-Schwinger integral equation*.

In order to investigate this equation we will investigate the integral operator $(-\Delta - k^2 - i0)^{-1}$ in some weighted spaces. As a matter of fact, $(-\Delta - k^2 - i0)^{-1}$ is not bounded in $L^2(\mathbb{R}^n)$ but it is bounded from $L^2_\delta(\mathbb{R}^n)$ to $L^2_{-\delta}(\mathbb{R}^n))$ for $\delta > \frac{1}{2}$ with the norm estimate

$$\|(-\Delta - k^2 - i0)^{-1}\|_{L^2_\delta \to L^2_{-\delta}} \leq \frac{c}{|k|}.$$

This fact was proved by S. Agmon in [5]. We will prove this estimate for $n \geq 3$ in Chapter 10.

10 Estimates for Laplacian and Hamiltonian

Let us recall Agmon's $(2,2)$-estimate for Laplacian:

$$\|(-\Delta - k^2 - i0)^{-1}\|_{L^2_\delta \to L^2_{-\delta}} \leq \frac{c}{|k|},$$

where $\delta > \frac{1}{2}$. In fact, this estimate allows us to consider the Hamiltonian with L^∞_{loc}-potentials only (if we want to preserve $(2,2)$-estimates). But we would like to consider Hamiltonian with L^p_{loc}-potentials. That's why we need to prove (p,q)-estimates. In this case we follow A. Ruiz.

We have proved in Example 4.9 that the limit $\lim_{\varepsilon \downarrow 0} \frac{1}{x - i\varepsilon} := \frac{1}{x - i0}$ exists in the sense of tempered distributions and

$$\frac{1}{x - i0} = \text{p.v.}\frac{1}{x} + i\pi\delta(x)$$

i.e.

$$\langle \frac{1}{x - i0}, \varphi \rangle = \lim_{\delta \to +0} \int_{|x| > \delta} \frac{\varphi(x)}{x} dx + i\pi\varphi(0).$$

In Example 4.3 we have considered the simple layer

$$\langle T, \varphi \rangle := \int_\sigma a(\xi)\varphi(\xi) d\sigma_\xi,$$

where σ is a hypersurface of dimension $n-1$ in $\mathbb{R}^n$ and $a(\xi)$ is a density. These examples can be extended as follows. If $H : \mathbb{R}^n \to \mathbb{R}$ and $|\nabla H| \neq 0$ at any point where $H(\xi) = 0$ then we can define the distribution

$$(H(\xi) - i0)^{-1} := \lim_{\varepsilon \to +0} \frac{1}{H(\xi) - i\varepsilon}$$

in $S'(\mathbb{R})$ and we can also prove that

$$(H(\xi) - i0)^{-1} = \text{p.v.}\frac{1}{H(\xi)} + i\pi\delta(H(\xi) = 0),$$

where $\delta(H(\xi) = 0)$ is defined as follows:

$$\langle \delta(H), \varphi \rangle = \int_{H(\xi)=0} \varphi(\xi) d\sigma_\xi, \quad \varphi \in S(\mathbb{R}^n).$$

The equality $H(\xi) = 0$ defines an $n-1$ dimensional hypersurface and σ_ξ is any $(n-1)$-form such that $d\sigma_\xi \wedge \frac{dH}{|\nabla H|} = d\xi$ (in local coordinates).

Exercise 48. Prove that

$$\delta(\alpha H) = \frac{1}{\alpha}\delta(H)$$

for any function α which does not vanish at any point ξ where $H(\xi) = 0$.

Due to Exercise 48 we may conclude that $\delta(H) = \frac{1}{|\nabla H|}\delta\left(\frac{H}{|\nabla H|}\right)$ if $|\nabla H| \neq 0$ for $H = 0$.

Let us consider now $H(\xi) := -|\xi|^2 + k^2, k > 0$. Then $H(\xi) = 0$ or $|\xi| = k$ is a sphere and $\nabla H(\xi) = -2\xi$ and $|\nabla H(\xi)| = 2k$ at any point on this sphere. If we change the variables then we obtain

$$\langle\delta(H), \varphi\rangle = \int_{H(\xi)=0} \varphi(\xi) d\sigma_\xi = \frac{1}{2k}\int_{\mathbb{S}^{n-1}} \varphi(k\theta) d\theta.$$

We know that $(-\Delta - k^2 - i0)^{-1} f$ can be represented as

$$(-\Delta - k^2 - i0)^{-1} f = \int_{\mathbb{R}^n} G_k^+(|x-y|) f(y) dy,$$

where $G_k^+(|x|) = \frac{i}{4}\left(\frac{|k|}{2\pi|x|}\right)^{\frac{n-2}{2}} H^{(1)}_{\frac{n-2}{2}}(|k||x|)$. On the other hand we can write

$$\begin{aligned}
(-\Delta - k^2 - i0)^{-1} f &= F^{-1}(F[(-\Delta - k^2 - i0)^{-1} f]) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \frac{\widehat{f}(\xi) e^{i(x,\xi)} d\xi}{|\xi|^2 - k^2 - i0} \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \text{p.v.}\frac{1}{|\xi|^2 - k^2} \widehat{f}(\xi) e^{i(x,\xi)} d\xi \\
&+ \frac{i\pi (2\pi)^{-\frac{n}{2}}}{2k} \int_{\mathbb{R}^n} \delta(H) \widehat{f}(\xi) e^{i(x,\xi)} d\xi \\
&= (2\pi)^{-\frac{n}{2}} \text{p.v.} \int_{\mathbb{R}^n} \frac{\widehat{f}(\xi) e^{i(x,\xi)} d\xi}{|\xi|^2 - k^2} + \frac{i\pi}{2k(2\pi)^{\frac{n}{2}}} \int_{\mathbb{S}^{n-1}} \widehat{f}(k\theta) e^{ik(x,\theta)} d\theta \\
&= (2\pi)^{-\frac{n}{2}} \text{p.v.} \int_{\mathbb{R}^n} \frac{\widehat{f}(\xi) e^{i(x,\xi)} d\xi}{|\xi|^2 - k^2} \\
&+ \frac{i\pi}{2k(2\pi)^n} \int_{\mathbb{R}^n} f(y) dy \int_{\mathbb{S}^{n-1}} e^{ik(\theta, x-y)} d\theta.
\end{aligned}$$

Our aim is to prove the following result.

Theorem 1. *Let $k > 0$ and $\frac{2}{n} \geq \frac{1}{p} - \frac{1}{p'} \geq \frac{2}{n+1}$ for $n \geq 3$ and $1 > \frac{1}{p} - \frac{1}{p'} \geq \frac{2}{3}$ for $n = 2$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Then there exists a constant C independent of k and f such that*

$$\|(-\Delta - k^2 - i0)^{-1} f\|_{L^{p'}(\mathbb{R}^n)} \leq C k^{n\left(\frac{1}{p} - \frac{1}{p'}\right) - 2} \|f\|_{L^p(\mathbb{R}^n)}.$$

Remark. In what follows we will use $\widehat{G}_k$ instead of $(-\Delta - k^2 - i0)^{-1}$.

Proof. First we prove that if the claim holds for $k = 1$ then it holds for any $k > 0$. So, let us assume that

$$\|\widehat{G}_1 f\|_{L^{p'}(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}.$$

Denote $T_\delta f := f(\delta x), \delta > 0$. It is clear that $\|T_\delta f\|_{L^p(\mathbb{R}^n)} = \delta^{-\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)}$. It is not so difficult to show that $\widehat{G}_k = k^{-2} T_k \widehat{G}_1 T_{\frac{1}{k}}$. Indeed, since

$$\widehat{G}_k f = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{e^{i(y,\xi)} f(x-y) d\xi dy}{|\xi|^2 - k^2 - i0}$$

we get

$$\widehat{G}_1 T_{\frac{1}{k}} f = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{e^{i(y,\xi)} f(\frac{x-y}{k}) d\xi dy}{|\xi|^2 - 1 - i0}.$$

It follows that

$$\begin{aligned} T_k \widehat{G}_1 T_{\frac{1}{k}} f &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{e^{i(y,\xi)} f(x - \frac{y}{k}) d\xi dy}{|\xi|^2 - 1 - i0}, \quad (z := y/k, \eta := k\xi) \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{e^{i(z,\eta)} k^{-n} f(x-z) k^n dz d\eta}{\frac{|\eta|^2}{k^2} - 1 - i0} \\ &= (2\pi)^{-n} k^2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{e^{i(z,\eta)} f(x-z) dz d\eta}{|\eta|^2 - k^2 - i0}. \end{aligned}$$

This proves that

$$k^2 \widehat{G}_k f \equiv T_k \widehat{G}_1 T_{\frac{1}{k}} f.$$

We use it to get

$$\begin{aligned} \|\widehat{G}_k f\|_{L^{p'}} &= k^{-2} \|T_k \widehat{G}_1 T_{\frac{1}{k}} f\|_{L^{p'}} = k^{-2} k^{-\frac{n}{p'}} \|\widehat{G}_1 T_{\frac{1}{k}} f\|_{L^{p'}} \\ &\leq Ck^{-2-\frac{n}{p'}} \|T_{\frac{1}{k}} f\|_{L^p} = Ck^{-2-\frac{n}{p'}} \left(\frac{1}{k}\right)^{-\frac{n}{p}} \|f\|_{L^p} = Ck^{n\left(\frac{1}{p}-\frac{1}{p'}\right)-2} \|f\|_{L^p}. \end{aligned}$$

That's why it is enough to prove this theorem for $k = 1$.

Lemma 1. *Let $\omega(x) \in S(\mathbb{R}^n), 0 < \varepsilon < 1$ and $\sigma_\varepsilon \omega(\xi) = \varepsilon^{-n} \omega\left(\frac{\xi}{\varepsilon}\right)$. Let us denote*

$$P_\varepsilon(\xi) := p.v. \left(\frac{1}{|\eta|^2 - 1} * \sigma_\varepsilon \omega\right)(\xi).$$

Then

$$|P_\varepsilon(\xi)| \leq \frac{c}{\varepsilon}.$$

Proof. For P_ε we have the following representation:

$$P_\varepsilon = \text{p.v.} \left(\int_{1-\varepsilon \leq |\eta| \leq 1+\varepsilon} + \int_{|\eta| < 1-\varepsilon} + \int_{|\eta| > 1+\varepsilon} \right) \frac{\sigma_\varepsilon \omega(\xi - \eta)}{|\eta|^2 - 1} d\eta = I_1 + I_2 + I_3.$$

The integrals I_2 and I_3 can be easily bounded by $\varepsilon^{-1} \|\omega\|_{L^1}$ because $|\eta| < 1 - \varepsilon$ implies that $\left|\frac{1}{|\eta|^2-1}\right| = \frac{1}{1-|\eta|^2} < \frac{1}{\varepsilon}$ and $|\eta| > 1 + \varepsilon$ implies that $\left|\frac{1}{|\eta|^2-1}\right| = \frac{1}{|\eta|^2-1} < \frac{1}{\varepsilon}$. By the definition of p.v. we have

$$I_1 = \lim_{\delta \to +0} \int_{\delta < |1-|\eta|| < \varepsilon} \frac{\sigma_\varepsilon \omega(\xi - \eta)}{|\eta|^2 - 1} d\eta = \lim_{\delta \to +0} \left(\int_{1-\varepsilon}^{1-\delta} + \int_{1+\delta}^{1+\varepsilon} \right) \int_{\mathbb{S}^{n-1}} \sigma_\varepsilon \omega(\xi - r\theta) \frac{r^{n-1}}{r^2 - 1} d\theta dr.$$

Replacing r with $2 - r$ in the latter integral we obtain

$$I_1 = \lim_{\delta \to +0} \int_{1-\varepsilon}^{1-\delta} \frac{F(r, \xi)}{r - 1} dr,$$

where

$$F(r,\xi)=\int_{\mathbb{S}^{n-1}}\left[\sigma_\varepsilon\omega(\xi-r\theta)\frac{r^{n-1}}{r+1}-\sigma_\varepsilon\omega(\xi-(2-r)\theta)\frac{(2-r)^{n-1}}{3-r}\right]d\theta.$$

If we observe that $F(1,\xi)=0$ then we get by the mean value theorem (Lagrange formulae) that

$$\begin{aligned}\left|\int_{1-\varepsilon}^{1-\delta}\frac{F(r,\xi)}{r-1}dr\right| &= \left|\int_{1-\varepsilon}^{1-\delta}\frac{F(r,\xi)-F(1,\xi)}{r-1}dr\right|\le(\varepsilon-\delta)\sup_{1-\varepsilon<r<1}\left|\frac{\partial F}{\partial r}(r,\xi)\right|\\ &\le \varepsilon\sup_{1-\varepsilon<r<1}\left|\frac{\partial F}{\partial r}\right|.\end{aligned}$$

But

$$\begin{aligned}\frac{\partial F}{\partial r} &= \left(\frac{r^{n-1}}{r+1}\right)'\int_{\mathbb{S}^{n-1}}\sigma_\varepsilon\omega(\xi-r\theta)d\theta\\ &- \frac{r^{n-1}}{r+1}\int_{\mathbb{S}^{n-1}}\theta\cdot\nabla(\sigma_\varepsilon\omega(\xi-r\theta))d\theta-\left(\frac{(2-r)^{n-1}}{3-r}\right)'\int_{\mathbb{S}^{n-1}}\sigma_\varepsilon\omega(\xi-(2-r)\theta)d\theta\\ &- \frac{(2-r)^{n-1}}{3-r}\int_{\mathbb{S}^{n-1}}\theta\cdot\nabla(\sigma_\varepsilon\omega(\xi-(2-r)\theta))d\theta=\theta_1+\theta_2+\theta_3+\theta_4.\end{aligned}$$

By the proof of Lemma 2 below we get $|\theta_1|\le c_1\varepsilon^{-1}$ and $|\theta_3|\le c_3\varepsilon^{-1}$, where the constants c_1 and c_3 depend on ω. The second integral θ_2 can be estimated as (see Lemma 2)

$$\varepsilon^{-1}\sum_{j=1}^{n}\frac{r^{n-1}}{r+1}\int_{\mathbb{S}^{n-1}}\theta_j\sigma_\varepsilon\left(\frac{\partial}{\partial x_j}\omega\right)(\xi-r\theta)d\theta\le c_2\varepsilon^{-2}.$$

The same estimate holds for θ_4. Thus, Lemma 1 is proved. □

Lemma 2. *Let us assume that $f\in L^\infty(\mathbb{S}^{n-1})$ and $\omega\in S(\mathbb{R}^n)$. Then*

$$\left\|\int_{\mathbb{S}^{n-1}}\sigma_\varepsilon\omega(\xi-\theta)f(\theta)d\theta\right\|_{L^\infty(\mathbb{R}^n)}\le C\varepsilon^{-1}.$$

Proof. We can reduce the proof to compactly supported ω, since $\overline{C_0^\infty}\overset{S}{=}S$. Let us take a C_0^∞-partition of unity in $\mathbb{R}^n$ such that $\sum_{j=0}^\infty\psi_j(\xi)=1$ or even $\sum_{j=0}^\infty\psi_j\left(\frac{1}{\xi}\right)=1$, where ψ_0 is supported in $|\xi|<1$ and $\psi_j=\psi(2^{-j}\xi)$ for $j=1,2,3,\dots$, with ψ supported in the annulus $1/2<|\xi|<2$. That's why we may write

$$\int_{\mathbb{S}^{n-1}}\sigma_\varepsilon\omega(\xi-\theta)f(\theta)d\theta=\sum_{j=0}^\infty\int_{\mathbb{S}^{n-1}}\varepsilon^{-n}\psi_j\left(\frac{\xi-\theta}{\varepsilon}\right)\omega\left(\frac{\xi-\theta}{\varepsilon}\right)f(\theta)d\theta.$$

For $j=1,2,3,\dots$, the function $\psi_j\left(\frac{\xi-\theta}{\varepsilon}\right)\omega\left(\frac{\xi-\theta}{\varepsilon}\right)$ is supported in the annulus $2^{j-1}\le|\cdot|\le 2^{j+1}$. Since ω is rapidly decreasing we have that, in this annulus,

$$\left|\omega\left(\frac{\xi-\theta}{\varepsilon}\right)\right|\le\frac{C_M}{(1+2^j)^M},$$

for all $M \in \mathbb{N}$. Hence

$$\left|\int_{\mathbb{S}^{n-1}} \varepsilon^{-n}(\psi_j\omega)\left(\frac{\xi-\theta}{\varepsilon}\right) f(\theta)d\theta\right| \leq C_M \frac{(2^{j+1}\varepsilon)^{n-1}}{(1+2^j)^M}\varepsilon^{-n} \leq C'_M \varepsilon^{-1}\frac{(2^j)^{n-1}}{(1+2^j)^M}.$$

Taking M large enough the sum in j converges to $C\varepsilon^{-1}$. To end the proof of Lemma 2 notice that the term for $j=0$ satisfes this inequality trivially. □

Exercise 49. Prove that $(-\Delta)^{-1} : L^2_\delta(\mathbb{R}^3) \to L^2_{-\delta}(\mathbb{R}^3)$ for $\delta > 1$.

Let us return to the proof of Theorem 1. We can rewrite $\widehat{G}_1 f$ in the form

$$\widehat{G}_1 f = C\text{p.v.} \int_{\mathbb{R}^n} \frac{\widehat{f}(\xi)e^{i(x,\xi)}d\xi}{|\xi|^2-1} + I_1 f,$$

where

$$I_1 f = C\int_{\mathbb{S}^{n-1}} \widehat{f}(\theta)e^{i(\theta,x)}d\theta.$$

Let us take a partition of unity $\sum_{j=0}^\infty \psi_j(x) = 1$ such that $\operatorname{supp}\psi_0 \subset \{|x| < 1\}$ and $\operatorname{supp}\psi_j \subset \{2^{j-1} < |x| < 2^{j+1}\}$, where $\psi_j = \psi(2^{-j}x)$ with a fixed function $\psi \in S$. Denote $\Psi_j := \psi_j G_1^+$ and $K_j f := \Psi_j * f$, where G_1^+ is the kernel of the integral operator $\widehat{G}_1$. Using the estimates of the Hankel function $H^{(1)}_{\frac{n-2}{2}}(|x|)$ for $|x| < 2$ we obtain

$$|\Psi_0| \leq C|x|^{2-n}, \quad n \geq 3$$

and

$$|\Psi_0| \leq C(|\log|x|| + 1), \quad n = 2.$$

Exercise 50 (Sobolev inequality)**.** Let $0 < \alpha < n, 1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then

$$\left\|\int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha}}\right\|_{L^q} \leq C\|f\|_{L^p}.$$

Hint: For $K := |x|^{-n+\alpha}$ use the representation $K = K_1 + K_2$, where

$$K_1 = \begin{cases} K, & |x| < \mu \\ 0, & |x| > \mu \end{cases} \quad \text{and} \quad K_2 = \begin{cases} 0, & |x| < \mu \\ K, & |x| > \mu. \end{cases}$$

From Sobolev inequality for $\alpha = 2$ we may conclude that the operator K_0 is bounded from $L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$ for the range $\frac{2}{n} \geq \frac{1}{p} - \frac{1}{p'} \geq 0$ if $n \geq 3$ and $1 > \frac{1}{p} - \frac{1}{p'} \geq 0$ if $n = 2$. From Lemma 1 and 2 with $\varepsilon = \frac{1}{2^j}$ we can obtain that

$$\|F(\Psi_j)\|_\infty = \|(|\xi|^2 - 1 - i0)^{-1} * \psi_j\|_\infty \leq C \cdot 2^j.$$

This inequality leads to

$$\|K_j\|_{L^2 \to L^2} \leq C \cdot 2^j,$$

because

$$\|K_j f\|_{L^2} = \|F(\Psi_j * f)\|_{L^2} = C\|\widehat{\Psi_j} \cdot \widehat{f}\|_{L^2} \leq \|\widehat{\Psi_j}\|_{L^\infty}\|\widehat{f}\|_{L^2} \leq C \cdot 2^j \|f\|_{L^2}.$$

On the other hand, due to the estimate of the fundamental solution at infinity we can obtain that $|\Psi_j(x)| \leq C \cdot 2^{-j \cdot \frac{n-1}{2}}$ and

$$\|K_j\|_{L^1 \to L^\infty} \leq C \cdot 2^{-j \cdot \frac{n-1}{2}}.$$

We have used here two facts:

$$\left| H^{(1)}_{\frac{n-2}{2}}(|x|) \right| \leq \frac{C}{|x|^{\frac{1}{2}}}, \quad |x| > 1$$

and $\operatorname{supp} \Psi_j(x) \subset \{x : 2^{j-1} < |x| < 2^{j+1}\}$. Interpolating these estimates we obtain the self-dual estimates:

$$\|K_j\|_{L^p \to L^{p'}} \leq C(2^j)^{2\left(1-\frac{1}{p}\right)-\frac{n-1}{2}\left(\frac{2}{p}-1\right)}.$$

For convergence of this series we need the condition $2(1-\frac{1}{p}) - \frac{n-1}{2}(\frac{2}{p}-1) < 0$ or $\frac{1}{p} - \frac{1}{p'} > \frac{2}{n+1}$. If we want to get the sharper inequality $\frac{1}{p} - \frac{1}{p'} \geq \frac{2}{n+1}$ we have to use Stein's theorem about interpolation. Thus, Theorem 1 is proved. □

It follows from Theorem 1 that if we consider the values of p from the interval

$$\frac{2n}{n+2} \leq p \leq \frac{2n+2}{n+3}, \quad n \geq 3$$

$$1 < p \leq 6/5, \quad n = 2,$$

then we have the self-dual estimate

$$\|\widehat{G}_k\|_{L^p \to L^{p'}} \leq \frac{C}{|k|^{2-n\left(\frac{1}{p}-\frac{1}{p'}\right)}}.$$

But we would like to extend the estimates for $\widehat{G}_k$ for $\frac{2n}{n+2} \leq p \leq 2, n \geq 3$, and $1 < p \leq 2, n = 2$. In order to do so we use interpolation of the Agmon's estimate and the latter estimate for $p = \frac{2n+2}{n+3}$. This process leads to the estimate

$$\|\widehat{G}_k\|_{L^p_\delta \to L^{p'}_{-\delta}} \leq \frac{C}{|k|^{1-(n-1)\left(\frac{1}{p}-\frac{1}{2}\right)}},$$

where $\frac{2n+2}{n+3} < p \leq 2, n \geq 2$ and $\delta > \frac{1}{2} - (n+1)\left(\frac{1}{2p} - \frac{1}{4}\right)$.

Theorem 2. *Assume that the potential $q(x)$ belongs to $L^p_\sigma(\mathbb{R}^n), n \geq 2$, with $\frac{n}{2} < p \leq \infty$ and $\sigma = 0$ for $\frac{n}{2} < p \leq \frac{n+1}{2}$ and $\sigma > 1 - \frac{n+1}{2p}$ for $\frac{n+1}{2} < p \leq +\infty$. Then for all $k \neq 0$ the limit*

$$\widehat{G}_q := \lim_{\varepsilon \to +0} (H - k^2 - i\varepsilon)^{-1}$$

exists in the uniform operator topology from $L^{\frac{2p}{p+1}}_{\frac{\sigma}{2}}(\mathbb{R}^n)$ to $L^{\frac{2p}{p-1}}_{-\frac{\sigma}{2}}(\mathbb{R}^n)$ with the norm estimate

$$\|\widehat{G}_q f\|_{L^{\frac{2p}{p-1}}_{-\frac{\sigma}{2}}} \leq C|k|^{-\gamma}\|f\|_{L^{\frac{2p}{p+1}}_{\frac{\sigma}{2}}}$$

for large k with p and σ as above and with $\gamma = 2 - \frac{n}{2}$ for $\frac{n}{2} < p \leq \frac{n+1}{2}$ and $\gamma = 1 - \frac{n-1}{2p}$ for $\frac{n+1}{2} < p \leq \infty$.

Proof. Let us prove first that the integral operator $\widehat{K}$ with the kernel

$$K(x, y) := |q|^{\frac{1}{2}}(x)G_k^+(|x-y|)q_{\frac{1}{2}}(y),$$

where $q_{\frac{1}{2}}(y) = |q(y)|^{\frac{1}{2}} \operatorname{sgn} q(y)$ maps from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ with the same norm estimate as in Theorem 2. Indeed, if $f \in L^2(\mathbb{R}^n)$ and $q \in L^p_\sigma(\mathbb{R}^n)$ then $|q|^{\frac{1}{2}} \in L^{2p}_{\frac{\sigma}{2}}(\mathbb{R}^n)$ and, therefore, $f|q|^{\frac{1}{2}} \in L^{\frac{2p}{p+1}}_{\frac{\sigma}{2}}(\mathbb{R}^n)$. Applying Theorem 1 we obtain

$$\|\widehat{G}_k(|q|^{\frac{1}{2}}f)\|_{L^{\frac{2p}{p-1}}_{-\frac{\sigma}{2}}} \leq C|k|^{-\gamma}\|f\|_{L^{\frac{2p}{p+1}}_{\frac{\sigma}{2}}},$$

where γ is as in Theorem 2. Then, by Hölder's inequality we have that $|q|^{\frac{1}{2}}\widehat{G}_k(q_{\frac{1}{2}}f) \in L^2(\mathbb{R}^n)$ as asserted.

Let us consider now the operator $\widehat{G}_q$. This operator satisfies the resolvent equation

$$\widehat{G}_q = \widehat{G}_k - \widehat{G}_k q \widehat{G}_q$$

which follows easily from $(H - k^2)\widehat{G}_q = I$. Denote by $\widehat{G}_l$ and $\widehat{G}_r$ the integral operators having the kernels $G_k^+(|x-y|)q_{\frac{1}{2}}(y)$ and $|q(x)|^{\frac{1}{2}}G_k^+(|x-y|)$, respectively. Then one can show that

$$\widehat{G}_q = \widehat{G}_k - \widehat{G}_l(1 + \widehat{K})^{-1}\widehat{G}_r$$

for large k. Since $\widehat{K} : L^2 \to L^2, \widehat{G}_r : L^{\frac{2p}{p+1}}_{\frac{\sigma}{2}} \to L^2$ and $\widehat{G}_l : L^2 \to L^{\frac{2p}{p-1}}_{-\frac{\sigma}{2}}$ then we may conclude that Theorem 2 is proved. □

There are some corollaries from these two theorems.

Corollary 1. *Assume that the potential $q(x)$ belongs to $L^p_\sigma(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with p and σ as in Theorem 2. Then for $\lambda > 0$ large enough there is a unique solution u of the equation*

$$(-\Delta + q - \lambda)u = 0$$

of the form

$$u(x, \overrightarrow{k}) = e^{i(\overrightarrow{k}, x)} + u_{sc}(x, \overrightarrow{k}),$$

where $\overrightarrow{k}, x \in \mathbb{R}^n, (\overrightarrow{k}, \overrightarrow{k}) = \lambda$ and

$$\||q|^{\frac{1}{2}}u_{sc}\|_{L^2(\mathbb{R}^n)} \leq \frac{C}{\lambda^{\frac{\gamma}{2}}}\|q\|^{\frac{1}{2}}_{L^1(\mathbb{R}^n)}$$

with γ as in Theorem 2.

Proof. Let us rewrite the Lippmann-Schwinger equation

$$u(x, \overrightarrow{k}) = e^{i(\overrightarrow{k},x)} - \int_{\mathbb{R}^n} G_\lambda^+(|x-y|)q(y)u(y, \overrightarrow{k})dy$$

in the form

$$|q(x)|^{\frac{1}{2}}u(x, \overrightarrow{k}) = |q(x)|^{\frac{1}{2}}e^{i(\overrightarrow{k},x)} - \int_{\mathbb{R}^n} K(x,y)|q(y)|^{\frac{1}{2}}u(y, \overrightarrow{k})dy$$

or

$$v = v_0 - \widehat{K}v,$$

where $v(x) = |q(x)|^{\frac{1}{2}}u(x, \overrightarrow{k}), v_0(x) = |q(x)|^{\frac{1}{2}}e^{i(\overrightarrow{k},x)}$ and $\widehat{K}$ is as above. Since $v_0 \in L^2$ if and only if $q \in L^1$ and $\widehat{K} : L^2 \to L^2$ with a good norm estimate, we may conclude that

$$v = v_0 + \widehat{K}v_0 + \widehat{K}^2 v_0 + \dots$$

for $|\overrightarrow{k}|$ large enough and

$$\|v - v_0\|_{L^2} \leq C\|\widehat{K}\|_{L^2\to L^2}\|v_0\|_{L^2}$$

i.e.

$$\||q(x)|^{\frac{1}{2}}u_{\text{sc}}\|_{L^2} \leq \frac{C}{\lambda^{\gamma/2}}\|q\|_{L^1}^{\frac{1}{2}}.$$

□

Corollary 2. *Let v be the outgoing solution of the inhomogeneous Schrödinger equation*

$$(H - k^2)v = f$$

i.e.

$$v = (H - k^2 - i0)^{-1}f,$$

where $f \in S(\mathbb{R}^n)$. Then the following representation holds:

$$v(x) = \widehat{G}_k(f - q\widehat{G}_q(f))(x).$$

Moreover, for $|x| \to \infty$ and fixed positive k,

$$v(x,k) = C_n \frac{e^{ik|x|}k^{\frac{n-3}{2}}}{|x|^{\frac{n-1}{2}}} A_f(k,\theta') + o\left(\frac{1}{|x|^{\frac{n-1}{2}}}\right),$$

where $\theta' = \frac{x}{|x|}$ and the function A_f is defined by

$$A_f(k,\theta') := \int_{\mathbb{R}^n} e^{-ik(\theta',y)}(f(y) - q(y)\widehat{G}_q(f))dy.$$

Proof. The first representation follows immediately from the definition of $\widehat{G}_q$. Indeed, since $v = \widehat{G}_q f$ then $\widehat{G}_k f = v + \widehat{G}_k qv$ or $v = \widehat{G}_k f - \widehat{G}_k qv = \widehat{G}_k(f - q\widehat{G}_q f)$.

In order to prove the asymptotic behavior for v let us assume that q and f have compact support, say in the ball $\{x : |x| \le R\}$. We will use the following asymptotic behavior of $G_k^+(|x|)$:

1) $k|x| < 1$:

   a) $G_k^+(|x|) \sim C|x|^{2-n}, \quad n \ge 3,$

   b) $G_k^+(|x|) \sim C\log(k|x|), \quad n = 2.$

2) $k|x| > 1$:

$$G_k^+(|x|) \sim C\frac{k^{\frac{n-3}{2}}}{|x|^{\frac{n-1}{2}}}e^{ik|x|}, \quad n \ge 2.$$

Since k is fixed, $|y| \le R$ and $|x| \to +\infty$ we may assume that $k|x-y| > 1$ for x large enough. That's why

$$\begin{aligned} v(x) &= C\frac{e^{ik|x|}k^{\frac{n-3}{2}}}{|x|^{\frac{n-1}{2}}}\int_{|y|\le R} e^{ik(|x-y|-|x|)}(f - q\widehat{G}_q f)dy \\ &+ \int_{|y|\le R} o\left(\frac{1}{|x-y|^{\frac{n-1}{2}}}\right)(f - q\widehat{G}_q f)dy = I_1 + I_2. \end{aligned}$$

It is clear that for I_2 the following is true

$$I_2 = o\left(\frac{1}{|x|^{\frac{n-1}{2}}}\int_{|y|\le R}(f(y) - q(y)\widehat{G}_q f(y))dy\right) = o\left(\frac{1}{|x|^{\frac{n-1}{2}}}\right),$$

because $f - q\widehat{G}_q f$ is an integrable function. Next, let us note that

$$|x-y| - |x| = \frac{|x-y|^2 - |x|^2}{|x-y| + |x|} = \frac{y^2 - 2(x,y)}{|x-y| + |x|} = -\left(\frac{x}{|x|}, y\right) + \mathcal{O}\left(\frac{1}{|x|}\right), \quad |x| \to +\infty.$$

That's why we can rewrite the integral appearing in I_1 as follows:

$$\begin{aligned} \int_{|y|\le R} e^{-ik\left(\frac{x}{|x|},y\right)+\mathcal{O}\left(\frac{1}{|x|}\right)}(f - q\widehat{G}_q f)dy &= \int_{|y|\le R} e^{-ik(\theta',y)}(f - q\widehat{G}_q f)dy \\ &+ \mathcal{O}\left(\frac{1}{|x|}\right)\int_{|y|\le R} e^{-ik(\theta',y)}(f - q\widehat{G}_q f)dy \\ &= \int_{|y|\le R} e^{-ik(\theta',y)}(f - q\widehat{G}_q f)dy + \mathcal{O}\left(\frac{1}{|x|}\right), \end{aligned}$$

where $\theta' = \frac{x}{|x|} \in \mathbb{S}^{n-1}$. Thus, Corollary 2 is proved when q and f have compact support. The proof in the general case is much more difficult and is therefore omitted. □

Remark. Hint for the general case: The integral over $\mathbb{R}^n$ might be divided in two parts: $|y| < |x|^\varepsilon$ and $|y| > |x|^\varepsilon$, where $\varepsilon > 0$ is chosen appropriately.

Lemma 3 (Optical lemma)**.** *For the function $A_f(k,\theta')$ the following equality holds:*

$$\int_{\mathbb{S}^{n-1}} |A_f(k,\theta')|^2 d\theta' = -\frac{1}{C^2 k^{n-2}} \int_{\mathbb{R}^n} \mathrm{Im}\,(f\overline{v})dx,$$

where C is the constant from the asymptotic representation of $v = (H - k^2 - i0)^{-1} f$.

Proof. Let ρ be a smooth real-valued function on $[0,+\infty)$ such that $0 \le \rho \le 1$ and $\rho(r) = 1$ for $0 \le r < 1$ and $\rho(r) = 0$ for $r \ge 2$. We set $\rho_m(r) = \rho\left(\frac{r}{m}\right)$. Multiplying f by $\overline{v}\rho_m(|x|)$, integrating over $\mathbb{R}^n$ and taking imaginary parts leads to

$$\mathrm{Im} \int_{\mathbb{R}^n} f(x)\rho_m(|x|)\overline{v}(x)dx = \mathrm{Im} \int_{\mathbb{R}^n} (-\Delta v)\rho_m(|x|)\overline{v}(x)dx.$$

As m tends to infinity, the left-hand side converges to $\mathrm{Im}\int_{\mathbb{R}^n} f(x)\overline{v}(x)dx$. To get the desired limit for the right-hand side, we integrate by parts and obtain

$$\begin{aligned}
\mathrm{Im} \int_{\mathbb{R}^n} (-\Delta v)\rho_m(|x|)\overline{v}(x)dx &= \mathrm{Im} \int_{\mathbb{R}^n} \frac{x}{|x|} \cdot \nabla v \rho_m'(|x|)\overline{v}(x)dx \\
&= \mathrm{Im} \int_{\mathbb{R}^n} \left[(\theta' \cdot \nabla v - ikv)\overline{v}(x)\rho_m'(|x|) + ik\rho_m'(|x|)|v|^2\right] dx \\
&= \mathrm{Im} \int_{\mathbb{R}^n} (\theta' \cdot \nabla v - ikv)\overline{v}(x)\rho_m'(x)dx \\
&+ k \int_{\mathbb{R}^n} \rho_m'(|x|)|v(x)|^2 dx = I_1 + I_2.
\end{aligned}$$

Since $v = (H - k^2 - i0)^{-1} f$ then using the asymptotical representation we may conclude that v satisfies the *Sommerfeld radiation condition*

$$\frac{\partial v}{\partial r} - ikv = o\left(\frac{1}{r^{\frac{n-1}{2}}}\right), \quad r = |x|$$

at the infinity. That's why $I_1 \to 0$ as $m \to \infty$. By Corollary 2 the second term I_2 is equal to

$$\begin{aligned}
k \int_{\mathbb{R}^n} \rho_m'(|x|)|v(x)|^2 dx &= k \int_{\mathbb{R}^n} \rho_m'(|x|) \cdot C^2 \frac{k^{n-3}}{|x|^{n-1}} |A_f(k,\theta')|^2 dx \\
&+ k \int_{\mathbb{R}^n} \rho_m'(|x|) o\left(\frac{1}{|x|^{n-1}}\right) dx \\
&= C^2 k^{n-2} \int_{\mathbb{S}^{n-1}} |A_f(k,\theta')|^2 d\theta' \int_m^{2m} \frac{r^{n-1}}{r^{n-1}} \rho_m'(r)dr \\
&+ k \int_{\mathbb{S}^{n-1}} d\theta \int_m^{2m} r^{n-1} o\left(\frac{1}{r^{n-1}}\right) \rho_m'(r)dr \\
&= C^2 k^{n-2} \int_{\mathbb{S}^{n-1}} |A_f(k,\theta')|^2 d\theta' \int_m^{2m} \rho_m'(r)dr + o_m(1) \\
&= C^2 k^{n-2} \int_{\mathbb{S}^{n-1}} |A_f(k,\theta')|^2 d\theta' [\rho(2) - \rho(1)] + o_m(1) \\
&= -C^2 k^{n-2} \int_{\mathbb{S}^{n-1}} |A_f(k,\theta')|^2 d\theta' + o_m(1).
\end{aligned}$$

Letting $m \to \infty$ we obtain

$$\operatorname{Im} \int_{\mathbb{R}^n} f(x)\overline{v}(x)dx = -C^2 k^{n-2} \int_{\mathbb{S}^{n-1}} |A_f(k, \theta')|^2 d\theta'.$$

Thus, Lemma 3 is proved. □

Exercise 51. Let $n = 2$ or $n = 3$. Assume that $q \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with $1 < p \le \infty$ if $n = 2$ and $3 < p \le \infty$ if $n = 3$. Prove that the generalized eigenfunctions $u(x, \overrightarrow{k})$ are uniformly bounded with respect to $x \in \mathbb{R}^n$ and $|\overrightarrow{k}|$ large enough.

We will obtain very important corollaries from Optical lemma. Let $A_q(k)$ denote the linear mapping that takes the inhomogeneity f to the corresponding scattering amplitude

$$A_q(k) : f(x) \to A_f(k, \theta').$$

Lemma 4. *Let the potential $q(x)$ satisfy the conditions from Theorem 2. Then A_q is a well-defined bounded operator from $L^{\frac{2p}{p+1}}_{\frac{\sigma}{2}}(\mathbb{R}^n)$ to $L^2(\mathbb{S}^{n-1})$ with the operator norm estimate*

$$\|A_q\|_{L^{\frac{2p}{p+1}}_{\frac{\sigma}{2}} \to L^2} \le \frac{C}{|k|^{\frac{\gamma}{2} + \frac{n-2}{2}}},$$

where p, σ and γ are as in Theorem 2.

Proof. By Lemma 3 and the definition of $A_q f$ we have that

$$\begin{aligned} \|A_q f\|^2_{L^2(\mathbb{S}^{n-1})} &= \int_{\mathbb{S}^{n-1}} |A_f(k, \theta')|^2 d\theta' = -\frac{1}{C^2 |k|^{n-2}} \int_{\mathbb{R}^n} \operatorname{Im}(f \cdot \overline{v}) dx \\ &\le \frac{1}{C^2 |k|^{n-2}} \|v\|_{L^{\frac{2p}{p-1}}_{-\frac{\sigma}{2}}(\mathbb{R}^n)} \|f\|_{L^{\frac{2p}{p+1}}_{\frac{\sigma}{2}}(\mathbb{R}^n)}. \end{aligned}$$

Further, since $v = \widehat{G}_q f$, we obtain from Theorem 2 that

$$\|A_q f\|^2_{L^2(\mathbb{S}^{n-1})} \le \frac{C}{|k|^{n-2}} \cdot |k|^{-\gamma} \|f\|^2_{L^{\frac{2p}{p+1}}_{\frac{\sigma}{2}}(\mathbb{R}^n)}.$$

Thus, Lemma 4 is proved. □

Let us denote by $A_0(k)$ the operator $A_q(k)$ which corresponds to the potential $q \equiv 0$ i.e.

$$A_0 f(\theta') = \int_{\mathbb{R}^n} e^{-ik(\theta', y)} f(y) dy.$$

It is not so difficult to see that

$$A_q f(\theta') = A_f(k, \theta') = \int_{\mathbb{R}^n} f(y)\overline{u(y, k, \theta')} dy,$$

where $u(\cdot, k, \theta')$ is the solution of Lippmann-Schwinger equation. Indeed, by Corollary 2 of Theorem 2 we have

$$\begin{aligned} A_f(k, \theta') &= \int_{\mathbb{R}^n} e^{-ik(\theta', y)}(f(y) - q(y)\widehat{G}_q(f))dy = ((I - q\widehat{G}_q)f, e^{ik(\theta', y)})_{L^2(\mathbb{R}^n)} \\ &= (f, (I - \widehat{G}_q(q))e^{ik(\theta', \cdot)})_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f(y)\overline{(I - \widehat{G}_q(q))(e^{ik(\theta', \cdot)})(y)}dy \\ &= \int_{\mathbb{R}^n} f(y)\overline{u(y, k, \theta')}dy, \end{aligned}$$

since $\widehat{G}_q$ is a self-adjoint operator.

Let us prove now that

$$u(y, k, \theta') := (I - \widehat{G}_q(q))(e^{ik(\theta', \cdot)})(y)$$

is the solution of Lippmann-Schwinger equation. Indeed,

$$\begin{aligned} (H - k^2)u &= (H - k^2)(e^{ik(\theta', y)}) - (H - k^2)\widehat{G}_q(q) \cdot (e^{ik(\theta', \cdot)})(y) \\ &= (-\Delta - k^2)e^{ik(\theta', y)} + qe^{ik(\theta', y)} - qe^{ik(\theta', y)} = 0, \end{aligned}$$

since $(-\Delta - k^2)e^{ik(\theta', y)} = 0$ and $(H - k^2)\widehat{G}_q = I$. It means that this $u(y, k, \theta')$ is the solution of the equation $(H - k^2)u = 0$.

Remark. Let us consider the Lippmann-Schwinger equation

$$u(x, k, \theta) = e^{ik(x, \theta)} - \int_{\mathbb{R}^n} G_k^+(|x - y|)q(y)u(y, k, \theta)dy.$$

Then for fixed $k > 0$ and $|x| \to \infty$ the solution $u(x, k, \theta)$ admits the asymptotical representation

$$u(x, k, \theta) = e^{ik(x, \theta)} + C_n \frac{e^{ik|x|}k^{\frac{n-3}{2}}}{|x|^{\frac{n-1}{2}}} A(k, \theta', \theta) + o\left(\frac{1}{|x|^{\frac{n-1}{2}}}\right),$$

where $\theta' = \frac{x}{|x|}$ and the function $A(k, \theta', \theta)$ is called the *scattering amplitude* and has the form

$$A(k, \theta', \theta) = \int_{\mathbb{R}^n} e^{-ik(\theta', y)}q(y)u(y, k, \theta)dy.$$

For $k < 0$ we set

$$A(k, \theta', \theta) = \overline{A(-k, \theta', \theta)}, \quad u(x, k, \theta) = \overline{u(x, -k, \theta)}.$$

Proof. If $(H - k^2)u = 0$ and $u = e^{ik(\theta, x)} + u_{\text{sc}}(x, k, \theta)$ then $u_{\text{sc}}(x, k, \theta)$ satisfies the equation

$$(H - k^2)u_{sc} = -qe^{ik(\theta, x)}.$$

That's why we may apply Corollary 2 of Theorem 2 with $v := u_{\text{sc}}$ and $f := -qe^{ik(\theta, x)}$ and obtain

$$u_{\text{sc}}(x, k, \theta) = C_n \frac{e^{ik|x|}k^{\frac{n-3}{2}}}{|x|^{\frac{n-1}{2}}} A_f(k, \theta') + o\left(\frac{1}{|x|^{\frac{n-1}{2}}}\right),$$

where

$$\begin{aligned} A_f(k,\theta') &= \int_{\mathbb{R}^n} e^{-ik(\theta',y)}(-qe^{ik(\theta,y)} + q\widehat{G}_q(qe^{ik(\theta,\cdot)})(y))dy \\ &= -\int_{\mathbb{R}^n} e^{-ik(\theta',y)}q(y)(e^{ik(\theta,y)} - \widehat{G}_q(qe^{ik(\theta,\cdot)})dy. \end{aligned}$$

But we have proved that $e^{ik(\theta,y)} - \widehat{G}_q(qe^{ik(\theta,\cdot)})(y)$ is solution of the equation $(H-k^2)u = 0$. That's why we may conclude that

$$A_f(k,\theta') = -\int_{\mathbb{R}^n} e^{-ik(\theta',y)}q(y)u(y,k,\theta)dy := -A(k,\theta',\theta).$$

Thus, this remark is proved. □

Now let $\Phi_0(k)$ and $\Phi(k)$ be the operators defined for $f \in L^2(\mathbb{S}^{n-1})$ as

$$(\Phi_0(k)f)(x) := |q(x)|^{\frac{1}{2}} \int_{\mathbb{S}^{n-1}} e^{ik(x,\theta)} f(\theta)d\theta \tag{10.1}$$

and

$$(\Phi(k)f)(x) := |q(x)|^{\frac{1}{2}} \int_{\mathbb{S}^{n-1}} u(x,k,\theta)f(\theta)d\theta. \tag{10.2}$$

Lemma 5. *The operators $\Phi_0(k)$ and $\Phi(k)$ are bounded from $L^2(\mathbb{S}^{n-1})$ to $L^2(\mathbb{R}^n)$ with the norm estimates*

$$\|\Phi_0(k)\|, \|\Phi(k)\| \le \frac{C}{k^{\frac{\gamma}{2}+\frac{n-2}{2}}}, \quad k > 0,$$

where γ is as in Theorem 2.

Proof. Let us prove that

$$(\Phi_0(k)f)(x) = |q(x)|^{\frac{1}{2}}(A_0^* f)(x) \tag{10.3}$$

and

$$(\Phi(k)f)(x) = |q(x)|^{\frac{1}{2}}(A_q^* f)(x), \tag{10.4}$$

where A_0^* and A_q^* are the adjoint operators for A_0 and A_q, respectively. Indeed, if $f \in L^2(\mathbb{S}^{n-1})$ and $g \in L^2(\mathbb{R}^n)$ then

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} f(\theta)\overline{(A_0 g)(\theta)}d\theta &= \int_{\mathbb{S}^{n-1}} f(\theta)d\theta \int_{\mathbb{R}^n} e^{ik(\theta,y)}\overline{g(y)}dy \\ &= \int_{\mathbb{R}^n} \overline{g(y)}dy \int_{\mathbb{S}^{n-1}} e^{ik(\theta,y)} f(\theta)d\theta \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{S}^{n-1}} e^{ik(\theta,y)} f(\theta)d\theta\right)\overline{g(y)}dy. \end{aligned}$$

It means that

$$A_0^* f(y) = \int_{\mathbb{S}^{n-1}} e^{ik(\theta,y)} f(\theta)d\theta$$

and (10.3) is immediate. Similarly for (10.4). Since (see Lemma 4)

$$\|A_0\|, \|A_q\|_{L^{\frac{2p}{p+1}}_{\frac{\sigma}{2}} \to L^2(\mathbb{S}^{n-1})} \leq \frac{C}{k^{\frac{\gamma}{2}+\frac{n-2}{2}}}$$

we have that

$$\|A_0^*\|, \|A_q^*\|_{L^2(\mathbb{S}^{n-1}) \to L^{\frac{2p}{p-1}}_{-\frac{\sigma}{2}}(\mathbb{R}^n)} \leq \frac{C}{k^{\frac{\gamma}{2}+\frac{n-2}{2}}}.$$

The proof is finished by

$$\begin{aligned} \|\Phi_0(k)f\|_{L^2(\mathbb{R}^n)} &= \||q|^{\frac{1}{2}}(A_0^* f)\|_{L^2(\mathbb{R}^n)} \leq \|q\|^{\frac{1}{2}}_{L^p_\sigma(\mathbb{R}^n)} \|A_0^* f\|_{L^{\frac{2p}{p-1}}_{-\frac{\sigma}{2}}(\mathbb{R}^n)} \\ &\leq \frac{C}{k^{\frac{\gamma}{2}+\frac{n-2}{2}}} \|q\|^{\frac{1}{2}}_{L^p_\sigma(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{S}^{n-1})}, \end{aligned}$$

where we have made use of Hölder inequality in the first estimate. It is clear that the same is true for $\Phi(k)$. □

11 Some inverse problems for the Schrödinger operator

The classical inverse scattering problem is to reconstruct the potential $q(x)$ from the knowledge of the far field data (scattering amplitude) $A(k,\theta',\theta)$, when k,θ' and θ are restricted to some given set.

If $q \in L^1(\mathbb{R}^n)$ then $q(y)u(y,k,\theta) \in L^1(\mathbb{R}^n)$ uniformly with respect to $\theta \in \mathbb{S}^{n-1}$ due to

$$q(y)u(y,k,\theta) = q(y)(e^{ik(\theta,y)} + u_{\text{sc}}(y,k,\theta)) = \underbrace{q(y)e^{ik(\theta,y)}}_{\in L^1} + \underbrace{|q|^{\frac{1}{2}}}_{\in L^2} \cdot \underbrace{q_{\frac{1}{2}} u_{\text{sc}}(y,k,\theta)}_{\in L^2}$$

and Hölder's inequality. That's why we may conclude that the scattering amplitude $A(k,\theta',\theta)$ is well-defined and continuous. Also the following representation holds:

$$\begin{aligned} A(k,\theta',\theta) &= \int_{\mathbb{R}^n} e^{-ik(\theta',y)} q(y)(e^{ik(\theta,y)} + u_{\text{sc}})dy = \int_{\mathbb{R}^n} e^{-ik(\theta'-\theta,y)} q(y)dy + R(k,\theta',\theta) \\ &= (2\pi)^{n/2}(Fq)(k(\theta'-\theta)) + R(k,\theta',\theta), \end{aligned}$$

where $R(k,\theta',\theta) \to 0$ as $k \to +\infty$ uniformly with respect to θ' and θ. This fact implies that

$$A(k,\theta',\theta) \approx (2\pi)^{n/2}(Fq)(k(\theta'-\theta))$$

or

$$q(x) \approx (2\pi)^{-n/2} F^{-1}(A(k,\theta',\theta))(x),$$

where the inverse Fourier transform must be understood in some special sense.

Let us introduce the cylinders $M_0 = \mathbb{R} \times \mathbb{S}^{n-1}$ and $M = M_0 \times \mathbb{S}^{n-1}$, and the measures μ_θ and μ on M_0 and M, respectively, as

$$d\mu_\theta(k,\theta') = \frac{1}{4}|k|^{n-1} dk |\theta - \theta'|^2 d\theta',$$

$$d\mu(k,\theta',\theta) = \frac{1}{|\mathbb{S}^{n-1}|} d\theta d\mu_\theta(k,\theta')$$

where $|\mathbb{S}^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$ is the area of the unit sphere $\mathbb{S}^{n-1}$ and $d\theta$ and $d\theta'$ denote the usual Lebesgue measures on $\mathbb{S}^{n-1}$. We shall define the inverse Fourier transform on M_0 and M as

$$(F_{M_0}^{-1}\varphi_1)(x) = \frac{1}{(2\pi)^{n/2}} \int_{M_0} e^{-ik(\theta-\theta',x)} \varphi_1(k,\theta') d\mu_\theta,$$

$$(F_M^{-1}\varphi_2)(x) = \frac{1}{(2\pi)^{n/2}} \int_M e^{-ik(\theta-\theta',x)} \varphi_2(k,\theta',\theta) d\mu.$$

If we write $\xi = k(\theta - \theta')$ then k and θ' are obtained by

$$k = \frac{|\xi|}{2(\theta,\widehat{\xi})}, \quad \theta' = \theta - 2(\theta,\widehat{\xi})\widehat{\xi}, \quad \widehat{\xi} = \frac{\xi}{|\xi|}. \tag{11.1}$$

Exercise 52. Let $u_\theta(k,\theta')$ be the coordinate mapping $M_0 \to \mathbb{R}^n$ given as

$$u_\theta(k,\theta') = k(\theta - \theta'),$$

where θ is considered as a fixed parameter. Prove that

1) the formulas (11.1) for k and θ' hold

2) the following is true:

$$\int_{M_0} \varphi \circ u_\theta(k,\theta') d\mu_\theta(k,\theta') = \int_{\mathbb{R}^n} \varphi(x)dx$$

if $\varphi \in S$ is even and

$$\int_{M} \varphi \circ u_\theta(k,\theta') d\mu(k,\theta',\theta) = \int_{\mathbb{R}^n} \varphi(x)dx$$

if $\varphi \in S$.

3) in addition:

$$F_{M_0}^{-1}(\varphi \circ u_\theta) = F^{-1}\varphi$$

if $\varphi \in S$ is even and

$$F_{M}^{-1}(\varphi \circ u_\theta) = F^{-1}\varphi$$

if $\varphi \in S$. Here F^{-1} is the usual inverse Fourier transform in $\mathbb{R}^n$.

Exercise 53. Prove that

1) $A(-k,\theta',\theta) = \overline{A(k,\theta',\theta)}$

2) $A(k,\theta',\theta) = A(k,-\theta,-\theta')$.

The approximation $q(x) \approx (2\pi)^{-\frac{n}{2}} F^{-1}(A(k,\theta',\theta)(x))$ for all θ' and θ and for sufficiently large k allows us to introduce the following definitions.

Definition. The inverse Born approximations $q_{\mathrm{B}}^\theta(x)$ and $q_{\mathrm{B}}(x)$ of the potential $q(x)$ are defined by

$$q_{\mathrm{B}}^\theta(x) = (2\pi)^{-n/2}(F_{M_0}^{-1}A)(x) = \frac{1}{(2\pi)^n}\int_{M_0} e^{-ik(\theta-\theta',x)}A(k,\theta',\theta)d\mu_\theta,$$

and

$$q_{\mathrm{B}}(x) = (2\pi)^{-n/2}(F_{M}^{-1}A)(x) = \frac{1}{(2\pi)^n}\int_{M} e^{-ik(\theta-\theta',x)}A(k,\theta',\theta)d\mu.$$

Remark. The equalities from the latter definition must be understood in the sense of distributions.

Theorem 1 (Uniqueness). *Assume that the potential $q(x)$ belongs $L^p_{\text{loc}}(\mathbb{R}^n)$, $\frac{n}{2} < p \leq \infty, n \geq 3$, and has the special behavior $|q(x)| \leq C|x|^{-\mu}, \mu > 2, |x| \to \infty$ at the infinity. Then the knowledge of $q_B^\theta(x)$ with θ restricted to an $(n-2)$ dimensional semisphere determines $q(x)$ uniquely.*

Proof. It is not so difficult to check that if $q(x)$ satisfies the conditions of present theorem then $q(x)$ will satisfy the conditions of Theorem 2 from Chapter 10:

$$q \in L^p(\mathbb{R}^n), \quad \frac{n}{2} < p \leq \frac{n+1}{2}$$

or

$$q \in L^p_\sigma(\mathbb{R}^n), \quad \frac{n+1}{2} < p \leq +\infty, \quad \sigma > 1 - \frac{n+1}{2p}.$$

Now we can represent $q_{\text{B}}^\theta(x)$ in the form

$$\begin{aligned} q_{\text{B}}^\theta(x) &= \frac{1}{(2\pi)^n} \int_{M_0} e^{-ik(\theta-\theta',x)} A(k,\theta',\theta) d\mu_\theta(k,\theta') \\ &= \frac{1}{(2\pi)^n} \int_{M_0} d\mu_\theta(k,\theta') \int_{\mathbb{R}^n} e^{-ik(\theta-\theta',x)} e^{-ik(\theta',y)} q(y) u(y,k,\theta) dy \\ &= \frac{1}{(2\pi)^n} \int_{M_0} d\mu_\theta \int_{\mathbb{R}^n} e^{-ik(\theta-\theta',x-y)} q(y) e^{-ik(\theta,y)} u(y,k,\theta) dy, \end{aligned}$$

where $u(y,k,\theta)$ is the solution of the Lippmann-Schwinger equation. Denoting

$$v(y,k,\theta) := e^{-ik(\theta,y)} u(y,k,\theta)$$

and making the change of variables $\xi = k(\theta - \theta')$ we obtain

$$q_{\text{B}}^\theta(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} e^{-i(\xi,x-y)} q(y) v\left(y, \frac{|\xi|}{2(\theta,\widehat{\xi})}, \theta\right) dy.$$

The usual Fourier transform of $q_{\text{B}}^\theta(x)$ is equal to

$$\widehat{q_{\text{B}}^\theta}(\xi) = \widehat{q}(\xi) + (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(\xi,y)} q(y) \left[v\left(y, \frac{|\xi|}{2(\theta,\widehat{\xi})}, \theta\right) - 1 \right] dy$$

and it implies that

$$|\widehat{q_{\text{B}}^\theta} - \widehat{q}| \leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} |q(y)| \left| v\left(y, \frac{|\xi|}{2(\theta,\widehat{\xi})}, \theta\right) - 1 \right| dy,$$

where the function $v(y,k,\theta)$ solves the equation

$$v(x,k,\theta) = 1 - \int_{\mathbb{R}^n} e^{-ik(x,\theta)} G_k^+(|x-y|) e^{ik(y,\theta)} q(y) v(y,k,\theta) dy$$

i.e.

$$v = 1 - \widehat{\widehat{G}}_k(qv),$$

where $\widetilde{G}_k = e^{-ik(x-y,\theta)} G_k^+$. For k large enough we may obtain that

$$v = (I + \widehat{\widetilde{G}}_k q)^{-1}(1)$$

or

$$v = 1 - \widehat{\widetilde{G}}_q(q), \tag{11.2}$$

where $\widehat{\widetilde{G}}_q$ is the integral operator with the kernel $\widetilde{G}_q = e^{-ik(x-y,\theta)} G_q$ and the integral operator $\widehat{G}_q$ with this kernel satisfies also the equation $(H - k^2)\widehat{G}_q = I$. In order to prove (11.2) we recall that

$$\widehat{G}_q = \widehat{G}_k - \widehat{G}_k q \widehat{G}_q$$

and, therefore

$$\widehat{\widetilde{G}}_q = \widehat{\widetilde{G}}_k - \widehat{\widetilde{G}}_k q \widehat{\widetilde{G}}_q$$

or

$$\widehat{\widetilde{G}}_q = (I + \widehat{\widetilde{G}}_k q)^{-1} \widehat{\widetilde{G}}_k.$$

The last equality implies that

$$\widehat{\widetilde{G}}_q(q) = (I + \widehat{\widetilde{G}}_k q)^{-1} \widehat{\widetilde{G}}_k(q) = -(v - 1)$$

because

$$(I + \widehat{\widetilde{G}}_k q)^{-1} \widehat{\widetilde{G}}_k(q) = -(v - 1)$$

is equivalent to

$$\begin{aligned} \widehat{\widetilde{G}}_k(q) &= -(I + \widehat{\widetilde{G}}_k q)(v - 1) = -(v - 1) - (\widehat{\widetilde{G}}_k q)(v) + (\widehat{\widetilde{G}}_k q)(1) \\ &= -v + 1 - 1 + v + \widehat{\widetilde{G}}_k(q) = \widehat{\widetilde{G}}_k(q). \end{aligned}$$

That's why we may apply Theorem 2 of Chapter 10 and get

$$\|v - 1\|_{L^{\frac{2p}{p-1}}_{-\frac{\sigma}{2}}(\mathbb{R}^n)} = \|\widehat{\widetilde{G}}_q(q)\|_{L^{\frac{2p}{p-1}}_{-\frac{\sigma}{2}}} \le \frac{C}{k^\gamma} \|q\|_{L^{\frac{2p}{p+1}}_{\frac{\sigma}{2}}},$$

where γ, p and σ are as in that theorem. It remains to check only that the potential $q \in L^p_{\text{loc}}(\mathbb{R}^n)$ with the special behavior at the infinity belongs to $L^{\frac{2p}{p+1}}_{\frac{\sigma}{2}}(\mathbb{R}^n)$. But it is a very simple exercise. Hence, the letter inequality leads to

$$|\widehat{q^\theta_{\mathrm{B}}}(\xi) - \widehat{q}(\xi)| \le C\|q\|^2_{L^{\frac{2p}{p+1}}_{\frac{\sigma}{2}}(\mathbb{R}^n)} \left(\frac{|(\widehat{\xi}, \theta)|}{|\xi|} \right)^\gamma, \quad \xi \neq 0$$

with the same γ. If q_1 and q_2 are as q then

$$\begin{aligned} |\widehat{q}_1(\xi) - \widehat{q}_2(\xi)| &= |\widehat{q}_1(\xi) - \widehat{q^\theta_{\mathrm{B}}} + \widehat{q^\theta_{\mathrm{B}}} - \widehat{q}_2(\xi)| \le |\widehat{q}_1(\xi) - \widehat{q^\theta_{\mathrm{B}}}| + |\widehat{q^\theta_{\mathrm{B}}} - \widehat{q}_2(\xi)| \\ &\le C\|q_1\|^2_{L^{\frac{2p}{p+1}}_{\frac{\sigma}{2}}(\mathbb{R}^n)} \left(\frac{|(\widehat{\xi}, \theta)|}{|\xi|} \right)^\gamma + C\|q_2\|^2_{L^{\frac{2p}{p+1}}_{\frac{\sigma}{2}}(\mathbb{R}^n)} \left(\frac{|(\widehat{\xi}, \theta)|}{|\xi|} \right)^\gamma = 0 \end{aligned}$$

if $(\widehat{\xi}, \theta) = 0$. Thus, this theorem is proved because $(\widehat{\xi}, \theta) = 0$ precisely when θ runs through an $(n-2)$-dimensional semisphere, see [6, 7]. □

Theorem 2 (Saito's formula). *Under the same assumptions for $q(x)$ as in Theorem 1,*

$$\lim_{k\to+\infty} k^{n-1}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} e^{-ik(\theta-\theta',x)}A(k,\theta',\theta)d\theta d\theta' = \frac{(2\pi)^n}{\pi}\int_{\mathbb{R}^n}\frac{q(y)dy}{|x-y|^{n-1}},$$

where the limit holds in classical sense for $n < p \le \infty$ and in the sense of the distributions for $\frac{n}{2} < p \le n$.

Proof. Let us consider only the case $n < p \le \infty$. The proof for $\frac{n}{2} < p \le n$ takes place with some changes.

By definition of the scattering amplitude,

$$\begin{aligned} I &:= k^{n-1}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} A(k,\theta',\theta)e^{-ik(\theta-\theta',x)}d\theta d\theta' \\ &= k^{n-1}\int_{\mathbb{R}^n} q(y)dy\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} e^{ik(\theta-\theta',y-x)}d\theta d\theta' \\ &+ k^{n-1}\int_{\mathbb{R}^n} q(y)dy\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} e^{-ik(\theta',y)}R(y,k,\theta)e^{-ik(\theta-\theta',x)}d\theta d\theta' := I_1 + I_2, \end{aligned}$$

where $R(y,k,\theta)$ is given by

$$R(y,k,\theta) = -\int_{\mathbb{R}^n} G_k^+(|y-z|)q(z)u(z,k,\theta)dz$$

and $u(z,k,\theta)$ is the solution of the Lippmann-Schwinger equation. Since

$$\begin{aligned} \int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} e^{ik(\theta-\theta',y-x)}d\theta d\theta' &= \left|\int_{\mathbb{S}^{n-1}} e^{ik(\theta,y-x)}d\theta\right|^2 \\ &= \frac{4\pi^{n-1}}{\Gamma^2(\frac{n-1}{2})}\left(\int_0^\pi e^{ik|y-x|\cos\psi}(\sin\psi)^{n-2}d\psi\right)^2 \\ &= (2\pi)^n\frac{J^2_{\frac{n-2}{2}}(k|x-y|)}{(k|x-y|)^{n-2}}, \end{aligned}$$

then I_1 can be represented in the form

$$I_1 = (2\pi)^n k\int_{\mathbb{R}^n}\frac{q(y)}{|x-y|^{n-2}}J^2_{\frac{n-2}{2}}(k|x-y|)dy.$$

We consider two cases: $k|x-y| < 1$ and $k|x-y| > 1$. In the first case using Hölder's inequality the integral I_1' over $\{y : k|x-y| < 1\}$ can be estimated by

$$\begin{aligned} |I_1'| &\le Ck\int_{|x-y|<\frac{1}{k}}\frac{|q(y)|(k|x-y|)^{n-2}}{|x-y|^{n-2}}dy \\ &\le Ck^{n-1}\left(\int_{|x-y|<\frac{1}{k}}|q(y)|^p dy\right)^{\frac{1}{p}}\left(\int_{|x-y|<\frac{1}{k}}1\cdot dy\right)^{\frac{1}{p'}} \\ &= Ck^{n-1}\cdot k^{-\frac{n}{p'}}\left(\int_{|x-y|<\frac{1}{k}}|q(y)|^p dy\right)^{\frac{1}{p}} = Ck^{\frac{n}{p}-1}\left(\int_{|x-y|<\frac{1}{k}}|q(y)|^p dy\right)^{\frac{1}{p}} \to 0 \end{aligned}$$

as $k \to +\infty$ since $n < p \le \infty$. This means that for every fixed x (or even uniformly with respect to x) I_1' approaches to zero as $k \to \infty$. Hence, we need only to estimate the integral I_1'' over $\{y : k|x-y| > 1\}$. The asymptotic behavior of the Bessel function $J_\nu(\cdot)$ for large argument implies that

$$\begin{aligned}
I_1'' &= (2\pi)^n k \int_{|x-y|>\frac{1}{k}} \frac{q(y)}{|x-y|^{n-2}} \\
&\times \left[\sqrt{\frac{2}{\pi k|x-y|}} \cos\left(k|x-y| - \frac{n\pi}{4} + \frac{\pi}{4}\right) + \mathcal{O}\left(\frac{1}{(k|x-y|)^{3/2}}\right)\right]^2 dy \\
&= (2\pi)^n k \int_{|x-y|>\frac{1}{k}} \frac{q(y)}{|x-y|^{n-2}} \left[\frac{2\cos^2(k|x-y| - \frac{n\pi}{4} + \frac{\pi}{4})}{\pi k|x-y|} + \mathcal{O}\left(\frac{1}{(k|x-y|)^2}\right)\right] dy \\
&= \frac{(2\pi)^n}{\pi} \int_{|x-y|>\frac{1}{k}} \frac{q(y)dy}{|x-y|^{n-1}} \\
&+ \frac{(2\pi)^n}{\pi} \int_{|x-y|>\frac{1}{k}} \frac{q(y)}{|x-y|^{n-1}} \cos\left(2k|x-y| - \frac{n\pi}{2} + \frac{\pi}{2}\right) dy \\
&+ \frac{1}{k} \int_{|x-y|>\frac{1}{k}} \frac{|q(y)|\mathcal{O}(1)}{|x-y|^n} dy \\
&= I_1^{(1)} + I_1^{(2)} + I_1^{(3)}.
\end{aligned}$$

It is clear that

$$\lim_{k\to+\infty} I_1^{(1)} = \frac{(2\pi)^n}{\pi} \int_{\mathbb{R}^n} \frac{q(y)dy}{|x-y|^{n-1}}$$

and

$$\lim_{k\to+\infty} I_1^{(2)} = 0.$$

The latter fact follows from the following arguments. Since q belongs to $L^p(\mathbb{R}^n)$ for $p > n$ and has the special behavior at the infinity then we may conclude that L^1-norm of the function $\frac{q(y)}{|x-y|^{n-1}}$ is uniformly bounded with respect to x. Hence it follows from the Riemann-Lebesgue lemma that $I_1^{(2)}$ approaches to zero uniformly with respect to x as $k \to +\infty$. For $I_1^{(3)}$ we have the estimate

$$|I_1^{(3)}| \le \frac{C}{k^{1-\delta}} \int_{\mathbb{R}^n} \frac{|q(y)|dy}{|x-y|^{n-\delta}}.$$

If we choose δ such that $1 > \delta > \frac{n}{p}$, then $\int_{\mathbb{R}^n} \frac{q(y)dy}{|x-y|^{n-\delta}}$ will be uniformly bounded with respect to x. Therefore, $I_1^{(3)} \to 0$ as $k \to \infty$ uniformly with respect to x. If we collect all estimates we obtain that

$$\lim_{k\to\infty} I_1 = \frac{(2\pi)^n}{\pi} \int_{\mathbb{R}^n} \frac{q(y)dy}{|x-y|^{n-1}}.$$

Our next task is to prove that $I_2 \to 0$ as $k \to \infty$. Since

$$I_2 = k^{n-1} \int_{\mathbb{R}^n} q(y)dy \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} e^{-ik(\theta',y)} R(y,k,\theta) e^{-ik(\theta-\theta',x)} d\theta d\theta',$$

where

$$R(y,k,\theta) = -\int_{\mathbb{R}^n} G_k^+(|y-z|)q(z)u(z,k,\theta)dz = -\widehat{G}_k(qu),$$

then one can check that $R(y,k,\theta) = -\widehat{G}_q(qe^{ik(\theta,z)})$. Hence, I_2 can be represented as

$$\begin{aligned} I_2 &= -k^{n-1}\int_{\mathbb{R}^n} q(y)dy\int_{\mathbb{S}^{n-1}} e^{ik(\theta',x-y)}d\theta' \cdot \widehat{G}_q\left(q(z)\int_{\mathbb{S}^{n-1}} e^{ik(\theta,z-x)}d\theta\right) \\ &= -k^{n-1}(2\pi)^n\int_{\mathbb{R}^n} q(y)\frac{J_{\frac{n-2}{2}}(k|x-y|)}{(k|x-y|)^{\frac{n-2}{2}}} \cdot \widehat{G}_q\left(q(z)\frac{J_{\frac{n-2}{2}}(k|x-y|)}{(k|x-y|)^{\frac{n-2}{2}}}\right)dy \\ &= (2\pi)^n k\int_{\mathbb{R}^n} q_{\frac{1}{2}}(y)\frac{J_{\frac{n-2}{2}}(k|x-y|)}{(|x-y|)^{\frac{n-2}{2}}} \cdot \widehat{K}_q\left(|q(z)|^{\frac{1}{2}}\frac{J_{\frac{n-2}{2}}(k|x-y|)}{(|x-y|)^{\frac{n-2}{2}}}\right)dy, \end{aligned}$$

where $\widehat{K}_q$ is the integral operator with the kernel

$$K_q(x,y) = -|q(x)|^{\frac{1}{2}}G_q(k,x,y)q_{\frac{1}{2}}(y).$$

It follows from Theorem 2 of Chapter 10 that $\widehat{K}_q : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with the norm estimate

$$\|\widehat{K}_q\|_{L^2\to L^2} \le \frac{C}{k^\gamma},$$

where γ is as in that theorem. That's why we can estimate I_2 using Hölder's inequality as

$$|I_2| \le \frac{C}{k^\gamma}k\int_{\mathbb{R}^n} |q(y)|\frac{J^2_{\frac{n-2}{2}}(k|x-y|)}{|x-y|^{n-2}}dy.$$

By the same arguments as in the proof for I_1 we can obtain that

$$k\int_{\mathbb{R}^n} |q(y)|\frac{J^2_{\frac{n-2}{2}}(k|x-y|)}{|x-y|^{n-2}}dy < \infty$$

uniformly with respect to x. It implies that

$$|I_2| \le \frac{C}{k^\gamma} \to 0, \quad k \to +\infty.$$

□

Remark. This proof holds also for $n = 2$. In dimension $n = 1$ there is an analogous result in which we replace the double integral in the left hand side by the sum of four values of the integrand at $\theta = \pm 1$ and $\theta' = \pm 1$.

Theorem 3. *Let us assume that $n \ge 2$. Under the same assumptions for $q_1(x), q_2(x)$ as in Theorem 2 let us assume that the corresponding scattering amplitudes A_{q_1} and A_{q_2} coincide for some sequence $k_j \to \infty$ and for all $\theta', \theta \in \mathbb{S}^{n-1}$. Then $q_1(x) = q_2(x)$ in the sence of L^p for $n < p \le \infty$ and in the sense of distributions for $\frac{n}{2} < p \le n$.*

Proof. Saito's formula shows that we only have to show that the homogeneous equation

$$\psi(x) := \int_{\mathbb{R}^n} \frac{q(y)dy}{|x-y|^{n-1}} = 0$$

has only the trivial solution $q(y) \equiv 0$. Let us assume that $n < p \leq \infty$. Introduce the space $S_0(\mathbb{R}^n)$ of all functions from the Schwartz space which vanish in some neighborhood of the origin. Due to the conditions for the potential $q(x)$ we may conclude (as it was before) that $\psi \in L^\infty(\mathbb{R}^n)$ and it defines a tempered distribution. Then for every function $\varphi \in S_0(\mathbb{R}^n)$ it follows that

$$0 = \langle \widehat{\psi}, \varphi \rangle = C_n \langle |\xi|^{-1} \widehat{q}(\xi), \varphi \rangle = C_n \langle \widehat{q}(\xi), |\xi|^{-1}\varphi \rangle.$$

Since $\varphi(\xi) \in S_0(\mathbb{R}^n)$ then $|\xi|^{-1}\varphi \in S_0(\mathbb{R}^n)$ also. Hence, for every $h \in S_0(\mathbb{R}^n)$ the following equation holds

$$\langle \widehat{q}, h \rangle = 0.$$

This means that the support of $\widehat{q}(\xi)$ is at most at the origin and therefore $\widehat{q}(\xi)$ can be represented as

$$\widehat{q}(\xi) = \sum_{|\alpha| \leq m} C_\alpha D^\alpha \delta.$$

Hence, $q(x)$ is a polynomial. But due to the behavior at the infinity we must conclude that $q \equiv 0$. This proves Theorem 3. □

Let us return now to the Born approximation of $q(x)$. A repeated use of the Lippmann-Schwinger equation leads to the following representation for the scattering amplitude $A(k, \theta', \theta)$:

$$\begin{aligned} A(k,\theta',\theta) &= \sum_{j=0}^{m} \int_{\mathbb{R}^n} e^{-ik(\theta',y)} q_{\frac{1}{2}}(y) \widehat{K}^j \cdot (|q|^{\frac{1}{2}} e^{ik(x,\theta)})(y)dy \\ &+ \int_{\mathbb{R}^n} e^{-ik(\theta',y)} q_{\frac{1}{2}}(y) \widehat{K}^{m+1}(|q|^{\frac{1}{2}}(u(x,k,\theta))(y)dy, \end{aligned}$$

where $u(x, k, \theta)$ is the solution of the Lippmann-Schwinger equation and $\widehat{K}$ is an integral operator with the kernel

$$K(x,y) = |q(x)|^{\frac{1}{2}} G_k^+(|x-y|) q_{\frac{1}{2}}(y).$$

The equality for A can be reformulated in the sense of integral operators in $L^2(\mathbb{S}^{n-1})$ as

$$\widehat{A} = \sum_{j=0}^{m} \Phi_0^*(k) \operatorname{sgn} q \widehat{K}^j \Phi_0(k) + \Phi_0^*(k) \operatorname{sgn} q \widehat{K}^{m+1} \Phi(k),$$

where Φ_0 and $\Phi(k)$ are defined by (10.1) and (10.2) and Φ_0^* is the L^2-adjoint of Φ_0.

Using this equality and the definition of Born's potential $q_{\mathrm{B}}(x)$ we obtain

$$q_{\mathrm{B}}(x) = \sum_{j=0}^{m} F_M^{-1} \left[\Phi_0^*(k) \operatorname{sgn} q \widehat{K}^j \Phi_0(k) \right] + F_M^{-1} \left[\Phi_0^*(k) \operatorname{sgn} q \widehat{K}^{m+1} \Phi(k) \right],$$

where the inverse Fourier transform is applied to the kernels of the corresponding integral operators. If we rewrite the latter formula as

$$q_{\mathrm{B}}(x) = \sum_{j=0}^{m} q_j(x) + \widetilde{q}_{m+1}(x),$$

then the term q_j has the form

$$\begin{aligned} q_j(x) &= F_M^{-1}\Big(\int_{\mathbb{R}^n} |q(z)|^{\frac{1}{2}} e^{-ik(z,\theta)} dz \int_{\mathbb{R}^n} \operatorname{sgn} q(z) K^j(z,y,k) |q(y)|^{\frac{1}{2}} e^{ik(\theta',y)} dy\Big) \\ &= F_M^{-1}\left(\Phi_0^* \operatorname{sgn} q \widehat{K}^j (|q|^{\frac{1}{2}} e^{ik(\theta',y)})\right) \\ &= \frac{1}{(2\pi)^n} \int_M e^{-ik(\theta-\theta',x)} d\mu(k,\theta',\theta) \left(\Phi_0^* \operatorname{sgn} q \widehat{K}^j (|q|^{\frac{1}{2}} e^{ik(\theta',y)})\right) \end{aligned}$$

and a similar formula holds for $\widetilde{q}_{m+1}$ with obvious changes.

In order to formulate the result about the reconstruction of singularities of the unknown potential $q(x)$ let us set $A(k,\theta',\theta) = 0$ for $|k| \le k_0$, where $k_0 > 0$ is arbitrarily large.

Theorem 4. *Assume that the potential $q(x)$ satisfies all conditions of Theorem 1 and also belongs to $L^1(\mathbb{R}^n)$. Then $q_j(x)$ and $\widetilde{q}_j(x)$ for any $j \ge 1$ belong to the Sobolev space $H^t(\mathbb{R}^n)$ for any $t < \gamma(j+\frac{1}{2}) - 1$ with γ as in Theorem 2 of Chapter 10.*

Proof. Using the change of variables $\xi = k(\theta - \theta')$ we obtain

$$\begin{aligned} \|q_j\|^2_{H^t(\mathbb{R}^n)} &= \|(1+|\xi|^2)^{\frac{t}{2}} F q_j(\xi)\|^2_{L^2(\mathbb{R}^n)} \\ &= C_n \int_{\mathbb{R}^n} (1+|\xi|^2)^t \left| \int_{\mathbb{S}^{n-1}} \left[\Phi_0^* \operatorname{sgn} q \widehat{K}^j \Phi_0\right] \left(\frac{|\xi|}{2(\widehat{\xi},\theta)}, \theta - 2(\widehat{\xi},\theta)\widehat{\xi}, \theta \right) d\theta \right|^2 d\xi, \end{aligned}$$

where in the brackets $[\cdot]$ there is the kernel of the corresponding integral operator. The last estimate can further be bounded by

$$\begin{aligned} &\le C_n \int_{\mathbb{R}^n} (1+|\xi|^2)^t \int_{\mathbb{S}^{n-1}} \left| \left[\Phi_0^* \operatorname{sgn} q \widehat{K}^j \Phi_0\right] (\ldots) \right|^2 d\theta d\xi \\ &= C_n \int_{M_0} (1+|k(\theta-\theta')|^2)^t \int_{\mathbb{S}^{n-1}} \left| \left[\Phi_0^* \operatorname{sgn} q \widehat{K}^j \Phi_0\right] (k(\theta',\theta)) \right|^2 d\mu \\ &\le C_n \int_{k_0}^{\infty} k^{n-1}(1+k^2)^t dk \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \left| \Phi_0^* (\operatorname{sgn} q \widehat{K}^j (|q|^{\frac{1}{2}} e^{ik(\theta',y)}) \right|^2 d\theta d\theta' \\ &\le C_n \int_{k_0}^{\infty} k^{n-1}(1+k^2)^t dk \int_{\mathbb{S}^{n-1}} \|\Phi_0^*\|^2 \|\widehat{K}^j (|q|^{\frac{1}{2}} e^{ik(\theta',y)})\|_{L^1} d\theta' \\ &\le C_n \int_{k_0}^{\infty} k^{n-1}(1+k^2)^t dk \int_{\mathbb{S}^{n-1}} \|\Phi_0^*\|^2 \|\widehat{K}\|^{2j} \|q\|_{L^1} d\theta' \\ &\le C_n \int_{k_0}^{\infty} \frac{k^{n-1+2t} dk}{k^{2\gamma(j+\frac{1}{2})+n-2}}. \end{aligned}$$

But the latter integral converges if and only if $t < \gamma(j+\frac{1}{2}) - 1$. This proves the theorem. □

Theorem 5 (Reconstruction of singularities)**.** *Assume that the potential $q(x)$ belongs to $L^p_{\mathrm{loc}}(\mathbb{R}^3)$ for $3 < p \leq \infty$ and has the special behavior $|q(x)| \leq C_0|x|^{-\mu}, \mu > 3, |x| \to \infty$ at the infinity. Then*

$$q_B(x) - q(x) - q_1(x) \in H^t(\mathbb{R}^3),$$

where $t < \frac{3}{2} - \frac{5}{2p}$ and $q_1(x)$ is continuous and bounded (more precisely, $q_1(x) \in W^1_p(\mathbb{R}^3)$ for $p > 3$).

Proof. The statement about the first nonlinear term $q_1(x)$ was proved by Lassi Päivärinta and Valery Serov, see [8]. It is also easy to check that $q_0(x)$ is simply the unknown potential $q(x)$. Hence, we can write

$$q_{\mathrm{B}} - q - q_1 = \widetilde{q}_2$$

and so the claim follows from Theorem 4. □

Remark. Let us assume that $3 < p < \infty$. Then the following embedding holds

$$H^t(\mathbb{R}^3) \subset W_p^{t-3\left(\frac{1}{2}-\frac{1}{p}\right)}(\mathbb{R}^3) = W_p^{\alpha}(\mathbb{R}^3),$$

where $\alpha = t - 3\left(\frac{1}{2} - \frac{1}{p}\right)$ and, therefore, $\alpha < \frac{1}{2p}$. It means that $\widetilde{q}_2$ belongs to the "smoother" space than the unknown potential $q(x)$ and so we can reconstruct main singularities of the potential $q(x)$ by the Born approximation.

Remark. Actually it is possible to prove (see [9]) that

$$q_2(x) \in C^{1-\frac{3}{p}}(\mathbb{R}^3)$$

for any $3 < p \leq \infty$. Using Theorem 4 we obtain that $q_{\mathrm{B}} - q - q_1 - q_2 \in H^t$ for any $t < \frac{5}{2} - \frac{7}{2p}$. But for $3 < p \leq \infty$ the following embedding holds: $H^t \subset C(\mathbb{R}^n)$. It means that for $3 < p \leq \infty$ we can reconstruct all singularities of the unknown potential because q_1 and q_2 are continuous in this case.

Remark. It follows from this theorem that by Born approximation we can reconstruct an arbitrary bounded domain D. In order to see this fact it is enough to consider $q(x) = \chi_D(x)$.

References

[1] Cycon H L, Froese R G, Kirsch W and Simon B, *Schrödinger Operators*, Academic Press, New-York, 1989

[2] Ikebe I, Arch. Rational Mech. Anal., 5 (1960), 1-34

[3] Kato T, Comm. Pure Appl. Math., 12 (1959), 403-425

[4] Ruiz A, lectures at the summer school in Oulu, June, 2002

[5] Agmon S, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 4 (1975), 151-218

[6] Saito Y, Osaka, J. Math.19 (1982), 527-547

[7] Päivärinta L and Somersalo E, SIAM J. Math. Anal., 22 (1991), 480-499

[8] Päivärinta L and Serov V, SIAM J. Math. Anal., 29 (1998), 697-711

[9] Päivärinta L and Serov V, Inverse Probl. Imaging, 1 (2007), 525-535