JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 19, Number 4, Winter 2007

SOLUTION OF A SIMPLE HYPERSINGULAR INTEGRAL EQUATION

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ABSTRACT. A direct function-theoretic method is developed to determine the solution of a simple hypersingular integral equation. The known form of the solution is recovered.

1. Introduction. The simple hypersingular integral equation

$$H\phi = \int_{-1}^{1} \frac{\phi(t)\,dt}{(t-x)^2} = f(x), \quad -1 < x < 1 \tag{1.1}$$

is considered for its solution for $\phi \in C^{1,\alpha}(-1,1)$ and $f \in C^{0,\alpha}(-1,1)$, $0 < \alpha < 1$, $C^{n,\alpha}(-1,1)$ denoting the class of functions having a Hölder continuous derivative of order n with α as the exponent, see [**5**]. The hypersingular integral $H\phi$ appearing in the equation (1.1) is understood to be equal to the Hadamard finite part, see [**4**], of this divergent integral, as given by the relation:

$$H\phi = \lim_{\varepsilon \to 0^+} \left[\int_{-1}^{x-\varepsilon} \frac{\phi(t)dt}{(t-x)^2} + \int_{x+\varepsilon}^{1} \frac{\phi(t)dt}{(t-x)^2} - \frac{\phi(x+\varepsilon)+\phi(x-\varepsilon)}{\varepsilon} \right]. \tag{1.2}$$

The equation (1.1) has been solved by Martin [**4**] and Chakrabarti and Mandal [**1**], under the circumstances when $\phi(-1) = 0 = \phi(1)$, in the following closed form:

$$\phi(x) = \frac{1}{\pi^2} \int_{-1}^{1} f(t) \log\left(\frac{|x-t|}{1 - xt + [(1-x^2)(1-t^2)]^{1/2}} \right) dt, \tag{1.3}$$

by utilizing known solution, see [**2**], of the Cauchy-type singular integral equation of the first kind, as given by the relation

$$T\psi = \int_{-1}^{1} \frac{\psi(t)dt}{t-x} = h(x), \quad -1 < x < 1, \tag{1.4}$$

Received by the editors on April 7, 2006, and in revised form on June 21, 2006.

where the Cauchy principal value of the singular integral $T\psi$ is to be understood.

It is observed that even though the original integral equation (1.1) involves hypersingular integrals, its solution, as given by formula (1.3) involves singular integrals possessing weaker singularities only, which can be avoided, because of the fact that once hypersingular integrals have been accepted there is no special need to deviate and bring in integrals with weaker singularities into the picture.

In the present note, we have developed a direct function theoretic method to solve the hypersingular integral equation (1.1), reducing it to a Riemann-Hilbert type boundary value problem of an unknown sectionally analytic function of a complex variable $z(= x+iy, i^2 = -1)$, in the complex z-plane, cut along the segment $(-1, 1)$ of the real x-axis. The resulting Riemann-Hilbert type problem is then solved by utilizing the Plemelj-type formulae directly which are developed for the type of sectionally analytic functions giving rise to the integral equation (1.1).

The final form of the solution is obtained in terms of hypersingular integrals as may be expected and, in special circumstances, when ϕ satisfies the end conditions $\phi(-1) = 0 = \phi(1)$, it is shown that the known form (1.3) of the solution is recovered, if only the finite parts of various divergent integrals are accepted in the analysis.

2. The detailed analysis. We first consider the sectionally analytic function

$$\Phi(z) = \int_{-1}^{1} \frac{\phi(t)dt}{(t-z)^2}, \quad z \notin [-1, 1]. \tag{2.1}$$

Then, if we utilize the following standard limiting values, see [**5**] and [**3**, page 104]:

$$\lim_{y\to\pm 0} \frac{1}{(x+iy)} = \mp\pi i\delta(x) + \frac{1}{x}, \quad -\infty < x < \infty, \tag{2.2}$$

giving on differentiation with respect to x

$$\lim_{y\to\pm 0} \frac{1}{(x+iy)^2} = \pm\pi i\delta'(x) + \frac{1}{x^2}, \quad -\infty < x < \infty, \tag{2.3}$$

where $\delta(x)$ denotes Dirac's delta function and $\delta'(x)$ denotes its derivative with respect to x, we obtain the following Plemelj-type formulae giving the limiting values of the function $\Phi(z)$, as z approaches a point on the cut $(-1,1)$ from above $(y \to 0^+)$ and below $(y \to 0^-)$, respectively:

$$\begin{aligned}\Phi(x \pm i0) \equiv \Phi^{\pm}(x) &= \mp\pi i \int_{-1}^{1} \phi(t)\delta'(t-x)dt + \int_{-1}^{1} \frac{\phi(t)dt}{(t-x)^2} \\ &= \pm\pi i\phi'(x) + \int_{-1}^{1} \frac{\phi(t)dt}{(t-x)^2}, \text{ for } -1 < x < 1.\end{aligned} \tag{2.4}$$

It may be noted that the limiting values (2.4) can also be derived by utilizing the standard Plemelj formulae involving the limiting values of the Cauchy type integral

$$\hat{\Phi}(z) = \int_{-1}^{1} \frac{\phi(t)dt}{t-z}, \text{ for } z \notin [-1,1], \tag{2.5}$$

giving

$$\hat{\Phi}(x \pm i0) = \pm\pi i\phi(x) + \int_{-1}^{1} \frac{\phi(t)dt}{t-x}, \tag{2.6}$$

and, by the aid of the relation

$$\Phi(z) = \frac{d\hat{\Phi}}{dz}, \tag{2.7}$$

along with the understanding that

$$H\phi = \frac{d}{dx}(T\phi). \tag{2.8}$$

Now, the two relations (2.4) can also be viewed as the following two equivalent relations:

$$\begin{aligned}\Phi^+(x) + \Phi^-(x) &= 2\int_{-1}^{1} \frac{\phi(t)dt}{(t-x)^2}, \\ \Phi^+(x) - \Phi^-(x) &= 2\pi i\phi'(x), \text{ for } -1 < x < 1.\end{aligned} \tag{2.9}$$

By utilizing the first of the above two relations (2.9), we now rewrite the given hypersingular integral equation (1.1) as

$$\Phi^+(x) + \Phi^-(x) = 2f(x), \quad -1 < x < 1, \tag{2.10}$$

which represents a special Riemann-Hilbert type boundary value problem, see [**2**] for the determination of the unknown function $\Phi(z)$.

We find that we can easily solve the above Riemann-Hilbert type problem (2.10) directly, as explained below.

If $\Phi_0(z)$ represents a nontrivial solution of the homogeneous problem (2.10), satisfying

$$\Phi_0^+(x) + \Phi_0^-(x) = 0, \quad -1 < x < 1, \tag{2.11}$$

then we may rewrite the inhomogeneous problem (2.10) as

$$\Psi^+(x) - \Psi^-(x) = \frac{2f(x)}{\Phi_0^+(x)}, \tag{2.12}$$

with

$$\Psi(z) = \Phi(z)/\Phi_0(z). \tag{2.13}$$

Then, the second of the relations (2.9) suggests that we can determine the function $\Psi(z)$ in the following form :

$$\Psi(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{g(t)dt}{(t-z)^2} + E_0(z) \tag{2.14}$$

where

$$\frac{dg}{dx} = g'(x) = \frac{2f(x)}{\Phi_0^+(x)}, \tag{2.15}$$

with $E_0(z)$ representing an entire function of z.

Next, by utilizing the form (2.14) of the function $\Psi(z)$, along with the relation (2.13) and the second of the Plemelj-type formulae (2.9), we obtain the following result:

$$\phi'(x) = -\frac{1}{2\pi^2}\Phi_0^+(x)\left[\int_{-1}^{1} \frac{g(t)dt}{(t-x)^2} + 2\pi i E_0(x)\right], \quad -1 < x < 1. \tag{2.16}$$

We find, at this stage, that we have been able to determine the first derivative of the unknown function $\phi(x)$, in terms of one unknown entire function $E_0(z)$, whose determination can be completed as explained below.

If we select

$$\Phi_0(z) = \left(\frac{z+1}{z-1}\right)^{1/2} \tag{2.17}$$

giving

$$\Phi_0^+(x) = -i\left(\frac{1+x}{1-x}\right)^{1/2} \quad \text{for} \quad -1 < x < 1, \tag{2.18}$$

we find that, because of the relations (2.13) and (2.14), we must select $E_0(z)$ to be equal to zero.

Then, using the relation (2.18), along with the relation (2.15), we obtain from the relation (2.16), the following result:

$$\phi'(x) = -\frac{1}{\pi^2}\left(\frac{1+x}{1-x}\right)^{1/2} \int_{-1}^{1} \frac{g_0(t)dt}{(t-x)^2} \tag{2.19}$$

with

$$g_0'(x) = f(x)\left(\frac{1-x}{1+x}\right)^{1/2}. \tag{2.20}$$

Finally, by integrating the relation (2.19), we can determine the solution of the given hypersingular integral equation (1.1), in the following form

$$\phi(x) = p(x) + D_0, \tag{2.21}$$

where

$$p'(x) = -\frac{1}{\pi^2}\left(\frac{1+x}{1-x}\right)^{1/2} \int_{-1}^{1} \frac{g_0(t)dt}{(t-x)^2}, \tag{2.22}$$

with D_0 being an arbitrary constant.

We note that the expression for $g_0(x)$ will involve an arbitrary constant of integration, arising out of the relation (2.20), and thus the form of the solution (2.21) will involve two arbitrary constants altogether.

This completes the method of solution of the hypersingular integral equation (1.1), in principle, once the hypersingular integral occurring in the relation (2.19) is evaluated, for a given forcing function $f(x)$.

We can derive the known form (1.3) of the solution of the equation (1.1), as obtained by Martin [**4**], by using a procedure as described below.

By integrating by parts, we obtain from the relation (2.22), that

$$\begin{aligned} p'(x) = &-\frac{1}{\pi^2}\left(\frac{1+x}{1-x}\right)^{1/2}\int_{-1}^{1}\frac{g_0'(t)}{(t-x)}\,dt \\ &+\frac{1}{\pi^2}\left[\frac{g_0(-1)-g_0(1)}{(1-x^2)^{1/2}}+\frac{2g_0(1)}{(1-x)(1-x^2)^{1/2}}\right]. \end{aligned}$$

Another integration gives, because of the relation (2.20):

$$\begin{aligned} p(x) = &\frac{1}{\pi^2}\int_{-1}^{1} f(t)K(x,t)\,dt + 4g_0(1)\left[\frac{1+(1-x^2)^{1/2}}{(1-x)+(1-x^2)^{1/2}}\right] \\ &+\frac{1}{\pi^2}\left[g_0(-1)-g_0(1)+\int_{-1}^{1}\left(\frac{1-t}{1+t}\right)^{1/2} f(t)\,dt\right]\sin^{-1}(x), \end{aligned} \tag{2.23}$$

ignoring an arbitrary constant, see (2.21), when the following results are used:

$$\left(\frac{1-t}{1+t}\right)^{1/2}\left(\frac{1}{t-x}\right) = \frac{(1-t^2)^{1/2}}{1+x}\left(\frac{1}{t-x}-\frac{1}{1+t}\right) \tag{2.24}$$

and

$$\frac{\partial K}{\partial x} = \left(\frac{1-t^2}{1-x^2}\right)^{1/2}\left(\frac{1}{x-t}\right). \tag{2.25}$$

We find that in the special circumstance, when $\phi(-1) = 0 = \phi(1)$, the solution of equation (1.1), as given by the formulae (2.19) and (2.21) is obtained in the form

$$\phi(x) = \frac{1}{\pi^2}\int_{-1}^{1} f(t)K(x,t)\,dt, \tag{2.26}$$

since we must have

$$\begin{aligned} g_0(1) &= 0 = D_0, \\ g_0(-1) &= -\int_{-1}^{1}\left(\frac{1-t}{1+t}\right)^{1/2} f(t)\,dt. \end{aligned} \tag{2.27}$$

The result (2.26) agrees with the form (1.3), involving a weakly singular integral.

The analysis presented above is believed to be self-contained and straightforward. We have thus avoided referring to the research papers dealing with more general hypersingular integral equations existing in the literature.

Acknowledgments. I thank the referees for many valuable comments and criticisms which have helped in revising the paper.

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