Global Existence and Uniqueness

of the Non-stationary 3D-Navier-Stokes

Initial-boundary Value Problem

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Abstract

Based on the Hilbert scale $H(\alpha)$, $\alpha \in R$, defined by the orthogonal set of eigenpairs of the Stokes operator we present a global unique weak $H(-1/2)-$solution of the generalized 3D Navier-Stokes initial value problem

$$(\dot{u}, v)_{-1/2} + (Au, v)_{-1/2} + (Bu, v)_{-1/2} = 0 \qquad \text{for all } v \in H_{-1/2}$$
$$(u(0), v)_{-1/2} = (u_0, v)_{-1/2} .$$

The global boundedness is a consequence of the Sobolevskii -estimate of the non-linear term ([SoP]) enabling the generalized energy inequality

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \le |(Bu,u)_{-1/2}| \le c \cdot \|u\|_{-1/2} \|u\|_1^2 \cdot$$

Putting $y(t) := \|u\|_{-1/2}^2$ one gets

$$y'(t) \le c \cdot \|u\|_1^2 \cdot y^{1/2}(t)$$

resulting into the a priori estimate

$$\|u(t)\|_{-1/2} \le \|u_0\|_{-1/2} + \int_0^t \|u\|_1^2 (s) ds \le c \left\{ \|u_0\|_{-1/2} + \|u_0\|_0^2 \right\} \quad .$$

§1 Introduction

This section is a summary from [CaM], [GaG], [GiY], [ShM] , [SoH], [TeR].

The Navier-Stokes Equations (NSE) describes a flow of incompressible, viscous fluid. The three key foundational questions of every PDE is existence, and uniqueness of solutions, as well as whether solutions corresponding to smooth initial data can develop singularities in finite time, and what these might mean. For the NSE satisfactory answers to those questions are available in two dimensions, i.e. 2D-NSE with smooth initial data possesses unique solutions which stay smooth forever. In three dimensions, those questions are still open. Only local existence and uniqueness results are known. Global existence of strong solutions has been proven only, when initial and external forces data are sufficiently smooth. Uniqueness and regularity of non-local Leray-Hopf solutions are still open problems.

Basically the existence of 3D solutions is proven only for “large” Banach spaces. The uniqueness is proven only in “small” Banach spaces. The question of global existence of smooth solutions vs. finite time blow up is one of the Clay Institute millennium problems.

The existence of weak solutions can be provided, essentially by the energy inequality. If solutions would be classical ones, it is possible to prove their uniqueness. On the other side for existing weak solutions it is not clear that the derivatives appearing in the inequalities have any meaning. Basically all existence proofs of weak solutions of the Navier-Stokes equations are given as limit (in the corresponding weak topology) of existing approximation solutions built on finite dimensional approximation spaces. The approximations are basically built by the Galerkin-Ritz method, whereby the approximation spaces are e.g. built on eigenfunctions of the Stokes operator or generalized Fourier series approximations.

It has been questioned whether the NSE really describes general flows: The difficulty with ideal fluids, and the source of the d'Alembert paradox, is that in such fluids there are no frictional forces. Two neighboring portions of an ideal fluid can move at different velocities without rubbing on each other, provided they are separated by a streamline. It is clear that such a phenomenon can never occur in a real fluid, and the question is how frictional forces can be introduced into a model of a fluid.

The question intimately related to the uniqueness problem is the regularity of the solution. Do the solutions to the NSE blow-up in finite time? The solution is initially regular and unique, but at the instant T when it ceases to be unique (if such an instant exists), the regularity could also be lost. Given a smooth datum at time zero, will the solution of the NSE continue to be smooth and unique for all time?

There is no uniqueness proof for weak solutions except for over small time intervals. The simplest possible model example how a singularity can appear, is the ODE

$$y'(t) = y^2(t)\,,\ \ y(0) = y_0$$

with the solution

$$y(t) = \frac{y_0}{1 - t \cdot y_0}$$

which becomes infinite in finite time.

Let P denote the orthogonal projection onto the kernel of the divergence operator and A the selfadjoint Stokes operator. If u is divergence free then

$$(Bv,v):=((u,\nabla)v,v)_0:=\iint_\Omega (u,\nabla)v\cdot vdx=0\,.$$

Therefore there is no contribution of the essential non-linear NSE term Bu to the a priori (energy) estimate

$$\|u(t)\|_0^2+2\int_0^t\|u\|_1^2(\tau)d\tau=\|u_0\|_0^2\,.$$

Especially there is no evidence that the energy norm $\|u(t)\|_1$ is bounded in case of space dimension n=3. This is due to the fact that the right hand side of the corresponding a priori estimate

$$\frac{1}{2}\frac{d}{dt}\|u\|_{2\sigma}^2+\|u\|_{2\sigma+11}^2\leq\left|(A^{\sigma-1/2}Bu,A^{\sigma+1/2}u)\right|\quad,\quad 0<\sigma\leq 1/2$$

depends on the space dimension. It holds

$$(v\cdot\nabla v,P\Delta v)\leq C\|v\|_q^{2q/(q-n)}|v|_{1,2}^2+\frac{\nu}{2}\|P\Delta v\|_2^2\quad\text{for all}\quad q\in(n,\infty)\quad.$$

The Gronwall inequality implies global boundedness in case of $n=2$.

For $n\geq 3$ there is up to now no global boundedness result.

§2 Stokes operator, Hilbert scales and a priori estimates

The content of this section is basically taken from [GiY]. With c we denote numeric constants which may have different values at different places.

Let P be the orthogonal projection operator of $(L_2(\Omega))^n$ onto the divergence free vector field H_ω consisting of all solenoidal vector functions u, i.e. the operator is an orthogonal projection onto the kernel of the divergence operator. It is a Pseudo-Differential operator (PDO) of degree zero [(EsG]). The Stokes operator A is a selfadjoint operator in H_σ, being the Friedrichs extension of the non-negative symmetric operator $-P\Delta$ in H_ω defined for all $u \in C^2$ with $divu = 0$ and $u_{n|\partial\Omega} = 0$. The Stokes operator enables the definition of a related Hilbert scale ($\alpha \in R$) with corresponding norm ([NiJ])

$$\|u\|_\alpha := \|A^{\alpha/2}u\| \quad .$$

Throughout this paper, if not explicitly mentioned, we assume $p = 2$ and $n = 3$ for $(L_p(\Omega))^n$.

Using the Stokes operator and its related Hilbert scale framework the Navier-Stokes equations can be represented as an evolution equation in H_0 . Since $P(gradp) = 0$ one gets

$$Au = Pf \quad \text{in } H_0$$

.

Putting $B(u) := P(u, grad)u)$ and assuming $Pu_0 = u_0$ the NSE initial-boundary equation is given by

(*) $$\frac{du}{dt} + Au + Bu = Pf \ , \ u(0) = u_0 \cdot$$

As u is divergence free and $u \cdot v$ identically vanishes on $\partial\Omega$ one gets

$$b(u,v,w) := ((u, grad)v, w) = \iint_\Omega (u, grad)v \cdot wdx = -b(u,w,v)$$

and especially $b(u,v,v) = 0$.

The linear homogeneous part of (*) enables the definition of a semigroup generated by the Stokes operator which leads to the representations

$$u(t) = e^{-tu_0} + \int_0^t e^{-(t-s)A} Pf(s)ds + \int_0^t e^{-(t-s)A} Bu(s)ds \cdot$$

Multiply u to both sides of the homogeneous equation of (*) and integrate over Ω one gets

$$(\dot{u}, u) + (Au, u) + (Bu, u) = 0 \ , \ u(x,0) = u_0 \cdot$$

This yields

$$\frac{d}{dt}\|u\|^2 + 2\|A^{1/2}u\|^2 = 0$$

respectively integrating over $[0,t]$

$$\|u(t)\|_0^2 + 2\int_0^t \|u\|_1^2(\tau)d\tau = \|u_0\|_0^2 \cdot$$

A priori estimates for higher space derivatives of u are derived by multiplying the homogeneous equation (*) with $A^{2\sigma}u$ leading to

$$(\frac{du}{dt}, A^{2\sigma}u) + (Au, A^{2\sigma}u) + (Bu, A^{2\sigma}u) = 0$$

resp.

$$\frac{1}{2}\frac{d}{dt}\left\|A^{\sigma}u\right\|^2 + \left\|A^{\sigma+1/2}u\right\| + (A^{\sigma-1/2}Bu, A^{\sigma+1/2}u) = 0 \cdot$$

Estimates of $A^{\sigma-1/2}Bu$ depend (heavily!) on the space dimension n, which is the root cause for the "still open NSE questions" for the space dimension $n=3$. The essential a priori estimates are e.g. given in [GiY] (see also appendix)

$$(**) \qquad \frac{1}{2}\frac{d}{dt}\|u\|_1^2 + \|u\|_2^2 \cong \frac{1}{2}\frac{d}{dt}\left\|A^{1/2}u\right\|^2 + \|Au\|^2 \le c\begin{cases} \|u\|^2\left\|A^{1/2}u\right\|^4 \cong \|u\|_0^2\|u\|_1^4 & n=2 \\ \left\|A^{1/2}u\right\|^6 \cong \|u\|_1^6 & n=3 \end{cases} .$$

Putting

$$y(t) := \begin{cases} \|u\|_{2\sigma}^2 & n=2 \quad 0<\sigma<1/2 \\ \|u\|_1^2 & n=3 \end{cases}$$

one gets

$$y'(t) \le c \cdot \begin{cases} \|u\|_1^2 \cdot y(t) & n=2 \\ \|u\|_1^6 \cdot y^3(t) & n=3 \end{cases}$$

For $n=2$ this leads to a global boundedness estimate in the form

$$z'(t) \le c \cdot \|u\|_1^2 \cdot z(t) \quad \text{resp.} \quad z(t) \le z(0) \cdot e^{c\int_0^t \|u\|_1^2(s)ds} .$$

The special case $\sigma = 1/2$ lead also to a global estimate, based on an argument to be found e.g. in ([TeR] 3.1).

For $n=3$ every positive solution of $y'(t) = cy^3(t)$ blows up, i.e. there is no global estimate by this method.

§3 A global unique weak $H(-1/2)$–solution of the NS initial value problem

Generalized functions on Hilbert spaces in combination with singular integral equations are successfully applied to problems of aerodynamics and electrodynamics ([LiL]). The corresponding framework is about Hilbert scales $H(-\alpha)$ with negative scale factor $-\alpha<0$ with corresponding Ritz-Galerkin approximation theory for Pseudo-Differential equations. In case of finite element approximation spaces this correlates to FEM resp. the BEM. The corresponding energy inner product with its related energy norm $\|u\|_{1/2}$ corresponds to the extended Green formulas, based on J. Plemelj's concept of an alternative normal derivative [PlJ]. Global and interior error estimates for the Ritz-Galerkin methods of Pseudo-Differential equations in the corresponding Hilbert space framework are e.g. provided in [BrK].

Hölder resp. Lipschitz spaces are the adequate ones in treating nonlinear elliptic problems. First a priori estimates and boundedness of the Ritz-Galerkin operator in Hölder resp. Lipschitz spaces for approximation spaces fulfilling the (AI) condition are given in [NiJ1].

The singular integral operators defined by the single-layer logarithmic and the normal derivative of the double layer logarithmic potential are bounded and selfadjoint with respect to the Hilbert spaces $H(\mp 1/2)$ resp. bounded with respect to the corresponding Hölder spaces $C^{0.\alpha}, C^{1.\alpha}$, which are dense in those Hilbert spaces ([KrR]). In this sense there is a relationship between the spaces $H(-1/2)\approx C^{0.\alpha}$ and $H(1/2)\approx C^{1.\alpha}$. The regulatory results of single layer and double layer potentials with uniformly Hölder continuous densities can be extended to the three-dimensional case ([CoD].

In [ChF] a Nitsche-based domain decomposition method for the normal derivative of the double layer (hypersingular integral) operator equation has been analyzed. Due to the low regularity of the underlying energy space there is still the problem of non-existence of a well-posed continuous counterpart for the discrete BEM formulation. The approach below is closing this kind of "regularity gap" problem.

The approach below (Hilbert scale framework defined by appropriate self-adjoint operator, Plemelj regularity electric & magnetic boundary data assumptions) can also be applied to the Maxwell initial-boundary equations ([WeP], [WeP1]). In ([CoM] the corresponding BEM has been analyzed.

The Stefan problem can be transformed to nonlinear initial-boundary problem for the heat equation in a fixed domain. In case the initial data fulfill certain compatibility conditions the solution is "regular" resulting in corresponding optimal FEM convergence. In case of non-regular initial value function the approach below can also be applied to improve the non-optimal finite element approximation convergence factor $\beta<1$ to the optimal factor $\beta=1$([NiJ2]).

With respect to the Navier-Stokes vorticity equation we note that the Biot-Savart singular integral operator is a Pseudo-Differential operator of order -1 (as $A^{-1/2}$, see below). In [SaT] the spectral method is applied to analyze the related 1D -Constantin-Lax-Majda (CLM) equation with respect to the $L_2=H_0$ – Hilbert space. A corresponding analysis with respect to any $H(\alpha)$ – Hilbert space is straight forward.

As $(Bu,u)=0$, the nonlinear term of the NSE makes no contribution to the energy equality. Since the solution of the associated linearized equation is already as smooth as the data allow a solution of the nonlinear NSE cannot be expected to be smoother than the corresponding linearized equations. Based on this we shift the Hilbert scale framework of the weak NSE representation appropriately from $\alpha=0$ to the left:

multiplying the homogeneous equation (*) with $A^{-1/2}u$ leads to

$$(\dot{u},u)_{-1/2}+(Au,u)_{-1/2}+(Bu,u)_{-1/2}=0 \ .$$

The corresponding generalized "energy" inequality is given by

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2+\|u\|_{1/2}^2 \le |(Bu,u)_{-1/2}| \le \|u\|_{-1/2}\|Bu\|_{-1/2} \cong \|u\|_{-1/2}\|A^{-1/4}Bu\|_0 \cdot$$

Applying lemma 3.2 of [GiY] (see also appendix and the original proof in [SoP]) with $p=2$, $\delta=1/4$, $\theta:=\rho:=1/2$ fulfilling

$$\theta+\rho \ge \frac{1}{4}(n+1)=1$$

it follows

$$\|A^{-\delta}P(u,grad)u\| \le c\|A^{\theta}u\|\cdot\|A^{\rho}u\| = c\|u\|_{2\theta}\cdot\|u\|_{2\rho} = c\|u\|_1^2 \quad ,$$

resp.

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2+\|u\|_{1/2}^2 \le |(Bu,u)_{-1/2}| \le c\cdot\|u\|_{-1/2}\|u\|_1^2$$

Putting $y(t):=\|u\|_{-1/2}^2$ one gets

$$y'(t) \le c\cdot\|u\|_1^2\cdot y^{1/2}(t)$$

resulting into the a priori estimate

$$\|u(t)\|_{-1/2} \le \|u(0)\|_{-1/2}+\int_0^t\|u\|_1^2(s)ds \le c\left\{\|u_0\|_{-1/2}+\|u_0\|_0^2\right\}$$

,

which ensures global boundedness by the a priori energy estimate provided that $u_0 \in H_0$.

By standard arguments the above estimate can be extended to the inhomogeneous case (*) based on corresponding shift theorems for the non-stationary Stokes operator (appendix)

$$\bar{A}w(t):=\dot{w}(t)+Aw(t)$$

$$w(0)=0$$

in the form

$$\left\|\left|t^{\beta/2}w(t)\right|\right\|_{\alpha+2}^2 \le c\left\|\left|t^{\beta/2}\bar{A}w(t)\right|\right\|_{\alpha}^2 \quad , \quad \beta>-1$$

whereby

$$\left\|\left|v(t)\right|\right\|_{\alpha}^2 := \int_0^t\|v(s)\|_{\alpha}^2 ds \quad , \quad \alpha \in R \, .$$

§4 NSE circulation modelling enabled by the Prandtl and a modified Stokes operators

The solution concept for the proof in the previous section is built on the weak $H(-1/2)-$ representation in the form

$$(\dot{u},u)_{-1/2}+(Au,u)_{-1/2}+(Bu,u)_{-1/2}=0 \ .$$

From an application point of view there is still the boundary layer circulation modelling challenge which is due to the model restriction allowing only potential flows. The lift of an airfoil in inviscid flow requires circulation in the flow around the airfoil, but a single potential function that is continuous throughout the domain around the airfoil cannot represent a flow with non-zero circulation. Therefore potential flow theory requires special treatment and an additional assumption which was formulated first by L. Prandtl.

For space dimension $n=2$ the concept of potential flows goes along with the concept of the Cauchy-Riemann differential equations. Those equations enable the definition of a complex function, by which the flow of an incompressible, vortex-free fluid can be modelled. In vector terminology this can be represented in the form

$$\nabla\cdot\vec{v}=0\ ,\ \ \nabla\times\vec{v}=0 .$$

At the same time the Stokes (Leray-Hopf) operator is applied which is a projector from

$$L_2\to L_\sigma^2:=\{v|v\in L_2\wedge div(v)=0\} \ .$$

enabling the related Hilbert scale theory analysis.

In [RuC] a generalization of the Cauchy-Riemann differential equations for space dimension $n=3$ is proposed in the form

$$\nabla\cdot\vec{v}=0 \quad , \quad (\nabla\times\vec{v})\times\vec{v}=(\vec{v}\cdot\nabla)\vec{v}-\nabla(\frac{\vec{v}\cdot\vec{v}}{2})=0$$

which allows also vortex flows with certain vortex line conditions. Those are related to the NSE by the formula

$$u\cdot\nabla u=(\nabla\times u)\times u+\nabla(\frac{u\cdot u}{2}) \ .$$

Runge's generalized 3D-Cauchy-Riemann differential equations are identical to the Cauchy-Riemann's differential equation for space dimension $n=2$, .e. when no vortex flow is required to be modelled.

In [StE] a generalization of the Cauchy-Riemann equation for space dimensions $n\geq 3$ is given built on representations arising from the spherical harmonics in the context of representations of the rotation group. It adds mathematical complexity w/o new conceptual solution elements regarding the boundary layer "zero" velocity vs. the airfoil uplift modelling problem. Nevertheless is the rotation group aspect an important one. We note in this context with respect to our proposal that also the Riesz operators are rotation invariant.

The term $(\vec{v}\cdot\nabla)\vec{v}$ is concerned with how the divergence affects the velocity (when a river converges, the narrowing acts like a funnel, and the overall velocity of the flow increases. Conversely, of the river diverges, the particles spread out, and the overall speed of the flow decreases). The term ∇p is concerned with the gradient of the pressure of the medium (the shear stress forces) which is an acceleration term. It therefore relates to the initial and boundary wall values of the velocity.

Our first proposal alternatively to today's NSE weak analysis approach is concerned with the range of the Stokes operator: we propose a smaller range for the 3D Stokes operator in the form

$$\Lambda : L_2 \to L_\sigma^2 := \{ v | v \in L_2 \wedge div(v) = 0 \wedge (\nabla \times v) \times v = 0 \}.$$

Our second proposal alternatively to today's NSE weak modelling and analysis is concerned with the construction of the weak $H(-1/2)-$ representation above (multiplying the homogeneous equation (*) with $A^{-1/2}u$) we propose the following second modification. In order take the boundary behavior of vortex flow into account we suggest an alternative multiplication with $\Pi^{-1}u : H_{-1/2} \to H_{1/2}$, whereby Π denotes the Prandtl operator. It is defined by the Neumann problem in the following way:

For a closed connected surface $S \subset R^3$ one considers the harmonic function (i.e. $\Delta u = \Delta(\prod(v)) = 0$)

$$(*) \quad u(x) := \Pi(v)(x) := \frac{1}{4\pi} \oiint_S v(y) \frac{\cos \varphi_{xy}}{|x-y|^2} dS_y \quad ,$$

whereby φ_{xy} is the angle between the vector $|x-y|$ and the normal n_y to the surface at the point y and $v(y)$ is the density of the double layer potential. One can seek the solution of the Neumann boundary value problem

$$\begin{aligned} \Delta u &= 0 && \text{in } R^3 - S \\ \frac{\partial u}{\partial n} &= f && \text{on } S \end{aligned}$$

as the double layer potential in the form $u = \prod(v)$, whereby the unknown function $v(y)$ is obtained by the equation

$$\Pi(v) = f \cdot$$

From [LiI] (and subsequently from [BrK2]) we recall the

Theorem: The Prandtl operator $\prod : H_{1/2} \to H_{-1/2}$ is bounded and coercive, the range $R(\prod) = H_1(R^3 - S)$ (*e.g. the velocity space of the airplane*) and the exterior Neumann problem admit one and only on generalized solution.

The usage of the Prandtl operator goes along with the current challenges of the representation of the NSE pressure p as a solution of the Neumann problem. In the following we give some further details on the above.

The initial boundary value problem of the three dimensional Navier-Stokes equations is given by

$$\begin{aligned} \partial_t u - \Delta u + (u\cdot\nabla)u &= -\nabla p + f \quad \text{in } \Omega\times(0,T) \\ div(u) &= 0 \quad \text{in } \Omega\times(0,T) \\ u(x,0) &= u_0(x) \ , \quad x\in\Omega \\ u(x,t) &= u_1(x,t) \ , \ (x,t)\in\partial\Omega\times(0,T) . \end{aligned}$$

With respect to the proposed weak $H(-1/2)-$ representation of this paper we note that

$$-(\Delta u, v)_{-1/2} + (\nabla p, v)_{-1/2} \cong (\nabla u, \nabla v)_{-1/2} + (\nabla p, v)_{-1/2} \cong (u,v)_{1/2} + (p,v)_0 .$$

The pressure p can be expressed in terms of the velocity by the formula

$$p = -\sum_{j,k=1}^{3} R_j R_k (u_j u_k)$$

where (R_1, R_2, R_3) is the Riesz transform. The Leray-Hopf projector is the matrix valued Fourier multiplier given by

$$P(\xi) = Id - \frac{\xi\otimes\xi}{|\xi|^2} = (\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2})_{1\le j,k\le n} \quad , \ P = Id - R\otimes R =: Id - Q$$

whereby Q is an orthogonal projector, i.e. it holds $Q := R\otimes R = (R_j R_k)_{1\le j,k\le 1} = Q^2$.

As a result the Leray-Hopf operator

$$P = Id - R\otimes R =: Id - Q = Id - \frac{D\otimes D}{D^2} Id - \Delta^{-1}(\nabla\times\nabla)$$

is also an orthogonal projection . We note that under rotation in R^n, the Riesz operators transform in the same manner as the components of a vector ([SteE1] III, 1.2).

The initial boundary value problem determines the initial pressure $p_0(x)$ by the Neumann problem

$$\begin{aligned} \Delta p_0 &= (f_0 - u_0\cdot\nabla u_0 \quad \text{in } \Omega \\ \frac{\partial p_0}{\partial n} &= \left[\Delta u_0 - u_0\cdot\nabla)u_0 + f_0\right]\cdot n \quad \text{at } \partial\Omega \end{aligned}$$

with $f_0 := \lim_{t\to 0} f(\cdot,t)$. Applying formally the div-operator to the classical NSE the pressure field must satisfy the following Neumann problem ([GaG])

$$\begin{aligned} \Delta p &= (u\cdot\nabla)u - f \quad \text{in } \Omega \\ \frac{\partial p}{\partial n} &= \left[\Delta u - (u\cdot\nabla)u + f\right]\cdot n \quad \text{at } \partial\Omega \end{aligned}$$

where n denotes the outward unit normal to $\partial\Omega$.

As it holds that

$$\left[\Delta u - (u\cdot\nabla)u + f\right]\cdot n\big|_{\partial\Omega} \to \left[\Delta u_0 - (u_{\cdot 0}\nabla)u_0 + f_0\right]\cdot n\big|_{\partial\Omega} \ \text{in}\, H_{-1/2}(\partial\Omega)$$

and

$$\nabla\cdot\left[f-u\cdot\nabla)u\right]_{\partial\Omega}\to\nabla\cdot\left[f_0-u_0\cdot\nabla)u_0\right]_{\partial\Omega}\ \text{in}\ H_{-1/2}(\partial\Omega)$$

the pressure p tends to p_0 in the sense that $\|\nabla(p(\cdot,t)-p_0\|\to 0$ as $t\to 0$.

From this it follows that in this framework the prescription of the pressure at the boundary walls or at the initial time independently of u , could be incompatible with and, therefore, could retender the problem ill-posed.

From [HeJ] we recall the counter example that there exists $u_0\in\{v\in L_2 | div(v)=0\}$ with

$$\lim_{t\to 0}\sup\|p(t)\|_{L_2(\Omega)/R}=\infty\cdot$$

As a consequence of the alternative $H(1/2)-$ (energy) Hilbert space there is a reduced requirement to the limit $p(\cdot,t)\to p_0$ as $t\to 0$ in the form

$$\|\nabla(p(\cdot,t)-p_0)\|_{-1/2}\cong\|p(\cdot,t)-p_0\|_{1/2}\to 0\ \text{as}\ t\to 0.$$

Multiplying

$$\partial_t u-\Delta u+\frac{1}{2}\Lambda(\nabla(u\cdot u))=-\nabla p+f$$

by $\Delta^{-1/2}u$ (or by $\Pi^{-1}u$) leads to

$$\partial_t\|u\|_{-1/2}^2+2\|u\|_{1/2}^2+\left(\Lambda(\nabla(u\cdot u),u)\right)_{-1/2}$$

whereby the later term defines a term in the form

$$\left(\Lambda(\nabla(u\cdot u),u)\right)_{-1/2}\approx\left(\Lambda((u\cdot u),u)\right)_0\cdot$$

As a consequence the weak representation can be handled as a quasi-linear partial differential equation applying monotone operator theory.

With respect to the pressure term multiplying by $\Pi^{-1}u\in H_{1/2}$ gives

$$(\nabla p,\Pi^{-1}u)_0\cong(p,u)_0$$

whereby the term is governed by

$$(p,u)_0\le\|p\|_{-1/2}\cdot\|u\|_{1/2}\cdot$$

The answer for the motion of a fluid in an infinite space ($x \in R^3$) such that it vanishes at infinity is given by the *Helmholtz-Hodge decomposition*. It is determined when one knows the values $div(v) = \nabla \cdot v$ and $curl(v) = \nabla \times v$. If the motion of a fluid is limited to a simple connected region $\Omega \subset R^3$ with boundary $\partial\Omega$, it is determined if $div(v), curl(v)$ and therefore the value of the flow normal to the boundary $\partial v / \partial n$ for $x \in \partial\Omega$ are known.

There is a related *inverse Helmholtz-Hodge decomposition* statement that a given vector field can be decomposed into its divergence-free (incompressible) and curl-free (irrotational) components, i.e. a vector field $v : R^3 \to R^3$ can be expressed as a sum of the gradient of a scalar potential and the curl of a vector potential in the form $v = \nabla A + curlB$ with

$$A(x) = -\frac{1}{4\pi} \int \frac{div(v)(y)}{|x-y|} dy \quad , \quad B(x) = \frac{1}{4\pi} \int \frac{curl(v)(y)}{|x-y|} dy$$

whereby $curl(grad)(A)) = div(curl(B)) = 0$.

§5 Related area

Optimal finite element approximation estimates for non-linear parabolic problems with not regular initial value data

In [BrK1] optimal finite element approximation estimates for non-linear parabolic problems with not regular initial value data are prove. The proof is based on same $H(1/2)-$ energy Hilbert space concept as above in the following way:

The free boundary Stefan problem with its solution $U(y,\tau)$ can be transformed into the non-linear parabolic equation ([NiJ1]) looking for a solution $u(x,t)=U(y,\tau)$ fulfilling

$$\dot{u}(y,\tau)-u''(x,t)+xu'(1,t)u'=0 \quad \text{in } Q=\{(x,t)|x\in(0,1),0<t\leq T\}$$

$$u'(0,t)=u(1,t)=0 \quad \text{for } t>0$$
$$u(x,0)=f(x) \quad \text{for } x\in(0,1)\ .$$

The proposed variation problem is given in the form

P_v : find $v\in\dot{H}_{1/2}$ with

$$(\dot{v},w)_{-1/2}+(v',w')_{-1/2}=v(1)(Hv,w)_0 \quad \text{for } w\in\dot{H}_{1/2} \text{ and } t>0$$

$$(v(\cdot,0),w)_{-1/2}=(f',w)_{-1/2}=(f,w)_0 \quad \text{for } w\in\dot{H}_{1/2} \text{ and } t=0\ .$$

The corresponding Galerkin approximation is given by

P_{v_h} : find $v_h\in S_h\subset\dot{H}_{1/2}$ $(v_h(1):=v_h(1,t))$ with

$$(\dot{v}_h,\chi)_{-1/2}+(v_h,\chi)_{1/2}=v_h(1)(Hv_h,\chi)_0 \quad \text{for } \chi\in S_h\subset\dot{H}_{1/2} \text{ and } t>0$$

$$(v_h(\cdot,0),\chi)_{-1/2}=(f,\chi)_0 \quad \text{for } \chi\in S_h\subset\dot{H}_{1/2} \text{ and } t=0\ .$$

The reduced Hilbert scale “regularity” requirements enable optimal finite element approximation estimates for not regular initial value data in the form:

Theorem: The problem P_v has an unique bounded solution $v\in H_{1/2}$ and an unique (linear splines) finite element approximation v_h of problem P_{v_h} in the neighborhood of v with optimal order of convergence

$$\|u-u_h\|_{L_\infty(0,t);L_\infty)}\leq O(h)\ .$$

Appendix

From [GaG] we recall the Navier-Stokes initial-boundary representation in the form

$$(**)\quad \frac{dv}{dt} - \Delta v + (v, grad)v) + gradp = f$$

$$divv = 0$$

$$v_{|\partial\Omega} = 0\ ,\ v_{|t=0} = v_0 .$$

We note that for $divv = \nabla \cdot v = 0$ it holds $(v, grad)v) = (v \cdot \nabla)v = \nabla \cdot (v \otimes v)$.

By formally dot-multiply through both sides of (**) above with $f \equiv 0$ by v, integrate by parts over Ω and take into account Green formula one obtains

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_2^2 + \nu |v(t)|_{1.2}^2 = 0$$

Integrating this from $s \geq 0$, $t \geq s$ one obtains the so-called energy equation

$$\|v(t)\|_2^2 + 2\nu \int_s^t |v(\rho)|_{1.2}^2 d\rho = \|v(s)\|_2^2 \quad , \quad 0 \leq s \leq t$$

As it holds $(v \cdot \nabla v, v) = 0$ the non-linear term $v \cdot \nabla v$ does not give any contribution to the energy equation. From the equations

$$(\frac{\partial v}{\partial t}, P\Delta v) = (\frac{\partial v}{\partial t}, \Delta v) = -\frac{1}{2}\frac{d}{dt}|v|_{1.2}^2$$

$$(\frac{\partial v}{\partial t}, P\Delta v) = 0$$

it follows

$$\frac{1}{2}\frac{d}{dt}|v|_{1.2}^2 + \nu \|P\Delta v\|_2^2 = (v \cdot \nabla v, P\Delta v) \neq 0$$

There is a loss of regularity if two (scalar) functions f and g are in H_1, their product $f \cdot g$ only belongs to $H_{1/2}$ and their derivative $\partial(fg)$ is even less regular as it belongs to $H_{-1/2}$. This is the root cause of some (positive) regularization effects by the heat semi-group $S(t)$ and the loss of regularity that comes from the differential (grad-) operator ∇ and from the pointwise multiplication $v \otimes u$ of the non-linear bilinear operator

$$B(v,u)(t) := -\int_0^t e^{-(s-t)\Delta} P\nabla \cdot (v \otimes u)(s) ds \ .$$

The estimate of the non-linear term is highly depending from the space dimension. It holds ([GaG])

$$(v \cdot \nabla v, P\Delta v) \leq C\|v\|_q^{2q/(q-n)} |v|_{1.2}^2 + \frac{\nu}{2}\|P\Delta v\|_2^2 \quad \text{for all} \quad q \in (n, \infty) \ .$$

From the Sobolev embedding theorems and properties of the projection operator P ([GaG]) one gets the a priori estimates

$$\frac{1}{2}\frac{d}{dt}|v(t)|_{1.2}^2+\frac{\nu}{2}\|P\Delta v\|_2^2\le\begin{cases}c_3\|v\|_2^2|v|_{1.2}^4 & , \ n=2\\ c_4|v|_{1.2}^6 & , \ n=3\end{cases} .$$

From [GiY] we recall the corresponding fundamental

Lemma 3.2 ([GiY]): let $0\le\delta<1/2+n\cdot(1-1/p)/2$. We have

$$\left|A^{-\delta}P(u,grad)v\right|_p\le M\cdot\left|A^{\theta}u\right|_p\cdot\left|A^{\rho}u\right|_p$$

with a constant $M:=M(\delta,\theta,\rho,p)$ if $\delta+\theta+\rho\ge n/2p++1/2$, $\theta,\rho>0$, $\theta+\rho>1/2$.

Example ([GiY]): when $p=n=2$, one gets

$$|P(u,grad)v|\le C\cdot|u|^{1/2}\cdot\left|A^{1/2}u\right|^{1/2}\cdot\left|A^{1/2}v\right|^{1/2}\cdot|Av|^{1/2} .$$

Stokes operator ([SoH]), IV15: Let $\Omega\subseteq R^n$ ($n\ge2$) denote an arbitrary domain and

$$A=\int_0^\infty\lambda dE_\lambda$$

the Stokes operator of Ω. Then the fractional powers

$$A^\alpha=\int_0^\infty\lambda^\alpha dE_\lambda \quad , -1\le\alpha\le1$$

are positive selfadjoint operators, and each operator $S(t)$ of the Stokes semigroup family

$$\left\{S(t):=e^{-tA}:=\int_0^\infty e^{-t\lambda}dE_\lambda \,\middle|\, \lambda\ge0,t\ge0\right\}$$

is bounded everywhere defined and positive selfadjoint in the Hilbert space $L_\sigma^2(\Omega)$.

For the orthogonal set $\{w_i,\lambda_i\}$ of eigenpairs of the non-stationary Stokes operator

$$\overline{A}:=\dot{w}+Aw=f \ , \ w(0)=0 \ , \ \tau\in[0,t]$$

one gets

$$w_i(\tau)=\int_0^\tau e^{-\lambda_i(\tau-s)}f_i(s)ds .$$

By changing the order of integration it follows for $\beta > -1$

$$\int_0^t \tau^\beta w_i^2(\tau)d\tau \le \int_0^t \left[\int_0^\tau e^{-\lambda_i(\tau-s)}ds\right]\left[\int_0^\tau s^\beta e^{-\lambda_i(\tau-s)} f_i^2(s)ds\right]d\tau \le \lambda_i^{-1}\int_0^t s^\beta f_i^2(s)\left[\int_\tau^t e^{-\lambda_i(\tau-s)}d\tau\right]ds \le \lambda_i^{-2}\int_0^t s^\beta f_i^2(s)ds \cdot$$

From this one gets

$$\left\|t^{\beta/2}w(t)\right\|_{\alpha+2}^2 \le c\left\|t^{\beta/2}\overline{A}w(t)\right\|_\alpha^2 \quad .$$

For $u_0 \in L_\sigma^2(\Omega)$ the function $u:[0,\infty)\to L_\sigma^2(\Omega)$ defined by $u(t):=S(t)u_0$ has the following properties:

i) $u(t)$ is strongly continuous for $t \ge 0$, $u(0)=u_0$

ii) $u'(t) = s-\lim_{\delta\to 0}\frac{1}{\delta}\left(u(t+\delta)-u(t)\right)$ for $t>0$

iii) $u(t)\in D(A)$ and $u'(t)+Au(t)=0$

iv) If $u_0 \in D(A)$ then $u'(0) = s-\lim_{\delta\to 0}\frac{1}{\delta}\left(u(\delta)-u(0)\right)$ exists and $u'(0)+Au(0)=0$

v) For all $t\ge 0$ it holds $\|u(t)\|_2 \le \|u_0\|_2$ and $s-\lim_{\delta\to 0} u(t)=0$.

From [PlJ] I, §5, §8, we quote:

"*Bisher war es üblich für das Potential $V(p)$ die Form*

$$-\int \log(r_{ps})\mu'(s)ds$$

vorauszusetzen, wobei dann $\mu'(s)$ die Massendichtigkeit der Belegung genannt wurde. Eine solche Annahme erweist sich aber als eine derart folgenschwere Einschränkung, daß dadurch dem Potentiale $V(p)$ der größte Teil seiner Leistungsfähigkeit genommen wird. Für tiefergehende Untersuchungen erweist sich das Potential V nur in der (Stieltjes integral-) *Form*

$$V(p) = -\int \log(r_{ps})d\mu_s$$

verwendbar. Die infinitesimale Größe $d\mu_s$ kann man dann das Massenelement nennen, von Massendichtigkeit wird man aber nur dann sprechen können, wenn der Differentialquotient

$$\frac{d\mu_s}{ds} = \mu'(s)$$

besteht."

The concept leads to generalized Green formula, based on a generalized normal derivative definition, which is basically about reduced regularity assumptions, i.e. there is no existence necessary of certain limits of the normal derivatives into the normal direction at the boundary.

"*Es handelt sich um eine Verallgemeinerung, wie es die Erweiterung differentiierbarer Funktionen auf die stetigen ist.*"

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