# An alternative entire Zeta function representation and its related duality equation

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The entire Zeta function and its related duality equation are given by ([EdH] 1.8)

$$\xi(s) := \frac{s}{2}\Gamma(\frac{s}{2})(s-1)\pi^{-s/2}\zeta(s) = \xi(1-s) \cdot$$

Let $H$ and $M$ denote the Hilbert and the Mellin transform operators. For the Gaussian function $f(x)$ it holds

$$M[f](s) = \frac{1}{2}\pi^{-s/2}\Gamma(\frac{s}{2}) \quad , \quad M[-xf'(x)](s) = \frac{s}{2}\pi^{-s/2}\Gamma(\frac{s}{2}) = \frac{1}{2}\pi^{-s/2}\Pi(\frac{s}{2})$$

leading to the entire Zeta function representation in the form

$$\xi(s) = (1-s)\cdot\zeta(s)M[-xf'(x)](s) \cdot$$

The central idea is to replace

$$M[-xf'(x)](s) \to M[f_H(x)](s)$$

with $f_H(x) := H[f](x)$, $\hat{f}_H(0) = 0$ and

$$M[f_H(x)](s) = 2\pi \cdot M\left[x \cdot {}_1F_1(1, \frac{3}{2}, -\pi x^2)\right](s) = \pi^{\frac{1-s}{2}}\Gamma(\frac{s}{2})\tan(\frac{\pi}{2}s) \quad .$$

A related alternative entire Zeta function (with same zeros as $\xi(s)$) can be defined by

$$\xi^*(s) := (1-s)\pi^{\frac{1-s}{2}}\Gamma(\frac{s}{2})\tan(\frac{\pi}{2}s)\cdot\zeta(s) = -\pi^{3/2}\frac{\tan(\frac{\pi}{2}s)}{\frac{\pi}{2}s}\xi(s) \quad .$$

With

$$\tan(\frac{\pi}{2}s) = \cot(\frac{\pi}{2}(1-s))$$

one gets the duality equation

$$\xi^{**}(1-s) := \frac{\pi}{2}(1-s)\cdot\cot(\frac{\pi}{2}(1-s))\cdot\xi^*(1-s) = \frac{\pi}{2}s\cdot\cot(\frac{\pi}{2}s)\cdot\xi^*(s) = \xi^{**}(s) \quad .$$

The replacement above is motivated by the challenge to formulate the term

$$\frac{2\xi(s)}{s(s-1)}$$

as a transform ([EdH] 10.5)

$$f(x) \to \int_0^\infty f(ux)G(u)du \quad .$$

The underlying issue is analog to the issue building the ("continuous") analog of Euler's ludicrous formula

$$\sum_{-\infty}^{\infty} x^n = (1+x+x^2+.....) + (x^{-1}+x^{-2}+....) = \frac{1}{1-x} + \frac{1}{x-1} = 0 \quad ,$$

which is

$$\int_0^\infty x^{-s}dx = \int_0^1 x^{-s}dx + \int_1^\infty x^{-s}dx = \frac{1}{1-s} - \frac{1}{1-s} = 0$$

The latter one is of course nonsense, because the values of $s$ for which the above integrals converge are mutually exclusively – the first integral being convergent for $\mathrm{Re}(s) < 1$ and the second integral being convergent for $\mathrm{Re}(s) > 1$ - but it does not suggest that the formal transform

$$f(x) \to \int_0^\infty f(ux)dx$$

is zero. The above issue leads to the well-defined analytical auxiliary function ([EdH] 10.3)

$$H(u) := \frac{d}{du}\left[u^2 \frac{d}{du}\left[\sum_{-\infty}^{\infty} e^{-\pi n^2 u^2}\right]\right]$$

fulfilling

$$-\int_0^\infty u^{1-s} H(u)\frac{du}{u} = -\int_0^\infty u^s H(u)\frac{du}{u} \ .$$

The underlying "trick" is basically about the replacement of

$$f(x) \to \tilde{f}(x) := -xf'(x)$$

in order to achieve $f(0) = \hat{f}(0) \neq \tilde{f}(0) = 0$, while keeping "somehow" the analytical properties of $f(x)$ (differentiate and multiply again with $x$). Unfortunately the corresponding self-adjoint operator with Mellin transform $2\xi(s)/(s(1-s))$ in the form

$$-\int_0^\infty u^{1-s}\left[\sum_{-\infty}^{\infty} e^{-\pi n^2 u^2}\right]\frac{du}{u} = -\int_0^\infty u^s\left[\sum_{-\infty}^{\infty} e^{-\pi n^2 u^2}\right]\frac{du}{u}$$

is only formally defined, as this operator has no transform at all, because the integral does not converge for any $s \in C$.

The central idea above is therefore nothing else then replacing $f(x)$ by an alternative function $\bar{f}(x)$, which ensures $\hat{\bar{f}}(0) = 0$ while keeping the analytical properties of $f(x)$ unchanged in a $L_2$ –sense. The underlying "purpose" of the "trick" above is fulfilled by appropriate properties of the Hilbert transform operator. The primary property is that the constant Fourier term of a Hilbert transformed function or distribution is vanishing.

The properties linking to the Berry conjecture are related to the Hermite polynomials and its corresponding Hilbert transforms. Both systems build an orthogonal system spanning the $L_2(-\infty,\infty)$ – Hilbert space, while each Hermite polynomial is orthogonal to its corresponding Hilbert transform. The basis Hermite polynomial $H_0(x)$ plays a specific role in the harmonic quantum oscillator (ground state energy) model, i.e. its corresponding Hilbert transform provides an appropriate alternative model.

As it further holds

$$\lim_{|x|\to\infty} xH[f](x) = \frac{1}{\pi}\int_R f(x)dx$$

it follows, that in case of $f(x)$ has non-zero mean, i.e. $\hat{f}(0) \neq 0$ then $H[f](x)$ only decays like $1/|x|$ at infinity. In particular it is not bounded on $L_1$. This property provides a link to the commutator concept, which plays also a key role in the theory of Pseudo-Differential Operators (PDO).

He calculation of the analogue series representation to the function $\xi(s)$ on the critical line ([EdH] 1.8)

$$\xi((\frac{1}{2}+it)=4\int_1^\infty \frac{d(x^{3/2}\left[\frac{d}{dx}\sum_1^\infty e^{-n^2\pi x}\right])}{dx}x^{-1/4}\cos(\frac{t}{2}\log x)dx=\sum_{n=0}^\infty a_{2n}(s-\frac{1}{2})^{2n}$$

whereby

$$a_{2n}:=4\int_1^\infty \frac{d(x^{3/2}\left[\frac{d}{dx}\sum_1^\infty e^{-n^2\pi x}\right])}{dx}x^{-1/4}\frac{(\frac{1}{2}\log x)^{2n}}{(2n)!}dx$$

is straightforward.

Remark: On the critical line it holds

$$\cot(\frac{\pi}{2}(\frac{1}{2}\pm it))=\frac{\cosh(\frac{\pi}{2}t)\mp i\sinh(\frac{\pi}{2}t)}{\cosh(\frac{\pi}{2}t)\pm i\sinh(\frac{\pi}{2}t)} \quad .$$

We further note the series representation ([GrI] 1.411, 1.518)

$$x\cdot\cot x=1-\sum_{k=1}^\infty\frac{2^{2k}|B_{2k}|}{(2k)!}x^{2k} \quad , \quad x^2<\pi^2$$

$$\log\frac{\tan x}{x}=\sum_{k=1}^\infty(-1)^{k+1}\frac{(2^{2k-1}-1)\cdot 2^{2k}|B_{2k}|}{k\cdot(2k)!}x^{2k} \quad , \quad x^2<\frac{\pi^2}{4} \quad .$$

From the duality equation one gets ([EdH] 1.13)

$$\log\zeta(s)=-\log(s-1)+\sum_\rho\log(1-\frac{s}{\rho})-\log(\Gamma(1+\frac{s}{2})+\log\xi(0)+\log(\pi^{s/2})$$

Riemann derived his famous approximation error function (see iii) below) from the identity

$$J(x):=\frac{1}{2}\left[\sum_{p^n<x}\frac{1}{n}+\sum_{p^n\le x}\frac{1}{n}\right]=-\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\log\zeta(s)}{s}\right]x^s ds$$

while proving the following mappings ([EdH] 1.13 ff.)

i) $-\log(s-1)\to Li(x)$

ii) $\sum_\rho\log(1-\frac{s}{\rho})\to\sum_{\mathrm{Im}(\rho)>0}\left[Li(x^\rho)+Li(x^\rho)\right]$

iii) $\log(\Gamma(1+\frac{s}{2})\to\int_x^\infty\frac{dt}{t\log t(t^2-1)}$

iv) $\log\xi(0)=-\log 2$

v) $\log(\pi^{s/2})\to 0 \quad .$

The alternative entire Zeta function representation enables a corresponding alternative approximation error function with alternative terms to iii) and iv) above.

## Reference

[EdH] Edwards H. M., Riemann´s Zeta Function, Dover Publications, Inc., Mineola, New York, 1974
[GrI] Gradshteyn I. S., Ryzhik I. M., Table of Integrals Series and Products, Fourth Edition, Academic Press, New York, San Francisco, London, 1965