

**The three Millennium problem solutions, RH, NSE, YME,
and a Hilbert scale based quantum geometrodynamics**

Klaus Braun

PART I

Riemann Hypothesis solutions

Klaus Braun
July 31, 2019
www.fuchs-braun.com

**A Kummer function based
alternative Zeta function theory
to solve the Riemann Hypothesis
and the binary Goldbach conjecture**

1. The tool set
2. An alternative entire Zeta function $\xi^*(s)$ based on Kummer / Dawson and $\tan\left(\pi\frac{s}{2}\right) = \cot\left(\pi\frac{1-s}{2}\right)$ functions
3. An alternative Zeta function $\zeta^*(s)$ based on the $\cot(\pi s) = \tan\left(\pi\left(\frac{1}{2} - s\right)\right)$ function
4. Alternative arithmetical functions based on the Kummer function zeros
5. New arithmetical functions for additive number theory
6. Dirichlet series and the (distributional) Hilbert space $H_{-1/2}^{\#} \cong l_2^{-1/2}$
7. A $H_{-1/2}^{\#} \cong l_2^{-1/2}$ based alternative circle method on the unit circle and a Kummer function zero based approach to prove the binary Goldbach conjecture
8. The $H_{-1/2}$ Hilbert space and corresponding arithmetic functions
9. Appendix: Formulas and properties

A Kummer function based alternative Zeta function theory to solve the Riemann Hypothesis and the binary Goldbach conjecture

1. The tool set

In order to prove the Riemann Hypothesis (RH) the Polya criterion can not be applied in combination with the Müntz formula ((TiE) 2.11). The Müntz formula is divergent in the critical stripe due to the asymptotics behavior of the exponential function. The conceptual challenge is about the not vanishing constant Fourier term of the Gaussian function and its related impact on the Poisson summation formula resp. on the corresponding Riemann duality equation ((EdH) 1.7). The proposed alternative "baseline" function is the Hilbert transform of the Gaussian function, which is the Dawson function ((GrI) 3.896, (OIF) p. 44)

$$F(x) := e^{-x^2} \int_0^x e^{t^2} dt = \int_0^\infty e^{-t^2} \sin(2xt) dt = x \cdot {}_1F_1\left(1, \frac{3}{2}; -x^2\right) = x e^{-x^2} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x^2\right).$$

In the context of the $Li(x)$ function RH approximation criterion we note that the Dawson function $F(\sqrt{x})$ enjoys an only polynomial asymptotics in the form $O(x^{-1/2})$. This, as well as the asymptotics $2dF(x) \sim \frac{dx}{x} = d \log x$, is a consequence of the identity $F'(x) + 2xF(x) = 1$ ((LeN) (9.13.3)).

The Mill's ration function of the standard normal law, given by

$$M(x) := e^{x^2} \int_x^\infty e^{-t^2} dt$$

is strictly log-convex ((BaÁ), (RuM)). It is related to the Dawson function by the solutions of the self-adjoint Whittaker operator (playing a key role in spectral and scattering theory, e.g. for the 3-D spherical, harmonic quantum oscillator, e.g. (DeJ)) by the formula (BuH) p. 209)

$$\int_0^z e^{\mp x^2} dx = z \cdot {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; \mp z^2\right) = c \cdot e^{\mp \frac{1}{2}z^2} \cdot z^{-\frac{1}{2}} \cdot M_{\pm \frac{1}{4}}(z^2), \quad c := \frac{\sqrt{\pi}}{2}$$

leading to

$$\begin{aligned} F(x) &= e^{-x^2} \int_0^x e^{t^2} dt = x \cdot e^{-x^2} \cdot {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x^2\right) = c \cdot e^{-\frac{1}{2}x^2} \cdot x^{-\frac{1}{2}} \cdot M_{-\frac{1}{4}}(x^2) \\ M(x) &= e^{x^2} \int_0^x e^{-t^2} dt = x \cdot e^{x^2} \cdot {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x^2\right) = c \cdot e^{\frac{1}{2}x^2} \cdot x^{-\frac{1}{2}} \cdot M_{\frac{1}{4}}(x^2) \end{aligned}$$

We note that the term $\Gamma\left(1 + \frac{s}{2}\right)$ was the originally introduced notation by Gauss for the Gamma function, appearing „to me much more natural and Riemann's use of it gives me a welcome opportunity to introduce it“ (quote, H. M. Edwards in (EdH), footnote, p. 8). The Riemann (exact) „approximation“ function is calculated out of the $\xi(s)$ –term $\Gamma\left(1 + \frac{s}{2}\right)$ given by ((EdH) 1.16)

$$\int_x^\infty \frac{dt}{t(t^2-1)\log t} = \int_x^\infty \frac{1}{t \log t} (\sum t^{-2n}) dt = \sum_{n=1}^\infty \int_x^\infty t^{-2n} \frac{dt}{t \log t} = \frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[\frac{\log \Gamma\left(1 + \frac{s}{2}\right)}{s} \right] x^s ds.$$

The relationship between $\Gamma\left(1 + \frac{s}{2}\right)$ and the Mellin transform of the considered Kummer function ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x^2\right)$ is given by ((GrI) 7.612)

$$\Gamma\left(1 + \frac{s}{2}\right) = s(1-s) \int_0^\infty x^s {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x^2\right) \frac{dx}{x}.$$

We note that the Mellin transform

$$\int_0^\infty x^s {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x^2\right) \frac{dx}{x} = \frac{\frac{1}{2}\Gamma\left(\frac{s}{2}\right)}{1-s} = \frac{\Gamma\left(1 + \frac{s}{2}\right)}{s(1-s)}, \quad 0 < \text{Re}(s) < 1.$$

is mirroring the only formally valid representation of the entire Zeta given by ((EdH) 10.3)

$$\int_0^\infty u^{-s} G(u) du = \frac{2\xi(s)}{s(s-1)}.$$

In ((BuH) p.184) the product representation of the Kummer functions

$${}_1F_1(a, c, z) = \frac{1}{\Gamma(c)} e^{\frac{z}{c}} \prod \left(1 - \frac{z}{\alpha_n}\right) e^{z/\alpha_n}$$

are provided, where α_n denotes the infinite set of zeros of ${}_1F_1(a, c, z)$, i.e. the Kummer functions are elements of the Laguerre-Polya class LP (CaD), which is related to the topic „zeros of certain trigonometric integrals in the context of entire transcendental higher genus 1 functions“ (PoG2).

The fractional part function related Zeta function theory is provided in ((TiE) II). The Hilbert transform of the fractional part function is given by the $\log(\sin x)$ –function. A $\log(\sin x)$ –function based Zeta function theory deals with a „Zeta“ function $\zeta^*(s)$ the form

$$\zeta^*(s) = \frac{\cot(\frac{\pi}{2}s)}{1-s} \cdot \zeta(s) = \frac{\tan(\frac{\pi}{2}(1-s))}{1-s} \cdot \zeta(s) .$$

The correspondingly defined distributional ("periodical") Hilbert space framework deals with the Hilbert spaces $H_0^\#(0,1)$ $H_{-1/2}^\#(0,1)$, $H_{-1}^\#(0,1)$, with its relationship to the Bagchi reformulation of the Nyman-Beurling RH criterion. This criterion basically becomes a „dense embedding“ argument of $H_{-1/2}^\#(0,1)$ into $H_{-1}^\#(0,1)$, as the \cot and the Zeta function on the critical line are $\in H_{-1}^\#(0,1)$.

The considered Hilbert space in [BaB] is about of all sequences $a = \{a_n | n \in \mathbb{N}\}$ of complex numbers such that

$$\sum_{n=1}^{\infty} \omega_n |a_n|^2 < \infty \quad \text{with} \quad \frac{c_1}{n^2} \leq \omega_n \leq \frac{c_2}{n^2}$$

which is isomorph to the Hilbert space $H_{-1} \cong l_2^{-1}$. Let $\gamma := \{11, 1, 1, \dots\}$, then it holds

$$\|\gamma\|_{-1}^2 = \sum_1^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

i.e. $\gamma \in l_2^{-1}$, resp. the Zeta function on the critical line $\mathcal{E} \in H_{-1}(-\infty, \infty)$.

Theorem (Bagchi-Nyman criterion, (BaB)): Let

$$\gamma_k := \left\{ \rho\left(\frac{n}{k}\right) \mid n = 1, 2, 3, \dots \right\} \quad \text{for } k = 1, 2, 3, \dots$$

and Γ_k be the closed linear span of γ_k . Then the Nyman criterion states that the following statements are equivalent:

- i) The Riemann Hypothesis is true
- ii) $\gamma \in \overline{\Gamma_k}$.

As $l_2^{-1/2}$ is dense in l_2^{-1} with respect to the l_2^{-1} –norm, then γ belongs to the closed linear span of a double infinite matrix $\{\gamma_k^H\}_{k \in \mathbb{N}}$, i.e.

$$\gamma \in l_2^{-1} = \overline{\text{span}}_{l_2^{-1}} \{\gamma_k^H\}_{k \in \mathbb{N}}$$

if $\gamma_k^H \in l_2^{-1/2}$.

The zeros of the considered Kummer functions (and therefore also the zeros of the solution of the corresponding Whittaker eigenvalue equations) enjoy appropriate behaviors, e.g. for the real part values ω_n of ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; 2\pi iz\right)$ it holds (SeA)

- i) $n - \frac{1}{2} < \omega_n < n, \frac{1}{2} < \omega_1 < a := s_n := \frac{\omega_n}{n} \rightarrow 1 \quad n \in \mathbb{N}$
- ii) $2n - 1 < 2\omega_n < 2n < \omega_n + \omega_{n+1} < 2n + 1 < 2\omega_{n+1} < 2(n + 1)$
- iii) the sequences $2\omega_n$ and $\omega_n + \omega_{n+1}$ fulfill the Hadamard gap condition

$$\frac{\omega_{n+1}}{\omega_n} > \frac{n+\frac{1}{2}}{n} = 1 + \frac{1}{2n} > q > 1 \quad \text{resp.} \quad \frac{\omega_{n+1} + \omega_{n+2}}{\omega_n + \omega_{n+1}} > \frac{2n+2}{2n+1} = 1 + \frac{1}{2n+1} > q > 1$$
- iv) $\theta := \frac{1}{4} < \frac{\omega_{n+1}}{2} - \frac{\omega_n}{2} < 1 - \frac{1}{4} = 1 - \theta.$

For the related sequences $a_n := \frac{2\omega_n}{2n} - \frac{1}{2}, b_n := \frac{\omega_{n+1} + \omega_n}{2n} - \frac{1}{2}$ it therefore follows

- i) $0 < a_1 = \omega_1 - \frac{1}{2} \leq a_n \rightarrow \frac{1}{2}, \frac{1}{2} \leftarrow b_n < b_1 = \frac{\omega_1 + \omega_2 - 1}{2} < 1$
- ii) $a_n, b_n - a_n \in (0, \frac{1}{2}), b_n, 1 - a_n \in (\frac{1}{2}, 1).$

The sequence $s_n := \frac{\omega_n}{n}$ fulfills the Hardy-Littlewood condition $|s_{n+1} - s_n| < \frac{K}{n}$, i.e. it has a defined Abel average ((EdH) 12.7)

$$\lim_{r \uparrow 1} \frac{s_1 r + s_1 r^2 + s_3 r^3 + \dots}{r + r^2 + r^3 + \dots} = L$$

and (with respect to the Goldbach conjecture below) the following inequalities are valid

$$c_1 \cdot \left[{}_1F_1\left(\frac{1}{2}, c; x\right) \right]^2 \leq {}_1F_1(a_n, c; x) \cdot {}_1F_1(1 - a_n, c; x) \leq \left[{}_1F_1\left(\frac{1}{2}, c; x\right) \right]^2.$$

The identities $n^2 = 1 + 3 + \dots + (2n - 1)$ and $n(n + 1) = 2 + 4 + \dots + 2n$ in combination with the imaginary parts of the zeros of the considered special Kummer function lead to the inequality

$$n - \frac{1}{2} < \frac{2}{n} \sum_{k=1}^n \omega_k < n + \frac{1}{2}$$

enabling an alternative Rogers-Ramanujan Continued Fraction with corresponding identities. We note that the integer "2" is not an element of the set $\mathbf{A} := \{[2\omega_n], n \in \mathbb{N}\}$, i.e. the Snirelmann density of \mathbf{A} is $\leq \frac{1}{2}$.

The Kummer function zeros related sequence ω_n enables the following replacements

$$\begin{array}{llll} 2n & \rightarrow & 2\omega_n & e^{i\pi(2n)x} \rightarrow e^{i\pi(2\omega_n)x} & S(x) := \sum_n e^{2\pi i n x} \rightarrow S^{(1)}(x) := \sum_n e^{i\pi(2\omega_n)x} \\ & & \omega_n + \omega_{n+1} & \rightarrow e^{i\pi(\omega_n + \omega_{n+1} - 1)x} & \rightarrow S^{(2)}(x) := \sum_n e^{i\pi(\omega_n + \omega_{n+1} - 1)x} \end{array}$$

The number $N(T)$ of zeros of the Zeta function with the ordinate axis values $0 < \gamma_1 \leq \gamma_2 \leq \dots$ within $0 < t \leq T$ in the critical stripe fulfills $N(T) = A \cdot T \log T + O(T)$, from which it follows the

Theorem 468 ((LaE4) p. 139): the series $S(\sigma) = \sum_{n=1}^{\infty} \frac{\sin(\gamma_n \sigma)}{\gamma_n} e^{-\gamma_n \sigma}$ is absolutely convergent

with $\lim_{\sigma \rightarrow 0} \frac{S(\sigma)}{\log \frac{1}{\sigma}} = \alpha_{135}$.

With respect to the zeros of the Kummer function above this gives, that the series $\sum_{p \in P} \frac{\sin(\omega_p \sigma)}{\omega_p} e^{-\omega_p \sigma}$ is absolutely convergent, as well. From (BeB) 5, Corollary 4, we recall that the representation ($(\text{Re}(s) > 1)$)

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s} = \prod_p \left(1 - \frac{\sin(\frac{\pi}{2}s)}{p^s}\right)^{-1}$$

implies the convergence of the series $\sum_p p^{-1} \sin(\frac{\pi}{2}p)$.

Let H and M denote the Hilbert and the Mellin transform operators. For the Gaussian function $f(x)$ it holds

$$M[f](s) = \frac{1}{2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right), \quad M[-xf'(x)](s) = \frac{s}{2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right) = \frac{1}{2}\pi^{-s/2}\Pi\left(\frac{s}{2}\right).$$

The corresponding entire Zeta function is given by ([EdH] 1.8)

$$\xi(s) := \frac{s}{2}\Gamma\left(\frac{s}{2}\right)(s-1)\pi^{-s/2}\zeta(s) = (1-s) \cdot \zeta(s)M[-xf'(x)](s) = \xi(1-s).$$

Replacing the Gaussian function by the Dawson function (which is the Hilbert transform of the Gaussian function) leads to an alternative entire Zeta function $\xi^*(s)$ in the form

$$\xi^*(s) := \frac{1}{2}(s-1)\pi^{\frac{1-s}{2}}\Gamma\left(\frac{s}{2}\right)\tan\left(\frac{\pi}{2}s\right) \cdot \zeta(s) = \zeta(s) \cdot M\left[\frac{d}{dx}[-x \cdot f_H(x)]\right](s)$$

with same zeros as $\xi(s)$, as it holds $s(1-s)\xi^*(s)\xi^*(1-s) = \pi\xi(s)\xi(1-s)$.

Replacing the Mellin transform of the Gaussian function in the definition of the $\xi(s)$ -function goes along with a replacement of the Gauss „Gamma“ function definition $\Gamma\left(1 + \frac{s}{2}\right)$ by

$$\rightarrow \quad \Gamma^*\left(\frac{s}{2}\right) := \Gamma\left(\frac{s}{2}\right)\tan\left(\frac{\pi}{2}s\right) = \frac{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} = \frac{\Gamma\left(1+\frac{s-1}{2}\right)\Gamma\left(1-\frac{s+1}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\Gamma\left(1+\frac{s}{2}\right)}{\left(k-\frac{1}{2}\right)^2 - \left(\frac{s}{2}\right)^2}.$$

It results into a correspondingly modified Riemann approximation error function with now newly three summation terms governed by product formula

$$\Gamma(1 \pm s) = \prod \left(1 + \frac{1}{n}\right)^{\pm s} \left(1 \pm \frac{s}{n}\right)^{-1}$$

with improved $(|li(x) - \pi(x)|)$ approximation behavior. We mention the product representation ((AbM) 6.1.3)

$$\Gamma^*\left(\frac{s}{2}\right) := \tan\left(\frac{\pi}{2}s\right)\Gamma\left(\frac{s}{2}\right) = \frac{\pi}{2}e^{-\gamma s} \prod \left(1 + \frac{s}{2n}\right)^{s/2n} \frac{\left(1 - \left(\frac{s}{2k}\right)^2\right)}{\left(1 - \left(\frac{s}{2k-1}\right)^2\right)}.$$

As the Hilbert transform defines a convolution operator, the Dawson function approach enables the „zeros of entire functions“ analysis techniques in the context of the Hilbert-Polya conjecture (e.g. (CaD)).

The Dawson function approach also enables an „analysis of zeros of certain trigonometric integrals and entire higher genus-1 functions“ (PoG2), based on the identity ((GrI) 3.896) $F(x) = e^{-x^2}H(x)$ with $H(x) := x {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x^2\right)$.

The proposed alternative entire Zeta function $\xi^*\left(\frac{1}{1-s}\right)$ is also suggested to verify the corresponding Li criterion (LiX). This criterion results from a necessary and sufficient condition, that the logarithmic of the function $\xi\left(\frac{1}{1-s}\right)$ be analytic in the unit disk resp. that the sequence of real constants $\sigma_n := \frac{1}{\Gamma(n)} \frac{d^n}{ds^n} [s^{n-1} \log \xi(s)]_{s=1}$ are not-negative (CoM). The proof of the Li criterion is built on the two representations of the Zeta function, its (product) representation over all its nontrivial zeros ((HdE) 1.10) and Riemann's integral representation derived from the Riemann duality equation, based on the Jacobi theta function ((EdH) 1.8). In (CoF), (KeJ) corresponding Li/Keiper constants are considered.

With respect to the binary Goldbach conjecture the above enables a new circle method on the unit circle, going along with corresponding *additive* number theory problem adequate arithmetic functions.

The Landau theorem to build appropriate arithmetic functions, is a special case of the (PoG1):

Generalized Landau Theorem: Let $w(x)$ a positive, non decreasing function with $\lim_{n \rightarrow \infty} \frac{w(\beta x)}{w(\alpha x)} = 1$ with α, β positive numbers. Then

$$\lim_{n \rightarrow \infty} \frac{w(x)}{x} \sum_{n \leq x} f\left(\frac{x}{n}\right) = \int_0^1 f(t) dt .$$

Applying this theorem in the context of the conceptual approach of this paper leads to the definitions (see also e.g. (BrK4) Notes O27/37)

$$\sigma^*(x) := \frac{1}{2} [\sigma^{(1)}(x) + \sigma^{(2)}(x)]$$

with

$$\sigma^{(1)}(x) := \frac{1}{x} h^{(1)}\left(\frac{1}{x}\right) \sum_{n \leq x} h^{(1)}\left(\frac{x}{2\omega_n}\right) , \quad \sigma^{(2)}(x) := \frac{1}{x} h^{(2)}\left(\frac{1}{x}\right) \sum_{n \leq x} h^{(2)}\left(\frac{x}{\omega_n + \omega_{n+1} - 1}\right)$$

an

$$h^{(1)}(x) := -(\pi x) \cot^{(1)}(\pi x) , \quad h^{(2)}(x) := -(\pi x) \cot^{(2)}(\pi x) .$$

This paper is concerned with completely different topics, but all of those are concerned with the very „big“ and the very „small“, what ever it means; the harmonic numbers $H_n = \sum_{k=1}^n \frac{1}{k}$ might give a first impression from a number theory point of view: the H_n are always fractions (except for $H_1 = 1, H_2 = 1.5, H_6 = 2.45$), the series is divergent, but the number n that the sum H_n past 100 is in the size of 10^{43} , i.e. a computer which takes 10^{-9} seconds to add each new term to the sum will have been completed in not less than 10^{17} (American) billion years (HaJ). The Fourier theory of cardinal functions enables a correspondingly absolute convergent cardinal series in the form $C(x) := \frac{\sin(\pi x)}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{H_{2n} - \frac{1}{2} H_n}{x-n}$ alternatively to $\frac{\sin(\pi x)}{\pi x}$ (WhJ2). What we propose in this paper is an alternative approximation term in the form $H_n^* := \frac{1}{2} H_n + \frac{\omega_n}{n}$ with $\lim_{n \rightarrow \infty} [H_n^* - \log \sqrt{e \cdot n}] = \frac{\gamma+1}{2}$.

For the harmonic numbers $2h_n = \sum_{k=1}^n \frac{2}{2k-1} = 2H_{2n} - H_n$, $H_n = \sum_{k=1}^n \frac{1}{k}$ resp. $c_n := \frac{2h_n}{n}$ it holds the Fourier series representation (EIL)

$$\frac{\pi}{2} \log \left(\tan \left(\frac{\pi}{2} x \right) \right) = - \sum \frac{2h_n}{n} \sin(2\pi n x) = - \sum c_n \sin(2\pi n x) \in L_2^\#(0,1)$$

i.e.

$$\sum_{n=1}^{\infty} c_n^2 < \infty .$$

The following two a priori estimates ((HaJ) p. 47, resp. (Goldbach conjecture) section 1g below)

$$\frac{1}{2n} < \frac{1}{2} (H_n - \log n) < \frac{1}{2} , \quad \frac{1}{2} - \frac{1}{2n} < a_n := s_n - \frac{1}{2} = \frac{\omega_n}{n} - \frac{1}{2} < \frac{1}{2}$$

leads to an alternative approximation term in the form

$$\frac{1}{2} < H_n^* - \log \sqrt{e \cdot n} = \frac{1}{2} H_n + \frac{\omega_n}{n} - \log \sqrt{e \cdot n} = \frac{1}{2} (H_n - \log n - 1) + \frac{\omega_n}{n} < 1$$

with

$$\lim_{n \rightarrow \infty} \left[\frac{1}{2} (H_n - \log n - 1) + \frac{\omega_n}{n} \right] = \frac{1}{2} (\gamma - 1) + 1 = \frac{\gamma+1}{2} .$$

2. An alternative entire Zeta function $\xi^*(s)$ based on Kummer / Dawson and $\tan\left(\pi\frac{s}{2}\right) = \cot\left(\pi\frac{1-s}{2}\right)$ functions

Let H and M denote the Hilbert and the Mellin transform operators. For the Gaussian function $f(x)$ it holds

$$M[f](s) = \frac{1}{2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right), \quad M[-xf'(x)](s) = \frac{s}{2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right) = \frac{1}{2}\pi^{-s/2}\Pi\left(\frac{s}{2}\right).$$

The corresponding entire Zeta function is given by ([EdH] 1.8)

$$\xi(s) := \frac{s}{2}\Gamma\left(\frac{s}{2}\right)(s-1)\pi^{-s/2}\zeta(s) = (1-s) \cdot \zeta(s)M[-xf'(x)](s) = \xi(1-s).$$

Putting

$$G(u) := \sum_{-\infty}^{\infty} e^{-\pi n^2 u^2} = \sum_{-\infty}^{\infty} f(nu)$$

Riemann's functional equation implies, that the invariant operator $g(x) \rightarrow \int_0^{\infty} g(ux)G(u)du$ is formally self-adjoint. A valid invariant operator would prove the Hilbert Polya conjecture. But the operator has no transform at all (due to the not vanishing constant Fourier term of the Gaussian function); that is the integral $\int_0^{\infty} u^{-s}G(u)du$, which is formally represented in the form ((EdH) 10.3)

$$\int_0^{\infty} u^{-s}G(u)du = \frac{2\xi(s)}{s(s-1)}$$

does not converge for any s . The central idea is to replace $M[-xf'(x)](s) \rightarrow M[-f_H(x)](s)$ with $f_H(x) := H[f](x)$ ($\hat{f}_H(0) = 0$) and

$$M[f_H(x)](s) = M\left[2\pi x {}_1F_1\left(1, \frac{3}{2}, -\pi x^2\right)\right](s) = M\left[2\sqrt{\pi}F(\sqrt{\pi}x)\right](s) = \frac{1}{2}\pi^{\frac{1-s}{2}}\Gamma\left(\frac{s}{2}\right)\tan\left(\frac{\pi}{2}s\right),$$

leading to the alternative *entire* Zeta function $\xi^*(s)$ in the form

$$\xi^*(s) := \frac{1}{2}(s-1)\pi^{\frac{1-s}{2}}\Gamma\left(\frac{s}{2}\right)\tan\left(\frac{\pi}{2}s\right) \cdot \zeta(s) = \zeta(s) \cdot M\left[\frac{d}{dx}[-x \cdot f_H(x)]\right](s).$$

The link to the Zeta functions $\xi(s)$ resp. $\zeta(s)$ is given by the equation

$$\xi^*(s)\xi^*(1-s) = \pi^{\frac{\xi(s)\xi(1-s)}{s(1-s)}},$$

i.e. $\xi(s)$ and $\xi^*(s)$ do have the same set of zeros in the critical stripe.

The corresponding invariant operator $f(x) \rightarrow \int_0^{\infty} f(ux)G_H(u)du$ is built on

$$G_H(u) := \sum_{-\infty}^{\infty} f_H(nu)$$

replacing Riemann's auxiliary function ((EdH) 10.3) $H(u) := \frac{d}{du}\left[u^2 \frac{d}{du}G(u)\right]$.

The Kummer function related Mellin transforms are given by the formula (GrI) 7.612

$$\int_0^{\infty} x^s {}_1F_1(\alpha, \beta; -x) \frac{dx}{x} = \frac{\Gamma(\beta)}{\Gamma(\alpha)} \Gamma(s) \frac{\Gamma(\alpha-s)}{\Gamma(\beta-s)}, \quad 0 < \operatorname{Re}(s) < \operatorname{Re}(\alpha),$$

leading to

$$\int_0^{\infty} x^{s/2} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x\right) \frac{dx}{x} = \frac{\Gamma\left(\frac{s}{2}\right)}{1-s} = \frac{2}{s(1-s)} \Gamma\left(1 + \frac{s}{2}\right), \quad 0 < \operatorname{Re}(s) < 1.$$

The link of the Hermite polynomials to the corresponding Hilbert transform of the Gaussian (the Dawson function) is given by (AbM), 7.1.15, resp. (RyG)

$$\frac{1}{\sqrt{\pi}}H[e^{-x^2}] = F(x) = \frac{1}{\sqrt{\pi}} \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{H_k^{(n)}}{x - x_k^{(n)}} = \frac{1}{\sqrt{\pi}} \lim_{h \rightarrow 0} \sum_{n \text{ odd}} \frac{e^{-(x-nh)^2}}{n} \approx \frac{1}{x}.$$

where $x_k^{(n)}$ and $H_k^{(n)}$ are the zeros and weight factors of the Hermite polynomials. The operator theory associated with the Hermite polynomials does not extend to the

generalized Hermite polynomials because the even and odd polynomials satisfy different equations. Nevertheless, in (KrA) a united spectral expansion is provided. In this context we note that the two sequences $\{2\omega_n\}, \{\omega_n + \omega_{n+1}\}$ (with $[2\omega_n] = 2n - 1$, $[\omega_n + \omega_{n+1}] = 2n$) of the zeros of ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; 2\pi iz\right)$ alternatively applied to the sequences of $e^{2\pi iz}$ satisfy the same differential equation.

The link to the normal distribution $\bar{F}(x) = \int_{-\infty}^{\infty} e^{-u^2} du$ resp. to the topic of „Convolution Operators and the Zeros of Entire Functions“ (CaD), is given by by the formula

$$h_n(z) = \int_{-\infty}^{\infty} (z - is)^n d\bar{F}(s) = \int_{-\infty}^{\infty} (z - is)^n e^{-s^2} ds = 2^{-n} H_n(z).$$

Riemann inverted the formula ((EdH), 1.13)

$$J(x) = \sum_{n \leq x} \Lambda(n) \log n = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log \zeta(s) x^s \frac{ds}{s} = Li(x) - \sum_{\rho} Li(x^{\rho}) - \log 2 + \int_x^{\infty} \frac{1}{t(t^2-1) \log t} dt$$

by means of the Möbius inversion formula, getting ((EdH), 1.17)

$$\pi(x) = Li(x) - \sum_{n=2}^{\infty} Li(x^{1/n}), \quad Li(x) := \int_2^x \frac{dt}{\log t} - Li(2), \quad Li(2) \sim 1.04 \dots$$

He suggested a more natural and better approximation in the form

$$\pi(x) \sim R(x) := Li(x) - \sum_{n=2}^{\infty} \frac{\mu(n)}{n} Li(x^{1/n}).$$

However, he was aware of the defects of this approximation and his analysis of it, which is basically due to fact, that he has no estimate at all of the size of the „periodic“ terms $\sum_{n=1}^{\infty} \sum_{\rho} Li(x^{\rho/n})$ ((EdH), 1.17). The $Li(x)$ function RH approximation criterion is given by

$$|Li(x) - \pi(x)| = O(\sqrt{x} \log x) = O\left(x^{\frac{1}{2} + \varepsilon}\right); \varepsilon > 0.$$

The challenge to prove the $Li(x)$ function approximation criterion is about the (exponential) asymptotics of the Gaussian function (which we propose to be replaced by the Dawson function). Based on the product representations of the Gamma function

$$\Gamma(1+s) = \prod \frac{(1+\frac{1}{n})^s}{(1+\frac{s}{n})}, \quad \Gamma(1-s) = \prod \frac{(1+\frac{1}{n})^{-s}}{(1-\frac{s}{n})}, \quad \Gamma\left(1+\frac{s}{2}\right) = \prod \frac{(1+\frac{1}{n})^{s/2}}{(1+\frac{s}{2n})}$$

the Riemann approximation error function ((EdH) 1.16)

$$\int_x^{\infty} \frac{dt}{t(t^2-1) \log t} = \int_x^{\infty} \frac{1}{t \log t} \left(\sum t^{-2n} \right) dt = \sum_{n=1}^{\infty} \int_x^{\infty} t^{-2n} \frac{dt}{t \log t} = \frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[\frac{\log \Gamma\left(1+\frac{s}{2}\right)}{s} \right] x^s ds$$

is calculated from the terms

$$\frac{d}{ds} \left[\frac{\log \Gamma\left(1+\frac{s}{2}\right)}{s} \right] \quad \text{and} \quad H(\beta) := \frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[\frac{\log(1-\frac{s}{\beta})}{s} \right] x^s ds, \quad (H(1) = Li(x) - i\pi).$$

As the alternative entire $\xi^*(s)$ function is going along with the replacement (*) above, this results into a correspondingly modified Riemann approximation error function with now newly three summation terms with improved ($|li(x) - \pi(x)|$ error) convergence behavior.

We mention the Kummer function based representation of the $li(x)$ -function in the form $li(x) = x {}_1F_1(1,1; -\log x)$ ((LeN) (9.13.7)) with the asymptotics

$$li(x) = -x {}_1F_1(1; 1, -\log x) = Ei(\log x) \approx \log^{-1}(x).$$

From (ObF), 2.4, we recall the following inverse Mellin transforms:

$$a^z \cot(\pi z), \quad -n < Re(z) < 1 - n, \quad n = 0, \pm 1, \pm 2, \dots \quad \text{Principle value} \quad \rightarrow \quad \pi^{-1} \left(\frac{x}{a}\right)^n \left(1 - \left(\frac{x}{a}\right)\right)^{-1}$$

$$a^z \tan(\pi z), \quad -n - 1/2 < Re(z) < 1/2 - n, \quad n = 0, \pm 1, \pm 2, \dots \quad \text{Principle value} \quad \rightarrow \quad -\pi^{-1} \left(\frac{x}{a}\right)^{n+1/2} \left(1 - \left(\frac{x}{a}\right)\right)^{-1}$$

3. An alternative Zeta function $\zeta^*(s)$ based on the $\cot(\pi s) = \tan\left(\frac{\pi}{2}(1-2s)\right)$ function

The fractional part function related Zeta function theory is provided in ((TiE) II). The Hilbert transform of the fractional part function is given by the $\log(\sin x)$ –function. The correspondingly defined distributional ("periodical") Hilbert space framework enables the Bagchi reformulation of the Nyman-Beurling RH criterion, which then becomes basically a „being dense embedded“ argument of $H_{-1/2}^\#(0,1)$ into $H_{-1}^\#(0,1)$. The $H_{-1}^\#(0,1)$ related counterpart of the $\log(\sin x)$ –function is given by the Clausen integral (ChJ) define by

$$Cl_2(x) := \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2} = - \int_0^x \log \left[2 \sin \frac{y}{2} \right] dy, \quad x \in R$$

with the periodic properties $Cl_2(2n\pi \pm \vartheta) = Cl_2(\pm \vartheta) = \pm Cl_2(\vartheta)$ and $Cl_2(\pi + \vartheta) = -Cl_2(\pi - \vartheta)$.

One proof of the Riemann functional equation is based on the fractional part function $\rho(x)$, whereby the zeta function $\zeta(s)$ in the critical stripe is given by the Mellin transform

$$\zeta(1-s) = M[-x \cdot \rho'(x)](s-1), \quad (\text{TiE}) \quad (2.1.5).$$

The functional equation is given by ((TiE) (2.1.12)

$$\zeta(s) = \chi(s)\zeta(1-s) = \pi^{s-1/2} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \zeta(1-s).$$

The Hilbert transform of the fractional part function $\rho(x) = x - [x] = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{\pi n} \in L_2^\#(0,1)$ is given by $\rho_H(x) = \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{\pi n} = -\frac{1}{\pi} \log(2 \sin(\pi x)) \in L_2^\#(0,1)$, resulting into $\rho'_H(x) = -\cot(\pi x) = -2 \sum_{n=1}^{\infty} \sin(2\pi nx) \in H_{-1}^\#(0,1)$. We note that the $\cot(\pi x)$ series representation is Cesàro summable (mean of order 1) (BrK4) Note O25, (ZyA) VI-3, VII-1).

The corresponding alternative $\zeta^*(s)$ function is given by ((BrK4) lemma 1.4, lemma 3.1 (GrI) 1.441, 3.761, 8.334, 8.335)

$$\zeta^*(1-s) = \frac{\tan(\frac{\pi s}{2})}{s} \cdot \zeta(1-s) \quad \text{resp.} \quad \zeta^*(s) = \frac{\cot(\frac{\pi s}{2})}{1-s} \cdot \zeta(s) = \frac{\tan(\frac{\pi}{2}(1-s))}{1-s} \cdot \zeta(s)$$

resp.

$$\log \zeta^*(s) = \log \left(\tan \frac{\pi}{2} (1-s) \right) + \log \left(\frac{1}{1-s} \right) + \log \zeta(s).$$

We note the series representation (TiE) 4.14,

$$\sum_{n>x} \frac{1}{n^s} = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} z^{1-s} (-\pi \cot(\pi z)) \frac{dz}{z}.$$

The functions $\log(2 \sin(\pi x))$, $\cot(\pi x)$, $\sin^{-2}(\pi x)$ are also related to the convolution kernels of the model adequate Pseudo-Differential operators in the context of the proposed new ground state energy model of the harmonic quantum oscillator model problem.

We recall the following inverse Mellin transform formulas ((NiN) §91/92):

$$\log\left(1 + \frac{1}{x}\right) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^{-s} \frac{\pi}{\sin(\pi s)} \frac{ds}{s},$$

$$\frac{\log(1+x)+\gamma}{1+x} = -\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \psi(s) x^{1-s} \frac{\pi}{\sin(\pi s)} \frac{ds}{s}$$

with $\psi(x) := \log' \Gamma(x) = \Gamma'(x)/\Gamma(x)$ and $0 < a < 1$.

We further mention the harmonic number related „Elementary RH criterion“ (LaJ):

$$\text{The RH holds true} \quad \text{iff} \quad \text{for each } n \geq 1 \quad \sum_{d|n} d \leq H_n + e^{H_n} \log(H_n).$$

The function $\pi \cot(\pi x)$ is holomorphic except the pole $z = 1$ and the following series representation is valid ((GrI) 1.442)

$$\log(\cot(\pi x)) = \sum_{k=1}^{\infty} \frac{2\cos(2k-1)\pi x}{(2k-1)}, \quad 0 < x < 1, \quad \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2}{2k-1} = \frac{\pi}{2}.$$

Replacing the harmonic numbers $H_n = \sum_{k=1}^n \frac{1}{k}$ by the „alternative“ harmonic numbers

$$2h_n := \sum_{k=1}^n \frac{2}{2k-1} = 2H_{2n} - H_n$$

and avering of the the two series

$$-\log(1-x) = \int_0^1 \frac{dt}{1-t} = \sum_{k=1}^{\infty} \frac{x^k}{k}, \quad +\log(1+x) = \int_0^1 \frac{dt}{1-t} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$$

results into the representation (ChH)

$$\frac{1}{2} \log\left(\frac{1+x}{1-x}\right) = \sum_{k=1}^{\infty} \frac{1}{2k-1} x^{2k-1} \quad \text{resp.} \quad \frac{1}{4} \log^2\left(\frac{1+x}{1-x}\right) = \sum_{k=1}^{\infty} \frac{h_k}{k} x^{2k}.$$

Based on this results in (EIL) the following series representation and corresponding regularity results are derived ($c_n := \frac{2h_n}{n}$)

$$T(x) := -\frac{\pi}{2} \log\left(\tan\left(\pi \frac{x}{2}\right)\right) = -\frac{\pi}{2} \log\left(\cot\left(\pi \frac{1-x}{2}\right)\right) = \sum \frac{2h_n}{n} \sin(\pi(2n)x) = \sum c_n \sin(2\pi n x)$$

with

$$T(x) \in L_2^{\#}(0,1) \quad \text{i.e.} \quad \sum_{n=1}^{\infty} c_n^2 < \infty.$$

While $T(x)$ is of same regularity as the fractional part function, its first derivative

$$\log'\left(\tan\left(\pi \frac{x}{2}\right)\right) = \frac{\pi}{\sin(\pi x)} \in H_{-1}^{\#}(0,1)$$

is of same regularity as the \cot – and the Zeta function on the critical line.

We note that the analysis in (EIL) is based on the („orthogonality“) formula

$$\int_0^1 \log(\tan(\frac{\pi}{2}x)) \cos(k\pi x) dx = \begin{cases} -1/k & k \text{ odd} \\ 0 & k \text{ even} \end{cases}.$$

To take advantage out of the convergent series $\sum_{n=1}^{\infty} c_n^2 < \infty$, we recall the

Lemma (KaM1): Let $\{n_k\}$ be a sequence of integers satisfying the “Hadamard” gap” condition, i.e. $\frac{n_{k+1}}{n_k} > q > 1$. Then the trigonometric gap series $\sum_{k=1}^{\infty} c_k \sin(2\pi n_k x)$ converges almost everywhere, if and only if, $\sum_{k=1}^{\infty} c_k^2 < \infty$

from which it follows the

Corollary: the series $\sum \frac{2h_k}{k} \sin(2\pi \omega_k x)$ and $\sum \frac{2h_k}{k} \sin(\pi(\omega_k + \omega_{k+1})x)$ converge almost everywhere.

The generalization of H_n to the real variable is given by

$$\psi(x) := \log' \Gamma(x) = \Gamma'(x)/\Gamma(x)$$

since $\psi(n+1) = H_n - \gamma$ ((AbM) (6.3.2)).

For the alternatively proposed $\psi^*(x) := \psi(x) + \frac{\pi}{\sin(\pi x)}$ we note the corresponding fomulas

$$\begin{aligned} \psi(1+z) &= \psi(z) + \frac{1}{z} & \rightarrow & \psi^*(1+z) = \psi(z) + \frac{1}{z} - \frac{\pi}{\sin(\pi z)} \\ \psi(1-z) &= \psi(z) + \pi \cot(\pi z) & \rightarrow & \psi^*(1-z) = \psi(1-z) + \pi \cot(\pi z) + \frac{\pi}{\sin(\pi z)}. \end{aligned}$$

For any natural number $n \geq 1$ the following inequality is valid (ViM):

$$\frac{1}{2n - \frac{1}{1-\gamma} - 2} \leq H_n - \log n \leq \frac{1}{2n + \frac{1}{3}}$$

i.e. the approximation formula $H_n \approx \log n + \gamma + \frac{1}{2n}$ is overestimating, which indicates a correction term in the form $H_n - \frac{\pi}{\sin(\pi\omega_n)}$.

The following two a priori estimates ((HaJ) p. 47, resp. (Goldbach conjecture) section 1g below)

$$\frac{1}{2n} < \frac{1}{2}(H_n - \log n) < \frac{1}{2}, \quad \frac{1}{2} - \frac{1}{2n} < a_n := s_n - \frac{1}{2} = \frac{\omega_n}{n} - \frac{1}{2} < \frac{1}{2}$$

leads to an alternative approximation term in the form

$$\frac{1}{2} < H_n^* - \log \sqrt{e \cdot n} = \frac{1}{2}H_n + \frac{\omega_n}{n} - \log \sqrt{e \cdot n} = \frac{1}{2}(H_n - \log n - 1) + \frac{\omega_n}{n} < 1$$

with

$$\lim_{n \rightarrow \infty} \left[\frac{1}{2}(H_n - \log n - 1) + \frac{\omega_n}{n} \right] = \frac{1}{2}(\gamma - 1) + 1 = \frac{\gamma+1}{2}.$$

In the context of arithmetical functions, q-series and Ramanujan's theta function we note the following (BeB1):

The Rogers-Ramanujan Continued Fraction $R(q)$ ((BeB1) (7.1.6)) is connected with the Rogers-Ramanujan functions

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q; q^5)_{\infty} (q^4; q^5)_{\infty}}, \quad H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_n} = \frac{1}{(q^2; q^5)_{\infty} (q^3; q^5)_{\infty}}$$

by

$$R(q) = q^{1/5} \frac{H(q)}{G(q)},$$

whereby

$$(a; q^j)_n := \prod_{k=0, \dots, (n-1)} (1 - aq^{jk}).$$

The Rogers-Ramanujan identities have "beautiful combinatorical interpretations, if in the definition of $G(q)$ write $n^2 = 1 + 3 + \dots + (2n - 1)$. Then the first identity is equivalent to the assertion that the number of partitions of a positive number N into distinct parts with differences at least 2 equals the number of partitions of N into parts congruent to either 1 or 4 modulo 5. For the second identity write $n(n + 1) = 2 + 4 + \dots + 2n$. Then this identity is an analytic statement of the fact that the number of partitions of N into distinct parts with differences at least 2 and with no 1's is equal to the number of partitions of N into parts congruent to either 2 or 3 modulo 5 ((BeB) 7.6).

The identities $n^2 = 1 + 3 + \dots + (2n - 1)$ and $n(n + 1) = 2 + 4 + \dots + 2n$ in combination with the imaginary parts of the zeros of the considered special Kummer function lead to the inequality

$$n - \frac{1}{2} < \frac{2}{n} \sum_{k=1}^n \omega_k < n + \frac{1}{2}$$

enabling an alternative Rogers-Ramanujan Continued Fraction with corresponding identities.

4. Alternative arithmetical functions based on the Kummer function zeros

The sequences ω_n (the imaginary parts of the Kummer function zeros) enjoy the following properties ((KaM1), (KoA) (ZyA), see also (BrK4) Notes S43-49, O5-7, O24-25, O27/37, Notes O5-O7, O13, O15-O17)

- i) $n - \frac{1}{2} < \omega_n < n$, $n + \frac{1}{2} < \omega_{n+1} < n + 1$, $\frac{1}{2} < \omega_1 < a := s_n := \frac{\omega_n}{n} \rightarrow 1$ $n \in \mathbb{N}$
- ii) $2n - 1 < 2\omega_n < 2n < \omega_n + \omega_{n+1} < 2n + 1 < 2\omega_{n+1} < 2(n + 1)$
- iii) the sequences $2\omega_n$ and $\omega_n + \omega_{n+1}$ fulfill the Hadamard gap condition
- iv) $\frac{\omega_{n+1}}{\omega_n} > \frac{n+\frac{1}{2}}{n} = 1 + \frac{1}{2n} > q > 1$ resp. $\frac{\omega_{n+1} + \omega_{n+2}}{\omega_n + \omega_{n+1}} > \frac{2n+2}{2n+1} = 1 + \frac{1}{2n+1} > q > 1$
 $\theta := \frac{1}{4} < \frac{\omega_{n+1}}{2} - \frac{\omega_n}{2} < 1 - \frac{1}{4} = 1 - \theta$.

For the related sequences $a_n := \frac{2\omega_n}{2n} - \frac{1}{2}$, $b_n := \frac{\omega_{n+1} + \omega_n}{2n} - \frac{1}{2}$ it therefore follows

- iii) $0 < a_1 = \omega_1 - \frac{1}{2} \leq a_n \rightarrow \frac{1}{2}$, $\frac{1}{2} \leftarrow b_n < b_1 = \frac{\omega_1 + \omega_2 - 1}{2} < 1$
- iv) $a_n, b_n - a_n \in (0, \frac{1}{2})$, $b_n, 1 - a_n \in (\frac{1}{2}, 1)$.

We note that the integer "2" is not an element of the set $\mathbf{A} := \{[2\omega_n], n \in \mathbb{N}\}$, i.e. the Sniirelmann density of \mathbf{A} is $\leq \frac{1}{2}$.

We further note the Mellin transform of the related Kummer function ((GrI) 7.612)

$$\int_0^\infty x^s {}_1F_1(a_n, a_n + 1; -x) \frac{dx}{x} = \frac{a_n}{a_n - s} \Gamma(s) \quad , \quad 0 < \text{Re}(s) < a_n$$

i.e.

$$\int_0^\infty x^s {}_1F_1(a_n, a_n + 1; -x) \frac{dx}{x} \rightarrow \int_0^\infty x^{s/2} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x\right) \frac{dx}{x} = \frac{\Gamma(\frac{s}{2})}{1-s} = \frac{\Gamma(1+\frac{s}{2})}{s(1-s)} \quad , \quad 0 < \text{Re}(s) < 1.$$

The sequence $s_n := \frac{\omega_n}{n}$ fulfills the Hardy-Littlewood condition $|s_{n+1} - s_n| < \frac{K}{n}$, i.e. it has a defined Abel average ((EdH) 12.7)

$$\lim_{r \uparrow 1} \frac{s_1 r + s_1 r^2 + s_3 r^3 + \dots}{r + r^2 + r^3 + \dots} = L \quad .$$

As a consequence, the sequence s_n enables a point measure, which is s_n at n and zero elsewhere, i.e.

$$d\sigma(x) = d\sigma_n(x) = \begin{cases} s_n & x = n \\ 0 & \text{else} \end{cases}$$

enjoying the identities

$$\lim_{r \uparrow 1} \frac{\int_0^\infty r^x d\sigma(x)}{\int_0^\infty r^x dx} = L \quad , \quad \lim_{A \rightarrow \infty} \frac{\int_0^A d\sigma(x)}{\int_0^A dx} = L \quad .$$

which is a restated Abel average representation of the sequence s_n . With respect to the relationship of series with Hadamard gaps, launary sequences, Abel summability and Tauberian theorems for distributional point values to local behavior of distributions we also refer to ((ViJ1) pp. 98, 119, 123, 125).

The (almost everywhere) convergence of the series $\sum \frac{2^{hk}}{k} \sin(2\pi\omega_k x)$ and $\sum \frac{2^{hk}}{k} \sin(\pi(\omega_k + \omega_{k+1})x)$ indicate the following replacements

1. $\cot(\pi x) = 2 \sum_{n=1}^{\infty} \sin(2\pi n x) \in H_{-1}^{\#}(0,1)$ Cesàro summable (ZyA) VI-3, VII-1)
 $\rightarrow \cot^{(1)}(\pi x) := \sum \sin(\pi(2\omega_n)x) \in H_0^{\#}(0,1)$ Abel summable
 $\rightarrow \cot^{(2)}(\pi x) := \sum \sin(\pi(\omega_n + \omega_{n+1})x) \in H_0^{\#}(0,1)$ Abel summable

2. $-\frac{\pi}{2} \log\left(\tan\left(\frac{\pi}{2}x\right)\right) = \sum \frac{2^{hn}}{n} \sin(2\pi n x) \in L_2^{\#}(0,1) \rightarrow$
 $\rightarrow -\frac{\pi}{2} \log\left(\tan^{(1)}\left(\frac{\pi}{2}x\right)\right) := \sum \frac{h_n}{n} \sin(\pi(2\omega_n)) \in H_1^{\#}(0,1)$
 $\rightarrow -\frac{\pi}{2} \log\left(\tan^{(2)}\left(\frac{\pi}{2}x\right)\right) := \sum \frac{h_n}{n} \sin(\pi(\omega_n + (\omega_{n+1}))) \in H_1^{\#}(0,1)$

3.
 $\rightarrow \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} z^{1-s} (-\pi \cot^{(1)}(\pi z)) \frac{dz}{z} \quad \text{resp.} \quad \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} z^{1-s} (-\pi \cot^{(2)}(\pi z)) \frac{dz}{z}.$

(EdH) 12.7: „The PNT is about the asymptotics equivalence of $\psi(x) \sim x$, which is equivalent to the statement that $d\psi(x) \sim dx$ as a Cesàro average in the context of Tauberian theorems. Hardy-Littlewood were able to prove the PNT by showing $d\psi(x) \sim dx$ as an Abel average, where a significant amount of work is done by a Tauberian theorem.”

The Mangoldt resp. Landau density functions are given by

$$\vartheta(x) = \sum_{n < x} \Lambda(n) \quad \text{resp.} \quad \vartheta(x) = \sum_{n=1}^{\infty} \Lambda(n) \log\left(\frac{x}{n}\right).$$

We propose an alternative density function (based on the sequence s_n) to enable a proof in which all of the work is done by a Tauberian theorem:

What cannot be derived from the PNT is the convergence of the series

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log\left(\frac{1}{n}\right) = 1.$$

„The corresponding theorem goes deeper than the PNT, and from it the PNT can be easily derived“ ((LaE) §160). In order to anticipate this finding we suggest to apply the point measures $d\sigma_n(x)$ and

$$\vartheta^*(x) = \sum_{n=1}^{\infty} \Lambda(n) \log\left(\frac{x}{\omega_n}\right), \quad \vartheta^*(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log\left(\frac{x}{\omega_n}\right) \quad \left(\frac{x}{n} < \frac{x}{\omega_n} = \frac{x}{n s_n} < 2 \frac{x}{n}\right)$$

(whereby $d\vartheta = d\vartheta^* = d\vartheta^{**}$), in combination with the Ikehara theorem ((EdH) 12.7). The convergence of the series

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log\left(\frac{1}{n}\right) = 1.$$

is related to the proposed Hilbert space $H_{-\frac{1}{2}}$ by the identity

$$1 = \sum_{n=1}^{\infty} \frac{1}{n} \mu(n) \log\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n} a_n b_n =: ((u, v))_{-\frac{1}{2}} := \lim_{\omega \rightarrow \infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} u\left(\frac{1}{2} + it\right) v\left(\frac{1}{2} - it\right) dt$$

where

$$\left(u\left(\frac{1}{2} + it\right)\right) := \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \in H_{-\frac{1}{2}}, \quad v\left(\frac{1}{2} - it\right) := \sum_{n=1}^{\infty} \frac{\log(1/n)}{n^s} \in H_{-\frac{1}{2}}, \quad s = \frac{1}{2} + it.$$

Property v) in combination with the theorem of E. Landau (LaE3) applied for $\alpha_n, \varphi_n := e^{2\pi i \alpha_n} = e^{\pi i \omega_n}$ leads to the estimate

$$\forall \varepsilon > 0 \quad \cot\left(\frac{\pi}{2}\theta\right) - \varepsilon = \cot\left(\frac{\pi}{8}\right) - \varepsilon < S_m := \left|\sum_{n=1}^m \varphi_n\right| = \left|\sum_{n=1}^m e^{\pi i \omega_n}\right| < \cot\left(\frac{\pi}{8}\right) = \cot\left(\frac{\pi}{2}\theta\right).$$

5. New arithmetical functions for additive number theory

From the section above we recall $2n-1 < 2\omega_n, \omega_n + \omega_{n+1} - 1 < 2n$, i.e.

$$g_n := \frac{1}{2\omega_n}, f_n := \frac{1}{\omega_n + \omega_{n+1} - 1} \in \left(\frac{1}{2n}, \frac{1}{2n-1}\right)$$

indicating a replacement of

$$\beta(x) := \sum_{n < x} a_n \log\left(\frac{x}{n}\right) \rightarrow \beta^*(x) := \frac{1}{2} \left[\sum_{\substack{n < x \\ n \text{ odd}}} a_n \log\left(\frac{x}{2\omega_n}\right) + \sum_{\substack{n < x \\ n \text{ even}}} a_n \log\left(\frac{x}{\omega_n + \omega_{n+1} - 1}\right) \right]$$

with $d\beta = \frac{dx}{x} = d\beta^*$ for a proper additive arithmetic function definition in line with an integer subset $\mathbf{A}^* := \{0\} \cup \{1\} \cup \{2\omega_n, \omega_n + \omega_{n+1} - 1\}$ with Snirelmann density $\frac{1}{2}$.

We note the inequality

$$4n-1 = \left(2n - \frac{1}{2}\right) + \left(2n + \frac{1}{2}\right) - 1 < \{2\omega_n\} + \{\omega_n + \omega_{n+1} - 1\} = \frac{1}{g_n} + \frac{1}{f_n} < \left(2n + \frac{1}{2}\right) + \left(2n + \frac{1}{2}\right) - 1 = 4n.$$

For the fractional part function $\rho(x) = x - [x] = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{\pi n} \in L_2^{\#}(0,1)$ we recall the (PoG1):

Landau theorem: Let q_n denote a divergent sequence of positive numbers $0 < q_1 \leq q_2 \leq q_3 \leq \dots$ $\lim_{n \rightarrow \infty} q_n = \infty$, $\tau(x)$ the corresponding counting function of the numbers of q_n less than $\leq x$ and $w(x)$ a positive, non decreasing function with

$$\lim_{n \rightarrow \infty} \frac{w(2x)}{w(x)} = 1, \text{ and } \lim_{n \rightarrow \infty} \frac{\tau(x)w(x)}{x} = 1.$$

Then

$$\lim_{n \rightarrow \infty} \frac{1}{\tau(x)} \sum_{q \leq x} \rho\left(\frac{x}{q}\right) = 1 - \gamma.$$

We mention that a consequence of the condition $\lim_{n \rightarrow \infty} \frac{w(2x)}{w(x)} = 1$ is $\lim_{n \rightarrow \infty} \frac{w(bx)}{w(ax)} = 1$ for any positive $a, b > 0$ and the formulas $\frac{\sin(2x)}{\sin x} = 2\cos x$ resp. $\frac{\cot(2x)}{\cot x} = \frac{1}{2} \frac{\cos(2x)}{\sin^2(x)}$.

The Landau theorem above is a special case of the (PoG1):

Generalized Landau Theorem: Let $w(x)$ a positive, non decreasing function with $\lim_{n \rightarrow \infty} \frac{w(\beta x)}{w(\alpha x)} = 1$ with α, β positive numbers. Then

$$\lim_{n \rightarrow \infty} \frac{w(x)}{x} \sum_{n \leq x} f\left(\frac{x}{n}\right) = \int_0^1 f(t) dt.$$

For $h(x) := (\pi x) \cot(\pi x)$ it holds $h(x) \rightarrow_{x \rightarrow 0} -1$ and $\int_0^1 h\left(\frac{x}{2}\right) dx = \log 2$ ((GrI) 3.747). Then for $w(x) := h\left(\frac{1}{x}\right)$ and $f(x) := h\left(\frac{x}{2}\right)$ one gets

$$\frac{w(x)}{x} f\left(\frac{x}{n}\right) = \frac{1}{x} h\left(\frac{1}{x}\right) h\left(\frac{x}{2n}\right) \text{ and } \int_0^1 f(t) dt = \log 2 = -\log \xi(0).$$

Putting $h^{(1)}(x) := -(\pi x) \cot^{(1)}(\pi x)$, $h^{(2)}(x) := -(\pi x) \cot^{(2)}(\pi x)$ this enables the following arithmetic function definitions

$$\sigma(x) := \frac{1}{x} h\left(\frac{1}{x}\right) \sum_{n \leq x} h\left(\frac{x}{2n}\right) \quad \sigma^{(1)}(x) := \frac{1}{x} h^{(1)}\left(\frac{1}{x}\right) \sum_{n \leq x} h^{(1)}\left(\frac{x}{2\omega_n}\right) \quad , \quad \sigma^{(2)}(x) := \frac{1}{x} h^{(2)}\left(\frac{1}{x}\right) \sum_{n \leq x} h^{(2)}\left(\frac{x}{\omega_n + \omega_{n+1} - 1}\right).$$

With respect to the additive number theory (and to be considered prime pairs (p, q)) we propose to deal with

$$\sigma^*(x) := \frac{1}{2} \left[\sigma^{(1)}(x) + \sigma^{(2)}(x) \right].$$

The regular varying function concept was introduced by Karamata (SeE).

Lemma ((VIV), I.3): if $v(x)$ is differentiable for $x \geq x_0$ and there is a limit $\lim_{x \rightarrow \infty} \frac{xu'(x)}{u(x)} \rightarrow \alpha$, then $v(x)$ is automodel (or regular varying) of order α , i.e. $\lim_{x \rightarrow \infty} \frac{u(bx)}{u(x)} = b^\alpha$.

Therefore, a further candidate for $w(x)$ is given by $k(x) := -\cot\left(\frac{\pi x}{2}\right)$, which is a slow regular varying (automodel) of order zero ((EsR) 3.9.8). The above criterion is fulfilled, as $\frac{xk'(x)}{k(x)} = \frac{\pi x}{\sin(\pi x)}$.

Further candidates for $f(x)$ are e.g.

$$\text{i) } f_1(x) := -\log(\sin \pi x) \text{ with } \int_0^1 f_1(t) dt = \log 2 \quad (\text{GrI}) \text{ 4.384}$$

$$\text{ii) } f_2(x) := -h\left(\frac{x}{2}\right) = -h(x) := \frac{1}{x} - \left(\frac{\pi}{2}x\right) \cot\left(\frac{\pi}{2}x\right) \text{ with } \int_0^1 f_2(t) dt = \log\left(\frac{\pi}{2}\right) \quad (\text{GrI}) \text{ 3.788}$$

$$\text{iii) } f_3(x) := \log\left(a \cdot \tan\left(\pi \frac{x}{2}\right)\right) \text{ with } \int_0^1 f_3(t) dt = \log a, \quad a > 0 \quad (\text{GrI}) \text{ 4.227.}$$

Typical examples of slow varying functions are positive constants or functions converging to a positive constant, logarithms and iterated logarithms. Specifically the function $-\log x$ is slowly varying at $x = 0^+$ ((SeE) p. 47), which will be applied in the following section.

Every regular varying function f of order α has a representation $f(x) = x^\alpha L(x)$, where L is some slow varying function (MiT).

A general representation of slow regular varying functions is given by the

Theorem 1.2 (SeE): A positive measurable function L on $[x_0, \infty[$ is a slow varying function if and only if it can be written in the form

$$L(x) = e^{v(x) + \int_{x_0}^x \varepsilon(y) \frac{dy}{y}}$$

where $v(\cdot)$ is a measurable bounded function, such that $\lim_{x \rightarrow \infty} v(x) = c$ ($|c| < \infty$) and $\varepsilon(x) \rightarrow 0$ as $x \rightarrow \infty$.

Comparison Tauberian theorems are about the asymptotics behavior of the ratio of some integral transforms of two functions (distributions), if the asymptotic behavior of the ratio of some other integral transform is given ((EsR), (VIV)). In ((VIV) the Abel and Cesaro series summation with respect to an automodel weight, as well as asymptotic properties of solutions of convolution equations are considered.

Karamata's Tauberian theorem involving regular variations are provided in ((SeE) 2.2)

Theorem 2.3: Let $U(x)$ be a monotone non-decreasing function on $[0, \infty[$ such that $w(x) = \int_0^\infty e^{-xu} dU(u)$ is finite for all $x > 0$. Then, if $\rho \geq 0$, and L is a slowly varying function,

- i) if $w(x) = x^{-\rho} L\left(\frac{1}{x}\right)$ as $x \rightarrow 0^+$ then $U(x) \sim x^\rho \frac{L(x)}{\Gamma(\rho+1)}$ all $x \rightarrow \infty$
- ii) if $w(x) = x^{-\rho} L(x)$ as $x \rightarrow \infty$ then $U(x) \sim x^\rho \frac{L\left(\frac{1}{x}\right)}{\Gamma(\rho+1)}$ all $x \rightarrow 0^+$

The corresponding „density“ extension of this theorem is provided in

Theorem 2.4: Let $U(x)$, defined and positive on $[A, \infty[$ for some given A sufficiently large, be given by $U(x) = \int_0^\infty u(y) dy$, where $u(y)$ is ultimately monotone. Then for $\rho \geq 0$, if $U(x) = x^\rho L(x)$ then $\frac{xu(x)}{U(x)} \rightarrow \rho$ as $x \rightarrow \infty$.

Another such theorem, using both parts i) and ii) of theorem 2.3 above is provided in theorem 2.5, which is about the asymptotics $\int_0^\infty \frac{dA(t)}{(t+x)^\rho} \sim x^\rho L(x)$.

6. Dirichlet series and the (distributional) Hilbert space $H_{-1/2}^{\#} \cong L_2^{-1/2}$

In this section we are concerned with Hilbert scales $H_{\alpha}^{\#} \cong L_2^{\alpha}$, $\alpha \in R$, which is built on the 2π -periodic Hilbert space $L_2^{\#}(\Gamma)$ with $\Gamma := S^1(R^2)$, i.e. Γ is the boundary of the unit circle sphere. Then for $u \in L_2^{\#}(\Gamma)$ and for real $\beta \in R$, $n \in Z$ the Fourier coefficients

$$u_n := \frac{1}{2\pi} \oint u(x) e^{inx} dx$$

enable the definition of the norms ((BrK), (BrK3,6,7))

$$\|u\|_{\beta}^2 := \sum_{n=-\infty}^{\infty} |n|^{2\beta} |u_n|^2.$$

With the notation of [LaE] §227, Satz 40, the for $s > 0$ convergent Dirichlet series (in a classical L_{∞} resp. C^0 sense, where H_k is a subset of C^0 for $k \geq \frac{1}{2} + \varepsilon$, $\varepsilon > 0$)

$$f(s) := \sum_1^{\infty} a_n e^{-s \log n} \quad g(s) := \sum_1^{\infty} b_n e^{-s \log n}$$

are linked to the (distributional) Hilbert space $H_{-1/2}^{\#} \cong L_2^{-1/2}$ by ((EdH) 9.8, (NaS))

$$((f, g))_{-1/2} := \lim_{\omega \rightarrow \infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(1/2 + it) g(1/2 - it) dt = \sum_1^{\infty} \frac{1}{n} a_n b_n.$$

Let $s = \sigma + it$ and $\mu_n = \log \lambda_n$ and the series $\sum_{n=1}^{\infty} a_n e^{-\mu_n s}$ convergent; then if $\sigma > 0$ the following definite integral representation is given (HaG)

$$\sum_{n=1}^{\infty} a_n e^{-\mu_n s} = \frac{1}{\Gamma(s)} \int_0^{\infty} x^s [a_n e^{-\lambda_n s}] \frac{dx}{x}$$

leading to the

Perron theorem ((HaG) theorem 13): if the series $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ is convergent for $s = \beta + i\gamma$ and $c > 0$, $c > \beta$, $\lambda_n < \omega < \lambda_{n+1}$ then

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s) e^{\omega s} \frac{ds}{s} = \sum_{k=1}^n a_k,$$

the path of integration being the line $\sigma = c$. At a point of discontinuity $\omega = \lambda_n$ the integral has a value half-way between ist limits on either side, but in this case the integral must be regarded as being defined by ist principal value.

For $\Gamma := S^1(\mathbb{R}^2)$, the operators defined by the the single layer potential, the normal derivative of the double layer potential and the Hilbert transform ((KrR) theorems 8.20, 8.21, (LiI), $\alpha = -\frac{1}{2}$) leads to the following representations ($\alpha \in \mathbb{R}$) (LiI1):

$$\begin{aligned} S_{-1}: H_{\alpha}^{\#} &\rightarrow H_{\alpha+1}^{\#} & S_{-1}[u](x) &:= 2 \int_0^1 \log \left| \frac{1}{2 \sin(\pi(x-y))} \right| u(y) dy, \quad -\log 2 \sin(\pi x) = \sum_{n=1}^{\infty} \frac{\cos(2n\pi x)}{n} \\ S_1: H_{\alpha+1}^{\#} &\rightarrow H_{\alpha}^{\#} & S_1[u](x) &:= \frac{1}{2} \int_0^1 \frac{u(y)}{\sin^2(\pi(x-y))} dy, \quad \left[\frac{\pi}{\sin(\pi x)} \right]^2 = \sum_{n=-\infty}^{\infty} \frac{1}{(x-n)^2} \\ S_0: H_{\alpha+1/2}^{\#} &\rightarrow H_{\alpha+1/2}^{\#} & S_0[u](x) &:= \int_0^1 \cot(\pi(x-y)) u(y) dy, \quad \pi \cot(\pi x) = \frac{1}{x} + \sum_{n \neq 0}^{\infty} \left[\frac{1}{x-n} + \frac{1}{n} \right]. \end{aligned}$$

For the two self-adjoint operators S_{-1}, S_1 it holds $(S_{-1}[u], v)_{\beta} \cong (u, v)_{\beta-1/2}$, $(S_1[u], v)_{\beta} \cong (u, v)_{\beta+1/2}$, for the Hilbert transform operator it holds $(S_0[u], v)_{\beta} = -(u, S_0[v])_{\beta}$, i.e. the three convolutions integrals are Pseudo-Differential operators of order -1, 1, 0, defining corresponding isomorphisms between the corresponding domains and ranges.

As the function $\log \frac{1}{x}$ is slowly varying at $x = 0^+$ ((SeE) p. 47), the kernel function of $S_{-1}[u]$ is slowly varying at

$$x = \begin{cases} 2\mu - 1 & +\mu \in \mathbb{N} \\ 2\mu & -\mu \in \mathbb{N} \end{cases}$$

With respect to the kernel functions of $S_1[u], S_0[u]$ we note the relationships

$$\frac{d^3}{dx^3} [-\log(\sin x)] = \frac{d^2}{dx^2} [-\cot x] = \frac{d}{dx} \left[\frac{1}{\sin^2 x} \right] = (-\cot x) \frac{1}{\sin^2 x} = (-\cot x) \frac{d}{dx} [-\cot x] = \frac{1}{2} \frac{d}{dx} [\cot^2 x]$$

with its relationship to the Clausen function ((AbM) 27.8, (BrK4) Notes O27,28).

For the (transformed) Zeta function on the critical line \mathcal{E} it holds $\mathcal{E} \in H_{-1}^{\#}$, i.e. there exists a *convolution* integral representation of the Zeta function by an unique $\omega \in H_0^{\#}$ with $S_1[\omega](x) := \mathcal{E}(x)$ (CaD). It leads to the corresponding variational representation in the form

$$(S_1[\omega], v)_{-1} = (\omega, v)_{-1/2} = (\mathcal{E}, v)_{-1}, \quad \forall v \in H_{-1}^{\#}$$

The distributional $H_{-1/2}$ Hilbert space framework enables the Bagchi RH criterion, which is a reformulation of the Nyman-Beurling RH criterion (BaB). It is basically a standard density argument of the Hilbert sub-space $H_{-1/2}^{\#}$, which is densely embedded into $H_{-1}^{\#}$ with respect to the $H_{-1}^{\#}$ - norm ((BrK4) remark 3.5, notes S21 & 24, see also „Riesz theory“ ((KrR) chapter 3). The corresponding approximating sequence \mathcal{E}_n^* of the Zeta function \mathcal{E} is defined by

$$(\omega_n^*, v)_{-1/2} = (\mathcal{E}_n^*, v)_{-1}, \quad \forall v \in H_{-1/2}^{\#}.$$

From (LiI) 1.2.34, we mention

$$S_1[a_n \cos(2\pi n x) + b_n \sin(2\pi n x)](x) = -2n[a_n \cos(2\pi n x) + b_n \sin(2\pi n x)]$$

With its relationship to the concept of logarithmic capacity of sets and convergence of Fourier series of functions fulfilling $\sum_1^{\infty} n[a_n^2 + b_n^2]$ and harmonic analysis ((ZyA) V-11, (BrK4) remarks 4.1, 4.2, Notes S37/38).

From the representation (EIL)

$$\frac{\pi}{2} \log \left(\tan \left(\pi \frac{x}{2} \right) \right) = \frac{\pi}{2} \log \left(\cot \left(\pi \frac{1-x}{2} \right) \right) = -\sum \frac{2h_n}{n} \sin(\pi(2n)x) \in H_0^\#(0,1)$$

with the harmonic numbers $2h_n := \sum_{k=1}^n \frac{2}{2k-1} = 2H_{2n} - H_n$ and $H_n = \sum_{k=1}^n \frac{1}{k}$ it follows

$$\frac{\pi}{2} \log' \left(\tan \left(\pi \frac{x}{2} \right) \right) = -2\pi \sum 2h_n \cos(\pi(2n)x) \in H_{-1}^\#(0,1).$$

From the equations $\log' \left(\tan \left(\frac{x}{2} \right) \right) = \frac{1}{\sin x}$, $\log'(2\sin(x)) = \cot x$, $\frac{1-\cos x}{\sin x} = \tan \left(\frac{\pi}{2} x \right)$ one gets

$$2 \frac{\pi}{2} \log' \left(\tan \left(\frac{\pi}{2} x \right) \right) = -\pi \log' \left(\cot \left(\frac{\pi}{2} x \right) \right) = \frac{\pi}{\sin(\pi x)}, \quad -\log'(\sin(\pi x)) = -\pi \cot(\pi x)$$

and

$$\pi \tan \left(\frac{\pi}{2} x \right) = 2 \frac{\pi}{2} \log' \left(\tan \left(\frac{\pi}{2} x \right) \right) - \log'(\sin(\pi x)) = \frac{\pi}{\sin(\pi x)} - \pi \cot(\pi x) = \pi \frac{1-\cos(\pi x)}{\sin(\pi x)}.$$

With

$$-\log'(\sin(\pi x)) = \sum \sin(\pi(2n)x) \in H_{-1}^\#(0,1)$$

one gets the following Fourier series representation

$$\tan \left(\frac{\pi}{2} x \right) = -\sum \{8h_n \cos(2\pi n x) - \sin((2\pi n x))\} \in H_{-1}^\#(0,1) .$$

With respect to the Bagchi criterion and its related Hilbert space $H_{-1}^\#$ we further mention the relationship to the Brownian motion $B(t) = \int_0^t dB(\tau)$, which is obtained as the integral of the white noise signal $dB(t)$, which is distribution. Its spectral density $E_0 = |\text{Fourier}[B](\omega)|^2 = \text{constant}$ is flat. Therefore the energy spectrum of the Brownian motion is given by $E(\omega) = |\text{Fourier}[B](\omega)|^2 = \frac{E_0}{\omega^2}$.

The Wiener-Ikehara theorem was devised to obtain a simple proof of the PNT. In this theorem the boundary behavior of a Laplace transform in the complex plane plays a crucial role. The distributional version of this theorem shows that local pseudofunction boundary behavior, which allows mild singularities, is necessary and sufficient for the desired asymptotic relation. It follows that the twin-prime conjecture is equivalent to a pseudofunction boundary behavior of a certain analytic function (KoJ).

In (ViJ), (ViJ1) a proof of the PNT is provided, based on the Dirac function $\delta \in H_{-1/2-\varepsilon}$ in combination with the concept of quasi-asymptotically bounded distributions defined by

$$\left\langle \frac{f(\vartheta)}{\rho(\vartheta)}, \varphi(\cdot) \right\rangle = O(1) \text{ for } \vartheta \rightarrow \infty .$$

The density of the Mangoldt function $\psi(x) = \sum_{n < x} \Lambda(n)$ is then given by ((BrK4) Note S17)

$$\psi'(x) = \sum_{n < x} \Lambda(n) \delta(x-n) \in H_{-\frac{1}{2}-\varepsilon}$$

We propose an alternative quasi-asymptotically bounded $H_{-1/2}$ distributions concept defined by

$$\lim_{\vartheta \rightarrow \infty} \left(\frac{f(\vartheta)}{\rho(\vartheta)}, \varphi(\cdot) \right)_{-1/2} = O(1), \quad \forall \varphi \in H_{-1/2} \rightarrow \infty$$

leading to a replacement in the form ((BrK4) Note S19)

$$\psi' \in H_{-\frac{1}{2}-\varepsilon} \rightarrow \tilde{S}_1[\psi] \in H_{-\frac{1}{2}}$$

defined by

$$(\tilde{S}_1[\psi], v)_{-1/2} \cong (\psi, v)_0 \quad \forall v \in H_{-1/2}.$$

The approximation by polynomials in a complex domain leads to several notions and theorems of convergence related to Newton-Gaussian and cardinal series. The latter one are closely connected with certain aspects of the theory of Fourier series and integrals. Under sufficiently strong conditions the cardinal function can be resolved by Fourier's integral. Those conditions can be considerably relaxed by introducing Stieltjes integrals resulting in (C,1) summable series ((WhJ1) theorems 16 & 17, (BrK4) Remarks 3.6/3.7). With respect to interpolations theory with points of sequences c_n (with respect to the Newton-Gauss and cardinal series), the corresponding cardinal series theory with certain aspects of the theory of Fourier series and integrals, especially Fourier-Stieltjes series and related convergent series in the form

$$\sum_{n=1}^{\infty} \frac{1}{n} (|a_n| + |a_{-n}|) < \infty$$

we refer to ((WhJ1), (WhJ2), (BrK4) Remarks 3.6 & 3.7, pp 105ff).

We recall the very „slow“ divergence of the monoton increasing (harmonic numbers) sequence $H_n = \sum_{k=1}^n \frac{1}{k}$, e.g. $[H_n] = 1$ for $n = 1, 2$, $[H_n] = 2$ for $n = 6$, $[H_n] = 5$ for $n \sim 10^2$, $[H_n] = 14$ for $n \sim 10^4$, $[H_n] = 100$ for $n \sim 43$ (HaJ).

Putting $a_n := h_n := \sum_{k=1}^n \frac{1}{2k-1} = H_{2n} - \frac{1}{2}H_n$, $a_{-n} := -h_n$ the theorem 2 in (WhJ2), then leads to the

Lemma: the cardinal series

$$C(x) := \frac{\sin(\pi x)}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{h_n}{x-n}$$

is absolutely convergent, and its sum is of the form

$$\int_0^1 \{ \cos(\pi x t) d\varphi(t) + \sin(\pi x t) d\omega(t) \}, \quad \varphi, \omega \text{ continuous functions.}$$

Given any function $f(x)$ of this form the series

$$\frac{\sin(\pi x)}{\pi} \cdot \left[\frac{f(0)}{x} + \sum_{n=1}^{\infty} (-1)^n \left\{ \frac{f(n)}{x-n} + \frac{f(-n)}{x+n} \right\} \right]$$

is (C,1) – summable to $f(x)$.

We propose to apply $C(x) := \frac{\sin(\pi x)}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{h_n}{x-n}$ alternatively to $\frac{\sin(\pi x)}{\pi x}$. In the context of the Zeta function related Ramanuja formulawe refer to (EdH) 10.10.

In the context of representations of coefficient sums by integrals we recall from (LaE2), §86 the

Lemma: Let $D(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$ denote an absolute convergent Dirichlet series ($a := \operatorname{Re}(s)$) and

$$f(x) := \begin{cases} \sum_{n=1}^x b_n & \text{for } x \text{ not integer} \\ \sum_{n=1}^x b_n - b_x/2 & \text{for } x \text{ integer} \end{cases}$$

Then

$$f(x) = \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{a-it}^{a+it} D(s) x^s \frac{ds}{s} \quad \text{resp.} \quad \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{D(s)}{s} x^s \frac{ds}{s} = \sum_{n=1}^x b_n \log \binom{x}{n} .$$

With respect to the physical aspects we refer to (NaS), where the $H_{-1/2}$ dual space of $H_{-1/2}$ on the circle (with its inner product defined by a Stieltjes integral) is considered in the context of Teichmüller theory and the universal period mapping via quantum calculus. For the corresponding Fourier series analysis we refer to ((ZyA) XIII, 11).

The common denominator of the alternatively proposed Hilbert space framework $H_{-1/2}$ goes along with the definition of a correspondingly defined "momentum" operator (of order 1) $P: H_{1/2} \rightarrow H_{-1/2}$ defined in a variational form.

In the one-dimensional case the Hilbert transform H (in the $n > 1$ case the Riesz operators R) is linked to such an operator given by $(Pu, v)_{-1/2} = (Hu, v)$. With respect to quantum theory this indicates an alternative Schrödinger momentum operator (where the gradient operator "grad" is basically replaced by the Hilbert transformed gradient, i.e. $\mathbf{P} := -i^* \mathbf{R}(\text{grad})$) and a corresponding alternative commutator representation $Q\mathbf{P} - \mathbf{P}Q$ in a weak $H_{-1/2}$ form.

We note that the Riesz operators \mathbf{R} commute with translations and homothesis and enjoy nice properties relative to rotations.

The theory of spectral expansions of non-bounded self-adjoint operator is connected with the notions "Lebesgue-Stieltjes integral" and "functional Hilbert equation for resolvents ((LuL) (7.8)). The corresponding Hilbert scale framework plays also a key role on the inverse problem for the double layer potential. The corresponding model problem (w/o any compact disturbance operator) with the Newton kernel enjoys a double layer potential integral operator with the eigenvalue $1/2$ (EbP).

The incomplete Gamma function play a key role to compute the action of the Leray projection operator on the Gaussian functions (LeN1). Those action formulas can be applied to derive in the context of the well-posedness topic of the NSE and related (based on tempered distribution and a Carleson measure characterization of the BMO space) estimates ((LeN1), (KoH), theorems 1 and 2, see also (BrK4) pp. 26, 58, 64, 99, 121).

For the related equations with respect to the incomplete Gamma function we refer to (OIF1) 7.2.2, 8.4.15, (AbM) 6.5.12, 13.6.10).

The RH is connected to the quantum theory via the Hilbert-Polya conjecture resp. the Berry-Keating conjecture. It is about the hypothesis, that the imaginary parts t of the zeros $1/2 + it$ of the Zeta function $Z(t)$ corresponds to eigenvalues of an unbounded self-adjoint operator, which is an appropriate Hermitian operator basically defined by $QP + PQ$, whereby Q denotes the location, and P denotes the (Schrödinger) momentum operator. In (BrK3) the corresponding model (convolution integral) operator S_1 (of order 1 with "density" $dcot$ for the one-dimensional harmonic quantum oscillator model is provided.

In the context of the Berry-Keating conjecture the Gaussian function $f(x)$ can be characterized as "minimal function" for the Heissenberg uncertainty inequality. Applying the same solution concept as above then leads to an alternative Hilbert operator based representation in $H_{-1/2}$, resp. to a H_{-1} based definition of the commutator operator with extended domain.

7. A $H_{-1/2}^{\#} \cong L_2^{-1/2}$ based alternative circle method on the unit circle and a Kummer function zero based approach to prove the binary Goldbach conjecture

The current tool trying to prove the tertiary and binary Goldbach conjecture is about the (Hardy/Littlewood) circle method. Vinogradov's „basic intervals“ correspond in principle to Hardy and Littlewood's „major arcs“ ((ViI) p.61). Hardy and Littlewood dissected the circle $x = e^{2\pi i\alpha}$, or rather a smaller concentric circle, into „Farey arcs“. The major arcs, or basic intervals, provide the main term in the asymptotic formula for the number of representations. Their treatment does not give rise to any very serious difficulties compared to the problems presented by the „minor arcs“, or „supplementary intervals“. The latter ones are analyzed by the Weyl (trigonometrical) sums

$$S(x) := \sum_n e^{2\pi i n x}.$$

On the one hand side, the probability that the binary Goldbach conjecture is true, is 100%, while on the other side the current (circle method) tool failed, because of not sufficient (already optimal) Weyl estimates. As those Weyl estimates are w/o any information (i.e. relevance) regarding the underlying (Goldbach) problem, the probability, that the circle method is not adequate is also 100%.

The natural Hilbert space framework (related to generalized Fourier analysis techniques and Dirichlet series, but also related to convergent Weyl sums series) is $\in H_{-1/2}^{\#}(0,1)$ (e.g. BrK9). The Cesàro summable $\cot(\pi x) = 2 \sum_{n=1}^{\infty} \sin(2\pi n x) \in H_{-1}^{\#}(0,1)$ (ZyA) VI-3, VII-1), is related to the eigenfunctions $e^{2\pi i n x} = e^{i\pi(2n)x}$, while the proposed alternative Abel summable functions

$$\begin{aligned} \rightarrow \quad \cot^{(1)}(\pi x) &:= \sum \sin(\pi(2\omega_n)x) \in H_0^{\#}(0,1) \\ \rightarrow \quad \cot^{(2)}(\pi x) &:= \sum \sin(\pi(\omega_n + \omega_{n+1} - 1)x) \in H_0^{\#}(0,1) \end{aligned}$$

are related to the eigenfunctions pair $e^{i\pi(2\omega_n)x}$ and $e^{i\pi(\omega_n + \omega_{n+1})x}$ resp. to the alternative Weyl sums

$$S_1^*(x) := \sum_n e^{i\pi(2\omega_n)x}, \quad S_1^*(x) := \sum_n e^{i\pi(\omega_n + \omega_{n+1} - 1)x}.$$

With the notation of (LaE1) the prime pair (p, q) counting function $H(x)$ with the condition $p + q \leq x$ is given by

$$H(x) = \sum_{n=1}^x G_n = \sum_{p \leq x} \pi(x-p) \sim \frac{x}{\log x} \int_0^{x/2} \frac{dt}{\log t} \sim \frac{1}{2} \left(\frac{x}{\log x} \right)^2$$

The (improved) Stäckel formula (based on the Euler $\varphi(n)$ –function) shows the asymptotics in the form

$$\tilde{G}_{2n} \sim \frac{1}{2} \frac{105 \cdot \zeta(3)}{\pi^4} \frac{n}{\log n} \cdot \frac{n}{\log n} \cdot \frac{1}{\varphi(n)} \sim 0,648 \dots \cdot \frac{1}{\varphi(n)} \cdot \frac{n}{\log n} \cdot \frac{n}{\log n}.$$

Therefore, Landau predicted a proof of the binary Goldbach conjecture „with high probability“ (LaE1).

Related to the Stäckel formula we recall from (ApM) p. 71, the following estimates

$$\frac{\sigma(n)}{n^2} \leq \frac{1}{\varphi(n)} \leq \frac{\pi^2}{6} \frac{\sigma(n)}{n^2} = \zeta(2) \frac{\sigma(n)}{n^2}, \quad n \geq 2$$

whereby $\sigma(n) = \sigma_1(n)$ denotes the sum of the divisors of n ((ApM) p. 38).

With respect to the factor $\zeta(2)$ we recall the related $\cot\left(\frac{\pi}{2}x\right)$ estimate from the previous section

$$\left|\frac{\pi}{2}\cot\left(\frac{\pi}{2}x\right) - \frac{1}{x}\right| \leq \frac{\pi^2}{6} = \zeta(2), \quad |x| \leq 1.$$

With respect to the Zeta function itself $\zeta(s)$ we recall the related $\cot(\pi z)$ representation from the previous section (TiE)

$$\sum_{n>x} \frac{1}{n^s} = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} z^{1-s} (-\pi \cot(\pi z)) \frac{dz}{z}, \quad s = \frac{1}{2} + ix$$

and its link to the Zeta function is given by

$$(1-s) \cdot \zeta^*(s) = \cot\left(\frac{\pi}{2}s\right) \cdot \zeta(s) = \tan\left(\frac{\pi}{2}(1-s)\right) \cdot \zeta(s)$$

whereby it holds

$$\pi \cot(\pi x) = \frac{\pi}{2} \left[\cot\left(\frac{\pi}{2}x\right) - \frac{1}{2} \cot\left(\frac{\pi}{2}(1-x)\right) \right] = \frac{\pi}{2} \left[\cot\left(\frac{\pi}{2}x\right) - \frac{1}{2} \tan\left(\frac{\pi}{2}x\right) \right].$$

and

$$\log'(\tan\left(\frac{\pi}{2}s\right)) = -\log'(\cot\left(\frac{\pi}{2}s\right)) = \frac{\pi}{\sin(\pi s)}.$$

In the following we shall deal with the special Kummer functions

$$K_a(z) := {}_1K_1(a, a+1; z) \text{ with } 0 < a \leq 1.$$

The asymptotics of ${}_1K_1(a, a+1; z)$ is given by ((OIF) 10.3, (AbM) 13.5.1.)

$${}_1K_1(a, a+1; z) \sim \frac{1}{\Gamma(a)} \frac{e^z}{z}, \quad z \rightarrow \infty, \text{ph}(z) = 0, \quad {}_1K_1(a, a+1; z) \sim \frac{1}{(-z)^a}, \quad z \rightarrow -\infty, \text{ph}(-z) = 0$$

where $\text{ph}(z)$ denotes the phase or the argument of z . For the real case ($x \in \mathbb{R}$) we deal with the function $F_a(x) := c \cdot {}_1F_1(a, a+1; x)$ with a given constant c fulfilling the following properties

- i) ${}_1F_1(a, c; x) \sim \frac{1}{\Gamma(a)} \frac{e^x}{x^{c-a}}$ resp. ${}_1F_1(a, a+1; x) \sim \frac{1}{\Gamma(a)} \frac{e^x}{x}, \quad x \rightarrow \infty$
- ii) $\frac{d}{dx} F_a(x) = \frac{a}{a+1} F_{a+1}(x)$
- iii) $F_a(x) \sim \frac{c}{\Gamma(a)} \frac{e^x}{x}, \quad x \rightarrow \infty$ ((OIF), 7 §10.1, (AbM) 13.5.1.)
- iv) $\frac{1}{2} \frac{d}{dx} F_a^2(x) = \frac{a}{a+1} F_a(x) \cdot F_{a+1}(x) \sim \frac{1}{a+1} \frac{1}{\Gamma^2(a)} \left(\frac{e^x}{x}\right)^2.$

From (AbM) 13.2.9, we mention the Barnes-type contour integral representation

$${}_1F_1(a, a+1; z) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{a}{a+s} \Gamma(1-s) (-z)^s \frac{ds}{s}, \quad |\arg(-z)| < \frac{\pi}{2}, \quad a \neq 0, -1, -2, \dots$$

where the contour must separate the poles of $\Gamma(-s)$ from those of $\Gamma(a+s)$.

The special Kummer functions $K_a(z) := {}_1K_1(a, a+1; z)$ go along with the Hurwitz generalization of the Zeta function ((TiE) 2.17)

$$\zeta(s, a) = \sum_{n=1}^{\infty} \frac{1}{(n+a)^s}, \quad \text{Re}(s) > 1, \quad 0 < a \leq 1$$

resp. the corresponding generalized Dirchlet series.

In the following we first shall deal with the choice

$$\frac{1}{2} \leq a := \tilde{a}_n := \frac{2n-1}{2n} \rightarrow 1$$

and

$$a := \frac{2n-1}{2n}, \quad \tilde{a}_n := \frac{1}{a+1} = \frac{2n}{4n-1}, \quad \tilde{b}_n := \frac{1}{r^2(a)} = \frac{1}{r^2\left(1-\frac{1}{2n}\right)}, \quad c := \sqrt{2/3}.$$

The corresponding definition of $H^*(x)$ is motivated by the replacement

$$H(x) \sim \frac{x}{\log x} \int_{\frac{x}{2}}^x \frac{dt}{\log t} \frac{d}{dx} \rightarrow H^*(x) \sim \frac{d}{dx} F_{\tilde{a}_n}(\log x) \cdot \int_{\log 2}^{\log(\frac{x}{2})} \frac{d}{dt} F_{\tilde{a}_n}(t) dt.$$

fulfilling the

Lemma („Chebychev“ inequality):

$$\frac{1}{3\pi} \left(\frac{x}{\log x}\right)^2 \leq H_n^*(x) \leq \frac{1}{2} \left(\frac{x}{\log x}\right)^2.$$

Proof: It holds $\tilde{a}_1 = \frac{3}{2}$, $\tilde{a}_\infty = \frac{1}{2}$, $\tilde{b}_1 = \frac{1}{\pi}$, $\tilde{b}_\infty = 1$ and therefore

$$\frac{1}{2\pi} \leq \tilde{a}_n \cdot \tilde{b}_n \leq \frac{3}{2} \text{ resp. } \frac{1}{3\pi} \leq \frac{2}{3} \tilde{a}_n \cdot \tilde{b}_n \leq 1.$$

Choosing $c := \sqrt{2/3}$ then proves the lemma.

We note that for $n \geq 4$ it holds $1 - \frac{1}{8} < 1 - \frac{1}{2n}$, $1 + \frac{1}{2n} < 1 - \frac{1}{8}$, with ist relationship to the Chebychev inequality

$$1 - \frac{1}{8} < \frac{\pi(x)}{\log x} < 1 + \frac{1}{8}.$$

The Kummer function zeros related inequality $2n - 1 < 2\omega_n < 2n < \omega_n + \omega_{n+1} < 2n + 1$ indicates a replacement

$$\frac{1}{2} \leq \tilde{a}_n \leq 1 \quad \rightarrow \quad \frac{1}{2} \leq \frac{2n-1}{2n} < d_n := \frac{\omega_n + \omega_n}{\omega_n + \omega_{n+1}} < \frac{2n}{2n+1} \leq 1$$

leading to

$$H_n^{**}(x) := \frac{1}{2} \sum_{n < x} \frac{d}{dx} F_{d_n}^2(\log x)$$

With respect to a Hilbert space framework $H_{-1/2}$ (see also below) we note the identities $((dK^2, v))_{\alpha-1/2} = (K^2, v)_\alpha$ for $K^2, v \in H_\alpha$.

Concerning the below with respect to the hypergeometric (Kummer) functions and their relationship to the Gamma function and the underlying Euler constant we recall from (BrR) the formula

$$\int_0^\infty x^s [e^{-x} - J_0(2\sqrt{x})] \frac{dx}{x} = \Gamma(s) - \frac{\Gamma(\frac{s}{2})}{\Gamma(1-\frac{s}{2})} = \frac{1}{s} \left[(1-\gamma s) - \frac{1-\gamma s}{1+\gamma s} + O(s^2) \right] \rightarrow 0 \quad s \rightarrow 0^+.$$

Analogue it holds

$$\int_0^\infty x^s [e^{-x} - J_0(2\sqrt{x})] \frac{dx}{x} = \Gamma(s) - \frac{\Gamma(s)}{\Gamma(1-s)} = \frac{1}{s} \left[(1-\gamma s) - \frac{1-\gamma s}{1+\gamma s} + O(s^2) \right] = \gamma \frac{1-\gamma s}{1+\gamma s} + O(s) \rightarrow \gamma \quad s \rightarrow 0^+.$$

From section 1d. we recall the following properties of the sequences ω_n (the imaginary parts of the Kummer function zeros):

- v) $n - \frac{1}{2} < \omega_n < n$, $n + \frac{1}{2} < \omega_{n+1} < n + 1$, $\frac{1}{2} < \omega_1 < a := s_n := \frac{\omega_n}{n} \rightarrow 1$ $n \in \mathbb{N}$
- vi) $2n - 1 < 2\omega_n < 2n < \omega_n + \omega_{n+1} < 2n + 1 < 2\omega_{n+1} < 2(n + 1)$
- vii) the sequences $2\omega_n$ and $\omega_n + \omega_{n+1}$ fulfill the Hadamard gap condition
- viii) $\frac{\omega_{n+1}}{\omega_n} > \frac{n+\frac{1}{2}}{n} = 1 + \frac{1}{2n} > q > 1$ resp. $\frac{\omega_{n+1} + \omega_{n+2}}{\omega_n + \omega_{n+1}} > \frac{2n+2}{2n+1} = 1 + \frac{1}{2n+1} > q > 1$
 $\theta := \frac{1}{4} < \frac{\omega_{n+1}}{2} - \frac{\omega_n}{2} < 1 - \frac{1}{4} = 1 - \theta$.

For the related sequences $a_n := \frac{2\omega_n}{2n} - \frac{1}{2}$, $b_n := \frac{\omega_{n+1} + \omega_n}{2n} - \frac{1}{2}$ it therefore follows

- v) $0 < a_1 = \omega_1 - \frac{1}{2} \leq a_n \rightarrow \frac{1}{2}$, $\frac{1}{2} \leftarrow b_n < b_1 = \frac{\omega_1 + \omega_2 - 1}{2} < 1$
- vi) $a_n, b_n - a_n \in (0, \frac{1}{2})$, $b_n, 1 - a_n \in (\frac{1}{2}, 1)$.

We note the Mellin transform of the related Kummer function ((GrI) 7.612)

$$\int_0^\infty x^s {}_1F_1(a_n, a_n + 1; -x) \frac{dx}{x} = \frac{a_n}{a_n - s} \Gamma(s) \quad , \quad 0 < \operatorname{Re}(s) < a_n$$

$$\int_0^\infty x^s {}_1F_1(a_n, a_n + 1; -x) \frac{dx}{x} \rightarrow \int_0^\infty x^{s/2} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x\right) \frac{dx}{x} = \frac{\Gamma\left(\frac{s}{2}\right)}{1-s} = 2 \frac{\Gamma\left(\frac{1+s}{2}\right)}{s(1-s)} \quad , \quad 0 < \operatorname{Re}(s) < 1.$$

For

$$g_n^{(1)}(x) := {}_1F_1(a_n, b_n; x) , \quad g_n^{(2)}(x) := {}_1F_1(1 - a_n, b_n; x) , \quad \varphi_n^{(k)}(x) := x g_n^{(k)}(x) \quad (k = 1, 2)$$

it holds

$$g_n^{(k)}(x) \rightarrow {}_1F_1\left(\frac{1}{2}, \frac{1}{2}; x\right) = e^x \quad \text{resp.} \quad \varphi_n^{(k)}(-\log x) \rightarrow \frac{\log x}{x} \quad \text{as } n \rightarrow \infty \quad (k = 1, 2).$$

From ((LeN) (9.13.7)) we recall the Kummer function based representation of the $li(x)$ -function in the form $li(x) = -x {}_1F_1(1, 1; -\log x) = Ei(\log x) = \int_0^x \frac{dt}{\log t} = \int_{-\infty}^{\log x} \frac{e^t}{t} dt$ with the asymptotics $li(x) \approx \log^{-1}(x)$, i.e. it holds $-x {}_1F_1\left(a_n + \frac{1}{2}, b_n + \frac{1}{2}; -\log x\right) \rightarrow -x {}_1F_1(1, 1; -\log x) = li(x)$.

The properties of the sequences above are proposed to build two different prime density functions for a prime pair in combination with the following two Kummer function related properties:

- i) $c_1 \cdot [{}_1F_1(a, c; x)]^2 \leq {}_1F_1(a - \mu, c; x) \cdot {}_1F_1(a + \mu, c; x) \leq [{}_1F_1(a, c; x)]^2$, $a > 0$, $c > a \geq \mu - 1$, $x \in \mathbb{R}$
- ii) $\int_0^\infty x^s {}_1F_1(\alpha, \beta; -x) \frac{dx}{x} = \frac{\Gamma(\beta)}{\Gamma(\alpha)} \Gamma(s) \frac{\Gamma(\alpha - s)}{\Gamma(\beta - s)}$, $0 < \operatorname{Re}(s) < \operatorname{Re}(\alpha)$, (GrI) 7.612 .

The second inequality of property i) for positiv integer μ is provided in (BaR) and for non-integer positive μ , complemented with a reverse inequality, in (KaD) (see also appendix).

Putting e.g. $a := \frac{1}{2}$, $\mu := \frac{1}{2} - a_n$ it holds $a - \mu = a_n$; $a + \mu = 1 - a_n$ resulting into the inequalities

$$c_1 \cdot \left[{}_1F_1\left(\frac{1}{2}, c; x\right) \right]^2 \leq {}_1F_1(a_n, c; x) \cdot {}_1F_1(1 - a_n, c; x) \leq \left[{}_1F_1\left(\frac{1}{2}, c; x\right) \right]^2 .$$

From the Kummer function Mellin transform property ii) above we get

- i) $\frac{1}{2} \int_0^\infty x^{s/2} {}_1F_1(a_n, b_n; -x) \frac{dx}{x} = \frac{\Gamma(b_n)}{\Gamma(a_n)} \Gamma(s) \frac{\Gamma(a_n - s)}{\Gamma(b_n - s)} \rightarrow \frac{1}{2} \int_0^\infty x^{s/2} e^{-x} \frac{dx}{x} = \frac{1}{2} \Gamma\left(\frac{s}{2}\right)$ $0 < \operatorname{Re}(s) < 1$
- ii) $\frac{1}{2} \int_0^\infty x^{s/2} {}_1F_1(a_n, b_n + 1; -x) \frac{dx}{x} \rightarrow \frac{1}{2} \int_0^\infty x^{s/2} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x\right) \frac{dx}{x} = \frac{\Gamma\left(\frac{1+s}{2}\right)}{s(1-s)}$ $0 < \operatorname{Re}(s) < 1$.

The circle method (based on the interior of the unit disk) is concerned with complex-valued functions $f(z) = f(r \cdot e(\varphi))$ ($e(\varphi) := e^{2\pi i \varphi}$, $\varphi \in (0,1)$) resp. $e_n(\varphi) := e(n\varphi)$ fulfilling

$$\int_0^1 e_n(\varphi) d\varphi = \begin{cases} 1 & n=0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad |1 - e(\varphi)|^2 = 4\sin^2(\pi\varphi)$$

It is about Fourier analysis of complex-valued power series functions

$$f(x) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < 1$$

i.e. defined for the interior unit circle domain. The underlying mathematical tool set is based on the formula ((ViI1) chapter I, lemma 4, Notes)

$$r^n a_n = \int_0^1 f(re^{2\pi i t}) e^{-2\pi i n t} dt, \quad 0 < r < 1.$$

The corresponding (convergence) requirements are ensured by the

Theorem ((OIF) 3.3): let $\sum_{n=0}^{\infty} a_n z^n$ converge when $|z| < r$. Then for fixed k

$$\sum_{n=k}^{\infty} a_n z^n = O(z^k) \quad \text{in any disk } |z| \leq \rho \text{ such that } \rho < r.$$

In line with the concepts of the previous section we propose a correspondingly modified circle method, which is about a Fourier analysis of complex-valued (generalized) Fourier series representations on the unit circle, based on a corresponding mathematical tool set enabled by the formula

$$-n f_n(x) = \frac{1}{2} \int_0^1 g(x-t) f_n(t) dt$$

with $f_n(t) = a_n \cos(2\pi n t) + b_n \sin(2\pi n t)$ and $g(x) := \sin^{-2}(\pi x)$. In the corresponding Hilbert scale framework $H_{\alpha}^{\#}(0,1)$ the mapping formula

$$v = S[u] \quad \text{with} \quad v(x) := \frac{1}{2} \int_0^1 g(x-t) u(t) dt$$

defines an (Pseudo-Differential) integral convolution operator $S: H_{\alpha}^{\#}(0,1) \rightarrow H_{\alpha-1}^{\#}(0,1)$ of order 1, where the Fourier transform of the kernel function is given by $\hat{g}(\omega) = -\omega$. It leads to a variational representation in the form

$$(S[v], w)_{\alpha-1/2} = (v, w)_{\alpha}, \quad \forall v, w \in H_{\alpha}^{\#}(0,1).$$

The corresponding convolution kernel of the inverse operator $S_{-1}: H_{\alpha-1}^{\#}(0,1) \rightarrow H_{\alpha}^{\#}(0,1)$ is given by $k(x) := -\log|2\sin(\pi x)|$ ((BrK3), (BrK4) remarks 3.6, 3.9, Notes S38, S48, O23-30).

Regarding orthogonal polynomials on the unit circle, built on a non-negative, integrable (in a Lebesgue-sense) function $f(\theta)$ of period 2π , we mention the case $f(\theta) = \{g(\theta)\}^{-1}$ where $g(\theta)$ is a positive trigonometric polynomial of degree m ((SzG) 11.2, (BrK4) Note S49).

For each positive real number x the Snirelmann density is defined for a subset \mathbf{A} of the set of positive integers \mathbf{N} by

$$0 \leq \sigma(\mathbf{A}) := \inf_n \frac{A(n)}{n} \leq 1 \quad \text{with} \quad A(x) := \sum_{\substack{a \in \mathbf{A} \\ a \leq x}} 1.$$

It holds $A(n) \geq kn$ for $\sigma(\mathbf{A}) = k$; $A(1) = 0$ (and therefore $\sigma(\mathbf{A}) = 0$), if 1 is not an element of \mathbf{A} ; and $A(n) = n$ (and therefore $\sigma(\mathbf{A}) = 1$), if $\mathbf{A} = \mathbf{N}$.

The Snirelmann-Goldbach theorem states, that the set $\mathbf{A} := \{0,1\} \cup \{p+q; p, q \text{ prim}\}$ has positive Snirelmann density. In case of a Snirelmann density $\frac{1}{2}$ the binary Goldbach conjecture would be proven.

For a subset \mathbf{A} of the set of integers \mathbf{N} integers, if

- i) the integer "1" is not an element of \mathbf{A} , the Snirelmann density of \mathbf{A} is = 0
- ii) if the integer "2" is not an element of \mathbf{A} , the Snirelmann density of \mathbf{A} is $\leq \frac{1}{2}$
- iii) if $\mathbf{A} = \mathbf{N}$, the Snirelmann density of \mathbf{A} is = 1.

For the smallest prime $p > n$, it holds $p < 2n$, i.e. $\pi(2n) - \pi(n) \geq 1$.

The set $\{2n - 1 = [2\omega_n]\}$ resp. $\{2n - 1 = [\omega_n + \omega_{n+1} - 1]\}$ of odd integers has Snirelmann density $\leq \frac{1}{2}$, while the set $\{2n = [\omega_n + \omega_{n+1}]\}$ of even integers has Snirelmann density = 0.

Therefore, the sequences $\{2\omega_n\}$ and $\{\omega_n + \omega_{n+1} - 1\}$ are suggested to build a problem adequate binary number theoretical function σ^* in sync with a corresponding set with Snirelmann density $\sigma^*(\mathbf{A}^*) = \frac{1}{2}$.

The „Snirelmann-Stieltjes integral“ density concept is related to another method to analyze binary additive problems, which is about the „dispersion method“, which is about a „correlation theory of binary problems“ (LiJ). Unfortunately, in its current form this method cannot be applied to the binary Goldbach problem ((LiJ), chapter X, §2).

We claim, that the proposed circle method on the unit circle with its underlying related distributional Hilbert space framework $H_{-\frac{1}{2}}^{\#}(0,1)$ (going along with the Stieltjes/Plemelj „differential“ potential density concept) provides the appropriate framework for a correspondingly adapted „truly“ „dispersion / variance method“, to solve the binary Goldbach conjecture. This is about the independence of the (number theoretical) „events“ related to the (positive integer resp. real number) sets

$$\mathbf{A} := \{0,1\} \cup \{p + q; p, q \text{ prim}\} \quad , \quad \mathbf{A}^* := \{0,1\} \cup \{2\omega_n\} \cup \{(\omega_n + \omega_{n+1} - 1)\} .$$

8. The $H_{-1/2}$ Hilbert space and corresponding arithmetic functions

In (ViJ) the Prime Number Theorem (PNT) is proven on a distributional way, applying the Dirac function to derive the first derivative of the Chebyshev function. In line with the proposal of section 4.d) below we propose to replace the Dirac distributions space by the $H_{-1/2}$ Hilbert space, resulting in corresponding representation of concerned arithmetic functions.

This section is about the building of a distributional density function $\theta(x)$ with

$$\theta'(x) = S[\theta](x) = \sum_{n < x} a(n) \in H_{-\frac{1}{2}} \text{ in a weak } H_0 \text{ -sense,}$$

alternatively to the Mangoldt resp. Landau distribution functions

$$\psi(x) = \sum_{n < x} \Lambda(n) \quad \text{resp.} \quad \vartheta(x) = \sum_{n=1}^{\infty} \Lambda(n) \log\left(\frac{x}{n}\right)$$

with

$$\psi'(x) = \sum_{n < x} \Lambda(n) \delta(x - n) \in H_{-\frac{1}{2}-\varepsilon} \quad \text{resp.} \quad \vartheta'(x) = \frac{1}{x} \sum_{n < x} \Lambda(n).$$

The Delta function representation of $\psi'(x)$ is applied in (ViJ) for „a quick distributional way to (prove) the prime number theorem“ (see also (BrK4) Note S19).

For

$$\theta(x) := \sum_{n < x} \Lambda(n) \cdot \log\left(\sin\left(\pi \frac{x}{2\omega_n}\right)\right) \in H_{1/2}$$

one gets with the abbreviation $p^*(x) := -\left(\frac{\pi}{2}x\right) \cot\left(\frac{\pi}{2}x\right)$

$$\theta'(x) = \frac{1}{x} \sum_{n < x} \Lambda(n) \left(-\pi \frac{x}{2\omega_n} \cot\left(\pi \frac{x}{2\omega_n}\right)\right) = \frac{1}{x} \sum_{n < x} \Lambda(n) \cdot \left(p^*\left(\frac{x}{\omega_n}\right)\right) \in H_{-1/2} \text{ in a strong sense.}$$

For the convergence $\lim_{n \rightarrow \infty} \theta'(x)$ we refer to the previous sections.

The baseline formulas for the following are the representations

- i) $p(x) := \left(\frac{\pi}{2}x\right) \cot\left(\frac{\pi}{2}x\right) = 1 + \sum_{n=1}^{\infty} \left[\frac{x}{x+n} + \frac{x}{x-n}\right]$
- ii) $w(x) := \frac{p(x)}{x} = \left(\frac{\pi}{2}\right) \cot\left(\frac{\pi}{2}x\right) = \frac{1}{x} + \sum_{n=1}^{\infty} \left[\frac{1}{x+n} + \frac{1}{x-n}\right]$
- iii) $\psi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[-\frac{\zeta'}{\zeta}(s)\right] x^s \frac{ds}{s} = \sum_{n < x} \Lambda(n)$ and $-\frac{\zeta'}{\zeta}(s) = \int_0^{\infty} x^{-s} d\psi = \sum_{n=2}^{\infty} \Lambda(n) n^{-s}$, ($Re(s) > 1$), (EdH) 3.2
- iv) $\vartheta(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[-\frac{\zeta'}{\zeta}(s)\right] x^s \frac{ds}{s} = \int_0^x \psi(t) d\log(t) = \sum_{n=1}^{\infty} \Lambda(n) \log\left(\frac{x}{n}\right)$ and $-\frac{\zeta'}{\zeta}(s) = \int_0^{\infty} x^{-s} d\vartheta$, ($Re(s) > 0$)

The Euler conjecture, i.e. the convergence of the series

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \dots = 0$$

can be derived from the PNT ((LaE) §159). This series is linked to the zeta function by the identities

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{n^s}$$

i.e.

$$-\frac{\zeta'(s)}{s\zeta(s)} = \left[\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right] \cdot \left[\sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{n^s}\right], \quad \frac{\zeta'(s)}{\zeta(s)} = \left[\sum_{n=1}^{\infty} \mu(n) \frac{1}{n^s}\right] \cdot \left[\sum_{n=1}^{\infty} \log\left(\frac{1}{n}\right) \frac{1}{n^s}\right].$$

What cannot be derived from the PNT is the convergence of the series

$$(*) \quad \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log\left(\frac{1}{n}\right) = 1.$$

„The corresponding theorem goes deeper than the PNT, and from it the PNT can be easily derived“ ((LaE) §160).

The Landau statement above corresponds to the proposed replacement of the Dirac distribution by appropriate distributions of the Hilbert space $H_{-\frac{1}{2}}$, going along with the following identities ((ApT), (BrK4), (LaE) §227), while at the same time enabling the Bagchi RH criterion (BaB),

$$1 = \sum_{n=1}^{\infty} \frac{1}{n} \mu(n) \log\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n} a_n b_n =: ((u, v))_{-\frac{1}{2}} := \lim_{\omega \rightarrow \infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} u\left(\frac{1}{2} + it\right) v\left(\frac{1}{2} - it\right) dt$$

with $(s = \frac{1}{2} + it)$

$$u\left(\frac{1}{2} + it\right) := \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \in H_{-\frac{1}{2}}, \quad v\left(\frac{1}{2} - it\right) := \sum_{n=1}^{\infty} \frac{\log(1/n)}{n^s} \in H_{-\frac{1}{2}}.$$

Let

$$z\left(\frac{1}{2} + it\right) := \sum_{n=1}^{\infty} \frac{1}{n^s} \in H_{-1}, \quad w\left(\frac{1}{2} - it\right) := \sum_{n=1}^{\infty} \log\left(\frac{1}{n}\right) \frac{\mu(n)}{n^s}$$

$$r\left(\frac{1}{2} - it\right) := \sum_{n=1}^{\infty} \frac{\varphi(n)}{n} \frac{1}{n^s}, \quad s\left(\frac{1}{2} - it\right) := \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n \log n} \frac{1}{n^s}.$$

Then it holds $((z, w))_{-\frac{1}{2}} = 1$. Because of $\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^2} < \infty$ and $z \in H_{-1}$, it follows that

$$((z, r))_{-\frac{1}{2}} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^2} < \infty, \quad ((w, s))_{-\frac{1}{2}} \sim \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} \frac{\mu(n)}{n} < \infty$$

and therefore $r \in H_0$, $w \in H_0$, $s \in H_{-1}$. For

$$\sigma(x) := \sum_{n \leq x} \frac{\mu(n)}{n} \log\left(\frac{x}{n}\right)$$

it holds

$$\sigma(xy) + 1 = \sigma(x) + \sigma(y), \quad \sigma'(x) = \frac{1}{x} \sum_{n \leq x} \frac{\mu(n)}{n}$$

and the inverse mapping is given by

$$\sigma^{-1}(x) = \sum_{n \leq x} \frac{1}{n} \log\left(\frac{x}{n}\right), \quad x \geq 1.$$

The asymptotics of a related arithmetical function is given by ((ApT) 3.12)

$$\sum_{n \leq x} \frac{1}{n} \frac{\varphi(n)}{n} + \sum_{n=1}^{\infty} \frac{1}{n} \frac{\mu(n) \log n}{n} = \frac{6}{\pi^2} (\log x + \gamma) + o\left(\frac{\log x}{x}\right).$$

For the relationship to the alternative Zeta function theory below we note that the function $\pi \cot(\pi x)$ is holomorphic except the pole $z = 1$ and it holds

$$\frac{1}{2} \cot\left(\frac{x}{2}\right) = \frac{d}{dx} \ln\left(2 \sin\left(\frac{x}{2}\right)\right) = -\frac{d}{dx} \sum_{n=1}^{\infty} \frac{\cos(nx)}{n} = \sum_{n=1}^{\infty} \sin(nx) \in H_{-1}^{\#}(0, 2\pi)$$

From the identity $(s = \frac{1}{2} + ix)$

$$\sum_{n \geq x} \frac{1}{n^s} = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} z^{1-s} (-\pi \cot(\pi z)) \frac{dz}{z} =: g(s)$$

it follows $((z, g))_{-1} < \infty$, i.e. $z, g \in H_{-1}$ resp. $((z, \omega))_{-1}, ((g, \omega))_{-1}, \forall \omega \in H_{-1}$. The latter representation enables an orthogonal projections $z^{(*)}, g^{(*)} \in H_{-1/2}$ in the form $((z^{(*)}, \omega))_{-1}, ((g^{(*)}, \omega))_{-1}, \forall \omega \in H_{-1/2}$.

9. Appendix: Formulas and properties

The proposed alternative "baseline" function is the Hilbert transform of the Gaussian function, which is the Dawson function ((OIF) p. 44)

$$F(x) := e^{-x^2} \int_0^x e^{-t^2} dt$$

Its relationship to the „analysis of zeros of certain trigonometric integrals and entire higher genus-1 functions“ (PoG2), is given by the identities ((GrI) 3.896)

$$F(x) := e^{-x^2} \int_0^x e^{-t^2} dt = \int_0^\infty e^{-t^2} \sin(2xt) dt = x {}_1K_1\left(1, \frac{3}{2}; -x^2\right) = e^{-x^2} H(x)$$

with $H(x) := x {}_1K_1\left(\frac{1}{2}, \frac{3}{2}; x^2\right)$. We note that the Hilbert transform of the $\sin(ax)$ –function is given by $-\cos(ax)$ –function and $\int_0^\infty e^{-t^2} \cos(2xt) dt = \frac{\sqrt{\pi}}{2} e^{-x^2}$ ((GrI) 3.896). From (GrI) 7.612) we mention the reciprocal formula

$$\int_0^\infty e^{-\frac{x^2}{2}} {}_1F_1\left(\frac{1}{2}, 1, \frac{x^2}{2}\right) \sin(yx) dx = \frac{\sqrt{\pi}}{2} e^{-\frac{x^2}{2}} {}_1F_1\left(\frac{1}{2}, 1, \frac{x^2}{2}\right).$$

The Kummer function related Mellin transforms can be derived from the following formulas ((GrI) 7.612):

$$\text{i) } \int_0^\infty x^s {}_1F_1(a, c, -x) \frac{dx}{x} = \frac{\Gamma(s)\Gamma(a-s)\Gamma(c)}{\Gamma(c-s)\Gamma(a)} \quad 0 < \text{Re}(s) < \text{Re}(a)$$

$$\text{ii) } \int_0^\infty x^s {}_1F_1(a, a+1, -x) \frac{dx}{x} = \frac{a}{(a-s)} \Gamma(s) \quad 0 < \text{Re}(s) < \text{Re}(a)$$

$$\text{iii) } \int_0^\infty x^{s-1/2} {}_1F_1\left(a, a+\frac{1}{2}, -x\right) \frac{dx}{x} = \frac{\Gamma\left(\frac{a+1}{2}\right)}{\Gamma(a)} \Gamma\left(s-\frac{1}{2}\right) \quad \frac{1}{2} < \text{Re}(s) < \text{Re}\left(a+\frac{1}{2}\right)$$

$$\text{iv) } \int_0^\infty x^{\frac{s}{2}} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}, -x\right) \frac{dx}{x} = \frac{\Gamma\left(\frac{s}{2}\right)}{1-s} \quad 0 < \text{Re}(s) < \text{Re}(1)$$

$$\text{v) } h(x) + 2xh'(x) = e^{-x} \quad \text{with } h(x) := {}_1F_1\left(\frac{1}{2}, \frac{3}{2}, -x\right)$$

$$\text{vi) } M[-xh']\left(\frac{s}{2}\right) = \frac{s}{2} M[h]\left(\frac{s}{2}\right) = \frac{\Gamma\left(1+\frac{s}{2}\right)}{1-s} \quad 0 < \text{Re}(s) < \text{Re}(1)$$

$$\text{vii) } M_{\frac{1}{4^4}}(z) = z^{\frac{3}{4}} e^{-z/2} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}, z\right), \quad e^{-\frac{z}{2}} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}, z\right) = z^{-\frac{3}{4}} M_{\frac{1}{4^4}}(z) = (-z)^{-\frac{3}{4}} M_{\frac{1}{4^4}}(-z)$$

for the Whittaker functions $M_{\frac{1}{4^4}}(z)$ ((GrI) 9.220, 9.231)

viii) For $-1/4 < \text{Re}(\vartheta) < 1/4$, $0 < 1/2 - 2\vartheta < 1$, $1/2 < 1 + 2\vartheta < 3/2$ it holds ((GrI) 7.612):

$$\int_0^\infty G_{2\vartheta}(x) \sin(yx) dx = \frac{\sqrt{\pi}}{2} G_{2\vartheta}(x) \quad \text{with } G_{2\vartheta}(x) := x^{4\vartheta} e^{-\frac{x^2}{2}} {}_1F_1\left(\frac{1}{2} - 2\vartheta, 1 + 2\vartheta, \frac{x^2}{2}\right)$$

resp.

$$\int_0^\infty e^{-\frac{x^2}{2}} {}_1F_1\left(\frac{1}{2}, 1, \frac{x^2}{2}\right) \sin(yx) dx = \frac{\sqrt{\pi}}{2} e^{-\frac{x^2}{2}} {}_1F_1\left(\frac{1}{2}, 1, \frac{x^2}{2}\right) \quad (\vartheta = 0)$$

ix) The asymptotics of the Kummer functions are given by ((OIF), 7 §10.1, (AbM) 13.5.1.)

$${}_1F_1(a, c; x) \sim \frac{1}{\Gamma(a)} \frac{e^x}{x^{c-a}} \quad \text{resp.} \quad {}_1F_1(a, a+1; x) \sim \frac{1}{\Gamma(a)} \frac{e^x}{x}, \quad x \rightarrow \infty$$

x) The simple zeros of ${}_1F_1\left(1, \frac{3}{2}; z\right)$ lie in the half-plane $\text{Re}(z) < -1/2$. The simple zeros of ${}_1F_1\left(\frac{3}{2}, \frac{3}{2}; z\right)$ lie in the half-plane $\text{Re}(z) > 1/2$. All zeros z_n of the Kummer function $K_a(z) := {}_1K_1(a, a+1; z)$ ($0 < a \leq 1$) are simple and satisfy the asymptotic formula (SeA)

$$z_n = 2\pi i n + \left[(1-a) + \frac{(1-a)^2}{2\pi i n} \right] \cdot \left[\log(2\pi|n|) + \frac{\log\Gamma(a)}{1-a} \pm \left(-\frac{\pi}{2} \right) \right] - \frac{3a+1}{2\pi i n} + O\left(\frac{\log|n|}{n^2}\right), \quad n \rightarrow \pm\infty,$$

The zeros all lie in the horizontal stripe

$$(2n-1)\pi < |\text{Im}(z_n)| := |2\pi\omega_n| < 2n\pi \quad .$$

((SeA), see also ((BrK4) lemma A3, Notes O5-17, O22, O23, (BrK7) Note 11)).

For the Fourier inverse of the Zeta function on the critical line (in a distributional sense) we refer to (BrK4) Notes S21, S24. The related (classical) Zeta approximation series representation is provided in (TiE) 4.14 resp. (BrK4) Notes S51, O9, O27. In (OIF1) 25.6.6, an integral value representation for $\zeta(2n + 1)$ is provided with $\cot(\pi x)$ "density" function.

We note the corresponding Gamma function equivalent in the form

$$\log' \Gamma(1-x) - \log' \Gamma(x) = \pi \cot(\pi x) = \frac{1}{x} [1 - 2 \sum_{k=0}^{\infty} s_{2k} x^{2k}] \quad (\text{NiN}) \text{ §14 (5), §19 (16)}.$$

The alternative „Gamma“ function $\Gamma^*(\frac{s}{2})$ fulfills the following properties

- i) $\Gamma^*(\frac{s}{2}) \Gamma^*(\frac{1-s}{2}) = \Gamma(\frac{s}{2}) \Gamma(\frac{1-s}{2})$, $\Gamma'(1) = -\gamma$, $\Gamma^{*'}(1) = \pi$ ($\tan(\pi) = 0$ and $\tan'(\pi) = \pi$)
- ii) $\Gamma(1 + \frac{s}{2}) \rightarrow \Gamma^*(\frac{s}{2}) := \Gamma(\frac{s}{2}) \tan(\frac{\pi}{2}s) = \frac{\Gamma(\frac{1+s}{2}) \Gamma(\frac{1-s}{2})}{\Gamma(1-\frac{s}{2})} = \frac{\Gamma(1+\frac{s-1}{2}) \Gamma(1-\frac{s+1}{2})}{\Gamma(1-\frac{s}{2})}$
- iii) $\log' \Gamma^*(s) = \log' \Gamma(s) + \log' \tan(\frac{\pi}{2}s) = \log' \Gamma(s) + \frac{\pi}{\sin(\pi s)}$
- iv) $\log' \Gamma^*(\frac{s}{2}) = \log' \Gamma(\frac{s}{2}) + \frac{1}{2} \frac{1}{\sin(\pi s)} = \log' \Gamma(\frac{s}{2}) + \frac{1}{2} \Gamma(s) \Gamma(1-s)$, $\log' \Gamma(\frac{3}{4}) - \log' \Gamma(\frac{1}{4}) = \pi$
- v) $\frac{\pi}{2} \Gamma^*(\frac{s}{2}) = \sum_{k=1}^{\infty} \frac{\Gamma(1+\frac{s}{2})}{(k-\frac{1}{2})^2 - (\frac{s}{2})^2}$ (because of $\tan \frac{\pi}{2} x = \frac{4x}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 - x^2}$).

The alternative „Gamma“ function properties and some related lemmata in section 2 below in combination with the „value“ property of Kummer resp. hypergeometric functions ((BeI), (WoJ)), might also enable a new tool to prove the irrationality or even the transcendence of e.g. the Euler constant γ . The approaches in ((BeI), (WoJ)) are both based on classical E-function theory, whereby the considered Kummer functions in (BeI) are explicitly excluding the considered Kummer function of this paper. The concept in (PoG4) (in combination with the conjecture in (PoG3) about power series with rational or integer coefficients in the context of a convergence radius one) might provide an alternative E- or G-function approach.

With respect to the alternative $\zeta^*(s)$ function

$$\zeta^*(1-s) = \frac{\tan(\frac{\pi}{2}s)}{s} \cdot \zeta(1-s) \quad \text{resp.} \quad \zeta^*(s) = \frac{\cot(\frac{\pi}{2}s)}{1-s} \cdot \zeta(s) = \frac{\tan(\frac{\pi}{2}(1-s))}{1-s} \cdot \zeta(s)$$

resp.

$$\log \zeta^*(s) = \log(\tan \frac{\pi}{2}(1-s)) + \log\left(\frac{1}{1-s}\right) + \log \zeta(s)$$

We onto the following properties

- i) $\frac{1}{1-s} = \int_1^{\infty} x^{-s} dx = 1 - s \int_1^{\infty} x^{-s-1} x dx$
- ii) $\frac{\pi}{2} \tan(\frac{\pi}{2}s) = \frac{1}{1-x} + O(|1-x|)$ in the neighborhood of $x = 1$
- iii) $\log(\tan(\frac{\pi}{2}s)) = -4 \sum h_n \frac{\sin(2\pi n s)}{\pi n}$
- iv) $\log' \Gamma\left(1 - \theta = \frac{3}{4}\right) - \log' \Gamma\left(\theta = \frac{1}{4}\right) = \pi$
- v) $\log'(\tan(\frac{\pi}{2}s)) = -\log'(\cot(\frac{\pi}{2}s)) = \frac{\pi}{\sin(\pi s)} = \frac{1}{x} [1 + 2 \sum_{k=0}^{\infty} s_{2k} x^{2k}] = \beta(x) + \beta(1-x)$
with $\beta(x) := \sum_{k=0}^{\infty} (-1)^k \frac{1}{x+k} = \sum_{k=0}^{\infty} (-1)^k s_{k+1} x^k$ (NiN) §14 (6), §19 (17).

Applying Riemann's building concept for the auxiliary function, defining a self-adjoint operator with Mellin transform $\xi(s)$ ((EdH) 10.3), results into a replacement of

$$\log(\sin x) = \log x - \frac{1}{6}x^2 - \frac{1}{180}x^4 - \frac{1}{2835}x^6 - \dots \rightarrow \log(\tan\left(\frac{\pi}{2}x\right)) = \log x + \frac{1}{3}x^2 + \frac{7}{90}x^4 + \frac{62}{2835}x^6 + \frac{127}{18900}x^8 + \dots$$

The density of prime numbers appears to be the Gaussian density $dg = \log\left(\frac{1}{t}\right) dt$ defining the corresponding prime number counting integral function ((EdH) 1.1 (3)). The Clausen density dw , based on the Clausen integral ((AbM) 27.8)

$$w(t) = \int_0^t \log\left(2\sin\frac{t}{2}\right) dt, \quad 0 \leq t \leq \pi$$

is related to the Hilbert transform of the fractional part function ((BrK4) Note O28).

For $T(x) := \log(\tan(\frac{\pi}{2}x))$, we summaries a few properties

i) $\frac{\pi}{2}T(x) = -\sum \frac{2h_n}{n} \sin(2\pi nx) \in L_2^\#(0,1)$ (EIL)

with $2h_n = \sum_{k=1}^n \frac{2}{2k-1} = 2H_{2n} - H_n$ and $H_n = \sum_{k=1}^n \frac{1}{k}$ (harmonic numbers) and

$$\int_0^1 T(x) \cos(k\pi x) dx = \begin{cases} -1/k & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

ii) the $\log(\tan x)$ -integral evaluated by series involving $\zeta(2n+1)$ is provided (EIL1)

iii) for the Hilbert transform evaluation of $T(x)$ we refer to (MaJ)

iv) from (GrI), 1.421, 1.518, we recall the series representations

$$T(x) = \log x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(2^{2k-1}-1)2^{2k} B_{2k} x^{2k}}{k(2k)!}, \quad x^2 < \left(\frac{\pi}{2}\right)^2$$

$$T(x) = \log x + \frac{1}{3}x^2 + \frac{7}{90}x^4 + \frac{62}{2835}x^6 + \frac{127}{18900}x^8 + \dots$$

$$\log(\sin x) = \log x - \frac{1}{6}x^2 - \frac{1}{180}x^4 - \frac{1}{2835}x^6 - \dots$$

v) For the related Fourier expansion of the $\log\Gamma(x)$ function we refer to (EsO) with coefficients $a_n = \frac{1}{2n}$, $b_n = \frac{A+\log n}{2cn}$ and $a_0 = \log\sqrt{2\pi}$.

vi) The counterpart of the asymptotics $\log(\tan\frac{\pi}{2}x) \sim \log(\sin(x)) \sim \log x$ with respect to the \cot -function is given by the estimate

$$\left| \frac{\pi}{2} \cot\left(\frac{\pi}{2}x\right) - \frac{1}{x} \right| \leq \frac{\pi^2}{6} = \zeta(2) \quad (|x| \leq 1),$$

which is a result of the following inequalities

$$\left| \frac{1}{2} \cot\left(\frac{x}{2}\right) - \frac{1}{x} \right| = \left| \sum_{k=1}^{\infty} \frac{1}{x+2\pi k} + \frac{1}{x-2\pi k} \right| \leq \left| \sum_{k=1}^{\infty} \frac{x}{\pi(x-2\pi k)} \right| \leq \frac{\pi}{2\pi^2} \sum_{k=1}^{\infty} \left| \frac{1}{1-\frac{x}{2\pi k}} \right| \frac{1}{k^2} \leq \frac{\pi}{6}, \quad |x| \leq \pi$$

With respect to the distributional Fourier series representation of the $\cot(\pi x)$ function we note the product representation ((GrI), 1.392)

$$\sin(nx) = \begin{cases} n \cdot \sin(x) \cdot \prod_{k=1}^{(n-1)/2} \left(1 - \frac{\sin^2(x)}{\sin^2(\pi \frac{k}{n})}\right) & n \text{ odd} \\ \frac{n}{2} \cdot \sin(2x) \cdot \prod_{k=1}^{(n-2)/2} \left(1 - \frac{\sin^2(x)}{\sin^2(\pi \frac{k}{n})}\right) & n \text{ even} \end{cases}$$

$$-\log \left| 2 \cos\left(\frac{x}{2}\right) \right| = -\sum_{n=1}^{\infty} (-1)^n \frac{\cos(nx)}{n} \quad -\log \left| 2 \sin\left(\frac{x}{2}\right) \right| = -\sum_{n=1}^{\infty} \frac{\cos(nx)}{n}$$

$$-\log \left| 2 \sin\left(\frac{x}{2}\right) \right| - \log \left| 2 \cos\left(\frac{x}{2}\right) \right| = -\log |2 \sin(x)| = -\sum_{n=1}^{\infty} \frac{\cos(2nx)}{n}.$$

In (ChK) VI, §2, two expansions of $\cot(z)$ are compared to prove that all coefficients of one of this expansion $\frac{\zeta(2n)}{\pi} e^{-2n}$ are rational. Corresponding formulas for odd integers are unknown.

In (EsR), 3.8 (example 78), resp. (BrK4) Notes S51, a "finite part"- "principle value" integral representation of the $\frac{\pi}{2} \cot\left(\frac{\pi}{2}x\right)$ – is given (which is zero also for positive or negative integers)

$$F.p.(P.v. \int_0^\infty t^x \frac{t}{t^2-1} dt) = \begin{cases} 0 \\ \left(\frac{\pi}{2}\right) \cot\left(\frac{\pi}{2}x\right) \end{cases} \text{ for } \begin{matrix} x \in 2\mathbb{Z} \\ \text{otherwise} \end{matrix}.$$

It is used as enabler to obtain the asymptotic expansion of the p.v. integral, defined by the "restricted" Hilbert transform integral of a function $u(x)$ over the positive x-axis, only. In case $u(x)$ has a structure $u(x) = v(x)\sqrt{x}$ the representation enjoys a remarkable form, where the numbers $n + 1/2$ play a key role.

In the context of Landau's „generalized number theoretical function theorem“ we note the following properties: for

$$g(x) := \frac{1}{\sin^2(\pi x)}, \quad p(x) := \left(\frac{\pi}{2}x\right) \cot\left(\frac{\pi}{2}x\right) = 1 + \sum_{n=1}^\infty \left[\frac{x}{x+n} + \frac{x}{x-n} \right], \quad h(x) := x \cdot p(x),$$

it holds

$$\text{i)} \quad \cot(x) = \frac{1}{2} \cot\left(\frac{x}{2}\right) - \frac{1}{2} \cot\left(\frac{\pi-x}{2}\right), \quad -\frac{1}{2} x \frac{g'(x)}{g(x)} = P(x) = -\pi \cot(\pi x) \xrightarrow{x \rightarrow 0} 1$$

$$\text{ii)} \quad \sin\left(\pi n + \frac{1}{n}\right) = O\left(\frac{1}{n}\right), \quad \sin\left(O\left(\frac{1}{n}\right)\right) = O\left(\frac{1}{n}\right), \quad \cos\left(O\left(\frac{1}{n}\right)\right) = O(1) \quad (\text{OIF}) \quad 3.1$$

$$\text{iii)} \quad \int_0^1 h(x) dx = \frac{1}{2} - \sum_{k=1}^\infty \frac{\zeta(2k)}{2^{2k}(2k+1)}, \quad \int_0^1 p(x) dx = \int_0^1 h(x) \frac{dx}{x} = \int_1^\infty h\left(\frac{1}{x}\right) \frac{dx}{x} = \log 2 \quad (\text{GrI}) \quad 3.747$$

$$\text{iv)} \quad \int_0^1 h(x) \frac{dx}{x} + \int_1^\infty h\left(\frac{1}{x}\right) \frac{dx}{x} = 2 \log 2 \quad \text{and} \quad h(x) \in L_1(0,1) \quad (\text{GrI}) \quad 3.748$$

$$\text{resp.} \quad \frac{1}{\log 2} \int_0^1 x \left(\frac{\pi}{2}x\right) \cot\left(\frac{\pi}{2}x\right) \frac{dx}{x} = \frac{1}{\log 2} \int_1^\infty \frac{1}{x} \left(\frac{\pi}{2} \frac{1}{x}\right) \cot\left(\frac{\pi}{2} \frac{1}{x}\right) \frac{dx}{x} = 1, \quad \int_0^1 x \left[\left(\frac{\pi}{2}x\right) \cot\left(\frac{\pi}{2}x\right)\right] dx = \frac{1}{2} - \sum_{k=1}^\infty \frac{\zeta(2k)}{2^{2k}(2k+1)}.$$

From (ZyA) V.2, we recall for $0 < \beta < 1$ and $0 < x \leq \pi$ the estimates

$$\text{i)} \quad \left| \sum_1^N \frac{\cos(nx)}{n^\beta} \right| \leq c \frac{x^\beta}{x}$$

$$\text{ii)} \quad \left| \sum_1^N \frac{\sin(nx)}{n^\beta} \right| \leq c \frac{x^\beta}{x}$$

$$\text{iii)} \quad \left| \sum_1^N \frac{\cos(nx)}{n} \right| \leq \log\left(\frac{1}{x}\right) + C.$$

The series $\sum_2^\infty \frac{\cos(nx)}{\log n}$ is divergent, its conjugate $\sum_2^\infty \frac{\sin(nx)}{\log n}$ is not a Fourier series, and ((ZyA) V.1)

$$\text{i)} \quad \sum_2^\infty \frac{\cos(nx)}{\log n} \sim \left[\frac{\pi}{2} \log^{-1}\left(\frac{1}{x}\right) \right]_x^1 \log^{-1}\left(\frac{1}{x}\right) \quad x \rightarrow +0$$

$$\text{ii)} \quad \sum_2^\infty \frac{\sin(nx)}{\log n} \sim \frac{1}{x} \log^{-1}\left(\frac{1}{x}\right) \quad x \rightarrow +0.$$

Because of $\int_0^1 f(x) \sin(\pi n x) dx \rightarrow 0, n \rightarrow \infty \quad \forall f \in L_2(0,1)$, the sequence $\{\sin(\pi n x)\}$ converges weakly to zero, but not strongly, as $\|\sin(\pi n x)\| = \frac{1}{\sqrt{2}}$. The same is true for

$$f(x) = \begin{cases} 1, & n < x < n+1 \\ 0, & \text{otherwise} \end{cases}$$

The specifically considered Kummer functions fulfill the following property

$$c_1 \cdot [{}_1F_1(a, c; x)]^2 \leq {}_1F_1(a - \mu, c; x) \cdot {}_1F_1(a + \mu, c; x) \leq [{}_1F_1(a, c; x)]^2, \quad a > 0, c > a \geq \mu - 1, x \in R$$

From (BaR) we recall the inequality

$${}_1F_1(a - \mu, c; x) \cdot {}_1F_1(a + \mu, c; x) \leq [{}_1F_1(a, c; x)]^2.$$

for $a > 0, c > a \geq \mu - 1$ and $x \in R$ or $a \geq \mu - 1, c > -1 (c \neq 0), x > 0$, and positiv integer μ .

In (KaD) this result is extended to non-integer positive μ and complemented with a reverse inequality given asymptotically precise lower bound for the quantity

$$\frac{{}_1F_1(a - \mu, c; x) \cdot {}_1F_1(a + \mu, c; x)}{[{}_1F_1(a, c; x)]^2}.$$

From (PeO) pp. 312, 352, we recall the following Continued fractions (CF) representations:

$$\log \frac{1+x}{1-x} \sim 2xF\left(\frac{1}{2}, 1, \frac{3}{2}; x^2\right) = \frac{2x}{|1} - \frac{1x^2}{|3} - \frac{4x^2}{|5} - \frac{9x^2}{|7} - \frac{16x^2}{|9} - \dots$$

$$\operatorname{actan}x \sim \frac{x}{|1} + \frac{1x^2}{|3} + \frac{4x^2}{|5} + \frac{9x^2}{|7} + \frac{16x^2}{|9} + \dots$$

$${}_1F_1(1, c; x) \sim 1 + \frac{x}{|c} - \frac{cx}{|c+1} + \frac{1x}{|c+2} - \frac{(c+1)x}{|c+3} + \frac{2x}{|c+4} - \frac{(c+2)x}{|c+5} + \frac{3x}{|c+6} \dots$$

$$\frac{1}{{}_1F_1(1, c; x)} \sim 1 - \frac{\frac{1}{c}x}{|1} + \frac{\frac{1x}{c(c+1)}}{|1} - \frac{\frac{cx}{(c+1)(c+2)}}{|1} + \frac{\frac{2x}{(c+2)(c+3)}}{|1} - \frac{\frac{(c+1)x}{(c+3)(c+4)}}{|1} + \frac{\frac{3x}{(c+4)(c+5)}}{|1} \dots$$

$$\frac{{}_1F_1(a, c; x)}{{}_1F_1(a+1, c+1; x)} \sim 1 - \frac{\frac{c-a}{c(c+1)}x}{|1} + \frac{\frac{a+1}{(c+1)(c+2)}x}{|1} - \frac{\frac{c-a+1}{(c+2)(c+3)}x}{|1} + \frac{\frac{a+2}{(c+3)(c+4)}x}{|1} \dots$$

with

$${}_1F_1(a+1, c+1; x) = \frac{c}{a} \frac{d}{dx} [{}_1F_1(a, c; x)].$$

and therefore

$$\frac{1}{{}_1F_1\left(1, \frac{3}{2}; x\right)} \sim 1 - \frac{\frac{2}{3}x}{|1} + \frac{\frac{1x}{3 \cdot 5}}{|1} - \frac{\frac{\frac{3}{2}x}{5 \cdot 7}}{|1} + \frac{\frac{2x}{7 \cdot 9}}{|1} - \frac{\frac{\frac{5}{2}x}{9 \cdot 11}}{|1} + \frac{\frac{3x}{11 \cdot 13}}{|1} \dots$$

$$= 1 - \frac{\frac{2 \cdot 2 \cdot 1}{1 \cdot 3 \cdot 2}x}{|1} + \frac{\frac{2 \cdot 2 \cdot 1}{3 \cdot 5}x}{|1} - \frac{\frac{2 \cdot 2 \cdot 3}{5 \cdot 7 \cdot 2}x}{|1} + \frac{\frac{2 \cdot 2 \cdot 2}{7 \cdot 9}x}{|1} - \frac{\frac{2 \cdot 2 \cdot 5}{9 \cdot 11 \cdot 2}x}{|1} + \frac{\frac{2 \cdot 2 \cdot 3}{11 \cdot 13}x}{|1} \dots$$

$$\frac{2}{3} \frac{d}{dx} [{}_1F_1\left(1, \frac{3}{2}; x\right)] \sim 1 - \frac{\frac{\frac{1}{35}x}{|1}}{|1} + \frac{\frac{2}{57}x}{|1} - \frac{\frac{\frac{3}{79}x}{|1}}{|1} + \frac{\frac{3}{79}x}{|1} = 1 - \frac{\frac{2 \cdot 2 \cdot 1}{3 \cdot 5 \cdot 2}x}{|1} + \frac{\frac{2 \cdot 2 \cdot 2}{5 \cdot 7}x}{|1} - \frac{\frac{2 \cdot 2 \cdot 3}{5 \cdot 7 \cdot 2}x}{|1} + \frac{\frac{2 \cdot 2 \cdot 3}{7 \cdot 9}x}{|1} \dots$$

From (ArE) we recall a few properties of log-convex functions:

Theorem 1.1:

The sum of convex functions is again convex. The limit function of a convex sequence of convex functions is convex. A convergent infinite series whose terms are all convex has a convex sum.

Theorem 1.3:

A function is convex, if, and only if, it is continuous and weakly convex.

Theorem 1.4: $f(x)$ is a convex function if, and only if, $f(x)$ has monotonically increasing one-side derivatives.

Theorem 1.5:

A function is convex, if, and only if, it is weakly convex.

Theorem 1.6:

A product of log-convex (weakly log-convex) functions is again log-convex (weakly log-convex). A convergent sequence of log-convex (weakly log-convex) functions has a log-convex (weakly log-convex) limit function, provided the limit is positive.

Theorem 1.7:

Suppose $f(x)$ is a twice differentiable function. If the inequalities

$$f(x)f''(x) - |f'(x)|^2 \geq 0$$

hold, then $f(x)$ is log-convex.

Theorem 1.8:

Suppose $f(x)$ and $g(x)$ are functions, defined on a common interval. If both are weakly log-convex, then their sum $f(x) + g(x)$ is also weakly log-convex.

Theorem 1.9:

If $\varphi(x)$ is a positive continuous function defined on the interior of the integration interval, then

$$\int_a^b \varphi(t)t^{x-1}dt$$

is a log-convex function of x for every interval on which the proper or improper integral exists.

Theorem 1.10:

If $f(x)$ is log-convex on a certain interval, and if c is any real number $\neq 0$, then both functions $f(x+c)$ and $f(cx)$ are log-convex on the corresponding intervals.

References

- (AbM) Abramowitz M., Stegun A., Handbook of mathematical functions, Dover Publications Inc., New York, 1970
- (AhJ) Ahner J. F., A scattering trinity: the reproducing kernel, null-field equations and modified Green's functions, *Q. J. Mech. Appl. Math.* Vol. 39, No. 1, 153-162, 1986
- (AnE) Anderson E., *The Problem of Time*, Springer, Cambridge, UK, 2017
- (AnJ) Annett J. F., *Superconductivity, Superfluids and condensates*, Oxford University Press, Oxford, 2004
- (AnM) Anderson M. T., Geometrization of 3-manifolds via the Ricci flow, *Notices Amer. Math. Soc.* 51, (2004) 184-193
- (ApT) Apostol T. M., *Introduction to Analytic Number Theory*, Springer Verlag, 2000
- (ArA) Arthurs A. M., *Complementary Variational Principles*, Clarendon Press, Oxford, 1970
- (ArE) Artin E., *The Gamma Function*, Holt, Rinehart and Winston, New York, Chicago, San Francisco, Toronto, London, 1964
- (ArN) Arcozzi N., Li X., Riesz transforms on spheres, *Mathematical Research Letters*, 4, 401-412, 1997
- (AzA) Aziz A. K., Kellogg R. B., Finite Element Analysis of Scattering Problem, *Math. Comp.*, Vol. 37, No 156 (1981) 261-272
- (AzT) Azizov T. Y., Ginsburg Y. P., Langer H., On Krein's papers in the theory of spaces with an indefinite metric, *Ukrainian Mathematical Journal*, Vol. 46, No 1-2, 1994, 3-14
- (BaB) Bagchi B., On Nyman, Beurling and Baez-Duarte's Hilbert space reformulation of the Riemann Hypothesis, *Indian Statistical Institute, Bangalore Centre*, (2005), www.isibang.ac.in
- (BaÁ) Baricz Á., Mills' ratio: Monotonicity patterns and functional inequalities, *J. Math. Anal. Appl.* 340, 1362-1370, 2008
- (BaÀ) Baricz Á., Pogány T. K., Inequalities for the one-dimensional analogous of the Coulomb potential, *Acta Polytechnica Hungarica*, Vol. 10, No. 7, 53-67, 2013
- (BaR) Barnard R. W., Gordy M., Richards K. C., A note on Turán type and mean inequalities for the Kummer function, *J. Math. Anal. Appl.* 349 (1), 259-263, 2009
- (BeA) Besse A., L., *Einstein Manifolds*, Springer-Verlag, Berlin, Heidelberg, 1987
- (BeB) Berndt B. C., *Ramanujan's Notebooks, Part I*, Springer Verlag, New York, Berlin, Heidelberg, Tokyo, 1985
- (BeB1) Berndt B. C., *Number Theory in the Spirit of Ramanujan*, AMS, Providence, Rhode Island, 2006
- (BeG) Besson G., The geometrization conjecture after R. Hamilton and G. Perelman, *Rend. Sem. Mat. Pol. Torino*, Vol. 65, 4, 2007, pp. 397-411
- (BeI) Belogrivov, I. I., On Transcendence and algebraic independence of the values of Kummer's functions, *Translated from Sibirskii Matematicheskii Zhurnal*, Vol. 12, No 5, 1971, 961-982
- (BeL) Bel L., Introduction d'un tenseur du quatrième ordre, *C. R. Acad. Sci. Paris*, 247, 1094-1096, 1959
- (BID) Bleecker D., *Gauge Theory and Variational Principles*, Dover Publications, Inc., Mineola, New York, 1981

- (BiP) Biane P., Pitman J., Yor M., Probability laws related to the Jacobi theta and Riemann Zeta functions, and Brownian excursion, Amer. Math. soc., Vol 38, No 4, 435-465, 2001
- (BoD) Bohm D., Wholeness and the Implicate Order, Routledge & Kegan Paul, London, 1980
- (BoJ) Bogner J., Indefinite Inner Product Spaces, Springer-Verlag, Berlin, Heidelberg, New York, 1974
- (BoM) Bonnet M., Boundary Integral Equations Methods for Solids and Fluids, John Wiley & Sons Ltd., Chichester, 1995
- (BrH) Bezis H., Asymptotic Behavior of Some Evolution Systems, In: Nonlinear Evolution Equations (M. C. Crandall ed.). Academic Press, New York, 141-154, 1978
- (BrK) Braun K., A new ground state energy model, www.quantum-gravitation.de
- (BrK1) Braun K., An alternative Schroedinger (Calderon) momentum operator enabling a quantum gravity model
- (BrK2) Braun K., Global existence and uniqueness of 3D Navier-Stokes equations
- (BrK3) Braun K., Some remarkable Pseudo-Differential Operators of order -1, 0, 1
- (BrK4) Braun K., A Kummer function based Zeta function theory to prove the Riemann Hypothesis and the Goldbach conjecture
- (BrK5) An alternative trigonometric integral representation of the Zeta function on the critical line
- (BrK6) Braun K., A distributional Hilbert space framework to prove the Landau damping phenomenon
- (BrK7=BrK1) Braun K., An alternative Schroedinger (Calderón) momentum operator enabling a quantum gravity model
- (BrK8) Braun K., Comparison table, math. modelling frameworks for SMEP and GUT
- (BrK9) Braun K., Interior Error Estimates of the Ritz Method for Pseudo-Differential Equations, Jap. Journal of Applied Mathematics, 3, 1, 59-72, 1986
- (BrK10) Braun K., J. A. Nitsche's footprints to NSE problems, www.navier-stokes-equations.com
- (BrK related papers) www.navier-stokes-equations.com/author-s-papers
- (BuH) Buchholtz H., The Confluent Hypergeometric Function, Springer-Verlag, Berlin, Heidelberg, New York, 1969
- (BrR) Brent R. P., An asymptotic expansion inspired by Ramanujan, rpb@cslab.anu.edu.au
- (CaD) Cardon D., Convolution operators and zeros of entire functions, Proc. Amer. Math. Soc., 130, 6 (2002) 1725-1734
- (CaH) Cao H.-D., Zhu X.-P., Hamilton-Perelman's Proof of the Poincare Conjecture and the Geometrization Conjecture, arXiv:math/0612069v1, 2006
- (CaH1) Cao H. D., Chow B., Chu S. C., Yau S. T., Collected Papers on Ricci Flow, International Press 2003
- (CaJ) Cao J., DeTurck D., The Ricci Curvature with Rotational Symmetry, American Journal of Mathematics 116, (1994), 219-241
- (ChD) Christodoulou D., Klainerman, Asymptotic properties of linear field equations in Minkowski space, Comm. Pure Appl. XLIII, 137-199, 1990
- (ChD1) Christodoulou D., Klainerman, The Global Nonlinear Stability of the Minkowski Space, Princeton University Press, New Jersey, 1993

- (ChF) Chen F., F., Introduction to Plasma Physics and Controlled Fusion, Volume I: Plasma Physics, Plenum Press, New York, 1984
- (ChH) Chen H., Evaluations of Some Variant Euler Sums, Journal of Integer Sequences, Vol. 9, (2006) 1-9
- (ChK) Chandrasekharan K., Elliptic Functions, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985
- (ChK1) Chaudhury K. N., Unser M., On the Hilbert transform of wavelets, arXiv:1107.4619v1
- (ChJ) Choi J., Srivastava H. M., The Clauen function and ist related Integrals, Thai J. Math., Vol 12, No 2, 251-264, 2014
- (CiI) Ciufolini I., Wheeler J. A., Gravitation and Inertia, Princeton University Press, Princeton, New Jersey, 1995
- (CoF) Coffey M. W., Polygamme theory, Li/Kneiper constantts, and validity of the Riemann Hypothesis, <http://arxiv.org>
- (CoP) Constatin P., Lax P. D., Majda A., A simple one-dimensional model fort he three-dimensional vorticity model, Communications on Pure and Applied Mathematics, Vol. XXXVIII, 715-724, 1985
- (CoR) Courant R., Hilbert D., Methods of Mathematical Physics Volume II, J. Wiley & Sons, New York, 1989
- (DeJ) Derezinski J., Richard S., On Radial Schrödinger Operators with a Coulomb Potential, Ann. Henri Poincaré 19 (2018), 2869-2917
- (DeR) Dendy R. O., Plasma Dynamics, Oxford Science Publications, Oxford, 1990
- (DrM) Dritschel M. A., Rovnyak, J., Operators on indefinite inner product spaces
- (EbP) Ebenfelt P., Khavinson D., Shapiro H. S., An inverse problem for the double layer potential, Computational Methods and Function Theory, Vol. 1, No. 2, 387-401, 2001
- (EdH) Edwards Riemann's Zeta Function, Dover Publications, Inc., Mineola, New York, 1974
- (EhP) Ehrlich P., Contemporary infinitesimalist theories of continua and their late 19th- and early 20th-century forerunners, arXiv.org 180.03345, Dec 2018
- (EiA) Einstein A., Grundzüge der Relativitätstheorie, Vieweg & Sohn, Braunschweig, Wiesbaden, 1992
- (EiA1) Einstein A., Äther und Relativitätstheorie, Julius Springer, Berlin, 1920
- (EiA2) Einstein A., Podolsky B., Rosen N., Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? Phsical Review, Vol. 47, 1935
- (EIL) Elaiassaoui L., Guennoun Z. El-Abidine, Relating log-tangent integrals with the Riemann zeta function, arXiv, May 2018
- (EIL1) Elaiassaoui L., Guennoun Z. El-Abidine, Evaluation of log-tangent integrals by series involving $\zeta(2n+1)$, arXiv, May 2017
- (EsG) Eskin G. I., Boundary Value Problems for Elliptic Pseudodifferential Equations, Amer. Math. Soc., Providence, Rhode Island, 1981
- (EsO) Esinosa O., Moll V., On some definite integrals involving the Hurwitz zeta function, Part 2, The Ramanujan Journal, 6, p. 449-468, 2002
- (EsR) Estrada R., Kanwal R. P., Asymptotic Analysis: A Distributional Approach, Birkhäuser, Boston, Basel, Berlin, 1994

- (EyG) Eyink G. L., Stochastic Line-Motion and Stochastic Conservation Laws for Non-Ideal Hydrodynamic Models. I. Incompressible Fluids and Isotropic Transport Coefficients, arXiv:0812.0153v1, 30 Nov 2008
- (FaK) Fan K., Invariant subspaces of certain linear operators, *Bull. Amer. Math. Soc.* 69 (1963), no. 6, 773-777
- (FaM) Farge M., Schneider K., Wavelets: application to turbulence, University Warnick, lectures, 2005
- (FaM1) Farge M., Schneider K., Wavelet transforms and their applications to MHD and plasma turbulence: a review, arXiv:1508.05650v1, 2015
- (FeR) Feynman R. P., Quantum Electrodynamics, Benjamin/Cummings Publishing Company, Menlo Park, California, 1961
- (FID) Fleisch D., A Student's Guide to Maxwell's Equations, Cambridge University Press, 2008
- (GaG) Galdi G. P., The Navier-Stokes Equations: A Mathematical analysis, Birkhäuser Verlag, Monographs in Mathematics, ISBN 978-3-0348-0484-4
- (GaL) Garding L., Some points of analysis and their history, *Amer. Math. Soc.*, Vol. 11, Providence Rhode Island, 1919
- (GaW) Gautschi W., Waldvogel J., Computing the Hilbert Transform of the Generalized Laguerre and Hermite Weight Functions, *BIT Numerical Mathematics*, Vol 41, Issue 3, pp. 490-503, 2001
- (GiY) Giga Y., Weak and strong solutions of the Navier-Stokes initial value problem, *Publ. RIMS, Kyoto Univ.* 19 (1983) 887-910
- (GöK) Gödel, K., An Example of a New Type of Cosmological Solutions of Einstein's Field Equations of Gravitation, *Review of Modern Physics*, Vol. 21, No. 3, 1949
- (GrI) Gradshteyn I. S., Ryzhik I. M., Table of integrals series and products, Academic Press, New York, San Francisco, London, 1965
- (GrJ) Graves J. C., The conceptual foundations of contemporary relativity theory, MIT Press, Cambridge, Massachusetts, 1971
- (GuR) Gundersen R. M., Linearized Analysis of One-Dimensional Magnetohydrodynamic Flows, Springer Tracts in Natural Philosophy, Vol 1, Berlin, Göttingen, Heidelberg, New York, 1964
- (HaE) Haidemenakis E. D., Physics of Solids in Intense Magnetic Fields, Plenum Press, New York, 1969
- (HaG) Hardy G. H., Riesz M., The general theory of Dirichlet's series, Cambridge University Press, Cambridge, 1915
- (HaJ) Havil J., Gamma, exploring euler's constant, Princeton University Press, Princeton and Oxford, 2003
- (HaJ1) Hartle J. B., Hawking S. W., Wave function of the Universe, *Physical Review, D*, Vol. 28, No. 12, 1983
- (HaR) Hamilton R. S., Non-singular solutions of the Ricci flow on three-manifolds, *Com. Anal. and Geometry*, Vol. 7, No. 4, pp. 695-729, 1999
- (HaR1) Hamilton R. S., Three manifolds with positive Ricci curvature, *Lour. Diff. Geom.* 17, pp. 255-306, 1982
- (HaS) Hawking S. Penrose R., The Singularities of Gravitational Collapse and Cosmology, The Royal Society, Vol. 314, Issue 1519, 1970

- (HeJ) Heywood J. G., Walsh O. D., A counter-example concerning the pressure in the Navier-Stokes equations, as $t \rightarrow 0^+$, Pacific Journal of Mathematics, Vol. 164, No. 2, 351-359, 1994
- (HeM) Heidegger M., Holzwege, Vittorio Klostermann, Frankfurt a. M., 2003
- (HeW) Heisenberg W., Physikalische Prinzipien der Quantentheorie, Wissenschaftsverlag, Mannheim, Wien, Zürich, 1991
- (HiP) Higgs P. W., Spontaneous Symmetry Breakdown without Massless Bosons, Physical Review, Vol. 145, No 4, p. 1156-1162, 1966
- (HoM) Holschneider M., Wavelets, An Analysis Tool, Clarendon Press, Oxford, 1995
- (HoA) Horvath A. G., Semi-indefinite-inner-product and generalized Minkowski spaces, arXiv
- (HoM) Holschneider M., Wavelets, An Analysis Tool, Clarendon Press, Oxford, 1995
- (IvV) Ivakhnenko, V. I., Smirnow Yu. G., Tyrtshnikov E. E., The electric field integral equation: theory and algorithms, Inst. Numer. Math. Russian of Academy Sciences, Moscow, Russia
- (JoF) John F., Formation of singularities in elastic waves, Lecture Notes in Phys., Springer-Verlag, 190-214, 1984
- (KaD) Karp D., Sitnik S. M., Log-convexity and log-concavity of hypergeometric-like functions, J. Math. Anal. Appl. 364, 384-394, 2010
- (KaM) Kaku M., Introduction to Superstrings and M-Theory, Springer-Verlag, New York, Inc., 1988
- (KaM1) Kac M., Probability methods in some problems of analysis and number theory, Bull. Am. Math. Soc., 55, 641-655, (1949)
- (KeL) Keiper J. B., Power series expansions of Riemann's Zeta function, Math. Comp. Vol 58, No 198, (1992) 765-773
- (KiA) Kirsch A., Hettlich F., The mathematical theory of time-harmonic Maxwell's equations, expansion-, integral-, and variational methods, Springer-Verlag, Heidelberg, New York, Dordrecht, London, 2015
- (KiA1) Kiselev A. P., Relatively Undistorted Cylindrical Waves Depending on Three Spatial Variables, Mathematical Notes, Vol. 79, No. 4, 587-588, 2006
- (KIB) Kleiner B., Lott J., Notes on Perelman's papers, Mathematics ArXiv
- (KiH) Kim H., Origin of the Universe: A Hint from Eddington-inspired Born-Infeld gravity, Journal of the Korean Physical Society, Vol. 65, No. 6, pp. 840-845, 2014
- (KIS) Klainermann S., Rodnianski, Regularity and geometric properties of solutions of the Einstein-Vacuum equations, Journées équations aux dérivées partielles, No. 14, p. 14, 2002
- (KIS1) Klainerman S., Nicolò, The Evolution Problem in General Relativity, Birkhäuser, Boston, Basel, Berlin, 1950
- (KIS2) Klainerman S., Remarks on the global Sobolev inequalities in Minkowski space, Comm. Pure. Appl. Math., 40, 111-117, 1987
- (KIS3) Klainerman S., Uniform decay estimates and the Lorentz invariance of the classical wave equation, Comm. Pure Appl. Math., 38, 321-332, 1985
- (KnA) Kneser A., Das Prinzip der kleinsten Wirkung von Leibniz bis zur Gegenwart, B. G. Teubner, Leipzig, Berlin, 1928
- (KoA) Kolmogoroff A., Une contribution à l'étude de la convergence des séries de Fourier, Fund. Math. Vol. 5, 484-489

- (KoH) Koch H., Tataru D., Well-posedness for the Navier-Stokes equations, *Adv. Math.*, Vol 157, No 1, 22-35, 2001
- (KoJ) Korevaar J., Distributional Wiener-Ikehara theorem and twin primes, *Indag. Mathem. N. S.*, 16, 37-49, 2005
- (KrA) Krall A. M., Spectral Analysis for the Generalized Hermite Polynomials, *Trans. American Mathematical Society*, Vol. 344, No. 1 (1994) pp. 155-172
- (KrR) Kress R. *Linear Integral Equations*, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, Hong Kong, 1941
- (KrR1) Kraußhar R. S., Malonek H. R., A characterization of conformal mappings in R^4 by a formal differentiability condition, *Bulletin de la Societè Royale de Liège*, Vol. 70, Vol. 1, 35-49, 2001
- (LaC) Lanczos C., *The variational principles of mechanics*, Dover Publications Inc., New York, 1970
- (LaC1) Langenhof C. E., Bounds on the norm of a solution of a general differential equation, *Proc. Amer. Math. Soc.*, 8, 615-616, 1960
- (LaE) Landau E., *Die Lehre von der Verteilung der Primzahlen Vol 1*, Teubner Verlag, Leipzig Berlin, 1909
- (LaE1) Landau E., Ueber die zahlentheoretische Function $\phi(n)$ und ihre Beziehung zum Goldbachschen Satz, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, Vol 1900, p. 177-186, 1900
- (LaE2) Landau E., *Die Lehre von der Verteilung der Primzahlen Vol 2*, Teubner Verlag, Leipzig Berlin, 1909
- (LaE3) Landau E., Über eine trigonometrische Summe, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, Vol 1928, p. 21-24, 1928
- (LaE4) Landau E. *Vorlesungen über Zahlentheorie, Erster Band, zweiter Teil*, Chelsea Publishing Company, New York, 1955
- (LaG) Lachaud G., Spectral analysis and the Riemann hypothesis, *J. Comp. Appl. Math.* 160, pp. 175190, 2003
- (LaJ) An Elementary Problem Equivalent to the Riemann Hypothesis, <https://arxiv.org>
- (LeN) Lebedev N. N., *Special Functions and their Applications*, translated by R. A. Silverman, Prentice-Hall, Inc., Englewood Cliffs, New York, 1965
- (LeN1) Lerner N., A note on the Oseen kernels, in *Advances in phase space analysis of partial differential equations*, Siena, pp. 161-170, 2007
- (LeP) LeFloch P. G., Ma Y., The global nonlinear stability of Minkowski space, arXiv: 1712.10045v1, 28 DEc 2017
- (LiI) Lifanov I. K., Poltavskii L. N., Vainikko G. M., *Hypersingular Integral Equations and their Applications*, Chapman & Hall/CRC, Boca Raton, London, New York, Washington, D. C., 2004
- (LiI1) Lifanov I. K., Nenashaev A. S., *Generalized Functions on Hilbert Spaces, Singular Integral Equations, and Problems of Aerodynamics and Electrodynamics*, *Differential Equations*, Vol. 43, No. 6, pp. 862-872, 2007
- (LiJ) Linnik J. V., *The dispersion method in binary additive problems*, American Mathematical Society, Providence, Rhode Island, 1963
- (LiP) Lions P. L., On Boltzmann and Landau equations, *Phil. Trans. R. Soc. Lond. A*, 346, 191-204, 1994

- (LiP1) Lions P. L., Compactness in Boltzmann's equation via Fourier integral operators and applications. III, *J. Math. Kyoto Univ.*, 34-3, 539-584, 1994
- (LiX) Li Xian-Jin, The Positivity of a Sequence of Numbers and the Riemann Hypothesis, *Journal of Number Theory*, 65, 325-333 (1997)
- (LoA) Lifanov I. K., Poltavskii L. N., Vainikko G. M., *Hypersingular Integral Equations and Their Applications*, Chapman & Hall/CRC, Boca Raton, London, New York, Washington, D. C. 2004
- (LuL) Lusternik L. A., Sobolev V. J., *Elements of Functional Analysis*, A Halsted Press book, Hindustan Publishing Corp. Delhi, 1961
- (MaJ) Mashreghi, J., Hilbert transform of $\log(\text{abs}(f))$, *Proc. Amer. Math. Soc.*, Vol 130, No 3, p. 683-688, 2001
- (MaJ1) Marsden J. E., Hughes T. J. R., *Mathematical foundations of elasticity*, Dover Publications Inc., New York, 1983
- (MeY) Meyer Y., Coifman R., *Wavelets, Calderón-Zygmund and multilinear operators*, Cambridge University Press, Cambridge, 1997
- (MiT) Mikosch T., *Regular Variation, Subexponentiality and Their Application in Probability Theory*, University of Groningen
- (MoC) Mouhot C., Villani C., On Landau damping, *Acta Mathematica*, Vol. 207, Issue 1, p. 29-201, 2011
- (MoJ) Morgan J. W., Tian G., Ricci Flow and the Poincare Conjecture, *Mathematics ArXiv*
- (NaS) Nag S., Sullivan D., Teichmüller theory and the universal period mapping via quantum calculus and the H space on the circle, *Osaka J. Math.*, 32, 1-34, 1995
- (NeD) Neuenschwander D. E., *Emmy Noether's Wonderful Theorem*, The John Hopkins University Press, Baltimore, 2011
- (NiJ) Nitsche J. A., lecture notes, approximation theory in Hilbert scales
- (NiJ1) Nitsche J. A., lecture notes, approximation theory in Hilbert scales, extensions and generalizations
- (NiJ2) Nitsche J. A., *Direct Proofs of Some Unusual Shift-Theorems*, *Anal. Math. Appl.*, Gauthier-Villars, Montrouge, 1988, pp. 383-400
- (NiJ3) Nitsche J. A., Free boundary problems for Stokes's flows and finite element methods, *Equadiff 6*, 2006, pp. 327-332
- (NiN) Nielsen N., *Die Gammafunktion*, Chelsea Publishing Company, Bronx, New York, 1965
- (ObF) Oberhettinger, *Tables of Mellin Transforms*, Springer-Verlag, Berlin, Heidelberg, New York, 1974
- (OIF) Olver F. W. J., *Asymptotics and special functions*, Academic Press, Inc., Boston, San Diego, New York, London, Sydney, Tokyo, Toronto, 1974
- (OIF1) Olver F. W. J., Lozier D. W., Boisvert R. F., Clark C. W., *NIST Handbook of Mathematical Functions*
- (OIR) Oloff R., *Geometrie der Raumzeit*, Vieweg & Sohn, Braunschweig/Wiesbaden, 1999
- (OsK) Oskolkov K. I., Chakhkiev M. A., On Riemann „Nondifferentiable“ Function and the Schrödinger Equation, *Proc. Steklov Institute of Mathematics*, Vol. 269, 2010, pp. 186-196
- (OsH) Ostmann H.-H., *Additive Zahlentheorie, erster Teil*, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1956

- (PeB) Petersen B. E., Introduction the the Fourier transform and Pseudo-Differential operators, Pitman Advanced Publishing Program, Boston, London, Melbourne, 1983
- (PeM) Perel M., Gorodnitskiy E., Representations of solutions of the wave equation based on relativistic wavelet, arXiv:1205.3461v1, 2012
- (PeO) Perron O., Die Lehre von den Kettenbrüchen, Volumes 1-2, Nabu Public Domain Reprint, copyright 1918 by B. G. Teubner in Leipzig
- (PeR) Penrose R., Cycles of Time, Vintage, London, 2011
- (PeR1) Peralta-Fabi, R., An integral representation of the Navier-Stokes Equation-I, Revista Mexicana de Fisica, Vol 31, No 1, 57-67, 1984
- (PeR2) Penrose R., Structure of space-time, Batelle Rencontre, C. M. DeWitt and J. M. Wheeler, 1967
- (PeR3) Penrose R., Zero rest mass fields including gravitation: asymptotic behaviours, Proc. Toy. Soc. Lond., A284, 159-203, 1962
- (PeR4) Penrose R., The Emperor's New Mind: Concerning Computers, Minds, and the Laws of Physics, Oxford Univ. Press, 1989
- (PeR5) Penrose R., Rindler W., Spinors and Space-Time, Cambridge University Press, Cambridge, 1984
- (PIJ) J. Plemelj, Potentialtheoretische Untersuchungen, B.G. Teubner, Leipzig, 1911
- (PoG) Polya G., Über Nullstellen gewisser ganzer Funktionen, Math. Z. 2 (1918) 352-383
- (PoG1) Polya G., Über eine neue Weise bestimmte Integrale in der analytischen Zahlentheorie zu gebrauchen, Göttinger Nachr. (1917) 149-159
- (PoG2) Polya G. Über die algebraisch-funktionentheoretischen Untersuchungen von J. L. W. V. Jensen, Det Kgl. Danske Videnskabernes Selskab., Mathematisk-fysiske Meddeleler. VII, 17, 1927
- (PoG3) Polya G., Über Potenzreihen mit ganzzahligen Koeffizienten, Math. Ann. 77, 1916, 497-513
- (PoG4) Polya G., Arithmetische Eigenschaften der Reihenentwicklung rationaler Funktionen, J. Reine und Angewandte Mathematik, 151, 1921, 1-31
- (PoD) Pollack D., Initial Data for the Cauchy Problem in General Relativity, General Relativity Spring School 2015, Junior Scientist Andrejewski Days, March 22nd to April 4th, 2015, Brandenburg an der Havel, Germany
- (PoP) Poluyan P., Non-standard analysis of non-classical motion; do the hyperreal numbers exist in the quantum-relative universe?
- (PrK) Prachar K., Primzahlverteilung, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1957
- (RoC) Rovelli C., Quantum Gravity, Cambridge University Press, Cambridge, 2004
- (RoC1) Rovelli C., The Order of Time, Penguin Random House, 2018
- (RoC2) Rovelli C., Reality is not what it seems, Penguin books, 2017
- (RoC3) Rovelli C., Seven brief lessons on physics, Penguin Books, 2016
- (RoJ) Roberts J. T., Leibniz on Force and Absolute Motion, Philosophy of Science, Vol 70, No 3, pp. 553-573, 2003
- (RuC) Runge C., Über eine Analogie der Cauchy-Riemannschen Differentialgleichungen in drei Dimensionen, Nach. v. d. Gesellschaft d. Wissenschaften zu Göttingen, Math-Phys. Klasse Vol 1992, 129-136, 1992

- (RuM) Ruskai M. B., Werner E., Study of a Class of Regularizations of using Gaussian Integrals, arXiv:math/990212v2
- (RyG) Rybicki, G. B., Dawson's integral and the sampling theorem, *Computers in Physics*, 3, 85-87, 1989
- (ScE) Schrödinger E., *Statistical Thermodynamics*, Dover Publications, Inc., New York, 1989
- (ScE1) Schrödinger E., *My View of the World*, Ox Bow Press, Woodbridge, Connecticut, 1961
- (ScE2) Schrödinger E., *What is Life? and Mind and Matter*, Cambridge University Press, Cambridge, 1967
- (ScP) Scott P., The Geometries of 3-Manifolds, *Bull. London Math. Soc.*, 15 (1983), 401-487
- (SeA) Sedletskii A. M., Asymptotics of the Zeros of Degenerated Hypergeometric Functions, *Mathematical Notes*, Vol. 82, No. 2, 229-237, 2007
- (SeE) Seneta E., Regularly Varying Functions, *Lecture Notes in Math.*, 508, Springer Verlag, Berlin, 1976
- (SeJ) Serrin J., *Mathematical Principles of Classical Fluid Mechanics*
- (ShF) Shu F. H., *Gas Dynamics, Vol II*, University Science Books, Sausalito, California, 1992
- (ShM) Scheel M. A., Thorne K. S., *Geodynamics, The Nonlinear Dynamics of Curved Spacetime*
- (ShM1) Shimoji M., Complementary variational formulation of Maxwell's equations in power series form
- (SiT) Sideris T., Formation of singularities in 3-D compressible fluids, *Comm. Math. Phys.*, 101, 47-485, 1985
- (SmL) Smolin L., *Time reborn*, Houghton Mifflin Harcourt, New York, 2013
- (SoH) Sohr H., *The Navier-Stokes Equations, An Elementary Functional Analytical Approach*, Birkhäuser Verlag, Basel, Boston, Berlin, 2001
- (SoP) Sobolevskii P. E., On non-stationary equations of hydrodynamics for viscous fluid. *Dokl. Akad. Nauk SSSR* 128 (1959) 45-48 (in Russian)
- (StE) Stein E. M., *Conjugate harmonic functions in several variables*
- (StE1) Stein E. M., *Harmonic Analysis, Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, New Jersey, 1993
- (SzG) Szegő, G., *Orthogonal Polynomials*, American Mathematical Society, Providence, Rhode Island, 2003
- (ThW) Thurston W. P., Three Dimensional Manifolds, Kleinian Groups and Hyperbolic Geometry, *Bulletin American Mathematical society*, Vol 6, No 3, 1982
- (TiE) Titchmarsh E. C., *The theory of the Riemann Zeta-function*, Clarendon Press, London, Oxford, 1986
- (TrH) Treder H.-J., Singularitäten in der Allgemeinen Relativitätstheorie, *Astron. Nachr. Bd. 301*, H. 1, 9-12, 1980
- (TsB) Al'Tsuler B. L., Integral form of the Einstein equations and a covariant formulation of the Mach's principle, *Soviet Physics Jetp*, Vol. 24, No. 4, 1967
- (VaM) Vainberg M. M., *Variational Methods for the Study of Nonlinear Operators*, Holden-Day, Inc., San Francisco, London, Amsterdam, 1964
- (VeW) Velte W., *Direkte Methoden der Variationsrechnung*, B. G. Teubner, Stuttgart, 1976

- (ViI) Vinogradov I. M., *The Method of Trigonometrical Sums in the Theory of Numbers*, Dover Publications Inc., Minelola, New York 2004
- (ViI1) Vinogradov, I. M., Representation of an odd number as the sum of three primes, *Dokl. Akad. Nauk SSSR* 15, 291-294 (1937)
- (ViJ) Vindas J., Estrada R., A quick distributional way to the prime number theorem, *Indag. Mathem.*, N.S. 20 (1) (2009) 159-165
- (ViJ1) Vindas J., Local behavior of distributions and applications, Dissertation, Department of Mathematics, Louisiana State University, 2009
- (ViM) Villarino M. B., Ramanujan's Harmonic Number Expansion Into Negative Powers of a Triangular Number, *Journal of Inequalities in pure and applied mathematics*, Vol. 9, No. 3 (2008), Art. 89, 12 pp.
- (ViV) Vladimirov V. S., Drozzinov Yu. N., Zavalov B. I., *Tauberian Theorems for Generalized Functions*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1988
- (WeD) Westra D. B., The Haar measure on $SU(2)$, March 14, 2008
- (WeH) Weyl H., *Space, Time, Matter*, Cosimo Classics, New York, 2010
- (WeH1) Weyl H., Matter, structure of the world, principle of action,, in (WeH) §34 ff.
- (WeH2) Weyl H., *Was ist Materie?* Verlag Julius Springer, Berlin, 1924
- (WeH3) Weyl H., *Philosophy of Mathematics and Natural Science*, Princeton University Press, Princeton and Oxford, 2009
- (WeP) Werner P., Self-Adjoint Extension of the Laplace Operator with Respect to Electric and Magnetic Boundary Conditions, *J. Math Anal. Appl.*, 70, 1979, pp. 131-160
- (WeP1) Werner P., Spectral Properties of the Laplace Operator with respect to Electric and Magnetic Boundary Conditions, *J. Math Anal. Appl.*, 92, 1983, pp. 1-65
- (WhJ1) Whittaker J. M., *Interpolatory Function Theory*, Cambridge University Press, Cambridge, 1935
- (WhJ2) Whittaker J. M., The „Fourier“ Theory of Cardinal Functions, *Proceedings of the Edinburgh Mathematical Society*, Vol. 1, Issue 3, pp. 169-176, 1928
- (WhJ) Wheeler J. A., *On the Nature of Quantum Geometrodynamics*
- (WhJ1) Wheeler J. A., *Awakening to the Natural State*, Non-Duality Press, Salisbury, 2004
- (WhJ2) Wheeler J. A., *At home in the universe*, American Institute of Physics, Woodbury, 1996
- (WoJ) Wohlfart J., Werte hypergeometrischer Funktionen, *Inventiones mathematicae*, Vol. 92, Issue 1, 1988, 187-216
- (WoW) Wong W. Kant's Conception of Ether as a field in the *Opus posthumum*, Proc Eighth Intern. Kant Congress, Marquette University Press, Vol. II, Memphis 1995
- (YeR) Ye R., Ricci flow, Einstein metrics and space forms, *Trans. Americ. Math. Soc.*, Vol. 338, No. 2, 1993
- (ZhB) Zhechev B., *Hilbert Transform Relations*
- (ZyA) Zygmund A., *Trigonometric series*, Volume I & II, Cambridge University Press, 1959