Global Existence and Uniqueness

of the Non-stationary 3D-Navier-Stokes

Initial-boundary Value Problem

Dr. Klaus Braun
June, 2016

Abstract

We present a global unique weak $H_{-1/2}-$solution of the generalized 3D Navier-Stokes initial value problem (for all $v \in H_{-1/2}$)

$$(\dot{u}, v)_{-1/2} + (Au, v)_{-1/2} + (Bu, v)_{-1/2} = 0$$
$$(u(0), v)_{-1/2} = (u_0, v)_{-1/2} \qquad .$$

The global boundedness is a consequence of the Sobolevskii -estimate of the non-linear term ([SoP]) enabling the generalized energy inequality

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \leq |(Bu,u)_{-1/2}| \leq c \cdot \|u\|_{-1/2}\|u\|_1^2$$

leading to the a priori estimate

$$\|u(t)\|_{-1/2} \leq \|u_0\|_{-1/2} + \int_0^t \|u\|_1^2(s)ds \leq c\left\{\|u_0\|_{-1/2} + \|u_0\|_0^2\right\} \qquad .$$

The H_0-based 3D NSE model provides only potential flows to model appropriate boundary layer circulation. The lift of an airfoil in inviscid flow requires circulation in the flow around the airfoil, but a single potential function that is continuous throughout the domain around the airfoil cannot represent a flow with non-zero circulation. In §4 we address this challenge by linking the Prandtl operator with physical $H_{-1/2}-$space concept.

Kolmogorov's turbulence theory is a purely statistical model which describes the qualitative behavior of turbulent flows. There is no linkage to the quantitative fluid behavior as it is described by the Euler or the Navier-Stokes equations. The physical counterpart of his low- and high-pass filtering Fourier coefficients analysis is a "local Fourier spectrum", which is a contradiction in itself, as, either it is non-Fourier, or it is nonlocal ([FaM]). In §5 we combine the wavelet based solution concept of [FaM] with a revisited CLM equation model in a physical $H_{-1/2}-$Hilbert space framework to enable a turbulent $H_{-1/2}-$signal which can be split into two components: coherent bursts and incoherent noise. The model enables a localized Heisenberg uncertainty inequality in the closed (noise) subspace $H_{-1/2} - H_0$, while the momentum-location commutator vanishes in the (coherent bursts) test space H_0.

In §6 we apply the underlying idea of §5 (a modified momentum operator with extended $H_{1/2}-$domain) to derive optimal FEM error estimates for the Stefan problem with non-regular initial value data.

§1 Introduction

This section is a summary from [CaM], [GaG], [GiY], [ShM] , [SoH], [TeR].

The Navier-Stokes Equations (NSE) describes a flow of incompressible, viscous fluid. The three key foundational questions of every PDE is existence, and uniqueness of solutions, as well as whether solutions corresponding to smooth initial data can develop singularities in finite time, and what these might mean. For the NSE satisfactory answers to those questions are available in two dimensions, i.e. 2D-NSE with smooth initial data possesses unique solutions which stay smooth forever. In three dimensions, those questions are still open. Only local existence and uniqueness results are known. Global existence of strong solutions has been proven only, when initial and external forces data are sufficiently smooth. Uniqueness and regularity of non-local Leray-Hopf solutions are still open problems.

Basically the existence of 3D solutions is proven only for "large" Banach spaces. The uniqueness is proven only in "small" Banach spaces. The question of global existence of smooth solutions vs. finite time blow up is one of the Clay Institute millennium problems.

The existence of weak solutions can be provided, essentially by the energy inequality. If solutions would be classical ones, it is possible to prove their uniqueness. On the other side for existing weak solutions it is not clear that the derivatives appearing in the inequalities have any meaning. Basically all existence proofs of weak solutions of the Navier-Stokes equations are given as limit (in the corresponding weak topology) of existing approximation solutions built on finite dimensional approximation spaces. The approximations are basically built by the Galerkin-Ritz method, whereby the approximation spaces are e.g. built on eigenfunctions of the Stokes operator or generalized Fourier series approximations.

It has been questioned whether the NSE really describes general flows: The difficulty with ideal fluids, and the source of the d'Alembert paradox, is that in such fluids there are no frictional forces. Two neighboring portions of an ideal fluid can move at different velocities without rubbing on each other, provided they are separated by a streamline. It is clear that such a phenomenon can never occur in a real fluid, and the question is how frictional forces can be introduced into a model of a fluid.

The question intimately related to the uniqueness problem is the regularity of the solution. Do the solutions to the NSE blow-up in finite time? The solution is initially regular and unique, but at the instant T when it ceases to be unique (if such an instant exists), the regularity could also be lost. Given a smooth datum at time zero, will the solution of the NSE continue to be smooth and unique for all time?

There is no uniqueness proof for weak solutions except for over small time intervals. The simplest possible model example how a singularity can appear, is the ODE

$$y'(t) = y^2(t) , \; y(0) = y_0$$

with the solution

$$y(t) = \frac{y_0}{1 - t \cdot y_0}$$

which becomes infinite in finite time.

Let P denote the orthogonal projection onto the kernel of the divergence operator and A the selfadjoint Stokes operator. If u is divergence free then

$$(Bv, v) := ((u, \nabla)v, v)_0 := \iint_\Omega (u, \nabla)v \cdot v\, dx = 0.$$

Therefore there is no contribution of the essential non-linear NSE term Bu to the a priori (energy) estimate

$$\|u(t)\|_0^2 + 2\int_0^t \|u\|_1^2 (\tau) d\tau = \|u_0\|_0^2.$$

Especially there is no evidence that the energy norm $\|u(t)\|_1$ is bounded in case of space dimension n=3. This is due to the fact that the right hand side of the corresponding a priori estimate

$$\frac{1}{2}\frac{d}{dt}\|u\|_{2\sigma}^2 + \|u\|_{2\sigma+11}^2 \leq \left|(A^{\sigma-1/2}Bu, A^{\sigma+1/2}u)\right| \quad, \quad 0 < \sigma \leq 1/2$$

depends on the space dimension. It holds

$$(v \cdot \nabla v, P\Delta v) \leq C\|v\|_q^{2q/(q-n)}|v|_{1.2}^2 + \frac{\nu}{2}\|P\Delta v\|_2^2 \quad \text{for all} \quad q \in (n, \infty) \quad .$$

The Gronwall inequality implies global boundedness in case of $n = 2$.

For $n \geq 3$ there is up to now no global boundedness result.

§2 Stokes operator, Hilbert scales and a priori estimates

The content of this section is basically taken from [GiY]. With c we denote numeric constants which may have different values at different places.

Let P be the orthogonal projection operator of $(L_2(\Omega))^n$ onto the divergence free vector field H_ω consisting of all solenoidal vector functions u, i.e. the operator is an orthogonal projection onto the kernel of the divergence operator. It is a Pseudo-Differential operator (PDO) of degree zero [(EsG)]. The Stokes operator A is a selfadjoint operator in H_σ, being the Friedrichs extension of the non-negative symmetric operator $-P\Delta$ in H_ω defined for all $u \in C^2$ with $div u = 0$ and $u_{n|\partial\Omega} = 0$. The Stokes operator enables the definition of a related Hilbert scale ($\alpha \in R$) with corresponding norm ([NiJ])

$$\|u\|_\alpha := \left\|A^{\alpha/2}u\right\| \; .$$

Throughout this paper, if not explicitly mentioned, we assume $p = 2$ and $n = 3$ for $(L_p(\Omega))^n$.

Using the Stokes operator and its related Hilbert scale framework the Navier-Stokes equations can be represented as an evolution equation in H_0. Since $P(grad p) = 0$ one gets

$$Au = Pf \quad \text{in } H_0$$

.

Putting $B(u) := P(u, grad)u$ and assuming $Pu_0 = u_0$ the NSE initial-boundary equation is given by

$$(*) \qquad \frac{du}{dt} + Au + Bu = Pf \;\; , \; u(0) = u_0 \; .$$

As u is divergence free and $u \cdot v$ identically vanishes on $\partial\Omega$ one gets

$$b(u, v, w) := ((u, grad)v, w) = \iint_\Omega (u, grad)v \cdot w dx = -b(u, w, v)$$

and especially $b(u, v, v) = 0$.

The linear homogeneous part of (*) enables the definition of a semigroup generated by the Stokes operator which leads to the representations

$$u(t) = e^{-tu_0} + \int_0^t e^{-(t-s)A} Pf(s)ds + \int_0^t e^{-(t-s)A} Bu(s)ds \; .$$

Multiply u to both sides of the homogeneous equation of (*) and integrate over Ω one gets

$$(\dot{u}, u) + (Au, u) + (Bu, u) = 0 \;\; , \; u(x,0) = u_0 \; .$$

This yields

$$\frac{d}{dt}\|u\|^2 + 2\left\|A^{1/2}u\right\|^2 = 0$$

respectively integrating over $[0,t]$

$$\|u(t)\|_0^2 + 2\int_0^t \|u\|_1^2(\tau)d\tau = \|u_0\|_0^2 .$$

A priori estimates for higher space derivatives of u are derived by multiplying the homogeneous equation (*) with $A^{2\sigma}u$ leading to

$$(\frac{du}{dt}, A^{2\sigma}u) + (Au, A^{2\sigma}u) + (Bu, A^{2\sigma}u) = 0$$

resp.

$$\frac{1}{2}\frac{d}{dt}\|A^\sigma u\|^2 + \|A^{\sigma+1/2}u\| + (A^{\sigma-1/2}Bu, A^{\sigma+1/2}u) = 0 .$$

Estimates of $A^{\sigma-1/2}Bu$ depend (heavily!) on the space dimension n, which is the root cause for the "still open NSE questions" for the space dimension $n = 3$. The essential a priori estimates are e.g. given in [GiY] (see also appendix)

$$(**) \qquad \frac{1}{2}\frac{d}{dt}\|u\|_1^2 + \|u\|_2^2 \cong \frac{1}{2}\frac{d}{dt}\|A^{1/2}u\|^2 + \|Au\|^2 \leq c\begin{cases} \|u\|^2\|A^{1/2}u\|^4 \cong \|u\|_0^2\|u\|_1^4 & n=2 \\ \|A^{1/2}u\|^6 \cong \|u\|_1^6 & n=3 \end{cases} .$$

Putting

$$y(t) := \begin{cases} \|u\|_{2\sigma}^2 & n=2 \quad 0<\sigma<1/2 \\ \|u\|_1^2 & n=3 \end{cases}$$

one gets

$$y'(t) \leq c \cdot \begin{cases} \|u\|_1^2 \cdot y(t) & n=2 \\ \|u\|_1^6 \cdot y^3(t) & n=3 \end{cases}$$

For $n = 2$ this leads to a global boundedness estimate in the form

$$z'(t) \leq c \cdot \|u\|_1^2 \cdot z(t) \quad \text{resp.} \quad z(t) \leq z(0) \cdot e^{c\int_0^t \|u\|_1^2(s)ds} .$$

The special case $\sigma = 1/2$ lead also to a global estimate, based on an argument to be found e.g. in ([TeR] 3.1).

For $n = 3$ every positive solution of $y'(t) = cy^3(t)$ blows up, i.e. there is no global estimate by this method.

§3 A global unique weak $H_{-1/2}$–solution of the NS initial value problem

Generalized functions on Hilbert spaces in combination with singular integral equations are successfully applied to problems of aerodynamics and electrodynamics ([LiL]). The corresponding framework is about Hilbert scales $H_{-\alpha}$ with negative scale factor $-\alpha < 0$ with corresponding Ritz-Galerkin approximation theory for Pseudo-Differential equations. In case of finite element approximation spaces this correlates to FEM resp. the BEM. The corresponding energy inner product with its related energy norm $\|u\|_{1/2}$ corresponds to the extended Green formulas, based on J. Plemelj's concept of an alternative normal derivative [PlJ]. Global and interior error estimates for the Ritz-Galerkin methods of Pseudo-Differential equations in the corresponding Hilbert space framework are e.g. provided in [BrK].

Hölder resp. Lipschitz spaces are the adequate ones in treating nonlinear elliptic problems. First a priori estimates and boundedness of the Ritz-Galerkin operator in Hölder resp. Lipschitz spaces for approximation spaces fulfilling the (AI) condition are given in [NiJ1].

The singular integral operators defined by the single-layer logarithmic and the normal derivative of the double layer logarithmic potential are bounded and selfadjoint with respect to the Hilbert spaces $H_{\mp 1/2}$ resp. bounded with respect to the corresponding Hölder spaces $C^{0,\alpha}, C^{1,\alpha}$, which are dense in those Hilbert spaces ([KrR]). In this sense there is a relationship between the spaces $H_{-1/2} \approx C^{0,\alpha}$ and $H_{1/2} \approx C^{1,\alpha}$. The regulatory results of single layer and double layer potentials with uniformly Hölder continuous densities can be extended to the three-dimensional case ([CoD].

In [ChF] a Nitsche-based domain decomposition method for the normal derivative of the double layer (hypersingular integral) operator equation has been analyzed. Due to the low regularity of the underlying energy space there is still the problem of non-existence of a well-posed continuous counterpart for the discrete BEM formulation. The approach below is closing this kind of "regularity gap" problem.

The approach below (Hilbert scale framework defined by appropriate self-adjoint operator, Plemelj regularity electric & magnetic boundary data assumptions) can also be applied to the Maxwell initial-boundary equations ([WeP], [WeP1]). In ([CoM] the corresponding BEM has been analyzed.

The Stefan problem can be transformed to nonlinear initial-boundary problem for the heat equation in a fixed domain. In case the initial data fulfill certain compatibility conditions the solution is "regular" resulting in corresponding optimal FEM convergence. In case of non-regular initial value function the approach below can also be applied to improve the non-optimal finite element approximation convergence factor $\beta < 1$ to the optimal factor $\beta = 1$ ([NiJ2]).

With respect to the Navier-Stokes vorticity equation we note that the Biot-Savart singular integral operator is a Pseudo-Differential operator of order -1 (as $A^{-1/2}$, see below). In [SaT] the spectral method is applied to analyze the related 1D -Constantin-Lax-Majda (CLM) equation with respect to the $L_2 = H_0$ –Hilbert space. A corresponding analysis with respect to any $H_{-\alpha}$ –Hilbert space is straight forward.

As $(Bu,u) = 0$, the nonlinear term of the NSE makes no contribution to the energy equality. Since the solution of the associated linearized equation is already as smooth as the data allow a solution of the nonlinear NSE cannot be expected to be smoother than the corresponding linearized equations. Based on this we shift the Hilbert scale framework of the weak NSE representation appropriately from $\alpha = 0$ to the left:

multiplying the homogeneous equation (*) with $A^{-1/2}u$ leads to

$$(\dot{u},u)_{-1/2}+(Au,u)_{-1/2}+(Bu,u)_{-1/2}=0 \ .$$

The corresponding generalized "energy" inequality is given by

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2+\|u\|_{1/2}^2\leq\left|(Bu,u)_{-1/2}\right|\leq\|u\|_{-1/2}\|Bu\|_{-1/2}\cong\|u\|_{-1/2}\left\|A^{-1/4}Bu\right\|_0 \ .$$

Applying lemma 3.2 of [GiY] (see also appendix and the original proof in [SoP] with $p=2$, $\delta=1/4$, $\theta:=\rho:=1/2$ fulfilling

$$\theta+\rho\geq\frac{1}{4}(n+1)=1$$

it follows

$$\left\|A^{-\delta}P(u,\,grad)u\right\|\leq c\left\|A^{\theta}u\right\|\cdot\left\|A^{\rho}u\right\|=c\|u\|_{2\theta}\cdot\|u\|_{2\rho}=c\|u\|_1^2 \ ,$$

resp.

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2+\|u\|_{1/2}^2\leq\left|(Bu,u)_{-1/2}\right|\leq c\cdot\|u\|_{-1/2}\|u\|_1^2$$

Putting $y(t):=\|u\|_{-1/2}^2$ one gets

$$y'(t)\leq c\cdot\|u\|_1^2\cdot y^{1/2}(t)$$

resulting into the a priori estimate

$$\|u(t)\|_{-1/2}\leq\|u(0)\|_{-1/2}+\int_0^t\|u\|_1^2(s)ds\leq c\left\{\|u_0\|_{-1/2}+\|u_0\|_0^2\right\}$$
,

which ensures global boundedness by the a priori energy estimate provided that $u_0\in H_0$.

By standard arguments the above estimate can be extended to the inhomogeneous case (*) based on corresponding shift theorems for the non-stationary Stokes operator (appendix)

$$\overline{A}w(t):=\dot{w}(t)+Aw(t)$$

$$w(0)=0$$

in the form

$$\left\|t^{\beta/2}w(t)\right\|_{\alpha+2}^2\leq c\left\|t^{\beta/2}\overline{A}w(t)\right\|_\alpha^2 \ , \quad \beta>-1$$

whereby

$$\left\|v(t)\right\|_\alpha^2:=\int_0^t\|v(s)\|_\alpha^2ds \ , \quad \alpha\in R.$$

§4 NSE circulation modelling enabled by the Prandtl and a modified Stokes operators

The solution concept for the proof in the previous section is built on the weak $H_{-1/2}$ – representation in the form

$$(\dot{u}, u)_{-1/2} + (Au, u)_{-1/2} + (Bu, u)_{-1/2} = 0 \ .$$

From an application point of view there is still the boundary layer circulation modelling challenge which is due to the model restriction allowing only potential flows. The lift of an airfoil in inviscid flow requires circulation in the flow around the airfoil, but a single potential function that is continuous throughout the domain around the airfoil cannot represent a flow with non-zero circulation. Therefore potential flow theory requires special treatment and an additional assumption which was formulated first by L. Prandtl.

For space dimension $n = 2$ the concept of potential flows goes along with the concept of the Cauchy-Riemann differential equations. Those equations enable the definition of a complex function, by which the flow of an incompressible, vortex-free fluid can be modelled. In vector terminology this can be represented in the form

$$\nabla \cdot \vec{v} = 0, \ \nabla \times \vec{v} = 0.$$

At the same time the Stokes (Leray-Hopf) operator is applied which is a projector from

$$L_2 \rightarrow L_\sigma^2 := \left\{ v \middle| v \in L_2 \wedge div(v) = 0 \right\} \ .$$

enabling the related Hilbert scale theory analysis.

In [RuC] a generalization of the Cauchy-Riemann differential equations for space dimension $n = 3$ is proposed in the form

$$\nabla \cdot \vec{v} = 0 \quad , \quad (\nabla \times \vec{v}) \times \vec{v} = (\vec{v} \cdot \nabla)\vec{v} - \nabla(\frac{\vec{v} \cdot \vec{v}}{2}) = 0$$

which allows also vortex flows with certain vortex line conditions. Those are related to the NSE by the formula

$$u \cdot \nabla u = (\nabla \times u) \times u + \nabla(\frac{u \cdot u}{2}) \ .$$

Runge's generalized 3D-Cauchy-Riemann differential equations are identical to the Cauchy-Riemann's differential equation for space dimension $n = 2$, .e. when no vortex flow is required to be modelled.

In [StE] a generalization of the Cauchy-Riemann equation for space dimensions $n \geq 3$ is given built on representations arising from the spherical harmonics in the context of representations of the rotation group. It adds mathematical complexity w/o new conceptual solution elements regarding the boundary layer "zero" velocity vs. the airfoil uplift modelling problem. Nevertheless is the rotation group aspect an important one. We note in this context with respect to our proposal that also the Riesz operators are rotation invariant.

The term $(\vec{v} \cdot \nabla)\vec{v}$ is concerned with how the divergence affects the velocity (when a river converges, the narrowing acts like a funnel, and the overall velocity of the flow increases. Conversely, of the river diverges, the particles spread out, and the overall speed of the flow decreases). The term ∇p is concerned with the gradient of the pressure of the medium (the shear stress forces) which is an acceleration term. It therefore relates to the initial and boundary wall values of the velocity.

Our first proposal alternatively to today's NSE weak analysis approach is concerned with the range of the Stokes operator: we propose a smaller range for the 3D Stokes operator in the form

$$\Lambda : L_2 \to L_\sigma^2 := \left\{ v \middle| v \in L_2 \wedge div(v) = 0 \wedge (\nabla \times v) \times v = 0 \right\}.$$

Our second proposal alternatively to today's NSE weak modelling and analysis is concerned with the construction of the weak $H_{-1/2} -$ representation above (multiplying the homogeneous equation (*) with $A^{-1/2}u$) we propose the following second modification. In order take the boundary behavior of vortex flow into account we suggest an alternative multiplication with $\Pi^{-1}u : H_{-1/2} \to H_{1/2}$, whereby Π denotes the Prandtl operator. It is defined by the Neumann problem in the following way:

For a closed connected surface $S \subset R^3$ one considers the harmonic function (i.e. $\Delta u = \Delta(\prod(v)) = 0$)

$$(*) \quad u(x) := \prod(v)(x) := \frac{1}{4\pi} \oiint_S v(y) \frac{\cos \varphi_{xy}}{|x-y|^2} dS_y \quad ,$$

whereby φ_{xy} is the angle between the vector $|x - y|$ and the normal n_y to the surface at the point y and $v(y)$ is the density of the double layer potential. One can seek the solution of the Neumann boundary value problem

$$\Delta u = 0 \qquad \text{in } R^3 - S$$
$$\frac{\partial u}{\partial n} = f \qquad \text{on } S$$

as the double layer potential in the form $u = \prod(v)$, whereby the unknown function $v(y)$ is obtained by the equation

$$\prod(v) = f \cdot$$

From [Lil] (and subsequently from [BrK2]) we recall the

Theorem: The Prandtl operator $\prod : H_{1/2} \to H_{-1/2}$ is bounded and coercive, the range $R(\prod) = H_1(R^3 - S)$ (*e.g. the velocity space of the airplane*) and the exterior Neumann problem admit one and only on generalized solution.

The usage of the Prandtl operator goes along with the current challenges of the representation of the NSE pressure p as a solution of the Neumann problem. In the following we give some further details on the above.

The initial boundary value problem of the three dimensional Navier-Stokes equations is given by

$$\partial_t u - \Delta u + (u \cdot \nabla)u = -\nabla p + f \quad \text{in } \Omega \times (0,T)$$
$$div(u) = 0 \qquad \text{in } \Omega \times (0,T)$$
$$u(x,0) = u_0(x) , \qquad x \in \Omega$$
$$u(x,t) = u_1(x,t) , \quad (x,t) \in \partial\Omega \times (0,T) .$$

With respect to the proposed weak $H_{-1/2}$ –representation of this paper we note that

$$-(\Delta u, v)_{-1/2} + (\nabla p, v)_{-1/2} \cong (\nabla u, \nabla v)_{-1/2} + (\nabla p, v)_{-1/2} \cong (u,v)_{1/2} + (p,v)_0 .$$

The pressure p can be expressed in terms of the velocity by the formula

$$p = -\sum_{j,k=1}^{3} R_j R_k (u_j u_k)$$

where (R_1, R_2, R_3) is the Riesz transform. The Leray-Hopf projector is the matrix valued Fourier multiplier given by

$$P(\xi) = Id - \frac{\xi \otimes \xi}{|\xi|^2} = (\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2})_{1 \le j,k \le n} \quad , \quad P = Id - R \otimes R =: Id - Q$$

whereby Q is an orthogonal projector, i.e. it holds $Q := R \otimes R = (R_j R_k)_{1 \le j,k \le 1} = Q^2$.

As a result the Leray-Hopf operator

$$P = Id - R \otimes R =: Id - Q = Id - \frac{D \otimes D}{D^2} Id - \Delta^{-1}(\nabla \times \nabla)$$

is also an orthogonal projection . We note that under rotation in R^n, the Riesz operators transform in the same manner as the components of a vector ([SteE1] III, 1.2).

The initial boundary value problem determines the initial pressure $p_0(x)$ by the Neumann problem

$$\Delta p_0 = (f_0 - u_0 \cdot \nabla u_0 \qquad \text{in } \Omega$$
$$\frac{\partial p_0}{\partial n} = [\Delta u_0 - u_0 \cdot \nabla)u_0 + f_0] \cdot n \quad \text{at } \partial\Omega$$

with $f_0 := \lim_{t \to 0} f(\cdot, t)$. Applying formally the div-operator to the classical NSE the pressure field must satisfy the following Neumann problem ([GaG])

$$\Delta p = (u \cdot \nabla)u - f \qquad \text{in } \Omega$$
$$\frac{\partial p}{\partial n} = [\Delta u - (u \cdot \nabla)u + f] \cdot n \quad \text{at } \partial\Omega$$

where n denotes the outward unit normal to $\partial\Omega$.

As it holds that

$$[\Delta u - (u \cdot \nabla)u + f] \cdot n \big|_{\partial\Omega} \to [\Delta u_0 - (u_{\cdot 0}\nabla)u_0 + f_0] \cdot n \big|_{\partial\Omega} \quad \text{in } H_{-1/2}(\partial\Omega)$$

and

$$\nabla \cdot \left[f - u \cdot \nabla) u \right]_{\partial \Omega} \rightarrow \nabla \cdot \left[f_0 - u_0 \cdot \nabla) u_0 \right]_{\partial \Omega} \text{ in } H_{-1/2}(\partial \Omega)$$

the pressure p tends to p_0 in the sense that $\left\| \nabla(p(\cdot,t) - p_0 \right\| \rightarrow 0$ as $t \rightarrow 0$.

From this it follows that in this framework the prescription of the pressure at the boundary walls or at the initial time independently of u, could be incompatible with and, therefore, could retender the problem ill-posed.

From [HeJ] we recall the counter example that there exists $u_0 \in \{ v \in L_2 | div(v) = 0 \}$ with

$$\limsup_{t \rightarrow 0} \left\| p(t) \right\|_{L_2(\Omega)/R} = \infty .$$

As a consequence of the alternative $H_{-1/2} -$(energy) Hilbert space there is a reduced requirement to the limit $p(\cdot,t) \rightarrow p_0$ as $t \rightarrow 0$ in the form

$$\left\| \nabla(p(\cdot,t) - p_0) \right\|_{-1/2} \cong \left\| p(\cdot,t) - p_0 \right\|_{1/2} \rightarrow 0 \text{ as } t \rightarrow 0 .$$

Multiplying

$$\partial_t u - \Delta u + \frac{1}{2}\Lambda(\nabla(u \cdot u)) = -\nabla p + f$$

by $\Delta^{-1/2} u$ (or by $\Pi^{-1} u$) leads to

$$\partial_t \left\| u \right\|_{-1/2}^2 + 2 \left\| u \right\|_{1/2}^2 + \left(\Lambda(\nabla(u \cdot u)), u) \right)_{-1/2}$$

whereby the later term defines a term in the form

$$\left(\Lambda(\nabla(u \cdot u)), u) \right)_{-1/2} \approx \left(\Lambda((u \cdot u), u) \right)_0 .$$

As a consequence the weak representation can be handled as a quasi-linear partial differential equation applying monotone operator theory.

With respect to the pressure term multiplying by $\Pi^{-1} u \in H_{1/2}$ gives

$$(\nabla p, \Pi^{-1} u)_0 \cong (p, u)_0$$

whereby the term is governed by

$$(p, u)_0 \leq \left\| p \right\|_{-1/2} \cdot \left\| u \right\|_{1/2} .$$

The answer for the motion of a fluid in an infinite space ($x \in R^3$) such that it vanishes at infinity is given by the *Helmholtz-Hodge decomposition*. It is determined when one knows the values $div(v) = \nabla \cdot v$ and $curl(v) = \nabla \times v$. If the motion of a fluid is limited to a simple connected region $\Omega \subset R^3$ with boundary $\partial \Omega$, it is determined if $div(v), curl(v)$ and therefore the value of the flow normal to the boundary $\partial v / \partial n$ for $x \in \partial \Omega$ are known.

There is a related *inverse Helmholtz-Hodge decomposition* statement that a given vector field can be decomposed into its divergence-free (incompressible) and curl-free (irrotational) components, i.e. a vector field $v : R^3 \to R^3$ can be expressed as a sum of the gradient of a scalar potential and the curl of a vector potential in the form $v = \nabla A + curlB$ with

$$A(x) = -\frac{1}{4\pi} \int \frac{div(v)(y)}{|x-y|} dy \quad , \quad B(x) = \frac{1}{4\pi} \int \frac{curl(v)(y)}{|x-y|} dy$$

whereby $curl(grad)(A)) = div(curl(B)) = 0$.

§5 A simple one-dimensional turbulent flow model
based on the revisited CLM vorticity equation with viscosity term

The trilinear form of the non-linear NSE term is antisymmetric. Therefore the energy inequality of the NSE with respect to the physical H_0 space does not take into account any contribution from the non-linear term. At the same time the regularity of the non-linear term cannot be smoother than the linear term. An alternative physical space $H_{-1/2}$ is the baseline of the unique 3D-NSE solution of this page.

Kolmogorov's turbulence theory is a purely statistical model, based on Brownian motion, which describes the qualitative behavior of turbulent flows. There is no linkage to the quantitative model of fluid behavior, as it is described by the Euler or Navier-Stokes equations.

Kolmogorov's famous 4/5 law is based on an analysis of low- and high-pass filtering Fourier coefficients. The physical counterpart to this is about a "local Fourier spectrum" which is (according to ([FaM]) nonsensical because, as, either it is non-Fourier, or it is nonlocal.

In Kolmogorov's spectral theory the two central concepts of a turbulent flow are homogeneous and isotropic flows (unfortunately they never encounter in nature). A flow is homogeneous if there is no "space" gradient in any averaged quantity, i.e. the statistics of turbulent flow is not a function of space. A flow is isotropic, if rotation and buoyancy are not relevant (they can be neglected) and there is no mean flow.

[FaM1] "*The definition of the appropriate "object" that composes a turbulent field is still missing. It would enable the study how turbulent dynamics transports these space-scale "atoms", distorts them, and exchanges their energy during the flow evolution. If the appropriate "object" has been defined that composes a turbulent field it would enable the study how turbulent dynamics transports these space-scale "atoms", distorts them, and exchanges their energy during the flow evolution*. …

Turbulent flows have non-zero vorticity and are characterized by a strong three-dimensional vortex generation mechanism (vortex stretching). Brownian motion describes the random motion of particles suspended in a fluid resulting from their collision with quick atoms or molecules in a gas or a liquid. In mathematics it is described by the Wiener process. It is related to the normal density function. A Brownian (=red) noise is produced by a Brownian motion (i.e. a random walk noise). It is obtained as the integral of a white noise signal.

[FaM1] "*The notion of "local spectrum" is antinomic and paradoxical when we consider the spectrum as decomposition in terms of wave numbers for as they cannot be defined locally. Therefore a "local Fourier spectrum" is nonsensical because, either it is non-Fourier, or it is nonlocal. There is no paradox if instead we think in terms of scales rather than wave numbers. Using wavelet transform then there can be a space-scale energy be defined with a correspondingly defined scale decomposition in the vicinity of location x and a correspondingly defined local wavelet energy spectrum. By integration this defines a local energy density and a global wavelet energy spectrum. The global wavelet spectrum can be expressed in terms of Fourier energy spectrum. It shows that the global wavelet energy spectrum corresponds to the Fourier spectrum smoothed by the wavelet spectrum at each scale. …
… The concept enables the definition of a space-scale Reynolds number, where the average velocity is being replaced by the characteristics root mean square velocity* $\mathrm{Re}(l,x)$ *at scale l and location x. At large scale (i.e.* $l \approx L$ *)* $\mathrm{Re}(L)$ *coincides with the usual large-scale Reynolds number, where* $\mathrm{Re}(L)$ *is defined as*

$$\mathrm{Re}(L) = \iint\limits_{R^n} \mathrm{Re}(L,x)dx \cdot \text{ "}$$

A wavelet series of a function $g(x)$ converges locally to $g(x)$, even if $g(x)$ is a distribution as long as the order of the distribution does not exceed the regularity of the analyzing wavelet. The admissibility condition ensures the validity of the inverse wavelet transform which then is valid for all Hilbert scale values.

Based on the re-revisited generalized CLM equation with viscosity term ([MaA] 5.2) we propose a turbulent flow model which allows non-stationary random functions with finite variance and related spectrum ([FrU] (4.54)) with respect to the $H_{1/2}-$ energy norm. The modification of current "revisited CLM proposals" is identical to the alternatively proposed auxiliary function definition of $v = H[u] = A[u_x]$ in §6 "Stefan problem". The model allows a wavelet synthesis according to [FaM], [FaM1] in a $H_{-1/2}-$ wavelet framework.

If the solution of the Euler equation is smooth then the solution to the slightly viscous NSE with same initial data is also smooth. Adding diffusion to the CLM model it makes the solution less regular [MuA]. As a consequence of this the CLM model lost most of the interest in the context of NSE analysis. In [MuA] a nonlocal diffusion term is proposed removing this drawback. The modification goes along with a reduced regularity of the "dissipation" term resulting in a reduced "energy" Hilbert scale of Hilbert scale factor -1/2. As this modification did not modify in same manner the non-linear term this leads to an unbalanced energy equation. As the non-linear term governs the dissipative term in case of turbulence, this is an argument to reject current revisited CLM model with viscosity term ([DeS], [MuA], [OkH], [SaT], [SaT1]). At the same time those suggested modifications being applied in same manner to the linear term would fit to the Stieltjes integral based Kolmogorov theory [ShA], as well as to the conceptual idea of this paper (i.e. an $H_{1/2}-$ energy inner product enabling an energy inequality which does not exclude any information from the non-linear term). Combining both conceptual ideas provides a functional analytical common framework ([BrK], [BrK3]) for a statistical fluid mechanics theory ([MoA]), a statistics of gases and highly turbulent fluid flows [HoE].

The building concept of the revisited generalized CLM model is therefore as follows: we consider periodic boundary condition and assume that the initial condition of ω is symmetric with respect to the origin ([SaT1]). We propose a weak $H_{-1/2}-$ variation representation of the extended Schochet-CLM model ([ScS]) in the form

$$(\dot{\omega}, v)_{-1/2} - \varepsilon(\omega_{xx}, v)_{-1/2} = (\omega H[\omega], v)_{-1/2} \ , \ \forall v \in H_{-1/2} \ .$$

With the notation of [BrK] this representation is equivalent to

$$(A\dot{\omega}, v)_0 + \varepsilon(H[\omega], v)_{1/2} = (A[\omega H[\omega]], v)_0 \ , \ \forall v = H[w] \in H_{-1/2} \ .$$

Taken into account that the Hilbert transform is an isometry on all Hilbert scales and that $H^2[v] = -v$ and putting $\omega_H := H[\omega]$ this can be reformulation in the form

$$(\dot{\omega}_H, w)_{-1/2} + \varepsilon(\omega, w)_{1/2} = (H[\omega H[\omega]], w)_{-1/2} \ , \ \forall w \in H_{-1/2} \ .$$

From [MaA] we recall the identity

$$2H[\omega H[\omega]] = \omega_H^2 - \omega^2$$

leading to

$$(\dot{\omega}_H - \varepsilon\omega', w)_{-1/2} = \frac{1}{2}(\omega_H^2 - \omega^2, w)_{-1/2} \ , \ \forall w \in H_{-1/2} \ .$$

The left hand side of the variation representations above is reflecting to current revisited proposals of the CLM model, while now the right hand side of the variation equation shows a modified non-linear CLM model operator (as the domain has changed).

In [SaT1] for periodic boundary conditions the Fourier (spectral) representation of the non-linear term $\omega H[\omega] = \omega A[\omega_x]$ is given, whereby ω denotes the vorticity and H the Hilbert transform operator. The spectral method analysis of the equation above follows the same way leading to:

$$\dot{\omega}_n \approx n(\varepsilon\omega_n + \sum_{k=1}^{n-1} \omega_k \omega_{n-k}) \quad , \quad \omega_n(0) = \frac{A_n}{2} \quad , \quad \dot{\omega}_0 = 0$$

whereby

$$\omega(x,0) = \sum_{1}^{\infty} A_n \sin(nx) \cdot$$

Following the concept of [FaM] the turbulent $H_{-1/2}$–signal can be split into two contributions: coherent bursts, corresponding to that part of the signal which can be compressed in a H_0–-wavelet basis, plus incoherent noise, corresponding to that part of the signal which cannot be compressed in a H_0–wavelet basis, but in the $H_{-1/2}$–wavelet basis. For the $n=1$ periodic case the later one corresponds to the alternative zero-state energy model of the harmonic quantum oscillator.

The spectral analysis above is also linked to the solution framework of [BrK3]. The Hilbert transform of the Gaussian is the Dawson function, which is norm equivalent to the Gaussian due to the related property of the Hilbert transform. Therefore a Dawson basis function based Hilbert space framework enables an alternative statistical hydromechanics:

Let $f(x)$ resp. $f_H(x)$ denote the Gaussian function resp. its Hilbert transform $f_H(x) = H[f](x)$ with

$$H[v](x) := \lim_{\varepsilon \to 0} \frac{1}{\pi} \oint_{|x-y|>\varepsilon} \frac{v(y)}{x-y} dy = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v(y)}{x-y} dy \quad \cdot$$

Then it holds $\hat{f}(0) = 1$ resp. $\hat{f}_H(0) = 0$, $f_H(x)$ is up to a constant identical with the Dawson function and $f(x)$ and $f_H(x)$ are norm-equivalent, i.e.

$$(f,w) = (f_H,w) \quad \forall w \in H_0(-\infty,\infty) \cdot$$

We further note the following properties of the Hilbert transform:

$$[xH - Hx][u](x) = \frac{1}{\pi} \int_{-\infty}^{\infty} u(y)dy = \hat{u}(0) \quad \text{and} \quad (u,Hu)_0 = 0 \text{ for } u, Hu \in L_2$$

(The insertion of a new variable $z = x - y$ yields

$$H[xu(x)](x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x-z)u(x-z)}{z} dz = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(xu(x-z)}{z} dz - \frac{1}{\pi} \int_{-\infty}^{\infty} u(x-z)dz = xH(u(x)) - \frac{1}{\pi} \int_{-\infty}^{\infty} u(y)dy \quad)$$

The Gaussian function is not a wavelet, as it does not fulfill the admissibility condition, but its first derivative and $f_H(x)$ do.

When considering $f'(x)$ in a $H_{-1/2} -$(physical) Hilbert space framework the following identities hold true:

$$(f', w)_{-1/2} = (Af', w)_0 = (f_H, w)_0 = (f, w)_0 \quad \forall w \in H_0(-\infty, \infty).$$

In other words, a Fourier synthesis (and a probability analysis) in a $H_0 -$Hilbert space framework can be transformed into a wavelet analysis ("white noise" signal appropriate, ([VoR])) $H_{-1/2} -$Hilbert space framework.

The corresponding central "distributional" functions in case of the $2\pi -$periodic function are given by the identities [BrK3]:

$$\frac{1}{2}\cot(\frac{x}{2}) = \frac{d}{dx}\ln(2\sin(\frac{x}{2})) = -\frac{d}{dx}\sum_{n=}^{\infty}\frac{\cos(nx)}{n} = \sum_{n=}^{\infty}\sin(nx) \in H_{-1/2}^{\#}(0, 2\pi).$$

We note that the Hilbert transform of a wavelet is again a wavelet ([WeJ]) and that the Hilbert transform is an isomorphism on any Hilbert scale H_β.

[FaM]: "*The turbulent regime develops when the non-linear term of the NSE strongly dominates the linear term. Superposition principle holds no more for non-linear phenomena. Therefore turbulent flows cannot be decomposed as a sum of independent subsystems that can be separately studied. A wavelet representation allows analyzing the dynamics in both space and scale, retaining those degrees of freedom which are essential to compute the flow evolution*".

[MeM]: "*Methods based on wavelet (Galerkin) expansions in $L_2 -$framework face the issue that in Galerkin methods the degrees of freedom are the expansion coefficients of a set of basis functions and these expansion coefficients are not in physical space (means in wavelet space). First map wavelet space to physical space, compute non-linear term in physical space and then back to wavelet space, is not very practical*".

The Galerkin method based on wavelet expansion requires (ongoing) mappings between wavelet and physical space (during computing process) in case both spaces are different. This is the case for most of current (weak) variation PDE representations in a $L_2 -$Hilbert space framework ([MeM]).

The proposed $H_{-1/2} -$ (weak) physical space concept enables identical wavelet and physical Hilbert spaces, while at the same time enabling the full power of Galerkin method computing non-linear terms in this (newly) physical space. This leads back to the "solution" section of this page with the weak NSE solution of the corresponding weak NSE representation in the $H_{-1/2} -$ Hilbert space and the ($H_0 -$based) physical principles of quantum theory [HeW].

Note 1: localized Heisenberg uncertainty inequality in $H_{-1/2} - H_0$ space

Motivated by the commutator property of the Hilbert transform for functions with vanishing constant Fourier term we propose a domain extension of the multiplication and differentiation operators (which are the infinitesimal generators of the modulation and translation group): let $\bar{u}(x) := x \cdot u(x)$

$$(Pu, v)_0 = (u', v)_0 \quad \forall v \in H_0 \quad \mapsto \quad (Pu, v)_{-1/2} = (u', v)_{-1/2} = (Au', v)_0 = (Hu, v)_0$$
$$(Qu, v)_0 = (\bar{u}, v)_0 \quad \forall v \in H_0 \quad \mapsto \quad (Qu, v)_{-1/2} = (\bar{u}, v)_{-1/2}$$

Then $PQu = P\bar{u} \mapsto H\bar{u}$ and $QPu \mapsto xHu$. In case u is a wavelet then this commutator vanishes in the $H_0 -$ test space.

Note 2: **Time-frequency and wavelet analysis of functions [RaH] and characterization of case space-time dimension** $n = 4$

Time-frequency and wavelet analysis of functions with symmetric properties is related to the Hankel (Fourier-Bessel) transform [RaH]. Let $n \geq 2$ and let ν be the uniform distribution of unit mass over the unit sphere with center at the origin. Suppose $g \in L_1(R^n)$ is radial, i.e. $g(x) = h(|x|)$. Then

$$\hat{\nu}(\xi) = 2^{(n-2)/2} \cdot |\xi|^{(2-n)/2} \cdot \Gamma(\frac{n}{2}) \cdot J_{(n-2)/2}(|\xi|)$$

and

$$\ddot{g}(\xi) = (2\pi)^{n/2} \cdot |\xi|^{(2-n)/2} \int_0^\infty J_{(n-2)/2}(r|\xi|)h(r)r^{n/2dr} \cdot$$

For $\text{Re}(\mu) > -1$ it holds

$$\int_0^\infty J_\mu(t)dt = 1 \quad \cdot$$

For the special case $n = 4$ we note the identity $J_0'(x) = -J_1(x)$, i.e.

$$\int_x^\infty J_1(t)dt = J_0(x) \quad \text{for } x > 0, \quad J_0(0) = 1$$

and the related Fourier resp. Hilbert transforms

$$\hat{J}_0(\xi) = \begin{cases} \dfrac{2}{\sqrt{1-\xi^2}} & for\ |\xi| < 1 \\ 0 & else \end{cases} \quad , \quad \hat{J}_1(\xi) = -i\xi\hat{J}_0(\xi)$$

$$H[J_0](\xi) = \frac{1}{\pi}\int_0^1 \hat{J}_0(t)\sin(\xi t)dt \quad , \quad H[J_1](\xi) = -\frac{1}{\pi}\int_0^1 \hat{J}_1(t)\cos(\xi t)dt \cdot$$

Therefore the Bessel function $J_0(x)$ is wavelet.

Note 3: **Kolmogorov, the Turbulence** [ShA] **and symmetry properties of the spectral density in** $L_2(R^n) - $**framework**

We note from [ShA] that a homogenous random field is a stationary process $X(s)$ with the correlation function $R(t) = EX(s+t)X(s)$ of the process X. Ist spectral (Stieltjes integral) representation based on a finite spectral measure F (spectral distribution function) with "spectral density" dF, where $dF(k)$ is the contribution to the "energy" of the harmonics whose frequencies are within the interval $(k, k+dk)$. $F(k)$ is characterized by the properties "symmetry" ($dF(-k) = dF(k)$, "monotonicity" ($F(k) \leq F(l)$ for $k \leq l$), $R(0)$ "boundedness"). If X is a real-valued process, then the spectral "function" is symmetric with respect to the point $k = 0$. As a consequence for $G(k) = F(k) - F(-k)$ the correlation function $R(t)$ is given by

$$R(t) = \int_0^\infty \cos(\lambda t)G(d\lambda) \cdot$$

This corresponds to a purely cos-Fourier series representation with respect to the $H_{-1/2} - $ norm of the $\cot(x)$.

§6 Optimal finite element approximation estimates for non-linear parabolic problems with not regular initial value data

In [BrK1] optimal finite element approximation estimates for non-linear parabolic problems with not regular initial value data are prove. The proof is based on same $H_{-1/2} -$ energy Hilbert space concept as above in the following way:

The free boundary Stefan problem with its solution $U(y,\tau)$ can be transformed into the non-linear parabolic equation ([NiJ1]) looking for a solution $u(x,t) = U(y,\tau)$ fulfilling

$$\dot{u}(y,\tau) - u''(x,t) + xu'(1,t)u' = 0 \quad \text{in } Q = \{(x,t) | x \in (0,1), 0 < t \le T\}$$

$$u'(0,t) = u(1,t) = 0 \quad \text{for } t > 0$$
$$u(x,0) = f(x) \quad \text{for } x \in (0,1) \ .$$

The proposed variation problem is given in the form

P_v : find $v \in \dot{H}_{1/2}$ with

$$(\dot{v},w)_{-1/2} + (v',w')_{-1/2} = v(1)(Hv,w)_0 \quad \text{for } w \in \dot{H}_{1/2} \text{ and } t > 0$$

$$(v(\cdot,0),w)_{-1/2} = (f',w)_{-1/2} = (f,w)_0 \quad \text{for } w \in \dot{H}_{1/2} \text{ and } t = 0.$$

The corresponding Galerkin approximation is given by

P_{v_h} : find $v_h \in S_h \subset \dot{H}_{1/2}$ $(v_h(1) := v_h(1,t))$ with

$$(\dot{v}_h,\chi)_{-1/2} + (v_h,\chi)_{1/2} = v_h(1)(Hv_h,\chi)_0 \quad \text{for } \chi \in S_h \subset \dot{H}_{1/2} \text{ and } t > 0$$

$$(v_h(\cdot,0),\chi)_{-1/2} = (f,\chi)_0 \quad \text{for } \chi \in S_h \subset \dot{H}_{1/2} \text{ and } t = 0.$$

The reduced Hilbert scale "regularity" requirements enable optimal finite element approximation estimates for not regular initial value data in the form:

Theorem: The problem P_v has an unique bounded solution $v \in H_{1/2}$ and an unique (linear splines) finite element approximation v_h of problem P_{v_h} in the neighborhood of v with optimal order of convergence

$$\|u - u_h\|_{L_\infty(0,t);L_\infty)} \le O(h).$$

Appendix

From [GaG] we recall the Navier-Stokes initial-boundary representation in the form

$$(**) \quad \frac{dv}{dt} - \Delta v + (v, grad)v) + grad p = f$$

$$div\, v = 0$$

$$v_{|\partial\Omega} = 0 \ , \ v_{|t=0} = v_0 .$$

We note that for $div\, v = \nabla \cdot v = 0$ it holds $(v, grad)v) = (v \cdot \nabla)v = \nabla \cdot (v \otimes v)$.

By formally dot-multiply through both sides of $(**)$ above with $f \equiv 0$ by v, integrate by parts over Ω and take into account Green formula one obtains

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_2^2 + \nu|v(t)|_{1,2}^2 = 0$$

Integrating this from $s \geq 0$, $t \geq s$ one obtains the so-called energy equation

$$\|v(t)\|_2^2 + 2\nu\int_s^t |v(\rho)|_{1,2}^2 d\rho = \|v(s)\|_2^2 \ , \quad 0 \leq s \leq t$$

As it holds $(v \cdot \nabla v, v) = 0$ the non-linear term $v \cdot \nabla v$ does not give any contribution to the energy equation. From the equations

$$(\frac{\partial v}{\partial t}, P\Delta v) = (\frac{\partial v}{\partial t}, \Delta v) = -\frac{1}{2}\frac{d}{dt}|v|_{1,2}^2$$

$$(\frac{\partial v}{\partial t}, P\Delta v) = 0$$

it follows

$$\frac{1}{2}\frac{d}{dt}|v|_{1,2}^2 + \nu\|P\Delta v\|_2^2 = (v \cdot \nabla v, P\Delta v) \neq 0$$

There is a loss of regularity if two (scalar) functions f and g are in H_1, their product $f \cdot g$ only belongs to $H_{1/2}$ and their derivative $\partial(fg)$ is even less regular as it belongs to $H_{-1/2}$. This is the root cause of some (positive) regularization effects by the heat semi-group $S(t)$ and the loss of regularity that comes from the differential (grad-) operator ∇ and from the pointwise multiplication $v \otimes u$ of the non-linear bilinear operator

$$B(v,u)(t) := -\int_0^t e^{-(s-t)\Delta}P\nabla \cdot (v \otimes u)(s)ds \ \cdot$$

The estimate of the non-linear term is highly depending from the space dimension. It holds ([GaG])

$$(v \cdot \nabla v, P\Delta v) \leq C\|v\|_q^{2q/(q-n)}|v|_{1,2}^2 + \frac{\nu}{2}\|P\Delta v\|_2^2 \quad \text{for all} \quad q \in (n,\infty) \ \ .$$

From the Sobolev embedding theorems and properties of the projection operator P ([GaG]) one gets the a priori estimates

$$\frac{1}{2}\frac{d}{dt}\big|v(t)\big|^2_{1,2} + \frac{\nu}{2}\big\|P\Delta v\big\|^2_2 \leq \begin{cases} c_3\big\|v\big\|^2_2\big|v\big|^4_{1,2} & ,\ n=2 \\ c_4\big|v\big|^6_{1,2} & ,\ n=3 \end{cases} \ .$$

From [GiY] we recall the corresponding fundamental

Lemma 3.2 ([GiY]): let $0 \leq \delta < 1/2 + n \cdot (1-1/p)/2$. We have

$$\big|A^{-\delta}P(u,grad)v\big|_p \leq M \cdot \big|A^\theta u\big|_p \cdot \big|A^\rho u\big|_p$$

with a constant $M := M(\delta,\theta,\rho,p)$ if $\delta + \theta + \rho \geq n/2p + +1/2$, $\theta,\rho > 0$, $\theta + \rho > 1/2$.

Example ([GiY]): when $p = n = 2$, one gets

$$\big|P(u,grad)v\big| \leq C \cdot \big|u\big|^{1/2} \cdot \big|A^{1/2}u\big|^{1/2} \cdot \big|A^{1/2}v\big|^{1/2} \cdot \big|Av\big|^{1/2} \ .$$

Stokes operator ([SoH]**)**, IV15: Let $\Omega \subseteq R^n (n \geq 2)$ denote an arbitrary domain and

$$A = \int_0^\infty \lambda dE_\lambda$$

the Stokes operator of Ω. Then the fractional powers

$$A^\alpha = \int_0^\infty \lambda^\alpha dE_\lambda \ ,\ -1 \leq \alpha \leq 1$$

are positive selfadjoint operators, and each operator $S(t)$ of the Stokes semigroup family

$$\left\{ S(t) := e^{-tA} := \int_0^\infty e^{-t\lambda} dE_\lambda \,\big|\, \lambda \geq 0, t \geq 0 \right\}$$

 is bounded everywhere defined and positive selfadjoint in the Hilbert space $L^2_\sigma(\Omega)$.

For the orthogonal set $\{w_i, \lambda_i\}$ of eigenpairs of the non-stationary Stokes operator

$$\overline{A} := \dot{w} + Aw = f \ ,\ w(0) = 0 \ ,\ \tau \in [0,t]$$

one gets

$$w_i(\tau) = \int_0^\tau e^{-\lambda_i(\tau-s)} f_i(s)ds \ \cdot$$

By changing the order of integration it follows for $\beta > -1$

$$\int_0^t \tau^\beta w_i^2(\tau)d\tau \leq \int_0^t \left[\int_0^\tau e^{-\lambda_i(\tau-s)}ds\right]\left[\int_0^\tau s^\beta e^{-\lambda_i(\tau-s)}f_i^2(s)ds\right]d\tau \leq \lambda_i^{-1}\int_0^t s^\beta f_i^2(s)\left[\int_\tau^t e^{-\lambda_i(\tau-s)}d\tau\right]ds \leq \lambda_i^{-2}\int_0^t s^\beta f_i^2(s)ds \quad.$$

From this one gets

$$\left\|t^{\beta/2}w(t)\right\|_{\alpha+2}^2 \leq c\left\|t^{\beta/2}\overline{A}w(t)\right\|_\alpha^2 \quad.$$

For $u_0 \in L_\sigma^2(\Omega)$ the function $u:[0,\infty) \to L_\sigma^2(\Omega)$ defined by $u(t) := S(t)u_0$ has the following properties:

i) $u(t)$ is strongly continuous for $t \geq 0$, $u(0) = u_0$

ii) $u'(t) = s - \lim_{\delta \to 0}\frac{1}{\delta}\left(u(t+\delta) - u(t)\right)$ for $t > 0$

iii) $u(t) \in D(A)$ and $u'(t) + Au(t) = 0$

iv)   If $u_0 \in D(A)$ then $u'(0) = s - \lim_{\delta \to 0}\frac{1}{\delta}\left(u(\delta) - u(0)\right)$ exists and $u'(0) + Au(0) = 0$

v)   For all $t \geq 0$ it holds $\|u(t)\|_2 \leq \|u_0\|_2$ and $s - \lim_{\delta \to 0}u(t) = 0$.

From [PlJ] I, §5, §8, we quote:

"*Bisher war es üblich für das Potential $V(p)$ die Form*

$$-\int \log(r_{ps})\mu'(s)ds$$

vorauszusetzen, wobei dann $\mu'(s)$ die Massendichtigkeit der Belegung genannt wurde. Eine solche Annahme erweist sich aber als eine derart folgenschwere Einschränkung, daß dadurch dem Potentiale $V(p)$ der größte Teil seiner Leistungsfähigkeit genommen wird. Für tiefergehende Untersuchungen erweist sich das Potential V nur in der (Stieltjes integral-) *Form*

$$V(p) = -\int \log(r_{ps})d\mu_s$$

verwendbar. Die infinitesimale Größe $d\mu_s$ kann man dann das Massenelement nennen, von Massendichtigkeit wird man aber nur dann sprechen können, wenn der Differentialquotient

$$\frac{d\mu_s}{ds} = \mu'(s)$$

besteht."

The concept leads to generalized Green formula, based on a generalized normal derivative definition, which is basically about reduced regularity assumptions, i.e. there is no existence necessary of certain limits of the normal derivatives into the normal direction at the boundary.

"*Es handelt sich um eine Verallgemeinerung, wie es die Erweiterung differentiierbarer Funktionen auf die stetigen ist.*"

"*Vom Integral $\oint \frac{\partial U}{\partial n}ds$ auf einer nichtgeschlossenen Kurve ergibt sich aus der Gleichung (6) eine Eigenschaft von grosser Wichtigkeit. Das Integral hängt nämlich nur von den Endpunkten ab und nicht von der näheren Form der sie verbindenden Integrationskurve in der Weise, dass die Integrale alle gleich einander gleich sind, welche Integrationswege entsprechen, die durch stetige Deformation im Regularitätsgebiete auseinander hervorgehen. Sind also p und q zwei Punkte im Regularitätsgebiete und verbindet man sie durch irgendeine Kurve (die Tangenten hat), so ist $\int_p^q \frac{\partial U}{\partial n}ds$ wohl definiert und hat einen von der näheren Form der Kurve nicht abhängigen Wert. ….*

… Das Integral zwischen zwei Punkten p und q

$$\overline{U}(q) = -\int_p^q \frac{\partial U}{\partial n}ds$$

ist, weil von der Kurve unabhängig, eine wohl definierte Funktion der Grenzen p und q und soll in seiner Abhängigkeit von q mit bezeichnet werden."

References

[BrK] Braun K., Interior Error Estimates of the Ritz Method for Pseudo-Differential Equations, Japan J. Appl. Math., 3 (1986), 59-72

[BrK1] Braun K., Optimal finite element approximation estimates for non-linear parabolic problems with not regular initial value data, http://www.navier-stokes-equations.com

[BrK2] Braun K.,The Prandtl (hyper singular integral) operator with double layer potential, http://www.navier-stokes-equations.com

[BrK3] Braun K., A Kummer function based Zeta function theory to prove the Riemann Hypothesis, http://www.riemann-hypothesis.de

[CaM] Cannone M., Harmonic Analysis Tools for Solving the Incompressible Navier-Stokes Equations, Handbook of Mathematical Fluid Dynamics, vol 3, eds. S. Friedlander and D. Serre, Elsevier, 2003

[ChF] Chouly F., Heuer N., A Nitsche-based domain decomposition method for hypersingular integral equations, Numer. Math. 121 (2012) 705-729

[CoD] Colton D, Kress R., Integral equation methods in scattering theory, Wiley, New York, 1983, Springer Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, Hong Kong, 1989

[CoM] Costabel M., A Coercive Bilinear Form for Maxwell's Equations, J. Math. Anal. Appl. 157, 2 (1991) 527-541

[DeS] De Gregorio S., On a One-Dimensional Model for the Three-Dimensional Vorticity Equation, J. Stat. Phys., 59, 1251-1263, 1990

[EsG] Eskin G., Boundary Value Problems for Elliptic Pseudodifferential Operators, Amer. Math. Soc., Providence, Rhode Island, translation of mathematical monographs, vol. 52, 1973

[FaM] Farge M., Schneider K., Wavelets: application to turbulence, University Warnick, lectures, 2005

[FaM1] Farge M., Wavelet transforms and their applications to turbulence, Annu. Rev. Fluid Mech. 24, 395-457, 1992

[FeC] Fefferman Ch. L., Existence & Smoothness of the Navier-Stokes Equations, Princeton University, Department of Mathematics, Princeton, New Jersey, May 1, 2000

[FrU] Frisch U., Turbulence, Cambridge University Press, 1995

[GaG] Galdi G. P., The Navier-Stokes Equations: A Mathematical analysis, Birkhäuser Verlag, Monographs in Mathematics, ISBN 978-3-0348-0484-4

[GiY] Giga Y., Weak and strong solutions of the Navier-Stokes initial value problem, Publ. RIMS, Kyoto Univ. 19 (1983) 887-910

[HeW] Heisenberg W., Physikalische Prinzipien der Quantentheorie, B. I. Hochschultaschenbuch, Bd. 1, Mannheim, Wien, Zürich, 1991

[HeJ] Heywood J. G., Walsh O. D., A counter-example concerning the pressure in the Navier-Stokes equations, as $t \to 0^+$, Pacific J. Math., 164 (1994), 351-359

[HoE] Hopf E., Statistical hydromechanics and functional calculus, J. Rat. Mech. Anal. 1, 1, 87-123, 1952

[KrR] Kress R., Linear Integral Equations, Springer Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, Hong Kong, 1989

[LiI] Lifanov I. K., Poltavskii L. N., Vainikko G. M., Hypersingular Integral Equations and their Applications, Chapman & Hall/CRC, Boca Rato, London, New York, Washington, D. C., 2004

[MaA] Majda A. J., Bertozzi A. L., Vorticity and Incompressible Flow, Cambridge University Press, Cambridge, 2002

[MoA] Monin A. S., Yaglom A. M., Statistical Fluid Mechanics, The MIT Press, Cambridge, Massachusetts, 1970

[MuA] Murthy A. S. V., The Constatin-Lax-Majda model for the vorticity equation revisited, J. Indian Inst. Sci, 78, 107-117, 1998

[NiJ] Nitsche, J.A., Lectures notes, Hilbert scales and Ritz-Galerkin approximation theory, Freiburg, 1983

[NiJ1] Nitsche, J.A., Schauder estimates for finite element approximations on second order elliptic boundary value problems, Proc. Of the Special Year in Numerical analysis, Lectures Notes no. 20, Univ. of Maryland, Babuska, Lin, Osborn eds., 290-343, 1981

[NiJ2] Nitsche, J.A., Approximation des eindimensionalen Stefan-Problems durch finite Elemente, Proceedings of the International Congress of Mathematics, Helsinki, 1978

[OkH] Okamoto H., Sakajo T., Wunsch M., On a generalization of the Constantin-Lax-Wajda equation, Nonlinearity, 21, 1-15, 2008

[PlJ] Plemelj J., Potentialtheoretische Untersuchungen, B.G. Teubner, Leipzig, 1911

[RaH] Rauhut H., Time-Frequency and Wavelet Analysis of Functions with Symmetry Properties, Logos Verlag, Berlin, 2005

[RuC] Runge C., Über eine Analogie der Cauchy-Riemannschen Differentialgleichungen in drei Dimensionen, Nachrichten von der Königl. Gesellschaft der Wissenschaften, Mathematisch-Physikalische Klasse, Bd. 1922 (1922) 129-136

[SaT] Sakajo T., Blow-up solutions of the Constatin-Lax-Majda equation with a generalized viscosity term, J. Math. Sci. Univ. Tokyo, (2003) 187-207

[SaT1] Sakajo T., On global solutions for the Constatin-Lax-Majda equation with generalized viscosity term, Nonlinearity, 16, 1319-1328, 2004

[ScS] Schochet S., Explicit solutions of the viscous model vorticity equation, Comm. Pure Appl. Math., 39, 531-537, 1986

[ShA] Shiryaev A. N., Kolmogorov and the Turbulence, May 20, 1999

[ShM] Shinbrot M., Lectures on Fluid Mechanics, Dover Publications, Inc., Mineola, New York, 2012

[SoP] Sobolevskii P. E., On non-stationary equations of hydrodynamics for viscous fluid. Dokl. Akad. Nauk SSSR 128 (1959) 45-48 (in Russian)

[SoH] Sohr H., The Navier-Stokes Equations, An Elementary Functional Analytical Approach, Birkhäuser Verlag, Basel, Boston, Berlin, 2001

[StE] Stein E. M., Weiss G., Generalizations of the Cauchy-Riemann Equations and Representations of the Rotation Group, Amer. Journal of Math., Vol. 90, No 1 (1968) 163-196

[StE1] Stein E. M., Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, new Jersey, 1970

[TeR] Teman R., Navier-Stokes Equations and Nonlinear Functional analysis, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1983

[VoR] Voss R. F., Fourier synthesis of Gaussian fractals, 1/f noises, landscapes, and flakes, 1982

[WeJ] Weiss J., The Hilbert Transform of Wavelets are Wavelets

[WeP] Werner P., Self-Adjoint Extensions of the Laplace Operator with Respect to Electric and Magnetic Boundary Conditions, J. Math. Anal. 70 (1979) 131-160

[WeP1] Werner P., Spectral Properties of the Laplace Operator with respect to Electric and Magnetic Boundary Conditions, J. Math. Anal. 92 (1983) 1-65

[WuM] Wunsch M., The generalized Constantin-Lax-Wajda equation revisted, Commun. Math. Sci., 9, No3, 929-936, 2011