The three Millennium problem solutions, RH, NSE, YME, and a Hilbert scale based quantum geometrodynamics

Overview

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September, 2019

The homepage www.fuchs-braun.com provides solutions to the following Millennium problems

- the Riemann Hypothesis
- well-posed 3D-nonlinear, non-stationary Navier-Stokes equations
- the mass gap problem of the Yang-Mills equations.

A common underlying distributional Hilbert space framework provides an answer to Derbyshine's question ((DeJ) p. 295)$^{(*)}$: „ *What on earth does the distribution of prime numbers have to do with the behavior of subatomic particles*?", enabling a quantum gravity theory based on an only Hamiltonian (*energy functional*) formalism. Due to reduced regularity assumptions to the domains of the concerned operators the "force" related Lagrange formalism is no longer valid; therefore the notion "force" plays no role anymore in the proposed quantum gravity theory.

The Bagchi Hilbert space reformulation of the Nyman, Beurling and Baez-Duarte RH criterion provides the link between the two solution areas above (BaB). The Zeta function on the critical line is an element of the distributional Hilbert space H_{-1}. Therefore, in order to verify the Hilbert-Polya conjecture any (weak) eigenfunction solution of a self-adjoint operator equation to verify the Hilbert-Polya conjecture needs to be an element of a $H_{-1/2}$. The imaginary part values ω_n of the zeros of the considered Kummer function ${}_1F_1\left(\frac{1}{2},\frac{3}{2};2\pi iz\right)$ (alternatively to $e^{2\pi inx}$) with its corresponding Mellin transform

$$M\left[{}_1F_1\left(\frac{1}{2},\frac{3}{2};-x\right)\right](s)=\int_0^\infty x^s\,{}_1F_1\left(\frac{1}{2},\frac{3}{2};-x\right)\frac{dx}{x}=\frac{\Gamma\left(1+\frac{s}{2}\right)}{s(1-s)}\quad,\quad 0<Re(s)<1$$

enjoy appropriate properties (SeA), e.g. $2n-1<2\omega_n<2n<\omega_n+\omega_{n+1}<2n+1$ satisfying the "Hadamard" gap" condition.

The corresponding analysis of the 2D-NSE for the 3D-NSE fails due to not appropriate Sobolev norm estimates. This is called the Serrin gap.

The classical Yang-Mills theory is the generalization of the Maxwell theory of electromagnetism where chromo-electromagnetic field itself carries charges. As a classical field theory it has solutions which travel at the speed of light so that its quantum version should describe massless particles (gluons). However, the postulated phenomenon of color confinement permits only bound states of gluons, forming massive particles. This is the Yang-Mills mass gap.

The current quantum state Hilbert space $H_0=L_2$ is extended to $H_{-1/2}$ to enable a Hilbert space based quantum gravity theory.

PART I:
Braun K., *A Kummer function based alternative Zeta function theory to solve the Riemann Hypothesis and the binary Goldbach conjecture*

PART II:
Braun K., *3D-NSE and YME mass gap solutions in a distributional Hilbert scale frame enabling a quantum gravity theory*

$^{(*)}$: ... "*The non-trivial zeros of Riemann's zeta function arise from inquiries into the distribution of prime numbers. The eigenvalues of a random Hermitian matrix arise from inquiries into the behavior of systems of subatomic particles under the laws of quantum mechanics. What on earth does the distribution of prime numbers have to do with the behavior of subatomic particles*?"

PART I

A Kummer/cot-function based alternative Zeta function theory to solve the Riemann Hypothesis

The Riemann Hypothesis states that the non-trivial zeros of the Zeta function all have real part one-half. The Hilbert-Polya conjecture states that the imaginary parts of the zeros of the Zeta function corresponds to eigenvalues of an unbounded self-adjoint operator. It is related to the Berry-Keating conjecture that the imaginary parts of the zeros of the Zeta function are eigenvalues of an „appropriate“ Hermitian operator $H = \frac{1}{2}(xp + px)$ where x and p are the position and conjugate momentum operators, respectively, and multiplicity is noncommunative. The operator H is symmetric, but might have nontrivial deficiency indices (W. Bulla, F. Gesztesy, J. Math. Phys. 26 (1), October 1985), i.e. in a mathematical sense H is not Hermitian.

The key ingredients of the Zeta function theory are the Mellin transforms of the Gaussian function and the fractional part function. To the author´s humble opinion, the main handicap to prove the RH is the not-vanishing constant Fourier term of both functions. The Hilbert transform of any function has a vanishing constant Fourier term.

Let H and M denote the Hilbert and the Mellin transform operators. Replacing the Gaussian function $f(x) := e^{-\pi x^2}$ and the fractional part function by its Hilbert transforms enables an alternative Zeta function theory.

The Mellin transform of the Gaussian function is given by

$$M[f](s) = \frac{1}{2}\pi^{-s/2}\Gamma(\frac{s}{2}) \ , \ \ M[-xf'(x)](s) = \frac{s}{2}\pi^{-s/2}\Gamma(\frac{s}{2}) = \frac{1}{2}\pi^{-s/2}\Pi(\frac{s}{2}) \ .$$

The related Theta function properties (based on the Poisson summation formula) of

$$G(x) \coloneqq \theta(x^2) := \sum_{-\infty}^{\infty} e^{-\pi n^2 x^2} = 1 + 2\sum_{1}^{\infty} e^{-\pi n^2 x^2} =: 1 + 2\psi(x^2) = \frac{1}{x}\sum_{-\infty}^{\infty} e^{-\pi\frac{n^2}{x^2}} = \frac{1}{x}G\frac{1}{x})$$

leads to the Riemann duality equation in the form (EdH) 1.8)

$$\xi(s) := \frac{s}{2}\Gamma\left(\frac{s}{2}\right)(s-1)\pi^{-\frac{s}{2}}\zeta(s) = (1-s)\cdot\zeta(s)M[-xf'(x)](s) = \zeta(s)\cdot M[-x(xf'(x))'](s) = \xi(1-s).$$

The Mellin transform for Riemann’s auxiliary function

$$H(x) \coloneqq -\frac{d}{dx}(x^2\frac{d}{dx})G(x)$$

is well defined and it holds

$$\int_0^\infty x^{1-s}H(x)\frac{dx}{x} = \int_0^\infty x^{s}H(x)\frac{dx}{x} \ .$$

(*) The Hilbert transform of the Gaussian function is given by the Dawson function

$$F(x) \coloneqq e^{-x^2}\int_0^x e^{t^2}dt = \int_0^\infty e^{-t^2}\sin(2xt)dt = x\ {}_1F_1\left(1,\frac{3}{2};-x^2\right) = xe^{-x^2}\ {}_1F_1\left(\frac{1}{2},\frac{3}{2};x^2\right)$$

The appropriate related Mellin transform formulas are given by ((GrI) 7.612)

$$\int_0^\infty x^s\ {}_1F_1(\alpha,\beta;-x)\frac{dx}{x} = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\Gamma(s)\frac{\Gamma(\alpha-s)}{\Gamma(\beta-s)} \ , \ 0 < Re(s) < Re(\alpha) \ ,$$

$$\int_0^\infty x^{\mu-1}e^{-\beta x^2}\sin(\gamma x)dx = \frac{\gamma e^{-\frac{\gamma^2}{4\beta}}}{2\beta^{\frac{\mu+1}{2}}}\ {}_1\Gamma(\frac{1+\mu}{2})F_1(1-\frac{\mu}{2};\frac{3}{2};\frac{\gamma^2}{4\beta}) \ , \ Re(\beta) > 0 \ , \ Re(\mu) > -1$$

leading to e.g., $\frac{1}{2}\int_0^\infty x^{s/2}\ {}_1F_1\left(\frac{1}{2},\frac{3}{2};-x\right)\frac{dx}{x} = \frac{\frac{1}{2}\Gamma\left(\frac{s}{2}\right)}{1-s} = \frac{\Gamma\left(1+\frac{s}{2}\right)}{s(1-s)} = \frac{\Pi\left(\frac{s}{2}\right)}{s(1-s)}$, $0 < Re(s) < 1$. It indicates a replacement of the Gauss „Gamma“ function definition ((EdH) p.8)

$$\Pi\left(\frac{s}{2}\right) \coloneqq \Gamma\left(1+\frac{s}{2}\right) = \frac{s}{2}\Gamma\left(\frac{s}{2}\right) \qquad \to \qquad \Gamma^*\left(1+\frac{s}{2}\right) \coloneqq \Gamma\left(\frac{s}{2}\right)\tan\left(\frac{\pi}{2}s\right) = \frac{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} = \frac{\Gamma\left(1+\frac{s-1}{2}\right)\Gamma\left(1-\frac{s+1}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} = \frac{2}{\pi}\sum_{k=1}^\infty\frac{\Gamma\left(1+\frac{s}{2}\right)}{(k-\frac{1}{2})^2-(\frac{s}{2})^2}.$$

We note the formula ((GRI) 3.511, 8.332) $\frac{2}{\pi}\int_{-\infty}^\infty\left|\Gamma\left(\frac{1}{2}+it\right)\right|^2 dt = \frac{2}{\pi}\int_{-\infty}^\infty\frac{\pi}{\cosh(\pi t)}dt = 1$, i.e. $\Gamma(\frac{1}{2}+it) \in L_2(-\infty,\infty)$.

Formally it also holds

$$\int_0^\infty x^{-s}\left[\left(-\frac{d}{dx}x^2\frac{d}{dx}\right)G(x)\right]dx = s(1-s)\int_0^\infty x^{-s}G(x)dx.$$

It implies that the invariant operator $x^{-s}\to\int_0^\infty x^{-s}G(x)dx$ is formally self-adjoint with the transform $2\xi(s)/(s(s-1))$. But this operator has no transform at all as the integrals do not converge, due to the not vanishing constant Fourier term of the Poisson summation formula ((EdH) 10.3). Replacing $f(x)\to\ f_H(x):=M[f](x)$ leads to an alternative entire Zeta function $\xi^*(s)$ in the form

$$\xi^*(s):=\frac{1}{2}(s-1)\pi^{\frac{1-s}{2}}\Gamma\left(\frac{s}{2}\right)tan\left(\frac{\pi}{2}s\right)\cdot\zeta(s)=\zeta(s)\cdot M\left[\frac{d}{dx}[-x\cdot f_H(x)]\right](s)$$

with same zeros as $\xi(s)$, as it holds $s(1-s)\xi^*(s)\xi^*(1-s)=\pi\xi(s)\xi(1-s)$.

A similar situation is valid, if the duality equation is built on the fractional part function ([TiE] 2.1).

The Mellin transforms in the critical stripe for the distributional Fourier series representation of the cot −function in a distributional H_{-1} −sense are given by [(*)]

$$M[Cot^*](s)=\boldsymbol{\zeta}(\mathbf{1}-\boldsymbol{s})\cdot tan\left(\frac{\pi}{2}s\right)=\boldsymbol{\zeta}(\mathbf{1}-\boldsymbol{s})\cdot cot\left(\frac{\pi}{2}(1-s)\right)$$

$$M\left[\frac{1}{x}Cot^*\left(\frac{1}{x}\right)\right](s)=M[Cot^*](1-s)=\boldsymbol{\zeta}(\boldsymbol{s})\cdot cot\left(\frac{\pi}{2}s\right).$$

[(*)] The Bagchi Hilbert space based RH criterion is dealing with the fractional part function. Its Hilbert transform is given by

$$g(x):=\ln\left(2\sin\left(\frac{x}{2}\right)\right)=-\sum_{n=1}^{\infty}\frac{\cos(nx)}{n},$$

which is an element of H_0. Therefore, its related Clausen integral ((AbM) 27.8) is an element of H_1, and its first derivative, $\frac{1}{2}\cot\left(\frac{x}{2}\right)$ resp. $\cot(\pi x)$, joins the Zeta function on the critical line as an element of H_{-1}. The H_{-1} Hilbert space corresponds to the weighted l_2^{-1} −space as considered in (BhB). As $g(x)\in H_0=H_0^*$, it holds

$$(g,v)_0\cong(g',v)_{-\frac{1}{2}}=(S^1[g],v)_{-\frac{1}{2}}=(Cot,v)_{-1/2}<\infty,\ \ \forall v\in H_0$$

i.e. the formally derived Fourier series representation of

$$Cot(x)=\sum_{n=1}^{\infty}\sin(nx)\quad\text{resp.}\quad Cot^*(x)=2\sum_{n=1}^{\infty}\sin(2\pi nx)$$

is defined in a distributional H_{-1} −sense (see also (BeB) (17.12) (17.13)). For $a>0$ and $0<|Re(s)|<1$ it holds ((GrI) 3.761)

$$\int_0^\infty x^s\sin(a)\frac{dx}{x}=\frac{\Gamma(s)}{a^s}\sin\left(\frac{\pi}{2}s\right)\quad,\quad\int_0^\infty x^s\cos(a)\frac{dx}{x}=\frac{\Gamma(s)}{a^s}\cos\left(\frac{\pi}{2}s\right).$$

Therefore the Mellin transforms of the H_{-1} − distributional Fourier series representation of the $Cot^{(*)}$- resp. $G_H(x)$ − functions are given by

$$M[Cot](s)=\Gamma(s)\sin\left(\frac{\pi}{2}s\right)\zeta(s)\text{ resp. }M[Cot^*](s)=2(2\pi)^{-s}\Gamma(s)\sin\left(\frac{\pi}{2}s\right)\zeta(s)$$

$$M[G_H(x)](s)=2\sqrt{\pi}\sum_1^\infty\int_0^\infty\int_0^\infty x^s e^{-\pi t^2}sin(2\pi nxt)\,dt\frac{dx}{x}=2M\left[\sum_1^\infty f_H(nx)\right](s)$$

$$=\sqrt{\pi}\int_0^\infty e^{-\pi t^2}\int_0^\infty x^sCot^*(tx)\frac{dx}{x}dt=\sqrt{\pi}\left[\int_0^\infty t^{1-s}e^{-\pi t^2}\frac{dt}{t}\right]\cdot\left[\int_0^\infty x^sCot^*(x)\frac{dx}{x}\right]=\pi^{\frac{s}{2}}\Gamma\left(\frac{1-s}{2}\right)\cdot M[Cot^*](s)$$

In combination with the functional equation of the entire Zeta function in the form $\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{\pi}{2}s\right)\Gamma(1-s)\zeta(1-s)$ ((TiE) (2.1.1)) this leads to

$$M[Cot^*](s)=\boldsymbol{\zeta}(\mathbf{1}-\boldsymbol{s})\cdot tan\left(\frac{\pi}{2}s\right)\ \ ,\ M\left[\frac{1}{x}Cot^*\left(\frac{1}{x}\right)\right](s)=M[Cot^*](1-s)=\boldsymbol{\zeta}(\boldsymbol{s})\cdot cot\left(\frac{\pi}{2}s\right).$$

On the critical line $s=\frac{1}{2}+it$ holds $M[Cot^*](s)\cdot M[Cot^*](1-s)=\zeta(s)\cdot\zeta(1-s)\cdot[1+tanh^2(\pi t)]$ because of

$$\sin\left(\frac{\pi}{2}s\right)=\frac{1}{\sqrt{2}}\left[\cosh\left(\frac{\pi}{2}t\right)+i\cdot\sinh\left(\frac{\pi}{2}t\right)\right]\text{ and }|\Gamma(s)|^2=\frac{\pi}{\cosh(\pi t)}$$

$$\cot\left(\frac{\pi}{2}s\right)=\tan\left(\frac{\pi}{2}(1-s)\right)=1-i\cdot\tanh(\pi t)=1-2i\cdot\sum_{k=1}^{\infty}(-1)^k e^{-2kt}\ \ (\ t>0\ )$$

$$\cot\left(\frac{\pi}{2}(1-s)\right)=\tan\left(\frac{\pi}{2}s\right)=1+i\cdot\tanh(\pi t)=1+2i\cdot\sum_{k=1}^{\infty}(-1)^k e^{-2kt}\ \ (\ t>0\ )$$

From (TiE) 4.14)), (ObF) p. 182, and (EsR) p. 139, we recall the formulas

$$\zeta(s)-\sum_{n<x}n^{-s}=\sum_{n>x}n^{-s}=-\frac{1}{2i}\int_{x-i\infty}^{x+i\infty}z^{1-s}\cot(\pi z)\frac{dz}{z}\ ,\ \ Re(s)>1;$$

$$M\left[\frac{1}{\pi}\frac{x^n}{1-x}\right](s)=cot(\pi s)\text{ (principle value) }\ -n<Re(s)<1-n\ ,\ n=0,\pm1,\pm2,\dots$$

$$F.p.(P.v.\int_0^\infty\frac{x^\alpha}{1-x}dx=\begin{cases}0 & ,\alpha\in Z\\ \pi\cot(\pi\alpha) & ,else\end{cases}.$$

They are related to the operator $x^{-s} \to \int_0^\infty x^{-s} G_H(x)dx$ (in a distributional H_{-1} –sense) by

$$M[G_H(x)](s) = \pi^{\frac{s}{2}} \Gamma(\frac{1-s}{2}) M[Cot^*](s)$$

whereby it holds

$$M[-xG'_H(x)](s) = s[G_H(x)](s) \ , \ M[(xG_H)'(x)](s) = (1-s)[G_H(x)](s).$$

The Polya criterion is about the approximation of the Mellin transform integral over the half-line $(0, \infty)$ by integrals over *finite intervals* to obtain a theorem about zeros of the Mellin transforms ((EdH) 12.5), (PoG). The Mellin transform $M[G_H(x)](s)$ is a Müntz type representation, i.e. in a classical framework the Polya criterion cannot be applied.

We note the similar structure between the Polya RH criterion the automodel criterion ((EsR) p.57). The functions $k(x) := cot(x)$ resp. $h(x) := \frac{1}{x} cot\left(\frac{1}{x}\right)$ are slow varying functions (automodels) of order zero ((EsR) p.57) (*). Other slow varying functions are $-logx$ at $x = 0^+$ or $-\log(1-x)$ at $x = 1$ (SeE).

The functional analysis approach to prove the Prime Number Theorem (PNT) is based on Tauberian theorems, which are derived from the celebrated Wiener Tauberian theorem, that „*the closed linear hull of translates of a function f is the whole space L_1 if and only if its Fourier transform never vanishes*" (**).

In (PiS) Tauberian theorems for integral transforms are provided, which are of Mellin convolution type and whose kernels belong to suitable test function spaces. The result is based on the Wiener-Tauberian theorems for distributions as proven in (PiS1). In (ViJ) a corresponding functional analysis scheme for Tauberian problems is provided based on the Dirac delta measure δ_a ($(\delta_a, \varphi) = \varphi(a)$).

It is proposed to replace the formal delta series $(f_x, \varphi) = \sum_{n=0}^\infty c_n \delta_{\frac{n}{x}}$ to the numerical series $\sum_{n=0}^\infty c_n$ by $(f_x, \varphi)_{-1/2} < \infty$,$\forall \varphi \in H_{-1/2}$. Conceptually this goes along with a replacement of the „dual" relationship $L_1 \leftrightarrow L_\infty$ by $H_{-1/2} \leftrightarrow H_{1/2}$ (***). The latter Hilbert spaces are the appropriate framework for central functions in current Zeta function theory (****). For a corresponding generalized Mellin (integral) transformation in the form $F(s) = (f(x), x^s)_{-1/2}$ we refer to (ZeA).

In (ViJ) for „a quick distributional way to *(prove)* the prime number theorem" is provided. It is built on the Delta function representation of

$$\psi'(x) = \sum_{n \le x} \Lambda(n) \delta(x-n) \in H_{-\frac{1}{2}-\varepsilon} \qquad \left(\psi(x) = \sum_{n \le x} \Lambda(n) = \int_{a-i\infty}^{a+i\infty} \left[-\frac{\zeta'(s)}{\zeta(s)}\right] x^s \frac{ds}{s} \approx x\right)$$

whereby the generalized Mellin transform of $\sum_{n=1}^\infty \delta(x-n)$ ($Re(s) < 0$) is given by $\zeta(1-s)$ ((ZeA) 4.3).

(*) For $k(x) := cot(x)$ resp. $h(x) := \frac{1}{x} cot\left(\frac{1}{x}\right)$ it holds $\frac{xk'(x)}{k(x)} = -\frac{2x}{\sin(2x)}$ resp. $\frac{xh'(x)}{h(x)} = -1 + \frac{2/x}{\sin(2/x)}$;

(**) It is about the behavior of the function f, where the limit for the convolution integral $K[f](x)$ when $x \to \infty$ corresponds to $\hat{k}(0)$ ($\hat{k}$ denotes the Fourier transform of the kernel function $k(x)$);

(***) There is a similar differentiator between a proof of the PNT (****) (from which the convergence of the series $\sum_{n=1}^\infty \frac{\mu(n)}{n}$ can be derived) and a proof of the convergence of the series $\sum_{n=1}^\infty \frac{\mu(n)}{n} log(\frac{1}{n}) = 1$. Ikehara showed a Tauberian theorem for Dirichlet series in a L_1- framework, which is equivalent to the statement that $d\psi(x) \sim dx$ as a Cesaro average.

„The corresponding theorem goes deeper than the PNT, and from it the PNT can be easily derived" ((LaE) §160).

(****)

$$\rho(x) = x - [x] = \frac{1}{2} + \sum_{n=1}^\infty \frac{sin(2\pi nx)}{\pi n}, \ \rho_H(x) = \sum_{n=1}^\infty \frac{cos(2\pi nx)}{\pi n} = -\frac{1}{\pi} log(2\, sin(\pi x)), log\left(tan\left(\pi\frac{x}{2}\right)\right) \in L_2^\#(0,1),$$

$$\rho'_H(x) = -cot(\pi x) = -2\sum_{n=1}^\infty sin(2\pi nx), log'\left(tan\left(\pi\frac{x}{2}\right)\right) = \frac{\pi}{sin(\pi s)} \in H_{-1}^\#(0,1),$$

$$\|\Xi\|_{-1}^2 = \sum_{n=1}^\infty \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} = \int_0^1 \frac{log\, x}{x-1} dx = \left[\sum_{n=1}^\infty \frac{\mu(n)}{n^2}\right]^{-1}, \text{ i.e. } \Xi \in l_2^{-1}$$

(****) (EdH) 12.7: „The PNT is about the asymptotics equivalence of $\psi(x) = \sum_{n<x} \Lambda(n) \sim x$, which is equivalent to the statement that $d\psi(x) \sim dx$ as a Cesaro average in the context of Tauberian theorems. Hardy-Littlewood were able to prove the PNT by showing $d\psi(x) \sim dx$ as an Abel average, where a significant amount of work is done by a Tauberian theorem."

With the notation of [LaE] §227, Satz 40, the convergent Dirichlet series

$$f(s) := \sum_1^\infty a_n e^{-s\log n} \quad g(s) := \sum_1^\infty b_n e^{-s\log n} \text{ , for } s > 0$$

are linked to the (distributional) Hilbert space $H_{-1/2}^{\#} \cong l_2^{-1/2}$ by ((EdH) 9.8, (NaS))

$$((f,g))_{-1/2} := \lim_{\omega\to\infty} \frac{1}{2\omega}\int_{-\omega}^{\omega} f(1/2+it)g(1/2-it)dt = \sum_1^\infty \frac{1}{n} a_n b_n .$$

A corresponding distributional counterpart of the Snirelmann density $\lim_{n\to\infty}\frac{A(n)}{n}$ could be $\sum_{n=1}^\infty \frac{1}{n} a_n^2 = \|A\|_{-1/2}^2$ with $A = (a_n)_{n\in N} \in l_2^{-1/2}$. It puts another light on the dispersion method in Binary Additive Problems, where the binary Goldbach problem is inaccessible in the given form (LiJ).

What cannot derived from the PNT is the convergence of the series $\sum_{n=1}^\infty \frac{\mu(n)}{n} log(\frac{1}{n}) = 1$. F
For the Zeta function on the critical line $\Xi(t) := \zeta\left(s = \frac{1}{2} + it\right) := \sum_{n=1}^\infty \frac{1}{n^s}$ it holds

$$\Xi \in H_{-\frac{1}{2}-\varepsilon} \quad \text{resp.} \quad \|\Xi\|_{-1/2}^2 = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\Xi(t)|^2 dt = \sum_{n=1}^\infty \frac{1}{n} = \zeta(1) = \infty.$$

Putting $w(t) := \sum_{n=1}^\infty \log\left(\frac{1}{n}\right)\frac{\mu(n)}{n^s}$ it holds $((\Xi, w))_{-1/2} = 1$ from which is follows $w \in H_{-\frac{1}{2}+\varepsilon}$.

A central piece of the alternatively proposed elements is the replacement of the Dirac function domain $H_{-\frac{1}{2}-\varepsilon}$ by $H_{-\frac{1}{2}}$. In order to avoid the usage of the Hardamard distribution function $\psi'(x)$ with the Dirac function domain $H_{-\frac{1}{2}-\varepsilon}$ for a distributional way to prove the PNT we shall use distribution functions with a $log(\frac{x}{n})$ structure in combination with point measures, defined by the imaginary part values of the zeros of the Kummer function ${}_1F_1\left(\frac{1}{2},\frac{3}{2};2\pi iz\right)^{(*)}$. The considered Hilbert space in (BaB) is about of all sequences $a = \{a_n | n \in N\}$ of complex numbers such that

$$\sum_{n=1}^\infty \omega_n |a_n|^2 < \infty \quad \text{with} \quad \frac{c_1}{n^2} \le \omega_n \le \frac{c_2}{n^2}$$

which is isomorph to the Hilbert space $H_{-1} \cong l_2^{-1}$. The real part values of the zeros of the considered Kummer function ${}_1F_1\left(\frac{1}{2},\frac{3}{2};2\pi iz\right)$ (alternatively to $e^{2\pi inx}$) enjoy appropriate behaviors.

Putting $0 < \omega_0 := 1 - \omega_1 < \frac{1}{2}$ we consider the sequences $s_n^{(1)} := \frac{2\omega_n}{2n}$, $s_n^{(2)} := \frac{\omega_{n-1}+\omega_n}{2n-1}$ fulfilling $^{(*)}$

$$min\left\{\frac{2n-1}{2n},\frac{2n}{2n+1}\right\} = 1 - \frac{1}{2n} < s_n^{(1)}, s_n^{(2)} < 1 .$$

The related integer subsets $F_{1;2n} := \{\{[\omega_{n-1}+\omega_n]\}|n \in N\} = \{\{1,\{2n\}\}|n \in N\}$, $F_{2n-1} := \{\{[2\omega_n]\}|n \in N\} = \{\{2n-1\}|n \in N\}$ do have the Snirelmann density $\sigma(F_{2n-1}) = \sigma(F_{2n}) = \frac{1}{2}$. They enable the definition of (binary additive) distributional functions in the form

$$\sum_{n\le x} a_n\, log(s_n^{(1)}\frac{x}{n}) + \sum_{n\le x} a_n\, log(s_n^{(2)}\frac{x}{n}) \quad \text{or} \quad \sum_{n\le x} a_n\left[\log\left(\frac{x}{2\omega_n}\right) + \log\left(\frac{x}{\omega_{n-1}+\omega_n}\right)\right].$$

We note that the Snirelmann density is sensitive to the first values of a set. This is why the subset of even integers has a Snirelmann density zero, while the subset of odd integers has Snirelmann density $\frac{1}{2}$ $^{(**)}$. For $\vartheta(x) = \sum_{n\le x}\Lambda(n)\, log(\frac{x}{n})$ we further note the asymptotics (KoJ) (ViJ)

$$\lim_{\lambda\to\infty}\vartheta'(\lambda x) = \frac{d}{dx}\left[\sum_{n=1}^\infty \Lambda(n)\, log(\frac{\lambda x}{n})\right] = \lim_{\lambda\to\infty}\frac{\psi(\lambda x)}{\lambda x} = 1 .$$

$^{(*)}$ For the real part values ω_n of the zeros of ${}_1F_1\left(\frac{1}{2},\frac{3}{2};2\pi iz\right)$ it holds (SeA) $2n-1 < 2\omega_n < 2n < \omega_n + \omega_{n+1} < 2n+1 < 2\omega_{n+1} < 2(n+1)$ and the sequences $2\omega_n$ and $\omega_n + \omega_{n+1}$ fulfill the Hadamard gap condition

$$\frac{\omega_{n+1}}{\omega_n} > \frac{n+\frac{1}{2}}{n} = 1 + \frac{1}{2n} > q > 1 \quad \text{resp.} \quad \frac{\omega_{n+1}+\omega_{n+2}}{\omega_n+\omega_{n+1}} > \frac{2n+2}{2n+1} = 1 + \frac{1}{2n+1} > q > 1.$$

$^{(**)}$
We mention the theorem of Kakeya (HuA) from which is follows that all zeros of $\sum_{k=1}^n s_k x^k = 0$ lie in the circular disk $\frac{1}{2} < |x| < 1$.
We further mention the relationship to the uniform distribution of numbers mod 1(WeH).

The arithmetical functions $\mu(n)$, $\varphi(n)$ are multiplicative, but not completely multiplicative ($\mu(4) = 0 \neq 1 = \mu(2)\mu(2)$, $\varphi(4) = 2 \neq 1 = \varphi(2)\varphi(2)$), ((ApT) pp. 36, 88).

The following related asymptotics are valid ($x \geq 2$), ((LaE1) (*), (ApT) (**), (ScW) p. 216),

$$\sum_{n\leq x}\frac{\Lambda(n)}{n} \sim \log x = \int_1^x d(\log t) \sim \Phi(x) = \sum_{n\leq x}\frac{1}{\varphi(n)} \sim \sum_{p\leq x}(1-\frac{1}{p})^{-1}$$

$$\sum_{n\leq x}\frac{\mu(n)}{n\log n} \sim \log(\log x) = \int_1^{\log x} d(\log t) \sim \sum_{p\leq x}\frac{1}{p} \quad .$$

For $\sigma(x) := \sum_{n\leq x}\frac{\mu(n)}{n}\log\left(\frac{x}{n}\right)$ and $A(x) := \sum_{n\leq x}\frac{\mu(n)}{n}$ it holds $A(x) = 0(1)$ ((ApT) p. 71) and

$$\sigma(xy) + 1 = \sigma(x) + \sigma(y) \quad , \quad \sigma'(x) = \frac{1}{x}\sum_{n\leq x}\frac{\mu(n)}{n} \sim \frac{1}{x}$$

and the inverse mapping is given by

$$\sigma^{-1}(x) = \sum_{n\leq x}\frac{1}{n}\log\left(\frac{x}{n}\right) \;,\; x \geq 1 \;.$$

Putting

$$\sigma^*(x) := \sum_{\substack{n\in F_{1;2n} \\ n\leq x}}\frac{\mu(n)}{n}\log\left(s_n^{(2)}\frac{x}{n}\right) + \sum_{\substack{n\in F_{2n-1} \\ n\leq x}}\frac{\mu(n)}{n}\log\left(s_n^{(1)}\frac{x}{n}\right)$$

$$= \log x + \sum_{\substack{n \text{ odd} \\ n\leq x}}\frac{\mu(n)}{n}\log\left(s_n^{(2)}\frac{x}{n}\right) + \sum_{\substack{n \text{ even} \\ n\leq x}}\frac{\mu(n)}{n}\log\left(s_n^{(1)}\frac{x}{n}\right) \;.$$

it follows ($x \geq 1$)

$$\sigma^*(xy) = \sigma^*(x) + \sigma^*(y) \quad , \quad \sigma^{*\prime}(x) = \frac{1}{x} + \sigma'(x) = \frac{1}{x} + \frac{1}{x}\sum_{n\leq x}\frac{\mu(n)}{n} \sim \frac{1}{x} \;.$$

(*) With the notation of (LaE1) the prime pair (p,q) counting function $H(x)$ with the condition $p + q \leq x$ is given by

$$H(x) = \sum_{n=1}^{x} G_n = \sum_{p\leq x}\pi(x-p) \sim \frac{x}{\log x}\int_0^{x/2}\frac{dt}{\log t} \sim \frac{1}{2}\left(\frac{x}{\log x}\right)^2$$

The (improved) Stäckel formula (based on the Euler $\varphi(n)$ –function) shows the asymptotics in the form

$$\tilde{G}_{2n} \sim \frac{1}{2}\frac{105\cdot\zeta(3)}{\pi^4}\frac{n}{\log n}\cdot\frac{n}{\log n}\cdot\frac{1}{\varphi(n)} \sim 0{,}648\ldots\cdot\frac{1}{\varphi(n)}\cdot\frac{n}{\log n}\cdot\frac{n}{\log n}$$

It is based on the analysis of

$$\Phi(x) := \sum_{n\leq x}\frac{1}{\varphi(n)} = \Phi_1(x) + \Phi_2(x) := \sum_{\substack{n\leq x \\ n\ odd}}\frac{1}{\varphi(n)} + \sum_{\substack{n\leq x \\ n\ even}}\frac{1}{\varphi(n)}$$

leading to the asymptotics

$$\Phi_1(x) = C\cdot\log x + c_1 + O\left(\frac{\log x}{x}\right) \;,\; \Phi_2(x) = 2C\cdot\log x + c_2 + O\left(\frac{\log x}{x}\right)$$

with

$$C := \frac{105\cdot\zeta(3)}{2\pi^4} \sim 0{,}648\ldots \sim \frac{1}{2} \;.$$

Related to the Stäckel formula we recall the following estimates

$$\frac{\sigma(n)}{n^2} \leq \frac{1}{\varphi(n)} \leq \frac{\pi^2}{6}\frac{\sigma(n)}{n^2} = \zeta(2)\frac{\sigma(n)}{n^2} \quad , n \geq 2$$

whereby $\sigma(n) = \sigma_1(n)$ denotes the sum of the divisors of n ((ApM) pp. 38, 71).

(**) We note that the product $\varphi(n) = n\prod_{p|n}(1-\frac{1}{p})$ is empty for $n = 1$ since there are no primes which divide 1; $\varphi(n)$ is even for $n \geq 3$ and $\varphi(p) = p - 1$.

Some values of arithmtical functions:

n :	1	2	3	4	5	6	7	8	9	10	….
$\Lambda(n)$:	0	$\log 2$	$\log 3$	$\log 2$	$\log 5$	0	$\log 7$	$\log 2$	$\log 3$	0	….
$\mu(n)$:	1	-1	-1	0	-1	1	-1	0	0	1	….
$\varphi(n)$:	1	1	2	2	4	2					….

$\varphi(12) = 4$, $\varphi(60) = 16$, $\varphi(81) = 54$

From (ApM) p. 38, 71, we recall the following estimates

$$\frac{\sigma(n)}{n^2} \leq \frac{1}{\varphi(n)} \leq \frac{\pi^2}{6}\frac{\sigma(n)}{n^2} = \zeta(2)\frac{\sigma(n)}{n^2} \quad , n \geq 2$$

whereby $\sigma(n) = \sigma_1(n)$ denotes the sum of the divisors of n.

The current tool trying to prove the tertiary and binary Goldbach conjecture is about the Hardy-Littlewood circle method. It is about a dissection of the circle $x = e^{2\pi i\alpha}$, or rather a smaller concentric circle, into „Farey arcs". The major arcs, or basic intervals, provide the main term in the asymptotic formula for the number of representations. Their treatment does not give rise to any very serious difficulties compared to the problems presented by the *„minor arcs"*, or *„supplementary intervals"*. The latter ones are analyzed by estimates of the Weyl (trigonometrical) sums

$$S(x) := \sum_n e^{2\pi inx}$$

without taking any (Goldbach) problem relevant information into account. We note that an asymptotic behavior in the form $O(N^{\frac{1}{2}+\varepsilon})$ of the Farey series is equivalent to to the Riemann Hypothesis (LaE5).

The Cesàro summable Fourier series representation (ZyA) VI-3, VII-1)

$$cot(\pi x) = 2\sum_{n=1}^{\infty} sin(2\pi nx)) \in H_{-1}^{\#}(0,1)$$

is related to the eigenfunctions $e^{2\pi inx} = e^{i\pi(2n)x}$. The proposed alternative Abel summable functions

$$cot^{(*)}(x) := \sum_{n=1}^{\infty} sin(\pi(2\omega_n)x) + sin\,(\pi(\omega_n + \omega_{n+1})x \in H_0^{\#}(0,1)$$

is related to the eigenfunctions pair $e^{i\pi(2\omega_n)x}$ and $e^{i\pi(\omega_n+\omega_{n+1})x}$ with corresponding alternative Weyl sums in the form

$$S_1^*(x) := \sum_n e^{i\pi(2\omega_n)x},\ \ S_1^*(x) := \sum_n e^{i\pi(\omega_n+\omega_{n+1})x}.$$

For the „weighted" $cot^{(*)}$ –function with the „alternative" harmonic numbers

$$2h_n := \sum_{k=1}^{n} \frac{2}{2k-1} = 2H_{2n} - H_n$$

the series

$$\sum \frac{2h_n}{n}(sin(2\pi\omega_n x) + sin\,(\pi(\omega_n + \omega_{n+1})x)$$

converges almost everywhere (*) (**).

The H_n are always fractions (except for $H_1 = 1$, $H_2 = 1.5$, $H_6 = 2.45$), the series is divergent, but the number n that the sum H_n past 100 is in the size of 10^{43}, i.e. a computer which takes 10^{-9} seconds to add each new term to the sum will have been completed in not less than 10^{17} (American) billion years (HaJ).

The extremly slow nondecreasing property on the interval $[1, n]$ might motivate the definition of an appropriate function to enable the corresponding Polya criterion (EdH) 12.5, (PoG)).

Alternatively to $\frac{\sin(\pi x)}{\pi x}$ the Fourier theory of cardinal functions enables a correspondingly absolute convergent cardinal series in the form (WhJ2)

$$C(x) := \frac{\sin(\pi x)}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{H_{2n} - \frac{1}{2}H_n}{x-n}\ .$$

(*) For $T(x) := -\frac{\pi}{2} log\left(tan\left(\pi\frac{x}{2}\right)\right)$ the following series representation holds true (ElL) $T(x) = \sum \frac{2h_n}{n} sin\,(\pi(2n)x) = \sum c_n\, sin\,(2\pi nx)$, whereby $\sum_{n=1}^{\infty} c_n^2 < \infty$ i.e. $T(x) \in L_2^{\#}(0,1)$ resp. the formal Fourier series representation of its first derivative $log'\left(tan\left(\pi\frac{x}{2}\right)\right) = \frac{\pi}{sin\,(\pi x)} \in H_{-1}^{\#}(0,1)$. The convergent series $\sum_{n=1}^{\infty} c_n^2 = \frac{\pi^4}{32} < \infty$ in combination with the

Lemma (KaM1): Let $\{n_k\}$ be a sequence of integers satisfying the "Hadamard gap" condition, i.e. $\frac{n_{k+1}}{n_k} > q > 1$. Then the trigonometric gap series $\sum_{k=1}^{\infty} c_k \sin(2\pi n_k x)$ converges almost everywhere, if and only if, $\sum_{k=1}^{\infty} c_k^2 < \infty$

then proves that the series $\sum \frac{2h_n}{n}(sin(2\pi\omega_n x) + sin\,(\pi(\omega_n + \omega_{n+1})x)$ converges almost everywhere.

(**) We note the related potency series in the form (ChH) $\frac{1}{2} log^2\left(\frac{1+x}{1-x}\right) = \sum_{n=1}^{\infty} \frac{2h_n}{n} x^{2n}$.

Additional references to PART I

(BiN) Bingham N. H., Goldie C. M., Teugels J. L., Regular variation, University Press, Cambridge, 1989

(DeJ) Derbyshire J., Prime Obsession, Joseph Henry Press, Washington, D. C., 2003

(HuA) Hurwitz A., Über einen Satz des Herrn Kakeya, Zürich, 1913

(LaE5) Landau E., Die Goldbachsche Vermutung und der Schnirelmannsche Satz, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Math.-Phys. Klasse, 255-276, 1930

(LaE6) Landau E., Über die Fareyreihe und die Riemannsche Vermutung, Göttinger Nachrichten (1932), 347-352

(ObF) Oberhettinger F., Tables of Mellin Transforms, Springer-Verlag, Berlin Heidelberg New York, 1974

(PiS) Pilipovic S., Stankovic B., Tauberian Theorems for Integral Transforms of Distributions, Acta Math. Hungar. 74, (1-2) (1997), 135-153

(PiS1) Pilipovic S., Stankovic B., Wiener Tauberian theorems for distributions, J. London Math. Soc. 47 (1993), 507-515

(ViJ) Vindas J., Introduction to Tauberian theory, a distributional approach, https://cage.ugent.be

(WeH) Weyl H., Über die Gleichverteilung von Zahlen mod. Eins, Math. Ann., 77, 1914, 313-352

(ZeA) Zemanian A. H., Generalized Integral Transformations, Dover Publications, Inc. New York, 1968

PART II

3D-NSE and YME mass gap solutions in a distributional Hilbert scale frame enabling a quantum gravity theory

A common Hilbert space framework for all quantum gravity related physical mathematical models requires common conceptual building principles for problem specific mathematical-physical PDE system models.

The following changes to current building principles are proposed:

1. a classical PDE system is an „only" approximation model to its corresponding physical relevant variational representation, and not the other way around

2. only the Hamiltonian formalism is valid, but not the Lagrange formalism (both formalisms are equivalent, if the Legendre transform is valid), because of only physical (energy related) relevant, but no longer mathematically (force related) assumed regularity assumptions to the variational solution. In this context *we note that „continuity" is one of the commonsense notions, which should be dropped out of the assumptions list of ground principles of the Universe (KaM) p. 12*)); consequently the physical concept of „*force*" stays to be a phenomenon of the considered PDE (problem specific physical model) system, but is no longer a conceptual element of the overall „physical world reality" (i.e. it is not a notion as part of the stage of theoretical physics).

3. The „Newton/Dirac" „point/particle *mass density*" concept (whereby the regularity of the Dirac „function" depends from the space dimension) ist being replaced by the „Leibniz/Plemelj" „ideal point/differential *mass element*" concept.

The proposed common distributional Hilbert space framework $H_{-1/2}$ resp. its corresponding energy dual space

$$\mathrm{H}_{1/2} = H_1 \otimes H_1^{\perp} = H_1^{repulsive} \otimes H_1^{attrac} \otimes H_1^{\perp}$$

enables a common (Zeta function and quantum gravity) spectral theory providing an answer to Derbyshine's question ((DeJ) p. 295) (*).

The („physical") Hilbert space pairs (H_0, H_1) resp. the („meta-physical") closed subspaces $H_0^{\perp}, H_1^{\perp})$ of (H_0, H_1) are being governed by Fourier waves resp. Calderón´s wavelets. The current "*symmetry break down*" model to generate matter is replaced by a "*self-adjointness break down*" effect defined by the orthogonal projection from $H_{1/2}$ onto H_1. Consequently, the (kinetic) energy driven „inverse" is a kind of entropy operator with a „discrete/compactly embedded" Hilbert space domain to its complementary closed subspace of $H_{1/2}$ (**).

We note that the $H_{1/2}$ space as first cohomology is fundamental to explain the properties of period mapping on the universal Teichmüller space (NaS).

We further note that the exterior Neumann problem admits one and only one generalized solution in case the related Prandtl operator of order one $\mathrm{P}: H_r \rightarrow H_{r-1}$ is defined for domains with $1/2 \leq r < 1$.

(*) "*The non-trivial zeros of Riemann's zeta function arise from inquiries into the distribution of prime numbers. The eigenvalues of a random Hermitian matrix arise from inquiries into the behavior of systems of subatomic particles under the laws of quantum mechanics. What on earth does the distribution of prime numbers have to do with the behavior of subatomic particles*?"

(**) *The continuous wavelet transform with the complex Shannon wavelet can be considered via solutions of Cauchy problems for PDE in the context of the construction of wavelets for an analysis of non-stationary wave propagation in inhomogeneous media ((PoE).*

The current quantum state Hilbert space $L_2 = H_0$ is extended to the Hilbert space $H_{-1/2}$, including „fluid", plasma", „fermion", „photon" and „boson" states. The dual space $\mathrm{H}_{1/2} = H_1 \otimes H_1^{\perp}$ of $H_{-1/2}$ provides the corresponding quantum energy space, whereby the „mass-less EPs" (hot plasma and photons/bosons) are (meta-physical, ground state (dark) energy) „elements" of the closed subspace $H_1^{\perp}$ of $\mathrm{H}_{1/2}$. The standard (variational) energy space H_1 is defined by the selfadjoint Friedrichs extension of the Laplacian operator in the standard $H_0 = L_2 -$ variational (statistics) framework. It keeps being valid for the quantum energy of the EPs *with* mass, including cold plasma.

The selfadjoint Friedrichs extension of the Laplacian operator defined on H_1 induces a decomposition of H_1 into the direct sum of subspaces, enabling the definition of a potential and a corresponding "*grad*" potential operator. Then a potential (barrier) criterion defines a manifold, which represents a hyperboloid in the Hilbert space $H_1 = H_1^{repulsive} \otimes H_1^{attrac}$ with corresponding hyperbolic and conical "fermions type" regions. The "attractive fermions" region might be interpreted as hyperspace. The „hyperboloid" concept enables a model of interacting fermions with velocity < light speed and interacting bosons with velocity > light speed.

We note that a vector space and any linear subspace are convex cones, i.e. the tool convex analysis and general vector spaces can be applied. Morse's calculus of variations in the large enables variation problems on manifolds resp. varifolds ((MoM), (SeH), (AlF)).

The central part to prove the well-posedness of the 2D non-linear, non-stationary Navier-Stokes equations is a proper energy norm inequality estimate. It do not lead to blow-up effects for $t = T$ and do not show a Serrin gap with respect to the corresponding Sobolev norm estimates. We note that the energy norm of the non-linear terms of the 2D-NSE vanishes, which is appreciated from a mathematical point of view, but seems to be questionable from a physical point of view. The corresponding analysis for the 3D-NSE fails due to not appropriate Sobolev norm estimates. The analog analysis in a $H_{-1/2}$ variational framework (including a not-vanishing non-linear energy term) works out well, due to the appropriate Sobolewski estimates.

The classical Yang-Mills theory is the generalization of the Maxwell theory of electromagnetism where chromo-electromagnetic field itself carries charges. As a classical field theory it has solutions which travel at the speed of light so that its quantum version should describe massless particles (gluons). However, the postulated phenomenon of color confinement permits only bound states of gluons, forming massive particles. This is the mass gap. Another aspect of confinement is asymptotic freedom which makes it conceivable that quantum Yang-Mills theory exists without restriction to low energy scales. A variational Maxwell equations representation in a $H_{-1/2}$ Hilbert space framework includes also "gluon" bosons and corresponding "*self-adjointness break downs*", i.e. there is no mass gap anymore.

The quantum gravity model also addresses the dilemma, as pointed out by E. Schrödinger: "*Since in the Bose case we seem to be faced, mathematically, with simple oscillator of Planck type, we may ask whether we ought not to adopt for half-odd integers quantum numbers rather than integers. Once must, I think, call that an open dilemma. From the point of view of analogy one would very much prefer to do so. For, the „zero-point energy*" of a Planck oscillator is not only borne out by direct observation in the case of crystal lattices, it is also so intimately linked up with the Heisenberg uncertainty relation that one hates to dispense with it.

The formalism of 2-"*spinors*" as an alternative to the standard vector-tensor calculus (Penrose R., Rindler W.) is proposed to be physically re-interpreted and mathematically applied in the context of a H_1-space decomposition into repulsive and attractive fermions subspaces, whereby it holds $spin(4) = SU(2)xSU(2)$.

„The two-component „spinor" calculus is a very specific calculus for studying the structure of space-time manifolds…. Space-time point themselves cannot be regarded as derived objects from spinor algebra, but a certain extension of it, namely the twistor algebra, can indeed be taken as more primitive than space-time itself. ... The programme of twistor theory, in fact, is to reformulate the whole of basic physics in twistor terms" (Penrose R., Rindler W. Volume II).

The point of departure for the twistor theory is the (classical) twistor equation (with a similar form as the continuity equation). Its corresponding weak variational represention with respect to the proposed $H_{-1/2}$ quantum state inner product leads to the Friedrichs extension of the classical Dirac spinor operator with domain $H_{-1/2}$, which is about the square root operator of order one of the Laplacian operator. The corresponding singular integral operator representation is about the Calderón-Zygmund integrodifferential operator ((EsG) example 3.5).

With respect to the NSE, the ground state (or dark) energy and quantum gravity topics we also refer to

http://www.navier-stokes-equations.com/

https://quantum-gravitation.de/

Additional references to PART II

(AlF) Almgren F. J., Plateau's Problem, An Invitation to Varifold Geometry, American Mathematical Society, New York, 2001

(AzT1) Azizov, T. Y., Iokhvidov, I. S., Dawson, E. R., Linear Operators in Spaces With an Indefinite Metric, Wiley, Chichester, New York, 1989

(DeJ) Derbyshire J. Prime Obsession, Joseph Henry Press, Washington, D. C., 2003

(EiA) Einstein A., Ether and the theory of relativity, An Address delivered on May 5th, 1920, in the University of Leyden

(EsG) Eskin G. I., Boundary Value Problems for Elliptic Pseudodifferential Equations, American Mathematical Society, New York, 1981

(GaA) Ganchev A. H., Greenberg W., van der Mee C. V. M., A class of linear kinetic equations in Krein space setting, Integral Equations and Operator Theory, Vol. 11, 518-535, 1988

(MoM) Morse M., Functional Topology and Abstract Variational Theory, Proc. N. A. S., 326-330, 1938

(PhR) Phillips R., Dissipative operators and hyperbolic systems of partial differential equations, Trans. Amer. Math. Soc. 90 (1959), 193-254

(PoE) Postnikov E. B., Singh V. K. Continuous wavelet transform with the Shannon wavelet from the point of view of hyperbolic partial differential equations, arXiv:1551.03082

(ScE) Schrödinger E., My View of the World, Cambridge University Press, Cambridge, 1961

(SeH) Seifert H., Threlfall W., Variationsrechnung im Grossen, Chelsea Publishing Company, New York, 1951

(HaJ) Hawley J., The Bhagavad Gita. A Walkthrough for Westerners, New World Library, Novato, California, USA, 2001