Some mathematical aspects of fluid-solid interaction

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Outline

1. Ideal fluids
   - D'Alembert's paradox (1752)
   - Boundary layer theory
2. Viscous fluids
   - Navier-Stokes type models
   - Weak and strong solutions
   - Drag computation and the no-collision paradox

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Statement of the paradox

"In an ideal incompressible fluid, bodies moving at constant speed do not experience any drag, or lift."

$\Rightarrow$ Failure of the Euler equation as a model for fluid-solid interaction.

The origin of the problem is the following:

Theorem ("Incompressible potential flows generate no force on obstacles")

Let $u = u(x)$ be a smooth 3D field, defined outside a smooth bounded domain $\mathcal{O}$.

Assume that u is a divergence-free gradient field, tangent at $\partial\mathcal{O}$, uniform at infinity. Then:

1. u is a (steady) solution of the Euler equation outside $\mathcal{O}$:

$$\partial_t u + u \cdot \nabla u + \nabla p = 0, \quad \operatorname{div} u = 0, \quad \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{O}}.$$

2. $F := \int_{\partial\mathcal{O}} p n d\sigma = 0.$

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2. $F := \int_{\partial\mathcal{O}} p n d\sigma \; = \; 0.$

Proof of the theorem: Assumptions on u:

$$u = u_\infty + \nabla\eta, \quad \Delta\eta = 0, \quad \nabla\eta \xrightarrow[|x|\to\infty]{} 0, \quad \partial_n\eta|_{\partial\mathcal{O}} = -u_\infty \cdot n.$$

1. u satisfies the Euler equation, due to the algebraic identity

$$u \cdot \nabla u = -u \times \operatorname{curl} u + \frac{1}{2}\nabla|u|^2 \quad (p := -\frac{1}{2}|u|^2).$$

2. To prove that the force is zero: one uses a representation formula:

$$\eta(x) = \eta_\infty + \int_{\partial\mathcal{O}} \partial_{n_y} G(x,y)\eta(y)d\sigma(y) + \int_{\partial\mathcal{O}} u_\infty \cdot n(y) G(x,y) d\sigma(y)$$

where $G(x,y) = -\frac{1}{4\pi|x-y|}$.

Allows to prove that: $u(x) = u_\infty + O(|x|^{-3}), \quad p = p_\infty + O(|x|^{-3})$.

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Back to the Euler equation:

$$\boxed{u \cdot \nabla u + \nabla p = 0}$$

the fast decay of $u - u_\infty$ and $p - p_\infty$ allows to integrate by parts "up to infinity":

$$\boxed{\int_{\mathbb{R}^3 \setminus \overline{\mathcal{O}}} (u \cdot \nabla u + \nabla p) \; = \; \int_{\partial \mathcal{O}} pn \; = \; 0.}$$

How does it imply the paradox ?

Example: A plane, initially at rest.

- Initially, the air around the plane is at rest, so curl-free.

- The curl-free condition is preserved by Euler.

- When the plane reaches its cruise speed, the conditions of the theorem are fulfilled (up to a change of frame).

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What is the flaw of the Euler model ? How to clear the paradox ?

Large consensus: in domains Ω with boundaries, one should add viscosity, and consider the *Navier-Stokes equations*:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} \;=\; 0, & \mathbf{x}\in\Omega\subset\mathbb{R}^2 \text{ ou } \mathbb{R}^3, \\ \nabla\cdot\mathbf{u} \;=\; 0, & \mathbf{x}\in\Omega. \end{cases} \tag{NS}$$

2 possible meanings for ν:

- Dimensionalized system: $\nu = \nu_K$, *kinematic viscosity*.
- Dimensionless system: $\nu = \nu_K\,/(U\,L)$,

  U, L : typical speed and length, $1/\nu$: *Reynolds number*.

Main point: The curl-free condition is not preserved by the Navier-Stokes equation in domains with boundaries.

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... but: in most experiments, ν is very small:

Example: Flows around planes: $\nu \approx 10^{-6}$.

Hence, Euler equations ($\nu = 0$) should be a good approximation !

Indeed, for smooth solutions in domains *without boundaries*, it is true !

But in domains *with boundaries*, not clear !

The problem comes from boundary conditions.

- For $\nu \neq 0$ (NS), classical *no-slip condition*:

$$\boxed{\mathbf{u}|_{\partial\Omega} = 0} \qquad \text{(D)}$$

- For $\nu = 0$ (Euler), one needs to relax this condition:

$$\boxed{\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0}$$

$\Rightarrow$ u_ν concentrates near $\partial\Omega$: *boundary layer*.

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Problem: Impact of this boundary layer on the asymptotics $\nu \to 0$?

This problem can be further specified:

Theorem [Kato, 1983]

Let Ω a bounded open domain. Let $\mathbf{u}_\nu$ and $\mathbf{u}_0$regular solutions of (NS)-(D) and Euler, with the same initial data. Then

$$\mathbf{u}_\nu \to \mathbf{u}_0 \quad \text{in } L^\infty(0, T; L^2(\Omega)) \quad \text{if and only if}$$

$$\nu \int_0^T \int_{d(\mathbf{x}, \partial\Omega) \leq \nu} |\nabla \mathbf{u}_\nu|^2 \to 0.$$

Remarks:

- Yields a quantitative and optimal criterium for convergence.
- The convergence is related to concentration at scale ν (and not at parabolic scale $\sqrt{\nu}$).

Still, the convergence from NS to Euler is (mostly) an open question.

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Prandtl's approach

Case $\Omega \subset \mathbb{R}^2$: we introduce

- curvilinear coordinates (x, y) near the boundary:

$$\mathbf{x} \;=\; \tilde{\mathbf{x}}(x) \;+\; y\,\mathbf{n}(x), \qquad \text{with } \tilde{\mathbf{x}} \in \partial\Omega,\; x \text{ arc length},\; y \geq 0.$$

- Frénet decomposition:

$$\mathbf{u}_\nu(t, \mathbf{x}) \;=\; u_\nu(t, x, y)\,\mathbf{t}(x) \;+\; v_\nu(t, x, y)\,\mathbf{n}(x),$$

Idea [Prandtl 1904]:

$$\boxed{\begin{aligned} u_\nu(t,x,y) \;&\approx\; u_0(t,x,y) \;+\; u_{BL}(t,x,y/\sqrt{\nu}), \\ v_\nu(t,x,y) \;&\approx\; v_0(t,x,y) \;+\; \sqrt{\nu}\, v_{BL}(t,x,y/\sqrt{\nu}), \end{aligned}} \qquad \text{(Ans)}$$

- $\mathbf{u}_0 = u_0\mathbf{t} + v_0\mathbf{n}$: solution of the Euler equation,
- $(u_{BL}, v_{BL}) = (u_{BL}, v_{BL})(t, x, Y)$: *boundary layer corrector.*

Prandtl equation: it is the equation satisfied formally by

$$\begin{aligned} u(t,x,Y) &:= u_0(t,x,0) + u_{BL}(t,x,Y), \\ v(t,x,Y) &:= Y\,\partial_y v_0(t,x,0) + v_{BL}(t,x,Y). \end{aligned}$$

Formally, for $Y > 0$:

$$\boxed{\begin{cases} \partial_t u + u\partial_x u + v\partial_Y u - \partial_Y^2 u = (\partial_t u_0 + u_0\partial_x u_0)\,|_{y=0}, \\ \partial_x u + \partial_Y v = 0, \\ (u,v)|_{Y=0} = (0,0), \qquad \lim_{Y\to+\infty} u \;=\; u_0|_{y=0}. \end{cases}}$$

Remarks:

- No curvature term in the operators ($\neq$ 3D).
- Curvature is involved through u_0 , and through the domain of definition of x. Classical choices:

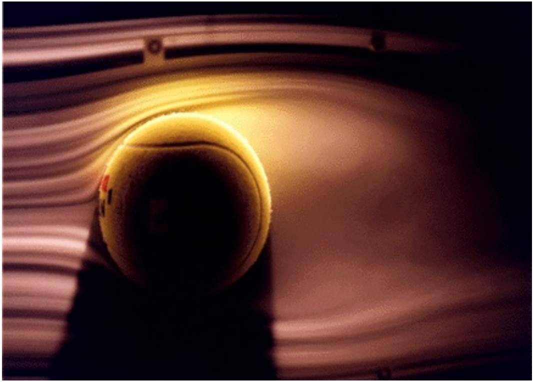
  a) $x \in \mathbb{R}, \mathbb{T}$ (local study in x, outside of a convex obstacle)

  b) $x \in (0, L)$, with an "initial" condition at $x = 0$.

Question: Is the Ansatz (Ans) justified ?

Credo: Yes, but only locally in space-time. Experiments show a lot of instabilities.

Example: Boundary layer separation



Mathematical results

Problem 1: Cauchy theory for Prandtl ?

Problem 2: Justification of (Ans) ?

For both pbs, *the choice of the functional spaces is crucial*.

Problem 1:

- $(x, Y) \in \mathbb{R} \times \mathbb{R}_+$, *analyticity in* x. *Well-posed locally in time* ([Sammartino 1998], [Cannone 2003]).
- $(x, Y) \in (0, L) \times \mathbb{R}_+$, *monotonicity in* y. *Well-posed locally in time, globally under further assumptions* ([Oleinik 1967], [Xin 2004]).

Remark: Without monotonicity, there are solutions that blow up in finite time: [E 1997].

Problem 2:

- Analytic framework: the asymptotics holds [Sammartino 1998].
- Sobolev framework: the asymptotics does not always hold in H^1 [Grenier,2000]. Relies on Rayleigh instability.

Natural question: Is Prandtl well-posed in Sobolev type spaces ?

We consider the case: $x \in \mathbb{T}$, $u^0 = 0$:

$$\boxed{\begin{cases} \partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u = 0, \quad (x,y) \in \mathbb{T} \times \mathbb{R}_+ \\ \partial_x u + \partial_y v = 0, \quad (x,y) \in \mathbb{T} \times \mathbb{R}_+ \\ (u,v)|_{y=0} = (0,0). \end{cases}} \tag{P}$$

Well- or ill-posed ?

Pb: To guess the correct answer !

No standard estimate available for the linearized system.

Example: Let $U(t,y)$ satisfying $\partial_t U - \partial_y^2 U \ = \ 0, \ U|_{y=0} = 0.$

The field $(U(t,y),0)$ satisfies (P).

Linearized equation:

$$\boxed{\begin{cases} \partial_t u + U\partial_x u + v\partial_y U - \partial_y^2 u = 0, & \text{in } \mathbb{T}\times\mathbb{R}^+. \\ \partial_x u + \partial_y v = 0, & \text{in } \mathbb{T}\times\mathbb{R}^+, \\ (u,v)|_{y=0} = (0,0), & \lim_{y\to+\infty} u = 0. \end{cases}} \qquad \text{(PL)}$$

L^2 estimate: the annoying term is $\int v\partial_y U\, u \ \sim \ O\left(\int |\partial_x u|\,|u|\right).$

A priori, loss of an x-derivative.

Another clue for ill-posedness: Freezing the coefficients, leads to the dispersion relation

$$\omega \;=\; k_x \; U \;+\; i\,\partial_y U \,\frac{k_x}{k_y} - ik_y^2.$$

Suggests that the equation is strongly ill-posed . . . But this is misleading !

Simpler situation: no vertical diffusion, $U = U_s'(y)$:

$$\begin{cases} \partial_t u + U_s \partial_x u + v\, U_s' = 0, & \text{in } \mathbb{T}\times\mathbb{R}^+. \\ \partial_x u + \partial_y v = 0, & \text{in } \mathbb{T}\times\mathbb{R}^+, \\ v|_{y=0} = 0. & \end{cases}$$

- Frozen coefficients: bad dispersion relation.
- But an explicit computation yields

$$u(t,x,y) \;=\; u_0(x - U_s(y)t, y) \;+\; t\, U_s'(y) \int_0^y \partial_x u_0(x - U_s(z)t, z)\, dz.$$

"Weakly" well-posed (loss of a finite number of derivatives).

Back to the nonlinear setting: *The inviscid Prandtl equation is weakly well-posed* [Hong 2003].

In fact, the solution is explicit through the methods of characteristics.

Conclusion: The study without diffusion suggests well-posedness of the Prandtl equation.

But . . .

We show: (P) is strongly ill-posed.

Tricky but violent instability mechanism.

Ingredients: diffusion and critical points of the velocity field.

Does not contradict the previous existing results.

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Theorems

The main theorem is on the linearization (PL) (around $U = U_s(y)$)

$$\begin{cases} \partial_t u + U_s \partial_x u + v U_s' - \partial_y^2 u = 0, & \text{in } \mathbb{T} \times \mathbb{R}^+. \\ \partial_x u + \partial_y v = 0, & \text{in } \mathbb{T} \times \mathbb{R}^+, \\ (u,v)|_{y=0} = (0,0), & \lim_{y \to +\infty} u = 0. \end{cases} \tag{PL}$$

Theorem (Linear ill-posedness in the Sobolev setting) (with E. Dormy)

There exists $U_s \in C_c^\infty(\mathbb{R}_+)$ such that: for all $T > 0$, one can find u_0 satisfying

1. $e^y u_0 \in H^\infty(\mathbb{T} \times \mathbb{R}_+)$
2. Equation (PL) has no distributional solution u with

$$u \in L^\infty(0, T; L^2(\mathbb{T} \times \mathbb{R}_+)), \quad \partial_y u \in L^2(0, T \times \mathbb{T} \times \mathbb{R}_+)$$

and initial data u_0.

'The k-th Fourier mode grows like $e^{c\sqrt{k}t}$"

Pondering on this linear result, one can establish a nonlinear result (joint work with T. NGuyen)

"If the nonlinear Prandtl equation (P) *generates a flow, this flow is not Lipschitz continous from bounded sets of* $e^{-y}H^m(\mathbb{T}\times\mathbb{R}_+)$ *to* $H^1(\mathbb{T}\times\mathbb{R}_+)$*, for arbitrarily small times."*

A few hints at the proof of the linear result

1. *The non-existence of solutions for some initial data amounts to the non-continuity of the semigroup.*

   Simple consequence on the closed graph theorem.

2. Proof of non-continuity.
   1. High frequency analysis of (PL) in the x variable:

      Construction of a quasimode, of WKB type. Allows to reduce the instability pb to a spectral problem for a differential operator on $\mathbb{R}$.
   2. Resolution of the spectral problem.
   3. Consequence on the semigroup.

High frequency analysis

Key Assumption: $\boxed{U_s'(y_c) = 0, \quad U_s''(y_c) < 0.}$

One looks for solutions that read $\begin{cases} u(t,x,y) \; = \; i\, e^{i\frac{\omega(\varepsilon)t+x}{\varepsilon}}\, v_\varepsilon'(y), \\ v(t,x,y) \; = \; \varepsilon^{-1}\, e^{i\frac{\omega(\varepsilon)t+x}{\varepsilon}}\, v_\varepsilon(y). \end{cases}$

System:

$$\boxed{\begin{cases} (\omega(\varepsilon)+U_s)v_\varepsilon' \; - \; U_s' v_\varepsilon \; + \; i\varepsilon v_\varepsilon^{(3)} \; = \; 0, \quad y>0, \\ v_\varepsilon|_{y=0} = 0, \quad v_\varepsilon'|_{y=0} = 0. \end{cases}}$$

Remark: Singular perturbation problem in y.

Simpler case: $\varepsilon = 0$ (inviscid version):

$$\begin{cases} (\omega+U_s)v' \; - \; U_s' v \; = \; 0, \quad y>0, \\ v|_{y=0} = 0. \end{cases}$$

One parameter family of eigenelements:

$$\omega = \omega_a \; := \; -U_s(a), \quad v = v_a \; := \; H(y-a)\,(U_s - U_s(a)).$$

Remarks:

- Wether a is a critical point or not, v_a is more or less regular at $y = a$.
- $\omega_a \in \mathbb{R}$: high frequency oscillations $e^{i\frac{\omega_a t}{\varepsilon}}$.

How are these oscillations affected by the singular perturbation $i\varepsilon v_\varepsilon^{(3)}$?

Remark: Similar question for the incompressible limit of the Navier-Stokes equation in bounded domains:

- The high frequency oscillations are the acoustic waves, $e^{i\lambda_k t/\varepsilon}$, $k \in \mathbb{N}$.
- The singular perturbation is the diffusion in Navier-Stokes.

[Desjardins et al 1999]: *Diffusion induces a correction $O(\sqrt{\varepsilon})$ of λ_k, with positive imaginary part*.

Leads to a damping of the waves, with typical time $\sqrt{\varepsilon}$.

Prandtl case: *For $a = y_c$, ω_a undergoes a correction of order $\sqrt{\varepsilon}$, but with negative imaginary part.*

Leads to exponential growth, with typical time $\sqrt{\varepsilon}$.

Ansatz:

- "Eigenvalue": correction of order $\sqrt{\varepsilon}$:

$$\boxed{\omega(\varepsilon) \approx -U_s(y_c) + \sqrt{\varepsilon}\tau}$$

- "Eigenvector": correction has two parts:

  - a "large scale" part, satisfying the equation up to $O(\varepsilon)$, away from $y = y_c$.

  - a "shear layer" part,which compensates for discontinuities at $y = y_c$.

$$\boxed{v_\varepsilon(y) \approx H(y - y_c)\left(U_s(y) - U_s(y_c) + \sqrt{\varepsilon}\tau\right) + \sqrt{\varepsilon} V\left(\frac{y - y_c}{\varepsilon^{1/4}}\right).}$$

Formally: $V = V(z),\ z \in \mathbb{R}$, satisfies:

$$\begin{cases} \left(\tau + U_s''(y_c)\dfrac{z^2}{2}\right) V' \,-\, U_s''(y_c)\, z\, V \,+\, i\, V^{(3)} \,=\, 0, \quad z \neq 0, \\ [V]_{|_{z=0}} \,=\, -\tau, \quad \left[V'\right]_{|_{z=0}} \,=\, 0, \quad \left[V''\right]_{|_{z=0}} = -U''(a), \\ \lim\limits_{\pm\infty} V \,=\, 0. \end{cases}$$

Remark: Too many constraints, so the parameter τ.

Idea: There is a solution (τ, V) with $\mathcal{I}m\, \tau < 0$.

Integrating factor:

$$V(z) \,=\, \left(\tau \,+\, U_s''(y_c)\frac{z^2}{2}\right) W(z) \,-\, \mathbf{1}_{\mathbb{R}_+}(z) \left(\tau + U_s''(y_c)\frac{z^2}{2}\right).$$

Change of variable:

$$\tau \;=\; \frac{1}{\sqrt{2}}|U''(y_c)|^{1/2}\,\tau', \quad z \;=\; 2^{1/4}|U_s''(y_c)|^{-1/4}\,z'.$$

Instability if

(SC) : there is $\tau \in \mathbb{C}$ with $\mathcal{I}m\,\tau < 0$, and a solution W of

$$\boxed{(\tau - z^2)^2\frac{d}{dz}W \;+ i\,\frac{d^3}{dz^3}\Big((\tau - z^2)W\Big) = 0,} \qquad \text{(ODE)}$$

such that $\lim_{z\to-\infty} W = 0$, $\lim_{z\to+\infty} W = 1$.

The spectral condition (SC)

Remark: (ODE) is an equation on $X = W'$:

$$\boxed{i(\tau - z^2)X'' \;-\; 6i\,z\,X' \;+\; \Big((\tau - z^2)^2 - 6i\Big)\,X \;=\; 0.} \qquad \text{(EDO2)}$$

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Step 1: consider an auxiliary eigenvalue problem:

$$Au \;:=\; \frac{1}{z^2+1}u'' \;+\; \frac{6z}{(z^2+1)^2}u' \;+\; \frac{6}{(z^2+1)^2}u \;=\; \alpha\, u$$

Proposition

$A : D(A) \mapsto \mathcal{L}^2$ selfadjoint, with

$$\begin{aligned} D(A) \;&:=\; \left\{ u \in \mathcal{H}^1,\; Au \in \mathcal{L}^2 \right\}, \\ \mathcal{L}^2 \;&:=\; \left\{ u \in L^2_{loc},\; \int_{\mathbb{R}} (z^2+1)^4 |u|^2 < +\infty \right\}, \\ \mathcal{H}^1 \;&:=\; \left\{ u \in H^1_{loc}\; \int_{\mathbb{R}} (z^2+1)^4 |u|^2 \;+\; \int_{\mathbb{R}} (z^2+1)^3 |u'|^2 < +\infty \right\}. \end{aligned}$$

Proposition

A has a positive eigenvalue.

Proof: One has $Au = A_1 u + A_2 u$, with

$$A_1 u := \frac{1}{z^2+1} u'' + \frac{6z}{(z^2+1)^2} u',$$

selfadjoint and negative in $\mathcal{L}^2$, and

$$A_2 u := \frac{6}{(z^2+1)^2} u$$

selfadjoint and A_1-*compact*. So $\Sigma_{ess}(A) = \Sigma_{ess}(A_1) \subset \mathbb{R}_-$.

Moreove, $(Au, u) > 0$ for $u(z) = e^{-2z^2}$.

<u>Change of variable</u>:There is $\tau < 0$, and Y solving

$$\left(\tau - z^2\right) Y\text{"} - 6zY' + ((\tau - z^2)^2 - 6)Y = 0.$$

Step 2:

Proposition

i) Y can be extended into a holomorphic solution in

$$U_\tau := \mathbb{C} \setminus \left(\left[-i\infty, -i|\tau|^{1/2} \right] \cup \left[i|\tau|^{1/2}, +i\infty \right] \right).$$

ii) In the sectors

$$\arg z \in (-\pi/4 + \delta, \pi/4 - \delta), \text{ and } \arg z \in (3\pi/4 + \delta, 5\pi/4 - \delta), \delta > 0,$$

$$|Y(z)| \leq C_\delta \exp(-z^2/4).$$

The proof relies on standard results of complex analysis. In each sector, one has even an asymptotic expansion of the solution as $|z| \to +\infty$.

Allows to consider

$$z := e^{-i\pi/8} z', \quad z' \in \mathbb{R}, \quad \tau := e^{-i\pi/4} \tau', \quad X(z') := Y(z).$$

Yields a solution (τ, X) *of* (EDO2),*with* $\mathcal{I}m\,\tau < 0$*, and* $X \underset{\pm\infty}{\longrightarrow} 0$.

Step 3:

To go from X to W through integration. One must check that $\int_{\mathbb{R}} X \neq 0$.

Reductio ad absurdum: if $\int X = 0$,

$$V := \left(\tau - z^2\right) \int_{-\infty}^{z} X$$

satisfies the energy estimate

$$\mathcal{I}m\,\tau \int_{\mathbb{R}} |V''|^2 = \int_{\mathbb{R}} |V^{(3)}|^2$$

Contradicts $\mathcal{I}m\,\tau < 0$.

Solids in a Navier-Stokes flow

The previous lecture has shown the limitations of the Euler model as regards fluid-solid interaction.

Idea: to consider the Navier-Stokes equations...

...but it raises modeling issues as well !

Example 1: The Stokes paradox

> *An infinite cylinder can not move at constant speed in a Stokes flow.*

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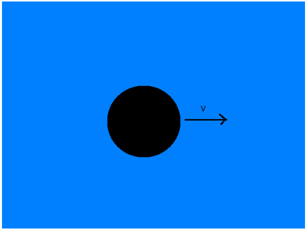

Theorem (Ladyzhenskaya 1969, Heywood 1974)

Let Ω be the exterior of the unit disk, and u be a weak solution of the Stokes equation satisfying

$$u|_{\partial\Omega} = V, \quad \int_{\Omega} |D(u)|^2 < +\infty$$

Then, $u \equiv V$ over Ω

In particular, u does not go to zero at infinity.

Proof: The field $v = u - V$ satisfies

$$\boxed{-\Delta v + \nabla p = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = 0.}$$

Hence,

$$\int_\Omega \nabla v \cdot \nabla \varphi = 0, \quad \forall \varphi \in \mathcal{D}_\sigma(\Omega).$$

But

$$\mathcal{D}_\sigma(\Omega) \quad \text{is dense in} \quad \{v \in \dot{H}^1(\Omega), \quad v|_{\partial\Omega} = 0\}.$$

so that $\int_\Omega |\nabla v|^2 = 0$.

Remarks:

- The density result does not hold in 3d, the same for Stokes paradox.
- The Stokes approximation is not justified: the low Reynolds number limit has no meaning (no typical scale in the problem).
- As soon as the Navier-Stokes flow, or the linear Oseen flow is considered, the paradox does not hold.

Example 2: The non-collision paradox

In a NS flow, rigid bodies sink, but never hit the bottom !

This paradox will be discussed later.

Governing equations

Framework:

- One rigid solid, in a cavity full of an incompressible viscous fluid.
- Both the solid and the fluid are homogeneous.

Cavity: domain Ω of $\mathbb{R}^d$, $d = 2$ or 3:

$$\Omega := \overline{S(t)} \cup F(t)$$

$S(t)$, $F(t)$: solid and fluid subdomains at time t.

- Navier-Stokes equations in $F(t)$:

$$\begin{cases} \rho_F \left(\partial_t u_F + u_F \cdot \nabla u_F\right) - \mu \Delta u_F = -\nabla p + \rho_F f, \\ \operatorname{div} u_F = 0. \end{cases} \tag{NS}$$

- Classical mechanics for the solid.
  - Rigid velocity field:

$$u_S(t,x) \;=\; \dot{x}(t) \;+\; \omega(t) \times (x - x(t))$$

  - Conservation of the linear momentum

$$m_S\,\ddot{x}(t) \;=\; \int_{\partial S(t)} \Sigma n \, d\sigma \;+\; \int_{S(t)} \rho_S \, f,$$

  - Conservation of the angular momentum

$$\frac{d}{dt}\left(J_S(t)\,\dot{\omega}(t)\right) \;=\; \int_{\partial S(t)} (x - x(t)) \times (\Sigma n)\, d\sigma + \int_{S(t)} (x - x(t)) \times \rho_S f$$

<u>Notations:</u> $x(t)$: center of mass, m_S : total mass of the solid,
Σ : stress tensor at the solid surface, J_S : inertial tensor.

$$J_S(t) = \int_{S(t)} |x - x(t)|^2 - (x - x(t) \otimes (x - x(t)))$$

Remark: $J_S(t) \;=\; Q(t) J_S(0) Q(t)^{-1}, \quad Q(t)$: orthogonal matrix.

- Continuity constraints at the fluid solid interface

$$\begin{cases} (\Sigma n)\,|_{\partial S(t)} \;=\; (2\mu D(u) n - p n)\,|_{\partial S(t)} \\ \quad u_F|_{\partial S(t)} \;=\; u_S|_{\partial S(t)} \end{cases}$$

- No slip condition at the boundary.

$$u_F|_{\partial\Omega} = 0.$$

1. Ideal fluids
   - D'Alembert's paradox (1752)
   - Boundary layer theory

2. Viscous fluids
   - Navier-Stokes type models
   - Weak and strong solutions
   - Drag computation and the no-collision paradox

Definitions

Many works on the well-posedness of viscous fluid-solid systems.

Key : *Global variational formulation over* Ω. Let

$$u(t,x) \; := \; u_S(t,x) \text{ si } \; x \in S(t), \quad u_F(t,x) \; \text{ if } \; x \in F(t),$$

$$\rho(t,x) \; = \; \rho_S \mathbf{1}_{S(t)}(x) \; + \; \rho_F \mathbf{1}_{F(t)}(x), \quad \chi^S(t,x) \; = \; \chi_S \mathbf{1}_{S(t)}(x).$$

- Constraints:

$$\nabla \cdot u = 0, \quad u|_{\partial\Omega} = 0, \quad \chi^S \, D(u) = 0. \tag{Co}$$

- Conservation of mass: for all $T > 0$

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \quad \partial_t \chi^S u + \operatorname{div}(\chi^S u) = 0. \tag{CM}$$

- Conservation of momentum in weak form: for all $T > 0$,

$$\int_0^T \int_\Omega \Big(\rho u \cdot \partial_t \varphi + \rho u \otimes u : D(\varphi) - \mu D(u) : D(\varphi) + \rho f \cdot \varphi \Big) dxds + \int_\Omega \rho_0 u_0 \cdot \varphi(0) \; = \; 0, \quad \text{(VF)}$$

for all φ in the test space

$$\mathcal{T} \; = \; \Big\{ \varphi \in \mathcal{D}([0,T) \times \Omega), \quad \nabla \cdot \varphi = 0, \quad \chi^S(t) D(\varphi) = 0, \, \forall t \Big\}$$

Remark: Close of the inhomogeneneous incompressible NS system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p - \mu \Delta u = \rho f, \quad \operatorname{div} u = 0 \end{cases}$$

Main difference: The test space depends on the solution itself.

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Main difference: The test space depends on the solution itself.

Data: $S(0) \Subset \Omega$, $\quad u^0 \in L^2_\sigma(\Omega), f \in L^2_{loc}\left(0, +\infty;\, L^2(\Omega)\right)$.

Definition (weak solution)

A *weak solution* over $(0, T)$, $T > 0$, is a triple (S, F, u) such that :

- $S(t)$ is a connected open set Ω, for all $0 < t < T$, and $F(t) = \Omega \setminus \overline{S(t)}$.
- The field u, and functions ρ, χ^S as above, satisfy

$$u \in L^\infty(0, T;\, L^2(\Omega)) \cap L^2(0, T;\, H^1_0(\Omega)), \quad \rho,\, \chi^S \in L^\infty(0, T \times \Omega)$$

  as well as equations (Co), (VF).
- The following energy inequality holds for a.e. $t \in (0, T)$

$$\frac{1}{2}\int_\Omega \rho(t)|u(t)|^2 + \mu \int_0^t \int_\Omega |\nabla u(s)|^2 ds \leq \frac{1}{2}\int_\Omega \rho_0|u_0|^2 + \int_0^t \rho f(s)\cdot u(s) ds$$

Definition (strong solution)

A *strong solution* over $(0, T)$, $T > 0$, is a weak solution with additional regularity:

$$u \in L^\infty\left(0, T; H_0^1(\Omega)\right) \cap L^2\left(0, T;\ W^{1,p}(\Omega)\right) \text{ for all finite } p,$$
$$\partial_t u \in L^2(0, T;\ L^2(\Omega)).$$

Remark: The situation is similar to the one of Navier-Stokes. Broadly

- Weak solutions are defined globally in time, even after possible collision between the solid and the boundary of the cavity.
- They are unique up to collision in 2d.
- They are not unique after collision (lack of a bouncing law).
- Strong solutions exist locally in time, up to collision in 2d.

Existence of weak solutions

Theorem

There exists a weak solution over $(0, T)$ for all T.

Refs : [Desjardins et al, 1999], [Hoffman et al, 1999], [San Martin et al, 2002], [Feireisl, 2003].

A few ideas from the proof.

Borrows to the inhomogeneous Navier-Stokes. Approximations are constructed by *relaxing the rigidity constraint inside the solid.*

Typically:

$$\begin{cases} \partial_t \rho^n + \mathrm{div}\,(\rho^n u^n) \;=\; 0, \quad \partial_t \chi_S^n \;+\; \mathrm{div}\,(\chi_S^n\, u^n) = 0 \\ \partial_t (\rho^n u^n) \;+\; \ldots \;-\mathrm{div}\;(\mu^n D(u^n)) \;=\; \ldots, \end{cases}$$

with $\mu^n \;:= \mu\,(1 - \chi_S^n) \;+\; n\,\chi_S^n$.

Energy estimates yield standard bounds on ρ^n, u^n, and weak limits ρ, u.

- Strong compactness of (ρ^n):

Follow from DiPerna-Lions results on the transport equation.

$\Rightarrow$ compactness in $C([0, T]; L^p)$ for all finite p.

$\Rightarrow$ the relaxation term yields the rigid constraint of u.

- Strong compactness of (u^n) ?

No control of the time derivative of $\rho^n u^n$, due to the penalized term.

Classical in singular perturbations problems: apply the projector on the kernel of the penalized operator.

Problem: The penalized operator depends on n. Requires some uniformity.

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One needs the Hausdorff convergence of S^n, not only the convergence of the characteristic functions.

$\Rightarrow$ The transport equation on χ_S^n must be modified.

Idea: Instead of transporting S_0 by u^n, one can:

- transport the δ-interior of S_0 by $\rho_\delta \star u^n$
- take the δ-exterior of the transported solid.

At fixed δ: smooth transport field. The Hausdorff convergence holds.

Asymptotically in n, one will have a rigid limit field u^δ over S^δ. Now:

$$\rho_\delta \star u^\delta \; = \; u^\delta \quad \text{in the } \delta-\text{interior of } S.$$

$\Rightarrow$ δ is not harmful !

Back to the strong compactness of u^n: let

$P^s_{S(\tau)}$ the projector in $H^s_\sigma(\Omega)$ on the subspace of all rigid fields over $S(\tau)$,

and $P^{s,*}_{S(\tau)}$ its dual operator.

- One proves, locally around each time τ, some some strong compactness for $\left(P^{s,*}_{S(\tau)}(\rho u^n)\right), \quad s < 1.$
- One shows that $P^s_{S(\tau)}(u^n)$ is "uniformly close" to u^n.

Combining both yields the strong convergence of (u^n).

Related Problem: slip boundary conditions

In link with the no-collision paradox, it can be a good idea *to allow for some slip at the solid boundaries.*

Idea: to replace the *Dirichlet conditions*

$$\boxed{(u_F - u_S)|_{\partial S(t)} \;=\; 0, \quad u_F|_{\partial\Omega} = 0.}$$

by the *Navier conditions*:

- No penetration: $\boxed{(u_F - u_S)\cdot n|_{\partial S(t)} = 0, \quad u_F \cdot n|_{\partial S(t)} = 0}$.
- Tangential stress

$$\boxed{\begin{cases} (u_F - u_S)\times n|_{\partial S(t)} \;=\; -2\;\beta_S\; D(u)n \times n|_{\partial S(t)}, \\ \qquad\qquad u_F \times n|_{\partial\Omega} \;=\; -2\;\beta_\Omega\; D(u)n\times n|_{\partial\Omega}. \end{cases}}$$

$\beta_S, \beta_P > 0$: slip lengths.

Existence of weak solutions ?

The main problem is the discontinuity of u across the fluid-solid interface.

$\Rightarrow$ the global velocity $u \notin H^1$.

$\Rightarrow$ No uniform H^1 bound on approximations u^n.

The same approach as before, based on an analogy with density dependent Navier-Stokes and DiPerna-Lions results, is not available as such.

Recent joint work with M. Hillairet: "Existence of weak solutions up to collision".

Approximate transport equation:

$$\partial_t \chi^{n,S} + \operatorname{div}(u_S^n \chi^{n,S}) = 0, \quad \rho^n := \rho_F(1-\chi_S^n) + \rho_S \chi_S^n.$$

where u_S^n is a rigid velocity field.

Namely, u_S^n is the orthogonal projection of u^n in $L^2(S^n)$ over the space of rigid velocity fields

Remark: The transport equation is nonlinear in the unknown χ_S^n.

Advantages

- Space regularity is not a problem: DiPerna-Lions theory applies
- Hausdorff convergence of S^n will be automatic.

Approximate momentum equation:

$$
\begin{aligned}
&-\int_0^T \int_\Omega \rho^n \left(u^n \partial_t \varphi + v^n \otimes u^n : \nabla\varphi\right) + \int_0^T \int_\Omega 2\mu^n D(u^n) : D(\varphi) \\
&+ \frac{1}{2\beta_S} \int_0^T \int_{\partial S^n(t)} ((u^n - u_S^n) \times \nu) \cdot ((\varphi - \varphi_S^n) \times \nu) \\
&+ \frac{1}{2\beta_\Omega} \int_0^T \int_{\partial\Omega} (u^n \times \nu) \cdot (\varphi \times \nu) + n \int_0^T \int_\Omega \chi_S^n (u^n - u_S^n) \cdot (\varphi - \varphi_S^n) = \dots
\end{aligned}
$$

- $\mu^n := \mu\,(1 - \chi_S^n) + \frac{1}{n^2}\,\chi_S^n$
- New penalization term.
- New jump terms at the boundary, due to the Navier condition
- in the convective term, u^n is replaced by a H^1 field

$$v^n := u_S^n \text{ in } S^n, \quad := u^n \quad \text{outside a } 1/n\text{-neighborhood of } S^n.$$

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The main problem is that u^n, v^n *concentrate at the solid boundary*, as $n \to \infty$.

The same problem holds for continuous test functions φ^n, converging to discontinous test functions φ.

$\Rightarrow$ One has to construct with care v^n, φ^n.

Although there is concentration near ∂S^n, u^n has still H^s uniform bounds for small s. Gives some compactness in space ...

Strong solutions

We restrict to 2d.

Theorem

Under the following regularity assumptions

- $u_0 \in H_0^1(\Omega), \quad \nabla \cdot u_0 = 0, \quad D(u_0) = 0$ in $S(0)$,
- $f \in L^2_{loc}\left(0, +\infty;\, W^{1,\infty}(\Omega)\right)$,
- Ω and $S(0)$ have $C^{1,1}$ boundaries.

there is a maximal T_* and a unique strong solution on $(0, T)$ for all $T < T_*$. Moreover,

a) either $T_* = +\infty$ and $\operatorname{dist}(S(t), \partial\Omega) > 0$, for all t.

b) or $T_* < +\infty$ and $\operatorname{dist}(S(t), \partial\Omega) > 0$, for all $t < T_*$,

$$\lim_{t \to T_*} \operatorname{dist}(S(t), \partial\Omega) \;=\; 0.$$

Refs : Existence : [Desjardins et al, 1999]. Uniqueness : [Takahashi, 2003].

Remark : Important $C^{1,1}$ assumption.

Used in the fluid domain $F(t)$:

$$-\Delta u + \nabla p \;=\; \mathcal{F} \;=\; f - \partial_t u - u \cdot \nabla u, \quad \nabla \cdot u \;=\; 0.$$

Elliptic regularity $L^2 \mapsto H^2$:

$$\begin{aligned}
\int_0^T \int_{F(t)} |\nabla^2 u|^2(t,\cdot) \;&\leq\; C \int_0^T \int_{F(t)} |\mathcal{F}(t,\cdot)|^2 \\
&\leq\; C\Big(\|f\|^2_{L^2 L^2} + \|\partial_t u\|^2_{L^2 L^2} \;+\; \int_0^T \|u\|^2_{L^4} \, \|\nabla u\|^2_{L^4} \Big) \\
&\leq\; C\Big(\|f\|^2_{L^2 L^2} + \|\partial_t u\|^2_{L^2 L^2} \;+\; \|u\|^2_{L^\infty H^1} \, \|\nabla u\|^2_{L^2 L^4} \;<\; +\infty
\end{aligned}$$

for a strong solution u over $(0, T)$. This *a priori* estimate (and gain of regularity) is a key ingredient for both existence and uniqueness.

In link with the no-collision paradox, it can be a good idea *to allow for some more irregular boundaries.*

Theorem (with M. Hillairet)

The result of existence and uniqueness of strong solutions up to collision is true for $C^{1,\alpha}$, $\forall 0 < \alpha \leq 1$.

Problem : The control $H^2(F(t))$ of $u(t,\cdot)$ does not hold anymore.

Idea 1 : $u(t,\cdot) \in H^2(F^\varepsilon(t))$, where

$$F^\varepsilon(t) = \{x \in F(t), \text{dist}(x, S(t)) \geq \varepsilon\}.$$

Remark : Implies that $u|_{F(t)}$ satisfies (NS) a.e.

Idea 2 : $\boxed{\nabla u(t,\cdot) \in \mathrm{BMO}(F(t))}$.

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Remark : Implies that $u|_{F(t)}$ satisfies (NS) a.e.

Idea 2 : $\boxed{\nabla u(t,\cdot) \in \mathrm{BMO}(F(t))}$.

In link with the no-collision paradox, it can be a good idea *to allow for some more irregular boundaries.*

Theorem (with M. Hillairet)

The result of existence and uniqueness of strong solutions up to collision is true for $C^{1,\alpha}$, $\forall 0 < \alpha \leq 1$.

Problem : The control $H^2(F(t))$ of $u(t,\cdot)$ does not hold anymore.

Idea 1 : $u(t,\cdot) \in H^2(F^\varepsilon(t))$, where

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Remark : Implies that $u|_{F(t)}$ satisfies (NS) a.e.

Idea 2 : $\boxed{\nabla u(t,\cdot) \in \mathrm{BMO}(F(t))}$.

Definition

$\mathcal{O}$ bounded open set. $\mathrm{BMO}(\mathcal{O})$ is the set of $f \in L^1(\mathcal{O})$ such that

$$\sup_B \frac{1}{|B|}\int_B |f(x) - \overline{f}_B| dx < +\infty, \quad \overline{f}_B = \frac{1}{|B|}\int_B f(x)dx,$$

where the supremum is taken over all open balls B in $\mathcal{O}$.

We denote

$$||f||_{\mathrm{BMO}(\mathcal{O})} := \sup_B \frac{1}{|B|} \int_B |f(x) - \overline{f}_B| \, dx \text{ (semi-norm)}.$$

Remark : $H^{d/2}(\mathcal{O}) \mapsto \mathrm{BMO}(\mathcal{O})$, $\mathcal{O}$ ouvert de $\mathbb{R}^d$.

Remark : $\|u\|_{L^q} \leq C \|u\|_{L^p}^{\theta} (\|u\|_{\mathrm{BMO}} + \|u\|_{L^1})^{1-\theta}, \quad \frac{1}{q} = \frac{\theta}{p}, \quad \theta \in (0,1).$

Proposition

Let $\mathcal{O}$ a bounded open set $C^{1,\alpha}$, $0 < \alpha \leq 1$. Let

$$F \in L^2(\mathcal{O}) \cap \mathrm{BMO}(\mathcal{O}), \quad g \in L^2(\mathcal{O}) \cap \mathrm{BMO}(\mathcal{O}).$$

Then, the weak solution (u, p) of the Stokes system

$$\begin{cases} -\Delta u + \nabla p \;=\; \operatorname{div} F, \quad x \in \mathcal{O}, \\ \qquad\quad \operatorname{div} u \;=\; g, \quad x \in \mathcal{O}, \\ \qquad\quad u|_{\partial\mathcal{O}} \;=\; 0, \end{cases}$$

satisfies

$$\| (\nabla u, p) \|_{\mathrm{BMO}(\mathcal{O})} \leq C \left(\| (F, g) \|_{\mathrm{BMO}(\mathcal{O})} + \| (F, g) \|_{L^2(O)} \right).$$

<u>Remark:</u> $\mathbb{R}^n$: use the continuity of Riesz transforms over BMO.

Remark : One can also show that $(\nabla u, p)(t, \cdot) \in W^{s,\tau}(F(t))$ for some s, τ with $s > 1/\tau$. Gives a sense to $\Sigma(t, \cdot)|_{\partial S(t)}$ in a strong form.

Proof of the Theorem :

Lagrangian type coordinates (based on the rigid velocity field)

$$x \in F(t) \cup S(t) \xrightarrow[Y(t,\cdot)]{} y \in F(0) \cup S(0).$$

The Navier-Stokes equation becomes

$$\boxed{(\partial_t + M)\, v + N(v) - \mu L v + G p = f, \quad y \in F(0).}$$

M, N, L, G : operators depending on ∇Y.

Analogue change for the other equations.

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M, N, L, G : operators depending on ∇Y.

Analogue change for the other equations.

Idea : As $\text{dist}(S(t), \partial\Omega) \geq 4\varepsilon$: Y can be chosen such that

$Nv = v \cdot \nabla v, \quad Lv = \Delta v, \quad Gp = \nabla p$ in an ε-neighborhood of the solid.

Fixed point argument. Write the previous equation as

$$\partial_t v - \mu\Delta v + \nabla p := \mathcal{F} = \mathcal{F}^\varepsilon(v) + \mathcal{F}(v) + f.$$

$$\begin{aligned} \text{with } \mathcal{F}^\varepsilon(v) &= -(Nv - v \cdot \nabla v) + \mu(L - \Delta)v - (G - \nabla p), \\ \mathcal{F}(v) &= -v \cdot \nabla v - Mv. \end{aligned}$$

The $H^1(F^\varepsilon(0))$ regularity of ∇v allows to control $\mathcal{F}^\varepsilon(v)$.

The regularity $\mathrm{BMO}(F(0))$ of ∇v allows to control $\mathcal{F}(v)$.

Key estimate :

$$\begin{aligned} \|v \cdot \nabla v\|_{L^2} &\leq \|v\|_{L^4} \|\nabla v\|_{L^4} \\ &\leq C\|v\|_{L^2}^{1/2} \|v\|_{H^1}^{1/2} (\|\nabla v\|_{L^2}^{1/2} \|\nabla v\|_{\mathrm{BMO}}^{1/2} + \|\nabla v\|_{L^2}) \end{aligned}$$

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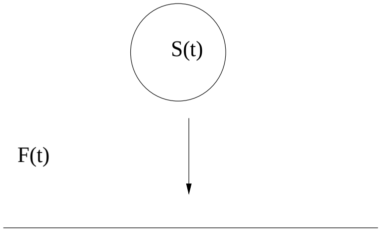
1. Ideal fluids
   - D'Alembert's paradox (1752)
   - Boundary layer theory

2. Viscous fluids
   - Navier-Stokes type models
   - Weak and strong solutions
   - Drag computation and the no-collision paradox

Motivations

One homogeneous rough solid, in a viscous fluid, above a wall.



Fluid and solid at time t : $F(t), S(t)$.

Aim : To describe solid's dynamics near the wall.

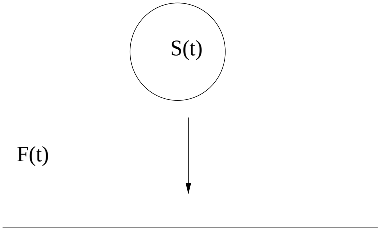
Question : Effect of solid roughness on the drag ?

At least two reasons to wonder about the roughness effect:

Reason 1: The no-collision paradox

Remark: Fluid-solid interaction is full of paradoxes !

Example: Immersed sphere, falling above a wall under the action of gravity.



Question: Will the sphere touch the wall ?

Archimedes ($\sim$ 265 B.C.): *If* $\rho_S > \rho_F$, *collision.*

Relies on the hydrostatic approximation :

$$\text{Stress tensor :} \quad \Sigma := (-p_{atm} - \rho_F \, g \, z) \, I_3.$$

Force on the disk :

$$\boxed{f = -\rho_S g e_z \, |S(t)| + \int_{\partial S(t)} \Sigma n = (\rho_F - \rho_S) \; g \, |S(t)| \, e_z.}$$

Pb : The drag due to molecular pressure and viscosity is neglected.

Refined model: The one we have seen :

- Stokes or Navier-Stokes for the liquid.
- Classical laws of mechanics for the solid.
- *The stress tensor at the solid surfaces includes the newtonian tensor of the fluid.*

Surprise : *In this framework, there is no collision between the sphere and the wall !!*

Refs: Stokes : [Brenner et al, 1963], [Cooley et al, 1969]. NS : [Hillairet, 2007]

Question : What is the flaw of the model ?

Refs : [Davis et al, 1986], [Barnocky et al, 1989], [Smart et al, 1989], [Davis et al, 2003].

Most popular idea:

> *Nothing is as smooth as a sphere. The irregularity of the solid surface can change the solids' dynamics.*

$\Rightarrow$ *Need to compute the drag, notably for rough boundaries.*

Reason 2: Microfluidics

Goal: To make fluids flow through very small devices.

Example: Microchannels with diameter $\sim \mu\mathrm{m}$.

Pb: The Reynolds number is very small.

To minimize (viscous) friction at the walls is crucial.

Many theoretical and experimental works.
Refs : [Tabeling, 2004], [Bocquet, 2007 and 2012], [Vinogradova, 2009 and 2012].

Summary: At such scales, the no-slip condition usually satisfied by a viscous fluid at a wall is not always satisfied. *Some rough surfaces (*hydrophobic*) increase the slip*.

Pb:

- To maximize slip (shape optimization).
- To derive an equivalent macroscopic boundary condition (*wall law*).

Idea [Vinogradova, 2009]

- measure of the drag exerted on a solid that gets closer and closer to the rough surface.
- comparison with the asymptotics predicted by the wall laws.

$\Rightarrow$ *To obtain an approximate expression for the drag, for various models of roughness.*

Main models and results

One rough solid above a rough wall.

$S(t)$: rough sphere. P: rough plane. Fluid: $F(t)$.

We denote $h(t) := \operatorname{dist}(S(t), P)$.

Restriction: the solid translates along a vertical axis.

Remarks: For this constraint to be preserved with time:

- One needs good symmetry properties for the solid and the wall. They will be satisfied in our models.
- The mathematical model must have a good Cauchy theory (uniqueness problem).

Remark: the geometry of the domain in characterized by h:

$$S(t) = S_{h(t)} = h(t)\, e_z + S, \quad F(t) = F_{h(t)},$$

$S_h = h\, e_z + S$, F_h: domains frozen at distance h.

Equations:

- Stokes equations in the fluid: $x \in F(t), t > 0$:

$$\boxed{-\Delta u + \nabla p = 0, \quad \operatorname{div} u = 0.}$$

- Classical mechanics for the solid:

$$\boxed{\ddot{h}(t) = \int_{\partial S(t)} (2D(u)n - pn)\, d\sigma \cdot e_z}$$

n : outward normal, $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^t)$.

Boundary conditions: will have the following general form:

- No penetration: $\boxed{u \cdot n|_P = 0, \quad \left(u - \dot{h}(t)\, e_z\right) \cdot n|_{\partial S(t)} = 0}$.

- Tangential stress

$$\boxed{\begin{cases} u \times n|_P = -2\, \beta_P\, D(u)n \times n|_P, \\ (u - \dot{h}(t)\, e_z) \times n|_{\partial S(t)} = -2\, \beta_S\, D(u)n \times n|_{\partial S(t)}. \end{cases}}$$

$\beta_S, \beta_P \geq 0$: slip lengths.

If $= 0$: no-slip (Dirichlet). If > 0: slip (Navier).

<u>Crucial remark</u>: This system turns into an ODE

$$\ddot{h}(t) \ = \ -\dot{h}(t)\, f_{h(t)}. \tag{ED}$$

with drag

$$\boxed{f_h \ = \ -\int_{\partial S_h} (2D(u_h)n - p_h n) \ d\sigma \cdot e_z}$$

where (u_h, p_h) solution of

$$\begin{cases} -\Delta u_h + \nabla p_h \ = \ 0, \quad \operatorname{div} u_h = 0, \\ u_h \cdot n|_P = 0, \quad (u_h - \ e_z) \cdot n|_{\partial S_h} = 0, \\ u_h \times n|_P \ = \ -2 \ \beta_P \ D(u_h)n \times n|_P \\ (u_h - \ e_z) \times n|_{\partial S_h} \ = \ -2 \ \beta_S \ D(u_h)n \times n|_{\partial S_h} \end{cases} \tag{S}$$

Remark: One can forget about the dynamics.

Goal: Study of f_h, h small, for various models of roughness.

Model 1: Non-smooth surface.

Cylindrical coordinates : (r, θ, z).

- $P : \{z = 0\}$
- S : ball of radius 1, perturbed near the south pole by a $C^{1,\alpha}$ "tip", $0 < \alpha < 1$. Locally, for $r < r_0$:

$$\boxed{z = 1 - \sqrt{1 - r^2} + \varepsilon r^{1+\alpha}}$$

- $\beta_P = \beta_S = 0$.

Remark: With this irregularity, $(\nabla u_h, p_h)$ is not H^1 near the boundary.

But one can show that : $(\nabla u_h, p_h) \in W^{s,\tau}$ for some s, τ with $s > 1/\tau$.
Allows to define f_h.

Model 2: Wall law of Navier type.

- $P : \{z = 0\}$.
- S : ball of radius 1.
- β_P or $\beta_S > 0$.

Model 3: Oscillations of small amplitude and wavelength.

- $P : \{z = \varepsilon\gamma\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)\}$,
  with γ periodic, smooth, ≤ 0, $\gamma(0,0) = 0$.
- S : ball of radius 1.
- $\beta_P = \beta_S = 0$.

Remark: The study is limited to the case $\varepsilon \ll h$.

Remark: Limit case : $\varepsilon \to 0$, $\beta_S, \beta_P \to 0$:

One recovers the well-known case of a sphere and a plane. Cooley-O'Neil, Cox-Brenner:

$$\boxed{f_h \sim \frac{6\pi}{h}, \quad h \to 0.}$$

(which implies no-collision).

Pb: Relies on the computation of the exact solution. Heavy and restricted to simple geometries.

The study of roughness effects requires another approach ...

Proposition (Expression of the drag for model 1):

Let $\beta := \varepsilon h^{\frac{\alpha-1}{2}}$.

- In the regime $h \to 0, \ \beta \to 0$:

$$\boxed{f_h \sim \frac{6\pi}{h}(1 + c\,\beta)} \quad c = c(\alpha) \ \text{explicit.}$$

- In the regime $h \to 0$, $\beta \to \infty$ (and $\varepsilon = O(1)$):
  - If $\alpha > \frac{1}{3}$,

$$\boxed{f_h \sim c\,\varepsilon^{\frac{-4}{1+\alpha}}\, h^{-\frac{3\alpha-1}{\alpha+1}}} \quad c = c(\alpha) \ \text{explicit.}$$

  - If $\alpha = \frac{1}{3}$,

$$\boxed{f_h \sim c\,\varepsilon^{-3}\,|\ln h|} \quad c \ \text{explicit.}$$

  - If $\alpha < \frac{1}{3}$,

$$\boxed{f_h = c\,\varepsilon^{\frac{-2}{1-\alpha}} + O(|\ln \varepsilon|)} \quad c = c(\alpha) \ \text{explicit.}$$

Remarks:

- Collisions are allowed by the model for all $\alpha < 1$. Not allowed for $C^{1,1}$ boundaries.
- The more the boundary is irregular, the less the drag is.
- One recovers the classical result as $\varepsilon = 0$ (with a much simpler proof).

Proposition (Expression of the drag for model 2):

- In the regime $h \to 0$, $\beta_S, \beta_P = O(1)$, with h/β_S or h/β_P uniformly lower bounded, one has

$$\boxed{\frac{c}{h} \leq f_h \leq \frac{C}{h}} \quad c, C > 0.$$

- In the regime $h \to 0$, $\beta_S, \beta_P = O(1)$, with $h/\beta_S \to 0$ and $h/\beta_P \to 0$, one has

$$\boxed{f_h = 2\pi \left(\frac{1}{\beta_S} + \frac{1}{\beta_P} \right) |\ln h| + O\left(\frac{1}{\beta_S} + \frac{1}{\beta_P} \right)}$$

Remark:

- This roughness model also allows for collision, if β_P *and* $\beta_S > 0$.
- Agrees with formal calculations of Hocking (1973)

Proposition (Expression of the drag for model 3):

In the regime $\varepsilon \ll h \ll 1$:

$$\boxed{\frac{6\pi}{h + c\,\varepsilon} + O(|\ln(h+\varepsilon)|) \leq f_h \leq \frac{6\pi}{h} + O(|\ln h|)}$$

Remark: With homogenization techniques, one has

$$\boxed{f_h \sim \frac{6\pi}{h + \alpha\varepsilon}}$$

(if $\varepsilon/h \to 0$ fast enough.)

α explicit, associated to some boundary layer problem.

Sketch of proof

Step 1: Variational characterization of the drag

$$f_h \;=\; \min_{u \in \mathcal{A}_h} \mathcal{E}_h(u) \;=\; \mathcal{E}_h(u_h).$$

for a good energy functional $\mathcal{E}_h$ and a good admissible set $\mathcal{A}_h$.

Dirichlet case (Models 1 and 3): $\mathcal{E}_h(u) \;:=\; \int_{F_h} |\nabla u|^2,$ and

$$\mathcal{A}_h \;:=\; \left\{ u \in H^1_{loc}(F_h), \quad \operatorname{div} u = 0, \quad u|_P = 0, \quad u|_{\partial S_h} = e_z \right\}.$$

Navier case (Model 2):

$$\mathcal{E}_h(u) \;:=\; \int_{F_h} |\nabla u|^2 \;+\; \frac{1}{\beta_P} \int_P |u \times n|^2 \;+\; \left(\frac{1}{\beta_S} + 1 \right) \int_{\partial S_h} |(u - e_z) \times n|^2,$$

$$\mathcal{A}_h \;:=\; \left\{ u \in H^1_{loc}(F_h),\; \operatorname{div} u = 0,\; u \cdot n|_P = (u - e_z) \cdot n|_{\partial S_h} = 0 \right\}.$$

Step 2: Approximate computation of f_h, *via* some relaxed minimization problem.

Rough idea: To find $\tilde{\mathcal{E}}_h \leq \mathcal{E}_h$, and $\tilde{\mathcal{A}}_h \supset \mathcal{A}_h$, such that:

1. $\min_{u \in \tilde{\mathcal{A}}_h} \tilde{\mathcal{E}}_h(u)$ and the associate minimizer can be computed easily.
2. The minimizer $\tilde{u}_h$ belongs to $\mathcal{A}_h$.

It will follow that:

$$\tilde{\mathcal{E}}_h(\tilde{u}_h) \leq f_h \leq \mathcal{E}_h(\tilde{u}_h)$$

If the relaxed pb is close enough to the original one, it will yield a good approximation of the drag.

Remark: this rough idea requires a few adaptations: modification of the minimizer $\tilde{u}_h$ to have it belong to $\mathcal{A}_h$, ...

Remark: The difficulty lies in the choice of the good relaxed problem.

Example: Model 1 ($C^{1,\alpha}$ tip).

Idea: Simplification due to axisymmetry. The minimizer $u = u_h$ reads

$$\boxed{u \;=\; -\partial_z\phi(r,z)e_r \;+\; \frac{1}{r}\partial_r(r\phi)e_z.} \qquad \text{(R)}$$

with $\phi = -\int_0^z u_r$. One restricts to fields in $\mathcal{A}_h$ of the type (R).

Boundary conditions on ϕ:

- Wall:

$$\partial_z\phi(r,0) = 0, \quad \phi(r,0) = 0, \qquad \text{(cl1)}$$

- Near the south pole:

$$\partial_z\phi(r, h + \gamma_\varepsilon(r)) = 0, \quad \phi(r, h + \gamma_\varepsilon(r)) = \frac{r}{2}, \quad r < r_0 \qquad \text{(cl2)}$$

where $\gamma_\varepsilon(r) = 1 - \sqrt{1 - r^2} + \varepsilon r^{1+\alpha}$.

$$\mathcal{E}_h(u) \;=\; \int_{F_h} |\partial_z^2 \phi|^2 \;+\; \int_{F_h} |\partial_{rz}^2 \phi|^2 \;+\; \ldots$$

Idea: The first term is the leading one. Only the zone near $r = 0$ matters.

Relaxed problem:

$$\tilde{\mathcal{A}}_h = \Big\{ u \in H^1_{loc}(F_h), \text{ satisfying (R)-(cl1)-(cl2) } \Big\},$$

$$\tilde{\mathcal{E}}_h(u) \;=\; \int_0^{r_0} \int_0^{\gamma_\varepsilon(r)} |\partial_z^2 \phi|^2 \, dz \, dr$$

1D minimization problems in z, parametrized by r. Minimizer:

$$\tilde{\phi}_h(r,z) = \frac{r}{2} \Phi\Big(\frac{z}{h + \gamma_\varepsilon(r)}\Big), \quad \Phi(t) = t^2(3 - 2t).$$

The minimum for the relaxed problem (lower bound for f_h) is

$$\begin{aligned}\tilde{f}_h &= 12\pi \int_0^1 \frac{r^3 dr}{(h+\gamma_\varepsilon(r))^3}\, dr \\ &= 12\pi \int_0^1 \frac{r^3 dr}{(h+\frac{r^2}{2}+\varepsilon r^{1+\alpha})^3}\, dr \;+\; ... \qquad\qquad = \mathcal{I}(\beta) \;+\; ...\end{aligned}$$

with $\beta := \varepsilon\, h^{\frac{\alpha-1}{2}}$, and

$$\boxed{\mathcal{I}(\beta) := \int_0^{+\infty} \frac{s^3 dr}{(1+\frac{s^2}{2}+\beta s^{1+\alpha})^3}.}$$

Integral with a parameter, the asymptotics of which can be computed in all regimes.

Similar drag computations are available for the other models.

Extension to Navier-Stokes (Dirichlet)

One solid $S(t)$ in a cavity Ω (bounded domains). Fluid: $F(t) := \Omega \setminus \overline{S(t)}$.

- *Navier-Stokes equations in $F(t)$*:

$$\begin{cases} \rho_F \left(\partial_t u_F + u_F \cdot \nabla u_F\right) - \Delta u_F \;=\; -\nabla p \;-\; \rho_F\, g e_z, \\ \operatorname{div} u_F \;=\; 0. \end{cases} \tag{NS}$$

- *Solid mechanics in $S(t)$*:

$$\begin{cases} u_S(t,x) \;=\; U(t) \;+\; \omega(t) \times (x - x(t)), \quad \text{with} \\ m_S\, \dot{U}(t) \;=\; \displaystyle\int_{\partial S(t)} \Sigma n \, d\sigma \;+\; \int_{S(t)} \rho_S\, g e_z, \\ J_S\, \dot{\omega}(t) \;=\; J_S \omega(t) \times \omega(t) + \displaystyle\int_{\partial S(t)} (x - x(t)) \times (\Sigma n)\, d\sigma \\ \;+\; \displaystyle\int_{S(t)} (x - x(t)) \times \rho_S g e_z \end{cases} \tag{MS}$$

- Conditions at the interface :

$$\begin{cases} (\Sigma n)\,|_{\partial S(t)} = (2D(u)n - pn)\,|_{\partial S(t)} - \rho_F g n|_{\partial S(t)} \\ u_F|_{\partial S(t)} = u_S|_{\partial S(t)} \end{cases} \tag{In}$$

- No slip conditions at the boundary of the cavity :

$$u_F|_{\partial\Omega} = 0. \tag{Pa}$$

Dynamics of the solid near $\partial\Omega$

One considers "model 1": $\partial\Omega$ is locally flat, the sphere $S(t)$ has a $C^{1,\alpha}$ tip and is in vertical translation.

Theorem

For any weak solution satisfying the assumptions of model 1, the solid touches the wall in finite time iff $\alpha < 1$.

Remark: Similar results in dimension 2. Collision in finite time iff $\alpha < 1/2$.

Idea for the proof

Choose $\varphi(t,x) = u_{h(t)}(x)$ in the variational formulation.

One has:

$$\boxed{-\mathcal{F}(h(t)) \, + \, (\rho_s - \rho_F)\, g|S(0)|\, t \, = \, R(t)}$$

where

$$\mathcal{F}(h) \, = \, \int_{h_0}^{h} f_{h'}\, dh'.$$

and $R(t)$ is a "remainder", coming from the transport in the Navier-Stokes equation.

Pb: u_h is not available.

Key: Replace u_h by $\tilde{u}_h$, minimizer of the relaxed problem.