Hilbert scales and approximation theory
Lecture notes

The Eigenvalue problem for compact symmetric operators

In the following H denotes an (infinite dimensional) real Hilbert space with scalar product $(.,.)$ and the norm $\|...\|$. We will consider mappings $K: H \to H$. Unless otherwise noticed the standard assumptions on K are:

i) K is symmetric, i.e. for all $x, y \in H$ it holds $(x, Ky) = (x, Ky)$

ii) K is compact, i.e. for any (infinite) sequence $\{x_n\}$ bounded in H contains a subsequence $\{x_{n'}\}$ such that $\{Kx_{n'}\}$ is convergent,

iii) K is injective, i.e. $Kx = 0$ implies $x = 0$.

A first consequence is

Lemma: K is bounded, i.e.

$$\|K\| := \sup_{x \neq 0} \frac{\|Kx\|}{\|x\|} \quad .$$

Lemma: Let K be bounded, and fulfill condition i) from above, but not necessarily the two other condition ii) and iii). Then $\|K\|$ equals

$$N(K) = \sup_{x \neq 0} \frac{|(x, Kx)|}{\|x\|} \quad .$$

Theorem: There exists a countable sequence $\{\lambda_i, \varphi_i\}$of eigenelements and eigenvalues

$K\varphi_i = \lambda_i \varphi_i$ with the properties

i) the eigenelements are pair-wise orthogonal, i.e. $(\varphi_i, \varphi_k) = \delta_{i,k}$

ii) the eigenvalues tend to zero, i.e. $\lim_{i \to \infty} \lambda_i$

iii) the generalized Fourier sums $S_n := \sum_{i=1}^{n} (x, \varphi_i)\varphi_i \to x$ with $n \to \infty$ for all $x \in H$

iv) the Parseval equation

$$\|x\|^2 = \sum_{i}^{\infty} (x, \varphi_i)^2$$

holds for all $x \in H$.

Hilbert Scales

Let H be a (infinite dimensional) Hilbert space with scalar product $(.,.)$, the norm $\|...\|$ and A be a linear operator with the properties

i) A is self-adjoint, positive definite

ii) A^{-1} is compact.

Without loss of generality, possible by multiplying A with a constant, we may assume

$$(x, Ax) \geq \|x\| \qquad \text{for all } x \in D(A)$$

The operator $K = A^{-1}$ has the properties of the previous section. Any eigenelement of K is also an eigenelement of A to the eigenvalues being the inverse of the first. Now by replacing $\lambda_i \to \lambda_i^{-1}$ we have from the previous section

i) there is a countable sequence $\{\lambda_i, \varphi_i\}$ with

$$A\varphi_i = \lambda_i \varphi_i \quad , \quad (\varphi_i, \varphi_k) = \delta_{i,k} \text{ and } \lim_{i\to\infty} \lambda_i$$

ii) any $x \in H$ is represented by

$$(*) \qquad x = \sum_{i=1}^{\infty} (x, \varphi_i)\varphi_i \quad \text{and} \quad \|x\|^2 = \sum_1^{\infty} (x, \varphi_i)^2 .$$

Lemma: Let $x \in D(A)$, then

$$(**) \quad Ax = \sum_{i=1}^{\infty} \lambda_i (x, \varphi_i)\varphi_i \quad , \quad \|Ax\|^2 = \sum_{i=1} \lambda_i^2 (x, \varphi_i)^2 ,$$

$$(Ax, Ay) = \sum_{i=1}^{\infty} \lambda_i^2 (x, \varphi_i)(y, \varphi_i) .$$

Because of (*) there is a one-to-one mapping I of H to the space $\hat{H}$ of infinite sequences of real numbers

$$\hat{H} := \{\hat{x} | \hat{x} = (x_1, x_2, ...)\}$$

defined by

$$\hat{x} = Ix \quad \text{with} \quad x_i = (x, \varphi_i) .$$

If we equip $\hat{H}$ with the norm

$$\|\hat{x}\|^2 = \sum_1^{\infty} (x, \varphi_i)^2$$

then I is an isometry.

By looking at (**) it is reasonable to introduce for non-negative α the weighted inner products

$$(\hat{x},\hat{y})_\alpha = \sum_i^\infty \lambda_i^\alpha (x,\varphi_i)(y,\varphi_i) = \sum_i^\infty \lambda_i^\alpha x_i y_i$$

and the norms

$$\|\hat{x}\|_\alpha^2 = (\hat{x},\hat{x})_\alpha$$

Let $\hat{H}_\alpha$ denote the set of all sequences with finite $\alpha -$ norm. then $\hat{H}_\alpha$ is a Hilbert space. The proof is the same as the standard one for the space l_2.

Similarly one can define the spaces H_α : they consist of those elements $x \in H$ such that $Ix \in \hat{H}_\alpha$ with scalar product

$$(x,y)_\alpha = \sum_i^\infty \lambda_i^\alpha (x,\varphi_i)(y,\varphi_i) = \sum_i^\infty \lambda_i^\alpha x_i y_i$$

and norm

$$\|x\|_\alpha^2 = (x,x)_\alpha .$$

Because of the Parseval identity we have especially

$$(x,y)_0 = (x,y)$$

and because of (**) it holds

$$\|x\|_2^2 = (Ax,Ax)_0 \;,\; H_2 = D(A) .$$

The set $\{H_\alpha | \alpha \geq 0\}$ is called a Hilbert scale. The condition $\alpha \geq 0$ is in our context necessary for the following reasons:

Since the eigen-values λ_i tend to infinity we would have for $\alpha < 0$: $\lim \lambda_i^\alpha \to 0$. Then there exist sequences $\hat{x} = (x_1, x_2, ...)$ with

$$\|\hat{x}\|_2^2 < \infty \;,\; \|\hat{x}\|_0^2 = \infty \;.$$

Because of Bessel's inequality there exists no $x \in H$ with $Ix = \hat{x}$. This difficulty could be overcome by duality arguments which we omit here.

There are certain relations between the spaces $\{H_\alpha | \alpha \geq 0\}$ for different indices:

Lemma: Let $\alpha < \beta$. Then

$$\|x\|_\alpha \leq \|x\|_\beta$$

and the embedding $H_\beta \to H_\alpha$ is compact.

Lemma: Let $\alpha < \beta < \chi$. Then

$$\|x\|_\beta \leq \|x\|_\alpha^\mu \|x\|_\gamma^\nu \text{ for } x \in H_\gamma$$

with $\mu = \frac{\gamma - \beta}{\gamma - \alpha}$ and $\nu = \frac{\beta - \alpha}{\gamma - \alpha}$.

Lemma: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

i) $\|x - y\|_\alpha \leq t^{\beta - \alpha} \|x\|_\beta$

ii) $\|x - y\|_\beta \leq \|x\|_\beta$, $\|y\|_\beta \leq \|x\|_\beta$

iii) $\|y\|_\gamma \leq t^{-(\gamma - \beta)} \|x\|_\beta$.

Corollary: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

i) $\|x - y\|_\rho \leq t^{\beta - \rho} \|x\|_\beta$ for $\alpha \leq \rho \leq \beta$

ii) $\|y\|_\sigma \leq t^{-(\sigma - \beta)} \|x\|_\beta$ for $\beta \leq \sigma \leq \gamma$.

Remark: Our construction of the Hilbert scale is based on the operator A with the two properties i) and ii). The domain $D(A)$ of A equipped with the norm

$$\|Ax\|^2 = \sum_{i=1} \lambda_i^2 (x, \varphi_i)^2$$

turned out to be the space H_2 which is densely and compactly embedded in $H = H_0$. It can be shown that on the contrary to any such pair of Hilbert spaces there is an operator A with the properties i) and ii) such that

$$D(A) = H_2 \quad R(A) = H_0 \text{ and } \|x\|_2 = \|Ax\|.$$

We give three examples of differential operator and singular integral operators, whereby the integral operators are related to each other by partial integration:

Example 1: Let $H = L^2(0,1)$ and

$$Au := -u''$$

with

$$D(A) = \overset{\bullet}{W}{}_2^2(0,1) := \overset{\circ}{W}{}_2^1(0,1) \cap W_2^2(0,1) \,.$$

Building on the orthogonal set of eigenpairs $\{\lambda_i, \varphi_i\}$ of A_i, i.e.

$$-\varphi_i'' = \lambda_i \varphi_i \qquad \varphi_i(0) = \varphi_i(1) = 0$$

it holds the inclusion

$$D(A) \subseteq H_A = H_1 = \overset{\circ}{W}{}_2^1(0,1) \subseteq L^2(0,1) \,.$$

Example 2: Let $H = L_{22}^*(\Gamma)$ with $\Gamma := S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Then H is the space of integrable periodic function in R. Let

$$(Au)(x) := -\oint \log 2\sin\frac{x-y}{2} u(y)dy =: \oint k(x-y)u(y)dy$$

and

$$D(A) = H = L_{22}^*(\Gamma) \,.$$

The Fourier coefficients of this convolution are

$$(Au)_\nu = k_\nu u_\nu = \frac{1}{2|\nu|} u_\nu$$

i.e. it holds $D(A) \subseteq H_A = H_{-1/2}(\Gamma)$.

A relation of this Fourier representation to the fractional function is given by

$$x - [x] - \frac{1}{2} = -\sum_1^\infty \frac{\sin 2\pi\nu x}{\pi\nu}$$

Remark: We give some further background and analysis of the even function

$$k(x) := -\ln\left|2\sin\frac{x}{2}\right| =: -\log\left|2\sin\frac{x}{2}\right| .$$

Consider the model problem

$$-\Delta U = 0 \qquad \text{in } \Omega$$

$$U = f \qquad \text{on } \Gamma := \partial\Omega \ ,$$

whereby the area Ω is simply connected with sufficiently smooth boundary. Let $y = y(s) - s \in (0,1]$ be a parametrization of the boundary $\partial\Omega$. Then for fixed $\bar{z}$ the functions

$$U(\bar{x}) = -\log|\bar{x} - \bar{z}|$$

Are solutions of the Lapace equation and for any $L_1(\partial\Omega) -$ integrable function $u = u(t)$ the function

$$(Au)(\bar{x}) := \oint_{\partial\Omega} \log|\bar{x} - u(t)| dt$$

is a solution of the model problem. In an appropriate Hilbert space H this defines an integral operator ,which is coercive for certain areas Ω and which fulfills the Garding inequality for general areas Ω . We give the Fourier coefficient analysis in case of $H = L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Let $x(s) := (\cos(s), \sin(s))$ be a parametrization of $\Gamma := S^1(R^2)$ then it holds

$$|x(s) - x(t)|^2 = \left|\begin{pmatrix} \cos(s) - \cos(t) \\ \sin(s) - \sin(t) \end{pmatrix}\right|^2 = 2 - 2\cos(s-t) = 2(1 - \cos(2\frac{s-t}{2})) = 2\left[2\sin^2\frac{s-t}{2}\right] = 4\sin^2\frac{s-t}{2}$$

and therefore

$$-\log|x(s) - x(t)| = -\log 2\left|\sin\frac{s-t}{2}\right| = k(s-t) \ .$$

The Fourier coefficients k_ν of the kernel $k(x)$ are calculated as follows

$$k_\nu := \frac{1}{2\pi}\oint k(x) e^{-i\nu x} dx = \frac{1}{2\pi}\int_0^{2\pi} \log\left|2\sin\frac{t}{2}\right| e^{-i\nu t} dt = \frac{2}{2\pi}\int_0^{\pi} \log\left|2\sin\frac{t}{2}\right| \cos(\nu t) dt = k_{-\nu}$$

As $\varepsilon \log 2\sin\frac{\varepsilon}{2} \underset{\varepsilon \to 0}{\to} 0$ partial integration leads to

$$k_\nu = \frac{1}{\nu\pi}\sin(\nu t)\Big|_0^{\pi} - \frac{1}{\nu\pi}\int_0^{\pi} \frac{2\sin(\nu t)\cos\frac{t}{2}}{2\sin\frac{t}{2}} dt = -\frac{1}{\nu\pi}\int_0^{\pi} \frac{\sin(\frac{2\nu+1}{2}t) - \sin(\frac{2\nu-1}{2}t)}{2\sin\frac{t}{2}} dt$$

$$k_\nu = -\frac{1}{\nu\pi}\int_0^{\pi} (\left[\frac{1}{2} + \cos(t) .. + \cos(\nu t)\right] - \left[\frac{1}{2} + \cos(t) .. + \cos((\nu-1)t)\right]) dt = -\frac{1}{\nu} \ .$$

Extension and generalizations

For $t > 0$ we introduce an additional inner product resp. norm by

$$(x, y)_{(t)}^2 = \sum_{i=1} e^{-\sqrt{\lambda_i} t}(x, \varphi_i)(y, \varphi_i)$$

$$\|x\|_{(t)}^2 = (x, x)_{(t)}^2 \quad .$$

Now the factor have exponential decay $e^{-\sqrt{\lambda_i} t}$ instead of a polynomial decay in case of λ_i^α.
Obviously we have

$$\|x\|_{(t)} \leq c(\alpha, t)\|x\|_\alpha \text{ for } x \in H_\alpha$$

with $c(\alpha, t)$ depending only from α and $t > 0$. Thus the $(t)-norm$ is weaker than any $\alpha - norm$. On the other hand any negative norm, i.e. $\|x\|_\alpha$ with $\alpha < 0$, is bounded by the $0 - norm$ and the newly introduced $(t) - norm$. It holds:

Lemma: Let $\alpha > 0$ be fixed. The $\alpha - norm$ of any $x \in H_0$ is bounded by

$$\|x\|_{-\alpha}^2 \leq \delta^{2\alpha}\|x\|_0^2 + e^{t/\delta}\|x\|_{(t)}^2$$

with $\delta > 0$ being arbitrary.

Remark: This inequality is in a certain sense the counterpart of the logarithmic convexity of the $\alpha - norm$, which can be reformulated in the form ($\mu, \nu > 0$, $\mu + \nu > 1$)

$$\|x\|_\beta^2 \leq \nu\varepsilon\|x\|_\gamma^2 + \mu e^{-\nu/\mu}\|x\|_\alpha^2$$

applying Young's inequality to

$$\|x\|_\beta^2 \leq (\|x\|_\alpha^2)^\mu (\|x\|_\gamma^2)^\nu \quad .$$

The counterpart of lemma 4 above is

Lemma: Let $t, \delta > 0$ be fixed. To any $x \in H_0$ there is a $y = y_t(x)$ according to

i) $\|x - y\| \leq \|x\|$

ii) $\|y\|_1 \leq \delta^{-1}\|x\|$

iii) $\|x - y\|_{(t)} \leq e^{-t/\delta}\|x\| \quad .$

Eigenfunctions and Eigendifferentials

Let H *be* a (infinite dimensional) Hilbert space with inner product$(.,.)$, the norm$\|...\|$ and A be a linear self-adjoint, positive definite operator, but we omit the additional assumption, that A^{-1} compact. Then the operator $K = A^{-1}$ does not fulfill the properties leading to a discrete spectrum.

We define a set of projections operators onto closed subspaces of H in the following way:

$$R \to L(H,H)$$

$$\lambda \to E_\lambda := \int_{\lambda_0}^{\lambda} \varphi_\mu (\varphi_\mu, *) d\mu \quad , \quad \mu \in [\lambda_0, \infty) \; ,$$

i.e.

$$dE_\lambda = \varphi_\lambda (\varphi_\lambda, *) d\lambda \; .$$

The spectrum $\sigma(A) \subset C$ of the operator A is the support of the spectral measure dE_λ. The set E_λ fulfills the following properties:

i) E_λ is a projection operator for all $\lambda \in R$

ii) for $\lambda \leq \mu$ it follows $E_\lambda \leq E_\mu$ i.e. $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$

iii) $\lim\limits_{\lambda \to -\infty} E_\lambda = 0$ and $\lim\limits_{\lambda \to \infty} E_\lambda = Id$

iv) $\lim\limits_{\substack{\mu \to \lambda \\ \mu > \lambda}} E_\mu = E_\lambda$.

Proposition: Let E_λ be a set of projection operators with the properties i)-iv) having a compact support $[a,b]$. Let $f : [a,b] \to R$ be a continuous function. Then there exists exactly one Hermitian operator $A_f : H \to H$ with

$$(A_f x, x) = \int_{-\infty}^{\infty} f(\lambda) d(E_\lambda x, x) \; .$$

Symbolically one writes

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda \; .$$

Using the abbreviation

$$\mu_{x,y}(\lambda) := (E_\lambda x, y) \quad , \quad d\mu_{x,y}(\lambda) := d(E_\lambda x, y)$$

one gets

$$(Ax, y) = \int_{-\infty}^{\infty} \lambda d(E_\lambda x, y) = \int_{-\infty}^{\infty} \lambda d\mu_{x,x}(\lambda) \qquad , \quad \|x\|_1^2 = \int_{-\infty}^{\infty} \lambda d\|E_\lambda x\|^2 = \int_{-\infty}^{\infty} \lambda d\mu_{x,x}(\lambda)$$

$$(A^2 x, y) = \int_{-\infty}^{\infty} \lambda^2 d(E_\lambda x, y) = \int_{-\infty}^{\infty} \lambda^2 d\mu_{x,x}(\lambda) \; , \quad \|Ax\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\|E_\lambda x\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\mu_{x,x}(\lambda) \; .$$

The function $\sigma(\lambda) := \|E_\lambda x\|^2$ is called the spectral function of A for the vector x. It has the properties of a distribution function.

It hold the following eigenpair relations

$$A\varphi_i = \lambda_i \varphi_i \qquad A\varphi_\lambda = \lambda\varphi_\lambda \qquad \|\varphi_\lambda\|^2 = \infty \text{ , } (\varphi_\lambda, \varphi_\mu) = \delta(\varphi_\lambda - \varphi_\mu) \text{ .}$$

The φ_λ are not elements of the Hilbert space. The so-called eigendifferentials, which play a key role in quantum mechanics, are built as superposition of such eigenfunctions.

Let I be the interval covering the continuous spectrum of A. We note the following representations:

$$x = \sum_1^\infty (x,\varphi_i)\varphi_i + \int_I \varphi_\mu (\varphi_\mu, x) d\mu \text{ , } \quad Ax = \sum_1^\infty \lambda_i (x,\varphi_i)\varphi_i + \int_I \lambda\varphi_\mu (\varphi_\mu, x) d\mu$$

$$\|x\|^2 = \sum_1^\infty |(x,\varphi_i)|^2 + \int_I |(\varphi_\mu, x)|^2 d\mu \text{ ,}$$

$$\|x\|_1^2 = \sum_1^\infty \lambda_i |(x,\varphi_i)|^2 + \int_I \lambda |(\varphi_\mu, x)|^2 d\mu$$

$$\|x\|_2^2 = \|Ax\|^2 = \sum_1^\infty \lambda_i^2 |(x,\varphi_i)|^2 + \int_I \lambda^2 |(\varphi_\mu, x)|^2 d\mu \text{ .}$$

Example: The location operator Q_x and the momentum operator P_x both have only a continuous spectrum. For positive energies $\lambda \geq 0$ the Schrödinger equation

$$H\varphi_\lambda(x) = \lambda\varphi_\lambda(x)$$

delivers no element of the Hilbert space H, but linear, bounded functional with an underlying domain $M \subset H$ which is dense in H. Only if one builds wave packages out of $\varphi_\lambda(x)$ it results into elements of H. The practical way to find Eigen-differentials is looking for solutions of a distribution equation.