

**Global Existence and Uniqueness
of the Non-stationary 3D-Navier-Stokes
Initial-boundary Value Problem**

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January, 2016

Abstract

Based on the Hilbert scale $H(\alpha)$, $\alpha \in \mathbb{R}$, defined by the orthogonal set of eigenpairs of the Stokes operator we present a global unique weak $H(-1/2)$ -solution of the generalized 3D Navier-Stokes initial value problem

$$\begin{aligned} (\dot{u}, v)_{-1/2} + (Au, v)_{-1/2} + (Bu, v)_{-1/2} &= 0 \\ &\text{for all } v \in H_{-1/2} \\ (u(0), v)_{-1/2} &= (u_0, v)_{-1/2} \end{aligned}$$

The global boundedness is a consequence of the Sobolevskii -estimate of the non-linear term ([SoP]) enabling the generalized energy inequality

$$\frac{1}{2} \frac{d}{dt} \|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \leq |(Bu, u)_{-1/2}| \leq c \cdot \|u\|_{-1/2} \|u\|_1^2.$$

Putting $y(t) := \|u\|_{-1/2}^2$ one gets

$$y'(t) \leq c \cdot \|u\|_1^2 \cdot y^{1/2}(t)$$

resulting into the a priori estimate

$$\|u(t)\|_{-1/2} \leq \|u_0\|_{-1/2} + \int_0^t \|u\|_1^2(s) ds \leq c \left\{ \|u_0\|_{-1/2} + \|u_0\|_0^2 \right\}.$$

§1 Introduction

This section is a summary from [CaM], [GaG], [GiY], [ShM], [SoH], [TeR].

The Navier-Stokes Equations (NSE) describes a flow of incompressible, viscous fluid. The three key foundational questions of every PDE is existence, and uniqueness of solutions, as well as whether solutions corresponding to smooth initial data can develop singularities in finite time, and what these might mean. For the NSE satisfactory answers to those questions are available in two dimensions, i.e. 2D-NSE with smooth initial data possesses unique solutions which stay smooth forever. In three dimensions, those questions are still open. Only local existence and uniqueness results are known. Global existence of strong solutions has been proven only, when initial and external forces data are sufficiently smooth. Uniqueness and regularity of non-local Leray-Hopf solutions are still open problems.

Basically the existence of 3D solutions is proven only for “large” Banach spaces. The uniqueness is proven only in “small” Banach spaces. The question of global existence of smooth solutions vs. finite time blow up is one of the Clay Institute millennium problems.

The existence of weak solutions can be provided, essentially by the energy inequality. If solutions would be classical ones, it is possible to prove their uniqueness. On the other side for existing weak solutions it is not clear that the derivatives appearing in the inequalities have any meaning. Basically all existence proofs of weak solutions of the Navier-Stokes equations are given as limit (in the corresponding weak topology) of existing approximation solutions built on finite dimensional approximation spaces. The approximations are basically built by the Galerkin-Ritz method, whereby the approximation spaces are e.g. built on eigenfunctions of the Stokes operator or generalized Fourier series approximations.

It has been questioned whether the NSE really describes general flows: The difficulty with ideal fluids, and the source of the d'Alembert paradox, is that in such fluids there are no frictional forces. Two neighboring portions of an ideal fluid can move at different velocities without rubbing on each other, provided they are separated by a streamline. It is clear that such a phenomenon can never occur in a real fluid, and the question is how frictional forces can be introduced into a model of a fluid.

The question intimately related to the uniqueness problem is the regularity of the solution. Do the solutions to the NSE blow-up in finite time? The solution is initially regular and unique, but at the instant T when it ceases to be unique (if such an instant exists), the regularity could also be lost. Given a smooth datum at time zero, will the solution of the NSE continue to be smooth and unique for all time?

There is no uniqueness proof for weak solutions except for over small time intervals. The simplest possible model example how a singularity can appear, is the ODE

$$y'(t) = y^2(t), \quad y(0) = y_0$$

with the solution

$$y(t) = \frac{y_0}{1 - t \cdot y_0}$$

which becomes infinite in finite time.

Let P denote the orthogonal projection onto the kernel of the divergence operator and A the selfadjoint Stokes operator. If u is divergence free then

$$(Bv, v) := ((u, \nabla)v, v)_0 := \iint_{\Omega} (u, \nabla)v \cdot v dx = 0.$$

Therefore there is no contribution of the essential non-linear NSE term Bu to the a priori (energy) estimate

$$\|u(t)\|_0^2 + 2 \int_0^t \|u\|_1^2(\tau) d\tau = \|u_0\|_0^2.$$

Especially there is no evidence that the energy norm $\|u(t)\|_1$ is bounded in case of space dimension $n=3$. This is due to the fact that the right hand side of the corresponding a priori estimate

$$\frac{1}{2} \frac{d}{dt} \|u\|_{2\sigma}^2 + \|u\|_{2\sigma+1}^2 \leq |(A^{\sigma-1/2}Bu, A^{\sigma+1/2}u)|, \quad 0 < \sigma \leq 1/2$$

depends on the space dimension. It holds

$$(v \cdot \nabla v, P\Delta v) \leq C \|v\|_q^{2q/(q-n)} \|v\|_{1,2}^2 + \frac{V}{2} \|P\Delta v\|_2^2 \quad \text{for all } q \in (n, \infty).$$

The Gronwall inequality implies global boundedness in case of $n = 2$.

For $n \geq 3$ there is up to now no global boundedness result.

§2 Stokes operator, Hilbert scales and a priori estimates

The content of this section is basically taken from [GiY]. With c we denote numeric constants which may have different values at different places.

Let P be the orthogonal projection operator of $(L_2(\Omega))^n$ onto the divergence free vector field H_ω consisting of all solenoidal vector functions u , i.e. the operator is an orthogonal projection onto the kernel of the divergence operator. It is a Pseudo-Differential operator (PDO) of degree zero [(EsG)]. The Stokes operator A is a selfadjoint operator in H_ω , being the Friedrichs extension of the non-negative symmetric operator $-P\Delta$ in H_ω defined for all $u \in C^2$ with $\operatorname{div} u = 0$ and $u_{n|\partial\Omega} = 0$. The Stokes operator enables the definition of a related Hilbert scale $(\alpha \in \mathbb{R})$ with corresponding norm [(Nij)]

$$\|u\|_\alpha := \|A^{\alpha/2}u\| .$$

Throughout this paper, if not explicitly mentioned, we assume $p = 2$ and $n = 3$ for $(L_p(\Omega))^n$.

Using the Stokes operator and its related Hilbert scale framework the Navier-Stokes equations can be represented as an evolution equation in H_0 . Since $P(\operatorname{grad} p) = 0$ one gets

$$Au = Pf \quad \text{in } H_0 .$$

Putting $B(u) := P(u, \operatorname{grad} u)$ and assuming $Pu_0 = u_0$ the NSE initial-boundary equation is given by

$$(*) \quad \frac{du}{dt} + Au + Bu = Pf \quad , \quad u(0) = u_0 .$$

As u is divergence free and $u \cdot \nu$ identically vanishes on $\partial\Omega$ one gets

$$b(u, v, w) := ((u, \operatorname{grad} v), w) = \iint_{\Omega} (u, \operatorname{grad} v) \cdot w \, dx = -b(u, w, v)$$

and especially $b(u, v, v) = 0$.

The linear homogeneous part of (*) enables the definition of a semigroup generated by the Stokes operator which leads to the representations

$$u(t) = e^{-tu_0} + \int_0^t e^{-(t-s)A} Pf(s) \, ds + \int_0^t e^{-(t-s)A} Bu(s) \, ds .$$

Multiply u to both sides of the homogeneous equation of (*) and integrate over Ω one gets

$$(\dot{u}, u) + (Au, u) + (Bu, u) = 0 \quad , \quad u(x, 0) = u_0 .$$

This yields

$$\frac{d}{dt} \|u\|^2 + 2 \|A^{1/2}u\|^2 = 0$$

respectively integrating over $[0, t]$

$$\|u(t)\|_0^2 + 2 \int_0^t \|u\|_1^2(\tau) d\tau = \|u_0\|_0^2 .$$

A priori estimates for higher space derivatives of u are derived by multiplying the homogeneous equation (*) with $A^{2\sigma}u$ leading to

$$\left(\frac{du}{dt}, A^{2\sigma}u\right) + (Au, A^{2\sigma}u) + (Bu, A^{2\sigma}u) = 0$$

resp.

$$\frac{1}{2} \frac{d}{dt} \|A^\sigma u\|^2 + \|A^{\sigma+1/2}u\|^2 + (A^{\sigma-1/2}Bu, A^{\sigma+1/2}u) = 0 .$$

Estimates of $A^{\sigma-1/2}Bu$ depend (heavily!) on the space dimension n , which is the root cause for the “still open NSE questions” for the space dimension $n = 3$. The essential a priori estimates are e.g. given in [GiY] (see also appendix)

$$(**) \quad \frac{1}{2} \frac{d}{dt} \|u\|_1^2 + \|u\|_2^2 \cong \frac{1}{2} \frac{d}{dt} \|A^{1/2}u\|^2 + \|Au\|^2 \leq c \begin{cases} \|u\|^2 \|A^{1/2}u\|^4 \cong \|u\|_0^2 \|u\|_1^4 & n=2 \\ \|A^{1/2}u\|^6 \cong \|u\|_1^6 & n=3 \end{cases} .$$

Putting

$$y(t) := \begin{cases} \|u\|_{2\sigma}^2 & n=2 \quad 0 < \sigma < 1/2 \\ \|u\|_1^2 & n=3 \end{cases}$$

one gets

$$y'(t) \leq c \cdot \begin{cases} \|u\|_1^2 \cdot y(t) & n=2 \\ \|u\|_1^6 \cdot y^3(t) & n=3 \end{cases}$$

For $n = 2$ this leads to a global boundedness estimate in the form

$$z'(t) \leq c \cdot \|u\|_1^2 \cdot z(t) \quad \text{resp.} \quad z(t) \leq z(0) \cdot e^{c \int_0^t \|u\|_1^2(s) ds} .$$

The special case $\sigma = 1/2$ lead also to a global estimate, based on an argument to be found e.g. in ([TeR] 3.1).

For $n = 3$ every positive solution of $y'(t) = cy^3(t)$ blows up, i.e. there is no global estimate by this method.

§3 A global unique weak $H(-1/2)$ -solution of the NS initial value problem

Generalized functions on Hilbert spaces in combination with singular integral equations are successfully applied to problems of aerodynamics and electrodynamics ([LiL]). The corresponding framework is about Hilbert scales $H(-\alpha)$ with negative scale factor $-\alpha < 0$ with corresponding Ritz-Galerkin approximation theory for Pseudo-Differential equations. In case of finite element approximation spaces this correlates to FEM resp. the BEM. The corresponding energy inner product with its related energy norm $\|u\|_{1/2}$ corresponds to the extended Green formulas, based on J. Plemelj's concept of an alternative normal derivative [PIJ]. Global and interior error estimates for the Ritz-Galerkin methods of Pseudo-Differential equations in the corresponding Hilbert space framework are e.g. provided in [BrK].

Hölder resp. Lipschitz spaces are the adequate ones in treating nonlinear elliptic problems. First a priori estimates and boundedness of the Ritz-Galerkin operator in Hölder resp. Lipschitz spaces for approximation spaces fulfilling the (A1) condition are given in [NIJ1].

The singular integral operators defined by the single-layer logarithmic and the normal derivative of the double layer logarithmic potential are bounded and selfadjoint with respect to the Hilbert spaces $H(\mp 1/2)$ resp. bounded with respect to the corresponding Hölder spaces $C^{0,\alpha}, C^{1,\alpha}$, which are dense in those Hilbert spaces ([KrR]). In this sense there is a relationship between the spaces $H(-1/2) \approx C^{0,\alpha}$ and $H(1/2) \approx C^{1,\alpha}$. The regulatory results of single layer and double layer potentials with uniformly Hölder continuous densities can be extended to the three-dimensional case ([CoD]).

In [ChF] a Nitsche-based domain decomposition method for the normal derivative of the double layer (hypersingular integral) operator equation has been analyzed. Due to the low regularity of the underlying energy space there is still the problem of non-existence of a well-posed continuous counterpart for the discrete BEM formulation. The approach below is closing this kind of "regularity gap" problem.

The approach below (Hilbert scale framework defined by appropriate self-adjoint operator, Plemelj regularity electric & magnetic boundary data assumptions) can also be applied to the Maxwell initial-boundary equations ([WeP], [WeP1]). In ([CoM]) the corresponding BEM has been analyzed.

The Stefan problem can be transformed to nonlinear initial-boundary problem for the heat equation in a fixed domain. In case the initial data fulfill certain compatibility conditions the solution is "regular" resulting in corresponding optimal FEM convergence. In case of non-regular initial value function the approach below can also be applied to improve the non-optimal finite element approximation convergence factor $\beta < 1$ to the optimal factor $\beta = 1$ ([NiJ2]).

With respect to the Navier-Stokes vorticity equation we note that the Biot-Savart singular integral operator is a Pseudo-Differential operator of order -1 (as $A^{-1/2}$, see below). In [SaT] the spectral method is applied to analyze the related one-dimensional Constantin-Lax-Majda (CLM) equation with respect to the $L_2 = H_0$ - Hilbert space. A corresponding analysis with respect to any $H(\alpha)$ - Hilbert space is straight forward.

As $(Bu, u) = 0$, the nonlinear term of the NSE makes no contribution to the energy equality. Since the solution of the associated linearized equation is already as smooth as the data allow a solution of the nonlinear NSE cannot be expected to be smoother than the corresponding linearized equations. Based on this we shift the Hilbert scale framework of the weak NSE representation appropriately from $\alpha = 0$ to the left:

multiplying the homogeneous equation (*) with $A^{-1/2}u$ leads to

$$(\dot{u}, u)_{-1/2} + (Au, u)_{-1/2} + (Bu, u)_{-1/2} = 0 .$$

The corresponding generalized “energy” inequality is given by

$$\frac{1}{2} \frac{d}{dt} \|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \leq |(Bu, u)_{-1/2}| \leq \|u\|_{-1/2} \|Bu\|_{-1/2} \cong \|u\|_{-1/2} \|A^{-1/4} Bu\|_0 .$$

Applying lemma 3.2 of [GiY] (see also appendix and the original proof in [SoP]) with $p = 2$, $\delta = 1/4$, $\theta := \rho := 1/2$ fulfilling

$$\theta + \rho \geq \frac{1}{4}(n+1) = 1$$

it follows

$$\|A^{-\delta} P(u, grad)u\| \leq c \|A^\theta u\| \cdot \|A^\rho u\| = c \|u\|_{2\theta} \cdot \|u\|_{2\rho} = c \|u\|_1^2 ,$$

resp.

$$\frac{1}{2} \frac{d}{dt} \|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \leq |(Bu, u)_{-1/2}| \leq c \cdot \|u\|_{-1/2} \|u\|_1^2$$

Putting $y(t) := \|u\|_{-1/2}^2$ one gets

$$y'(t) \leq c \cdot \|u\|_1^2 \cdot y^{1/2}(t)$$

resulting into the a priori estimate

$$\|u(t)\|_{-1/2} \leq \|u(0)\|_{-1/2} + \int_0^t \|u\|_1^2(s) ds \leq c \left\{ \|u_0\|_{-1/2} + \|u_0\|_0^2 \right\}$$

which ensures global boundedness by the a priori energy estimate provided that $u_0 \in H_0$.

By standard arguments the above estimate can be extended to the inhomogeneous case (*) based on corresponding shift theorems for the non-stationary Stokes operator (appendix)

$$\bar{A}w(t) := \dot{w}(t) + Aw(t)$$

$$w(0) = 0$$

in the form

$$\|t^{\beta/2} w(t)\|_{\alpha+2}^2 \leq c \|t^{\beta/2} \bar{A}w(t)\|_{\alpha}^2 , \quad \beta > -1$$

whereby

$$\|v(t)\|_{\alpha}^2 := \int_0^t \|v(s)\|_{\alpha}^2 ds , \quad \alpha \in \mathbb{R} .$$

Appendix

From [GaG] we recall the Navier-Stokes initial-boundary representation in the form

$$\begin{aligned}
 (**) \quad & \frac{dv}{dt} - \Delta v + (v, \text{grad})v + \text{grad}p = f \\
 & \text{div}v = 0 \\
 & v|_{\partial\Omega} = 0, \quad v|_{t=0} = v_0.
 \end{aligned}$$

We note that for $\text{div}v = \nabla \cdot v = 0$ it holds $(v, \text{grad})v = (v \cdot \nabla)v = \nabla \cdot (v \otimes v)$.

By formally dot-multiply through both sides of (**) above with $f \equiv 0$ by v , integrate by parts over Ω and take into account Green formula one obtains

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|_2^2 + \nu \|v(t)\|_{1,2}^2 = 0$$

Integrating this from $s \geq 0$, $t \geq s$ one obtains the so-called energy equation

$$\|v(t)\|_2^2 + 2\nu \int_s^t \|v(\rho)\|_{1,2}^2 d\rho = \|v(s)\|_2^2, \quad 0 \leq s \leq t$$

As it holds $(v \cdot \nabla v, v) = 0$ the non-linear term $v \cdot \nabla v$ does not give any contribution to the energy equation. From the equations

$$\begin{aligned}
 \left(\frac{\partial v}{\partial t}, P\Delta v\right) &= \left(\frac{\partial v}{\partial t}, \Delta v\right) = -\frac{1}{2} \frac{d}{dt} \|v\|_{1,2}^2 \\
 \left(\frac{\partial v}{\partial t}, P\Delta v\right) &= 0
 \end{aligned}$$

it follows

$$\frac{1}{2} \frac{d}{dt} \|v\|_{1,2}^2 + \nu \|P\Delta v\|_2^2 = (v \cdot \nabla v, P\Delta v) \neq 0$$

There is a loss of regularity if two (scalar) functions f and g are in H_1 , their product $f \cdot g$ only belongs to $H_{1/2}$ and their derivative $\partial(fg)$ is even less regular as it belongs to $H_{-1/2}$. This is the root cause of some (positive) regularization effects by the heat semi-group $S(t)$ and the loss of regularity that comes from the differential (grad-) operator ∇ and from the pointwise multiplication $v \otimes u$ of the non-linear bilinear operator

$$B(v, u)(t) := -\int_0^t e^{-(s-t)\Delta} P \nabla \cdot (v \otimes u)(s) ds.$$

The estimate of the non-linear term is highly depending from the space dimension. It holds ([GaG])

$$(v \cdot \nabla v, P\Delta v) \leq C \|v\|_q^{2q/(q-n)} \|v\|_{1,2}^2 + \frac{\nu}{2} \|P\Delta v\|_2^2 \quad \text{for all } q \in (n, \infty).$$

From the Sobolev embedding theorems and properties of the projection operator P ([GaG]) one gets the a priori estimates

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|_{1,2}^2 + \frac{\nu}{2} \|P\Delta v\|_2^2 \leq \begin{cases} c_3 \|v\|_2^2 \|v\|_{1,2}^4, & n=2 \\ c_4 \|v\|_{1,2}^6, & n=3 \end{cases} .$$

From [GiY] we recall the corresponding fundamental

Lemma 3.2 ([GiY]): let $0 \leq \delta < 1/2 + n \cdot (1 - 1/p)/2$. We have

$$|A^{-\delta} P(u, \text{grad})v|_p \leq M \cdot |A^\theta u|_p \cdot |A^\rho u|_p$$

with a constant $M := M(\delta, \theta, \rho, p)$ if $\delta + \theta + \rho \geq n/2p + 1/2$, $\theta, \rho > 0$, $\theta + \rho > 1/2$.

Example ([GiY]): when $p = n = 2$, one gets

$$|P(u, \text{grad})v| \leq C \cdot |u|^{1/2} \cdot |A^{1/2} u|^{1/2} \cdot |A^{1/2} v|^{1/2} \cdot |Av|^{1/2} .$$

Stokes operator ([SoH]), IV15: Let $\Omega \subseteq R^n$ ($n \geq 2$) denote an arbitrary domain and

$$A = \int_0^\infty \lambda dE_\lambda$$

the Stokes operator of Ω . Then the fractional powers

$$A^\alpha = \int_0^\infty \lambda^\alpha dE_\lambda, \quad -1 \leq \alpha \leq 1$$

are positive selfadjoint operators, and each operator $S(t)$ of the Stokes semigroup family

$$\left\{ S(t) := e^{-tA} := \int_0^\infty e^{-t\lambda} dE_\lambda \mid \lambda \geq 0, t \geq 0 \right\}$$

is bounded everywhere defined and positive selfadjoint in the Hilbert space $L^2_\sigma(\Omega)$.

For the orthogonal set $\{w_i, \lambda_i\}$ of eigenpairs of the non-stationary Stokes operator

$$\bar{A} := \dot{w} + Aw = f, \quad w(0) = 0, \quad \tau \in [0, t]$$

one gets

$$w_i(\tau) = \int_0^\tau e^{-\lambda_i(\tau-s)} f_i(s) ds .$$

By changing the order of integration it follows for $\beta > -1$

$$\int_0^t \tau^\beta w_i^2(\tau) d\tau \leq \int_0^t \left[\int_0^\tau e^{-\lambda_i(\tau-s)} ds \right] \left[\int_0^\tau s^\beta e^{-\lambda_i(\tau-s)} f_i^2(s) ds \right] d\tau \leq \lambda_i^{-1} \int_0^t s^\beta f_i^2(s) \left[\int_\tau^t e^{-\lambda_i(\tau-s)} d\tau \right] ds \leq \lambda_i^{-2} \int_0^t s^\beta f_i^2(s) ds \cdot$$

From this one gets

$$\|t^{\beta/2} w(t)\|_{\alpha+2}^2 \leq c \|t^{\beta/2} \bar{A} w(t)\|_{\alpha}^2 \cdot$$

For $u_0 \in L^2_\sigma(\Omega)$ the function $u : [0, \infty) \rightarrow L^2_\sigma(\Omega)$ defined by $u(t) := S(t)u_0$ has the following properties:

- i) $u(t)$ is strongly continuous for $t \geq 0$, $u(0) = u_0$
- ii) $u'(t) = s - \lim_{\delta \rightarrow 0} \frac{1}{\delta} (u(t + \delta) - u(t))$ for $t > 0$
- iii) $u(t) \in D(A)$ and $u'(t) + Au(t) = 0$
- iv) If $u_0 \in D(A)$ then $u'(0) = s - \lim_{\delta \rightarrow 0} \frac{1}{\delta} (u(\delta) - u(0))$ exists and $u'(0) + Au(0) = 0$
- v) For all $t \geq 0$ it holds $\|u(t)\|_2 \leq \|u_0\|_2$ and $s - \lim_{\delta \rightarrow 0} u(t) = 0$.

From [PIJ] I, §5, §8, we quote:

“Bisher war es üblich für das Potential $V(p)$ die Form

$$-\int \log(r_{ps}) \mu'(s) ds$$

vorauszusetzen, wobei dann $\mu'(s)$ die Massendichtigkeit der Belegung genannt wurde. Eine solche Annahme erweist sich aber als eine derart folgenschwere Einschränkung, daß dadurch dem Potentiale $V(p)$ der größte Teil seiner Leistungsfähigkeit genommen wird. Für tiefergehende Untersuchungen erweist sich das Potential V nur in der (Stieltjes integral-) Form

$$V(p) = -\int \log(r_{ps}) d\mu_s$$

verwendbar. Die infinitesimale Größe $d\mu_s$ kann man dann das Massenelement nennen, von Massendichtigkeit wird man aber nur dann sprechen können, wenn der Differentialquotient

$$\frac{d\mu_s}{ds} = \mu'(s)$$

besteht.“

The concept leads to generalized Green formula, based on a generalized normal derivative definition, which is basically about reduced regularity assumptions, i.e. there is no existence necessary of certain limits of the normal derivatives into the normal direction at the boundary.

“Es handelt sich um eine Verallgemeinerung, wie es die Erweiterung differenzierbarer Funktionen auf die stetigen ist.“

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