A $H_{1/2}-$ (energy) Hilbert space framework for an integrated electro (kinetic) & magnetic (dynamic) (Coulomb & Lorentz potential) plasma field model

Ischia,

Spiaggia dei Maronti

September 27 – October 4, 2018

Plasma is the fourth state of matter, where from general relativity and quantum theory it is known that all of them are fakes resp. interim specific mathematical model items. An adequate model needs to take into account the *axiom of (quantum) state* (physical states are described by vectors of a separable Hilbert space ***H***) and the *axiom of observables* (each physical observable ***A*** is represented as a linear Hermitian operator A of the state Hilbert space). The corresponding mathematical model and its solutions are governed by the Heisenberg uncertainty inequality. As the observable space needs to support statistical analysis the L_2 −Hilbert space, this Hilbert space needs to be at least a subspace of ***H***. At the same point in time, if plasma is considered as sufficiently collisional, then it can be well-described by fluid-mechanical equations. There is a hierarchy of such hydrodynamic models, where the magnetic field lines (or magneto-vortex lines) at the limit of infinite conductivity is "frozen-in" to the plasma. The "mother of all hydrodynamic models is the *continuity equation* treating observations with macroscopic character, where fluids and gases are considered as continua. The corresponding infinitesimal volume "element" is a volume, which is small compared to the considered overall (volume) space, and large compared to the distances of the molecules. The displacement of such a volume (a fluid particle) then is a not a displacement of a molecule, but the whole volume element containing multiple molecules, whereby in hydrodynamics this fluid is interpreted as a mathematical point.
Our approach below is based on the common Hilbert space framework and the proposed alternative Schrödinger momentum operator (see [BrK] and the references cited there) given by

$$u(x) \to P^*[u](x) := -i\frac{d}{dx}H[u](x) := -i\frac{d}{dx}{}_xH[u](x) = -iH[u_x](x)$$

with domain $H_{1/2} = H_1 + H_1^{\neg}$ and corresponding quantum state Hilbert space $H_{-1/2} = H_0 + H_0^{\neg}$. The Hilbert transform is related to the Laplace equation by the concept of conjugate functions. The corresponding generalization with respect to the Yukawan potential is provided in [DuR].

In quantum mechanics, a boson is a "particle" that follows the Bose-Einstein statistics ("photon gases"). A characteristic of bosons is that their statistics do not restrict the number of them that occupy the same quantum state. All bosons can be brought into the energetically lowest quantum state, where they show the same "collective" behavior. Unlike bosons, two identical fermions cannot occupy the same quantum state. Fermions follow the Fermi statistics (e.g. [AnJ]). With respect to the above extended domain of the Schrödinger momentum operator, we propose to identify H_0 as quantum state space for the fermions (which is compactly embedded into $H_{-1/2}$), and $H_0^{\neg}$ as quantum state space for the bosons. *The "fermions quantum state" Hilbert space H_0 is dense in $H_{-1/2}$ with respect to the $H_{-1/2}$ −norm, while the (orthogonal) "bosons quantum state" Hilbert space $H_0^{\neg}$ is a closed subspace of $H_{-1/2}$, resp. the "mass/energy fermions" Hilbert space H_1 is dense in $H_{1/2}$ with respect to the $H_{1/2}$ −norm, while the "mass/energy bosons" Hilbert space is a closed subspace of $H_{1/2}$.* The concept of "vacuons" (i.e. the vacuum expectation values of scalar fields) in the context of "spontaneous" breakdown of symmetry [HiP] then corresponds to the orthogonal projection $H_{1/2} \to H_1$.

In ([BrK]) a distributional variational Hilbert space framework for Landau type equations is provided being enriched by an additional norm with an "exponential decay" behavior in the form ($t > 0$) (in line with the statistics above) given by

$$(x,y)_{\alpha.(t)} = \sum_k \sigma_k^\alpha e^{-\sqrt{\sigma_k}t}(x,\varphi_k)(y,\varphi_k) \quad , \quad \|x\|_{\alpha.(t)}^2 := (x,x)_{\alpha.(t)} .$$

It is based on appropriately defined eigen-pair solutions of a problem adequate linear operator A with the properties (1) A selfadjoint, positive definite, (2) A^{-1} compact.

An element $x = x_0 + x_0^\neg \in H_{-1/2} = H_0 + H_0^\neg$ with $\|x_0\|_0 = 1$ is governed by the norm of its (observation) subspace H_0 in combination with the norm $|||x|||_{(t)}^2 := (x,x)_{(t)}$ in the form

$$\|x\|_{-1/2}^2 \le \theta\|x\|_0^2 + \sum_{k=1}^\infty e^{1-\sqrt{\sigma_k}\theta} x_k^2 \quad \text{with} \quad \theta := \|x_0^\neg\|_{-1/2}^2 ,$$

which is a special case of the general inequality ($\alpha > 0$ be fixed)

$$\|x\|_{-\alpha}^2 \le \delta^{2\alpha}\|x\|_0^2 + e^{t/\delta}\|x\|_{(t)}^2 .$$

Plasma is an ionized gas consisting of approximately equal numbers of positively charged ions and negatively charged electrons. One of the key differentiator to neutral gas is the fact that its electrically charged particles are strongly influenced by electric and magnetic fields, while neutral gas is not. The continuity equation of ideal magneto-hydrodynamics is given by ([DeR] (4.1))

$$\frac{\partial}{\partial t}\rho + \nabla\cdot(\rho\boldsymbol{v}) = 0$$

with $\rho = \rho(x,t)$ denoting the mass density of the fluid and $\boldsymbol{v}$ denoting the bulk velocity of the macroscopic motion of the fluid. The corresponding microscopic kinetic description of plasma fluids leads to a continuity equation of a system of (plasma) "particles" in a phase space $(\boldsymbol{x},\boldsymbol{v})$ (where $\rho(\boldsymbol{x},t)$ is replaced by a function $f(\boldsymbol{x},\boldsymbol{v},t)$) given by ([DeR] (5.1))

$$\frac{\partial}{\partial t}f + v\cdot\nabla_x f + \frac{dv}{dt}\cdot\nabla_v f + f\frac{\partial}{\partial v}\cdot\frac{dv}{dt} = 0 .$$

In case of a Lorentz force the last term is zero, leading to the so-called collisions-less (kinetic) Vlasov equation ([ShF] (28.1.2)).

In fluid description of plasmas (MHD) one does not consider velocity distributions (e.g. [GuR]). It is about number density, flow velocity and pressure. This is about moment or fluid equations (as NSE and Boltzmann/Landau equations). In [EyG] it is proven that smooth solutions of non-ideal (viscous and resistive) incompressible magneto-hydrodynamic (plasma fluid) equations satisfy a stochastic (conservation) law of flux. It is shown that the magnetic flux through the fixed surface is equal to the average of the magnetic fluxes through the ensemble of surfaces at earlier times for any (unit or general) value of the magnetic Prandtl number. For divergence-free $\vec{z} = (\vec{\mathrm{u}},\vec{B}) \in C([t_0,t_f], C^{k.\alpha}), (\vec{\mathrm{u}}(0),\vec{B}(0)) \in C^{k.\alpha}$ the key inequalities are given by

- unit magnetic Prandtl number:

$$e^{-2\gamma(t_f-t_0)}\left\|\vec{z}(t_f)\right\|_2^2 + 2\int_{t_0}^{t_f} e^{-2\gamma(t-t_0)}[\varepsilon\|\vec{z}(t)\|_2^2 + \mu\|\nabla\vec{z}(t)\|_2^2]dt \le \|\vec{z}(0)\|_2^2$$

- general magnetic Prandtl number (→ stochastic Lundquist formula):

$$e^{-2\gamma(t_f-t_0)}\left\|\vec{B}(t_f)\right\|_2^2 + 2\int_{t_0}^{t_f} e^{-2\gamma(t-t_0)}\left[\varepsilon\left\|\vec{B}(t)\right\|_2^2 + \mu\left\|\nabla\vec{\mathrm{B}}(t)\right\|_2^2\right]dt \le \left\|\vec{B}(0)\right\|_2^2 .$$

The corresponding situation of the fluid flux of an incompressible viscous fluid leads to the Navier-Stokes equations. They are derived from continuum theory of non-polar fluids with three kinds of balance laws: (1) conservation of mass, (2) balance of linear momentum, (3) balance of angular momentum ([GaG]). Usually the momentum balance conditions are expressed on problem adequate "force" formula derived from the Newton formula $= m \cdot \frac{dv}{dt}$. For getting any well-posed (evolution equation) system is it necessary to define its corresponding initial-boundary value conditions.

The NSE are derived from the (Cauchy) stress tensor (resp. the shear viscosity tensor) leading to liquid pressure force $\frac{\partial p}{\partial x_i} = -\frac{\partial T_{ji}}{\partial x_j} + \mu \Delta v_i$. In electrodynamics & kinetic plasma physics the linear resp. the angular momentum laws are linked to the electrostatic (mass "particles", collision, static, quantum mechanics, displacement related; "fermions") Coulomb potential resp. to the magnetic (mass-less "particles", collision-less, dynamic, quantum dynamics, rotation related; "bosons") Lorentz potential.

In [PlJ] a mathematical mass element concept is considered, which replaces the mathematical "mass" object x (real number) by a "differential" object dx. It leads to alternative unit outer normal derivative definition enabling a Newton potential, where a density function is replaced by its differential. This goes along with Plemelj's definition of a double layer potential. From a mathematical point in view this means that a Lebesgue integral is replaced by a Stieltjes integral. This goes along with Plemelj's double layer potential definition. From a physical interpretation perspective it means that Newton's "long distance test particle" defining a potential function is influenced by the enclosed mass elements and no longer by their corresponding density function, only.

When one wants to treat the time-harmonic Maxwell equations with variational methods, one has to face the problem that the natural bilinear form is not coercive on the whole Sobolev space H_1 ([KiA]). On can, however, make it coercive by adding a certain bilinear form on the boundary of the domain (vanishing on a subspace of H_1), which causes a change in the natural boundary conditions ([CoM]).

The mathematical tool to distinguish between unperturbed cold and hot plasma is about the Debye length and Debye sphere ([DeR]). The corresponding interaction (Coulomb) potential of the non-linear Landau damping model is based on the (Poisson) potential equation with corresponding boundary conditions. A combined electro-magnetic plasma field model needs to enable "interaction" of cold and hot plasma "particles", which indicates Neumann problem boundary conditions. The corresponding double layer (hyper-singular integral) potential operator of the Neumann problem is the Prandtl operator $\overline{\mathrm{P}}$, fulfilling the following properties ([LiI] Theorems 4.2.1, 4.2.2, 4.3.2):

i) the Prandtl operator $\overline{\mathrm{P}}: \mathrm{H_r} \to \widehat{\mathrm{H}}_{\mathrm{r}-1}$ is bounded for $0 \leq \mathrm{r} \leq 1$

ii) the Prandtl operator $\overline{\mathrm{P}}: \mathrm{H_r} \to \widehat{\mathrm{H}}_{\mathrm{r}-1}$ is Noetherian for $0 < \mathrm{r} < 1$

iii) for $1/2 \leq \mathrm{r} < 1$, the exterior Neumann problem admits one and only one generalized solution.

Therefore, the Prandtl operator enables a combined (conservation of mass & (linear & angular) momentum balances) integral equations system, where the two momentum balances systems are modelled by corresponding momentum operator equations with corresponding domains according to $H_{1/2} = H_1 \times H_1^{\neg}$. For a correspondingly considered variational representation (e.g. for the (Neumann) potential equation or the corresponding Stokes equation) it requires a less regular Hilbert space framework than in standard theory. Basically, domain $H_1$of the standard (Dirichlet integral based) "energy" (semi) inner product $\mathrm{a(u,v)} = (\nabla u, \nabla v)$ is extended to $H_{1/2}$ with a corresponding alternative (semi) inner product in the form $a(u,v)=(\nabla u,\nabla v)_{-1/2}=(u,v)_{1/2}$. It enables e.g. the method of Noble ([VeW] 6.2.4), [ArA] 4.2), which is about two properly defined operator equations, to analyze (nonlinear) complementary extremal problems. The Noble method leads to a “Hamiltonian” function $W(\cdot,\cdot)$ which combines the pair of underlying operator equations (based on the “Gateaux derivative” concept)

$$Tu=\frac{\partial W(\acute{u},u)}{\partial \acute{u}} \quad , \quad T^{*}\acute{u}=\frac{\partial W(\acute{u},u)}{\partial u} \quad u\in E=H_{1/2} \quad , \quad \acute{u}\in \acute{E}=H_{-1/2}.$$

The proposed (Hilbert space based) model provides a truly infinitesimal geometry. Variational integrators were originally developed for geometric time integration, particularly to simulate dynamical systems in Lagrange mechanics. In [StA] the concept of (electromagnetic) “variational integrators for the Maxwell equations with sources”, including free sources of charge and current in non-dissipative media, is provided in the framework of differential form. For the correspondence of the inner products of the distributional Hilbert spaces $H_{-1/2}$, H_{-1} in the context of a proper ground state energy model we refer to the references cited in [BrK].

The alternative Hilbert space framework can also be applied for correspondingly modified Maxwell equations, which govern the electromagnetic field for given distributions of the electric charges and currents. The laws how those charges and currents behave are unknown. What’s known, is, that electricity exists within the elementary particles (electron, positron), but the appropriate mathematical model, which is consistent with the Maxwell equations and the related Einstein field equations, is still missing. Only the energy tensor of electromagnetic fields *outside* of elementary particles is known. Modelling the elementary particles as singularities should be considered as interim “solution”, only, as well as applying hydrodynamic equations (the classical mechanics approach) to describe “matter” by terms like the density of the ponderable substance (and the corresponding “Ruhemasse”) and hydrodynamic pressure forces (area forces). The alternative Hilbert space framework preserves the electrostatics and magnetostatics equations ([ArA] 3.6, 3.7), while replacing Maxwell’s electromagnetic “mass density / flux/flow” density concept (enabled by the Maxwell displacement current (density) concept to extend the Ampere law to the Ampere-Maxwell law) by Plemelj’s “mass element / flux/flow strength” concept ([PlJ] §8). The latter one is enabled by Plemelj’s alternative normal derivative concept ([PlJ] §5: space dimension n=2 & logarithmic potential), where du/dn is replaced by $\bar{u}(\sigma):=-\int_{\sigma_0}^{\sigma}\frac{du}{dn}ds$ ($\bar{u}(\sigma)$ denotes the conjugate potential to u). The latter one might be well defined, while the standard normal derivative might not be defined. We emphasis that $\bar{u}(\sigma)$ is defined with purely boundary values, i.e. it requires no information about the interior or exterior domain of a related vector field. For the generalization of the Cauchy-Riemann equations to space dimension n=3 and related translation or rotation groups we refer to [StE2]. The corresponding physical interpretations are about “source density” or “invertebrate density/rotation” with its related mathematical formulas $rot(u)=rot(\pm grad\varphi)=0$ or $div(rot(u))=0$ (which are the 3-space interpretations of the Poincare lemma $d(d\omega))=0$). With respect to the below we also mention that the replacement of the displacement current concept avoids to “calibration (Eichung)” need to ensure well-posed PDE systems. As Plemelj’s flow strength definition requires only information from the boundary/surface it can be applied to both, the Gaussian law (based on normal directions to the surface) and the Stokes law (based on tangential directions to the surface). This enables a new “double layer potential***s***” concept with two different potentials on each side of the double layer of the boundary/surface. The corresponding electrostatic and magnetostatic systems are linked by common flow strength values at each point of the surface.

In [CoM] a coercive bilinear form of the Maxwell equations in combination with appropriate boundary conditions is provided. The underlying "energy" inner product is based on the standard (energy) Hilbert space $H_1 x H_1$. We propose the extended domain $H_{1/2} x H_{1/2}$ in combination with a "boundary bilinear form" (based on the Prandtl operator T above with $r = 1/2$) in the form

$$a(u,v) := (divu, divv)_{-1/2} + \langle \overline{P}u, \overline{P}v \rangle_{-1/2} + (curlu, curlv)_{-1/2} \quad \forall u, v \in \mathring{H}_{1/2}$$

In [BrK] the Leray-Hopf (Helmholtz-Weyl) operator $P : H_\beta \to H_\beta^\sigma$ is considered (whereby H_β^σ denotes the divergence-free (solenoidal) H_β). It is linked to the tensor product of the Riesz operators $Q := RxR$ (which is selfadjoint and a projection operator, i.e. $Q = Q^2$) and the curl operator by the following identities ([LeN]): $P + Q = Id$, $curl = curl^*$, $curl^2 = -\Delta P$, $[P, curl]$=0, $Pcurl = curlP = curl$, and P is also a projection operator (i.e. $P = P^2$), if $divu = 0$.

The physical interpretation is as follows: The "energy" space $\mathring{H}_{1/2} = \mathrm{H}_1 \times \mathrm{H}_1^{\neg}$ is built on the "charged electrical particles" Hilbert space H_1 and its related distributions (which is dense in $\mathrm{H}_{1/2}$ with respect to the $\mathring{H}_{1/2} -$ norm), and the "orthogonal" Hilbert space $\mathrm{H}_1^{\neg}$ generating those "charged electrical particles" (which is a closed subspace of $\mathrm{H}_{1/2}$). The Hilbert space $\mathrm{H}_1^{\neg}$ "acts" on the "particles" in H_1 (governed by the Prandtl operator), but also "binary collisions" in H_1 acts on $\mathrm{H}_1^{\neg}$, giving "back" energy into the Hilbert space $\mathrm{H}_1^{\neg}$, w/o affecting corresponding (statistical) distribution function in H_1. Without additional energy from "outside" the "system" the probability, that two "particles" collide, is zero.

The modified Maxwell equations are proposed as non-standard model of elementary particles (NMEP). Electromagnetic waves propagation in vacuum can be described by the source-free Maxwell equations w/o specifying anything about charges or currents that might have produced them. It provides an alternative model for spontaneous symmetry breakdown with massless bosons ([HiP]). In [WeP], [WeP1], self-adjoint extensions and spectral properties of the Laplace operator with respect to electric and magnetic boundary conditions are provided.

The ("Pythagoras") split of the newly proposed energy norm $\|x\|_{1/2}^2 = \|x_0\|_{H_0}^2 + \|x_0^{\neg}\|_{H_0^{\neg}}^2$ goes along with two corresponding groups of transformations, the group of translations and the group of rotations. The corresponding theory of generalized Cauchy-Riemann equations are given in [StE1] III, 4.2, and [StE2]. There are basically two characterizations of all possible generalizations of the Cauchy-Riemann equations: (1) the existence of a harmonic function H on R_+^{n+1} so that $u_j = \frac{\partial}{\partial x_j} H$, $j = 0,1,2, \dots n$; (2) rotation group theory based, building on the (complex unitary) vector space of symmetric tensors of $2x2$ matrices. The latter one includes the "electron equation of Dirac" in the case w/o external forces, with zero mass, and independent of time ([StE2]).

The same (extended Maxwell equations, combined transformation group for kinetic and potential energy interaction) concept as above can be applied to the Einstein field equations. In this context we note that the gravity "force" is an only attractive one. At the same point in time the magnetic field of the earth is the result of the gravity "force". The electromagnetic waves propagation in vacuum of the Maxwell equations corresponds to the Weyl (curvature) tensor, while the Ricci flow equation governs the evolution of a given metric to an Einstein metric ([AnM]). The Einstein vacuum field equations allow wave solutions describing a propagating gravitation field in a geometrical Minkowski space. The wave front of this gravitation field, which is the boundary of the curved and the plane space propagates with light speed.

The distributional Hilbert space framework above is also proposed as an appropriate framework for Wheeler's geometrodynamics ([WhJ]), which is about the attempt to describe space-time and associated phenomena completely in terms of geometry. It especially would enable a consistent model for "graviton" quanta dynamics regarding gravitation waves in Einstein's vacuum field equations and corresponding "graviton" quanta properties in quantum field theory.

A Hilbert space based variational representation of the Einstein field equations goes along with a geometric structure on a 3-manifold, which is a complete, locally homogeneous Riemannian metric g.

A geometric 3-manifold (i.e. a 3-manifold admitting a geometric structure) admits eight simply connected geometries with compact quotients G/H. Those eight geometric structures are rigid in that there are no geometries which interpolate continuously between them. Among them the constant curvature geometries H^3 and S^3 are by far the most important to understand (in terms of characterizing which manifolds are geometric). The stationary points of the volume-normalized Ricci flow are exactly the class of Einstein metrics, i.e. metrics of constant Ricci curvature. Einstein metrics are of constant curvature and so give the geometries H^3, R^3 and S^3 geometries. The Ricci flow is a non-linear heat-type equation for g_{ij} ([AnM]).

The Ricci curvature is a symmetric bilinear form, as it is the metric. In [GöK] a 4-D space S is provided, where matter everywhere rotates relative to the compass of inertia with the angular velocity with the properties: (1) S is homogeneous, (2) there exists a one-parametric group of transformations of S into itself which carries each world line of matter into itself, so that any two world lines of matter are equidistant, (3) S has rotational symmetry, (4) a positive direction of time can consistently be introduced in the whole solution.

With respect to long term stability questions we note, that both, Euler and NS equations, with smooth initial data possess unique solutions which stay smooth forever in case of space dimension n=2. For 3D-NSE the question of global existence of smooth solution vs. finite time blow up is one of the Clay Institute Millennium problems. Having in mind the setting $y(t) := \|v(t)\|^2$ the ODE $\frac{\partial}{\partial t}y(t) = y^2(t)$, $y(0) = y_0$, shows the solution $y(t) = \frac{y_0}{1-t\cdot y_0}$, which, for some initial data $y_0 > 0$, becomes infinite in finite times. An $H_{-1/2}$ inner product based variational representation of the 3D-NSE enjoys a corresponding (evolution) ODE in the form $\frac{\partial}{\partial t}y(t) = c_{\|v(t)\|_1} \cdot y^{1/2}(t)$ ([BrK4]).

We note that for an initial value function $a_0 \in L_\infty(R^n)$ there is an unique local in time solution for the NSE with

$$p = \sum_{i,j=1}^{n} R_i R_j v_i v_j \quad , \quad R_i \text{ denotes the Riesz operator.}$$

In [GiY] it is shown that in R^2 this solution can be extended globally in time.

In [HeJ] it is shown for existing solutions of the NSE, for which the Dirichlet norm $a(v,v) = (\nabla v, \nabla v)$ with domain H_1 of the velocity is continuous as $t = 0$, this is not the case for the corresponding normalized L_2norm of the pressure. At the same point in time, the pressure can be characterized as solution of a Neumann problem by formally operating with "div" on both sides of the NSE ([GaG]). If follows that the prescription of the pressure p at the bounding walls or at the initial time independently of v, could be incompatible with the initial-boundary data from the NSE system, and, therefore, could render the problem ill-posed. Both cases above supports the proposed extended Dirichlet norm of the Hilbert space $H_{1/2}$.

Nonlinear evolution equations are also analyzed in a Schauder/Lipschitz function space framework. The space of Lipschitz continuous functions is defined by the norm ([StE] V.4, proposition 6)

$$||f||_{\alpha} := \|f\|_{L_\infty} + sup_{|t|>0} \frac{\|f(x+t)-f(x)\|_{L_\infty}}{|t|^{\alpha}} \quad (0 < \alpha < 1).$$

Proposition 6: Every Lipschitz continuous function may be modified on a set of measure zero so that it becomes continuous.

We note that the spaces $C^{k.\alpha}$ are compactly embedded on C^0 and there is a general principle bounding the norm in $C^{k.\alpha}$ of a linear projection operator by means of the norm of in C^0 ([NiJ], [NiJ1]). Hölder- resp. Lipschitz spaces are the adequate ones in treating nonlinear elliptic problems ([NiJ]). Hölder space based Cauchy problem solution(s) of nonlinear evolutions equations are considered in [HöK]. Integral inequalities for the Hilbert transform applied to a nonlocal transport equation are provided in [CoA].

The Gromov compactness theorem (in the context of the study of the deformation and degeneration of general Riemann metrics with merely bounded curvature in place of constant curvature) is based on $C^{1.\alpha}$ topology ([AnM]).

References

[AnJ] Annett J. F., Superconductivity, Superfluids and condensates, Oxford University Press, Oxford, 2004

[AnM] Anderson M. T., Geometrization of 3-Manifolds via the Ricci Flow, Notices of the AMS, Vol 51, No 2, 2004, 184-193

[ArA] Arthurs A. M., Complementary Variational Principles, Clarendon Press, Oxford, 1970

[AzA] Aziz A. K., Kellogg R. B., Finite Element Analysis of a Scattering Problem, Math. Comp. Vol. 37, No 156, (1981) 261-272

[CoM] Constabel M., A Coercive Bilinear Form of Maxwell's Equations, J. Math. Anal. And Appl., Vol 157, No 2, p. 527-541, 1991

[DeR] Dendy R. O., Plasma Dynamics, Oxford Science Publications, Oxford, 1990

[DuR] Duffin R. J., Hilbert Transforms in Yukawan Potential Theory, Proc. Nat. Acad. Sci., Vol 69, No 12, pp. 3677-3679, 1972

[EyG] Eyink G. L., Stochastic Line-Motion and Stochastic Conservation Laws for Non-Ideal Hydrodynamic Models. I. Incompressible Fluids and Isotropic Transport Coefficients, arXiv:0812.0153v1, 30 Nov 2008

[GaG] Galdi G. P., The Navier-Stokes-equations: A Mathematical Analysis, http://www.navier-stokes-equations.com

[GöK] Gödel K., An Example of a New Type of Cosmological Solutions of Einstein's Field Equations of Gravitation, Rev. Modern Physics, Vol 21, No 3, 1949, 447-449

[GuR] Gundersen R. M., Linearized Analysis of One-Dimensional Magnetohydrodynamic Flows, Springer Tracts in Natural Philosophy, Vol 1, Berlin, Göttingen, Heidelberg, New York, 1964

[HeJ] Heywood J. G., Walsh O. D., A counter-example concerning the pressure in the Navier-Stokes Equations, as $t \to 0^+$

[HiP] Higgs P. W., Spontaneous Symmetry Breakdown without Massless Bosons, Physical Review, Vol. 145, No 4, 1966

[HöK] Höllig K., Linear parabolic equations with a singular lower order coefficient, University of Winsconsin – Madison Mathematics Research Center, Technical Summary Report #2453, Nov. 1982

[KiA] Kirsch A., Hettlich F., The Mathematical Theory of Time-Harmonic Maxwell's Equations, Springer, Cham, Heidelberg, New York, Dordrecht, London

[LeN] Lerner N., A Note on the Oseen Kernels, A. Bove et. Al. (eds.), Advances in Phase Space Analysis of Partial Differential Equations, DOI 10.1007/978-0-8161-9_10, Birkhäuser Verlag, Boston, 2009

[LiI] Lifanov I. K.,Poltavskii L. N., Vainikko G. M., Hypersingular Integral Equations And Their Applications, Chapman & Hall/CRC, Boca Raton, London, New York, Washington D. C., 2004

[NiJ] Nitsche J. A., Schauder estimates for finite element approximations on second order elliptic boundary value problems, Proc. of the Special Year in Numerical Analysis (I. Babuska, T.-P. Liu and J. Osborn, eds.), Lecture Notes 20, Univ. of Maryland, 1981, pp. 290-343

[NiJ1] Nitsche J. A., Direct Proofs of Some Unusual Shift-Theorems, Anal. Math. Appl., Gauthier-Villars, Montrouge, pp.383-400, 1988

[PlJ] Plemelj J., Potentialtheoretische Untersuchungen, Teubner Verlag, Leipzig, 1911

[ShF] Shu F. H., Gas Dynamics, Vol II, University Science Books, Sausalito, California, 1992

[StA] Stern A., Tong Y., Desbrun M., Marsden J. E., Variational Integrators for Maxwell's Equations with Sources, PIERS ONLINE, Vol. 4, No. 7, 2008

[StE] Stein E. M., Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, New Jersey, 1970

[StE1] Stein E. M., Harmonic Analysis, Princeton University Press, Princeton, New Jersey, 1993

[StE2] Stein E. M., Weiss G., Generalization of the Cauchy-Riemann equations and representations of the rotation group, Amer. J. Math., 90 (1968), 163-196

[VeW] Velte W., Direkte Methoden der Variationsrechnung, Teubner Studienbücher, B. G. Teubner, Stuttgart, 1976

[WhJ] Wheeler J., A., Geometrodynamics, Academic Press, New York, London, 1962

[WeP] Werner P., Self-Adjoint Extensions of the Laplace Operator with Respect to Electric and Magnetic Boundary Conditions, J. Math. Anal. 70 (1979) 131-160

[WeP1] Werner P., Spectral Properties of the Laplace Operator with respect to Electric and Magnetic Boundary Conditions, J. Math. Anal. 92 (1983) 1-65