‘Optimal’ FEM approximation for non-linear parabolic problems with not regular initial value data

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On the occasion of the20th anniversary year of death of
J. A. Nitsche

Abstract

We provide an ‘optimal’ finite element approximation error estimates for a one-dimensional non-linear parabolic model problem with non-regular initial value data. The solution concept is based on corresponding results from J. A. Nitsche for finite element approximation error estimates for the one-dimension Stefan problem ([NiJ], [NiJ1], [NiJ2]) leveraging on the NSE solution concept in [BrK1]. As the approach is not depending from the space dimension it can also be applied to the 3-D non-stationary, non-linear Navier-Stokes equations to improve the ‘not-optimal’ finite element approximation error estimates in [HeJ].

For a related elegant role of the $H_{1/2}$ (energy) Hilbert space in universal Teichmüller theory we refer to [NaS].

§ 1 Introduction, the non-linear parabolic Stefan problem

The free boundary Stefan problem with its solution $U(y,\tau)$ can be transformed into the non-linear parabolic equation ([NiJ1]) looking for a solution $u(x,t)=U(y,\tau)$ fulfilling

$$\dot{u}(y,\tau)-u''(x,t)+xu'(1,t)u'=0 \qquad \text{in } Q=\{(x,t)|x\in(0,1),0<t\le T\}$$

$$u'(0,t)=u(1,t)=0 \qquad \text{for } t>0$$
$$u(x,0)=f(x) \quad \text{for } x\in(0,1) \ .$$

Putting

$$\dot{H}_1 := \{(w|w\in H_1(0,1),>0, w(0)=0\}=\{(w|w'\in L_2(0,1),>0, w(0)=0\}$$

the approach in [NiJ], ([NiJ1], ([NiJ2] is defining an auxiliary function $v:=u'\in\dot{H}_1$ with

$$u(x,t)=-\int_x^1 v(z,t)dz\in\dot{H}_1$$

in order to satisfies the boundary condition above. Multiplying the differential equation above with w' ($w\in\dot{H}_1$) and integration gives the variation equation

$$\int_0^1 u''w'+\dot{u}'wdx=u'(1,t)\int_0^1 xu'w'dx \ .$$

This leads to a variation representation in the form

Problem P_v: find v such that $v(\cdot,t)\in\dot{H}_1$ and

$$(\dot{v},w)+(v',w')=v(1)(xv,w') \qquad \text{for } w\in\dot{H}_1 \text{ and } t>0$$
$$(v(\cdot,0),w)=(f',w) \qquad \text{for } w\in\dot{H}_1 \text{ and } t=0.$$

This is the weak formulation of

$$\dot{v}(y,\tau)-v''(x,t)+v(1,t)(xv)'=0 \text{ in } Q$$
$$v(0,t)=0$$
$$v'(1,t)=v^2(1,t)$$
$$v=f' \quad \text{for } t=0 \ .$$

Obviously the compatibility condition $v'(1)=v^2(1)$ is required in order to ensure the (too high) regularity requirements to the auxiliary function v as indicated by the setting $v:=u'\in\dot{H}_1$ for $u\in\dot{H}_1$.

For higher regularity assumptions further compatibility assumptions are required in the form $f''(1)=f'^2(1)$. In [NiJ1] for sufficiently high regularity assumptions for the solution v ‘optimal’ FEM error estimates are derived. The prize to be paid for the imbalanced regularity relationship between the solutions u and v are at least quadratic splines instead of expected linear splines for the FE approximation space.

§ 2 The 'not-regular' Stefan problem case, 'not-optimal' FEM error estimates

In case of reduced regularity assumption of the initial value function $g:=f'\in L_2$ and not fulfilled compatibility conditions the problem is called "non-regular". If only $g\in L_2$ is assumed then even $\|v'\|$ and hence $|v|$ is not necessarily bounded for $t\to 0$ ([NiJ2]). In [NiJ] for this case a 'not-optimal' FE error estimate of order h^α with $0<\alpha<1$ has been derived. The proof of those estimates needs to govern a singularity for $t\to 0$ of the energy norm $\|v'\|$ and hence for the term $v(1)$. It applies the following a priori estimates which are derived in ([NiJ2])

$$\|z\|^2+\int_0^t\|z'\|^2d\tau\le 2\|g\|^2 \quad , \quad t\|z'\|^2+\int_0^t\tau\|\dot z\|^2d\tau\le c=c(\|g\|)$$

$$\sup_{0\le t\le T}\left\{t^{2k}\|\partial_t^k v_h\|^2+\int_0^t\tau^{2k}\|\partial_t^k v_h'\|^2d\tau\right\}\le c_{2k}^2 \quad , \quad \sup_{0\le t\le T}\left\{t^{2k+1}\|\partial_t^k v_h'\|^2+\int_0^t\tau^{2k+1}\|\partial_t^{k+1}v_h\|^2d\tau\right\}\le c_{2k+1}^2 .$$

The technique is built on the Young inequality and the Gronwall lemma being applied to inequalities of the following types

$$\lambda:=\lambda(t)\le\|g\|^2+c\int_0^t\lambda^3(\tau)d\tau \quad , \quad \lambda:=\lambda(t)\le k_1+k_2\int_0^t\tau^{-1/}\lambda^{3/2}(\tau)d\tau \quad \text{with} \quad \lambda(t):=\|z\|^2 .$$

From those estimates the specific a priori estimates for the supremum norm of v and v_h can be derived which are

$$|v|,|v_h|\le ct^{-1/4} .$$

The key inequality from which the FE errow estimate is derived is given by

$$\frac{d}{dt}(t\|\Phi\|^2)+t\|\Phi'\|^2+\frac{1}{2}\frac{d}{dt}\|\varphi'\|^2\le|a(\varepsilon,\varphi)|+|v(1)||(x\Phi,\varphi')|+|\Phi'(1)||(xv_h,\varphi')|+ct\left\{\|\varepsilon'\|^2+t^{-1/2}\|\varepsilon\|^2\right\} .$$

It is built on a duality argument according to $-w''=\Phi,\ w(0)=w'(0)=0$ and its related Ritz approximation $\varphi:=R_h w\in S_h$.

Corresponding a priori estimates and similar proof techniques are applied to prove similar "not-optimal' FEM error estimates for the non-stationary, non-linear NSE ([HeJ]).

We claim that the inadequate (too high) regularity requirements for the auxiliary function $v\in H_1$ in relation to $u\in H_1$ is the root cause of the singularity behavior in the size of $\approx t^{-1/4}$. Building on the solution concept in [BrK1] we propose the definition of an alternative auxiliary function in the form

$$v:=Au'=-Hu \quad \text{resp.} \quad u=Hv$$

leading to a variation representation in a $H_{-1/2}$ – framework. The operators A and H denote the one-dimensional Symm resp. Hilbert transform singular integral operator (next §, [BrK], [NiJ6]). The properties of the Hilbert transform operator H (next §) ensure same regularity assumption for $u,v\in H_{1/2}$. From

$$|z|^2\le c(z.z')_{L_2}\le c\|z\|_{1/2}\cdot\|z'\|_{-1/2}\le c\|z\|_{1/2}^2$$

it follows that $v(1)$ is bounded if $v\in H_{1/2}$.

This means that in the $H_{-1/2}$ – framework the singularity behavior above no longer exists. Even in case of a modified model problem with singular coefficients in the size of $\approx t^{-1/4}$ the $H_{-1/2}$ – framework enables optimal FE error estimates (appendix).

As a consequence the technique from the regular case ([NiJ1]) can also be applied to the 'non-regular' case [NiJ1]. At the same time the required finite element approximation spaces can be linear splines instead of quadratic splines ([NiJ1], [NiJ7]).

§ 3 A non-linear parabolic model problem, 'optimal' FEM/BEM error estimates

In order to enable the full power of approximation theory in a Hilbert scale framework we consider the $2\pi-$periodic continuation of the solution of the Stefan problem. It enables the definition of problem adequate Hilbert spaces defined per appropriately chosen self-adjoint linear operators with corresponding domains.

Let $H=L_2^*(\Gamma)$ with $\Gamma:=S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Let $u(s)$ being a $2\pi-$periodic function and $\oint$ denotes the integral from 0 to 2π in the Cauchy-sense. Then for $u\in H:=L_2^*(\Gamma)$ with $\Gamma:=S^1(R^2)$ and for real β the Fourier coefficients

$$\hat{u}(\nu):=u_\nu:=\frac{1}{\sqrt{2\pi}}\oint u(x)e^{i\nu x}dx=:\oint u(x)\psi_n(x)dx$$

enable the definitions of the norms (see e.g. [BrK], [KrR], [LiI] Remark 11.1.5)

$$\|u\|_\beta^2:=\sum_{-\infty}^{\infty}|\nu|^{2\beta}|u_\nu|^2$$

defining corresponding Hilbert spaces $H_\beta^\#$. Then H_0 is the space of L_2-periodic functions in R. The definition of negative scaled Hilbert scales is enabled by appropriate self-adjoint singular integral operators ([KrR], [LiI]). We build our analysis on the Symm operator A ([BrK]) and the Hilbert transform (conformal mapping) operator H defined by

$$Au(x):=-\frac{1}{\pi}\oint\log 2\left|\sin\frac{x-y}{2}\right|u(y)dy$$

$$Hu(x):=\frac{1}{2\pi}\oint\cot(\frac{x-y}{2})u(y)dy\ .$$

Both operators are related to each other by $Au'(x)=-Hu(x)$. Some essential properties of the Hilbert transform operator are summaries in

Lemma: The Hilbert transform (conformal mapping) operator H fulfills the following properties:

i) $(Hv,w)=-(v,Hw)$, $\|Hv\|=\|v\|$,
ii) if $v\in H_0$ then $Hv\in H_0$ and $(Hv,v)=0$
iii) $[xH-Hx]v=\hat{v}(0)$, i.e. for odd functions v it holds $[xH-Hx]v=0$.

Galerkin approximation analysis in the different Sobolev, $L_\infty-$ and Hölder/Lipschitz spaces frameworks are given in the appendix. For instance the Galerkin approximation error in case of finite element approximation spaces $S_h^{k.t}$ of the Symm operator equation is 'optimal' with respect to the complete possible Hilbert scale range, i.e. it holds ([BrK])

$$\|e\|_\kappa\le ch^{\tau-\kappa}\|u\|_\tau\quad,\ -(t+1)\le\kappa\le-1/2\quad,\ \tau\le t\ .$$

The corresponding operators of the one-dimensional Hilbert transform operator H are the Riesz operators R_i ($i=1,...n$) which also play a key role in NSE theory ([CaM]) e.g. in order to define the pressure p as function of the velocity u. There is another use of the Hilbert transform in the context of a 1D Constantin-Lax-Majda model for the 3D vorticity equation building on the relationships $u=B\omega$ resp. $\omega=\nabla\times u$ for the velocity u and the vorticity ω whereby B denotes the Biot-Savart integral operator ([MaA], 5.2).

In order to avoid technical inconvenience we omit the term x from the Stefan problem. This is without loss of generality as this term is always estimated by a constant. With respect to non-parabolic problems for space dimensions $n>1$ this term potentially can even support to gain additional regularity (and therefore an additional order of convergence for FE approximation error estimates) enabled by corresponding (regularization) commutator properties (appendix, e.g. [BrK]) in combination with finite element super-approximation properties ([NiJ4], [NiJ5], [NiJ6]).

We consider the non-linear parabolic equation

$$\dot{u}-u''=-u'(1)u' .$$

Putting $v:=Au'=-Hu$ resp. $u=Hv$ leads to

$$A\dot{u}-Au''=-Au'(1)Au' \quad \text{resp.} \quad AH\dot{v}-v'=-v(1)v .$$

Multiplying with $Hw\in L_2^{\#}$ leads to the variation problem
P_v : find $v\in\dot{H}_{1/2}$ with

$$(\dot{v},w)_{-1/2}+(v',w')_{-1/2}=v(1)(Hv,w)_0 \quad \text{for } w\in\dot{H}_{1/2} \text{ and } t>0$$

$$(v(\cdot,0),w)_{-1/2}=(f',w)_{-1/2}=(f,w)_0 \quad \text{for } w\in\dot{H}_{1/2} \text{ and } t=0 .$$

The corresponding Galerkin approximation is given by
P_{v_h} : find $v_h\in S_h\subset\dot{H}_{1/2}$ ($v_h(1):=v_h(1,t)$) with

$$(\dot{v}_h,\chi)_{-1/2}+(v_h,\chi)_{1/2}=v_h(1)(Hv_h,\chi)_0 \quad \text{for } \chi\in S_h\subset\dot{H}_{1/2} \text{ and } t>0$$

$$(v_h(\cdot,0),\chi)_{-1/2}=(f,\chi)_0 \quad \text{for } \chi\in S_h\subset\dot{H}_{1/2} \text{ and } t=0 .$$

For later use we define the related trilinear form ([NiJ1])

$$b(\xi,\eta,\zeta):=\frac{1}{2}\xi(1)(\eta,\zeta')_{-1/2}+\frac{1}{2}\eta(1)(\xi,\zeta')_{-1/2}=\frac{1}{2}\xi(1)(\eta,\zeta)_0+\frac{1}{2}\eta(1)(\xi,\zeta)_0 .$$

With analog arguments than in [NiJ1] the corresponding bilinear form defined by

$$a_v(\xi,\eta):=(\xi,\eta)_{1/2}-2b(v,\xi,\eta) \quad \text{for fixed } v\in\dot{H}_{1/2}$$

is bounded and coercive in $\dot{H}_{1/2}$, i.e. it holds

i) $\quad |a_v(\xi,\eta)|\le M\cdot\|\xi\|_{1/2}\cdot\|\eta\|_{1/2}$

ii) $\quad a_v(\xi,\xi)\ge m\|\xi\|_{1/2}^2-\Lambda\|\xi\|_{-1/2}^2 .$

The above coerciveness property ii) is a Garding type inequality which enables the so-called "weak inf-sup condition" (appendix). This results into a positive bilinear form in case the domain is restricted to the approximation spaces $S_h \subset \dot{H}_{1/2}$. Therefore the Galerkin approximation $R_h v$

$$a_v(v - R_h v, \chi) = 0 \quad \text{for} \quad \chi \in S_h \subset \dot{H}_{1/2}$$

is 'optimal', i.e. it holds

$$\|v - R_h v\|_\kappa \le c h^{\tau-\kappa} \|v\|_\tau \quad , \; -(t+1) \le \kappa \le -1/2 \quad , \; \tau \le t \; .$$

For our analysis we will apply the analog Galerkin approximation $\tilde{v}_h := R_h v$ for the linear parabolic case, i.e.

$$(\dot{v} - \dot{\tilde{v}}_h, \chi)_{-1/2} + a_v(v - \tilde{v}_h, \chi) = 0 \quad \text{for} \quad \chi \in S_h \subset \dot{H}_{1/2}$$

which is again ‚optimal' with respect to Sobolev- and L_∞ –norms ([JNi5]), i.e. for $\varepsilon := v - \tilde{v}_h$ it holds e.g.

$$\|\varepsilon\|_{L_2(H_\kappa)} \le c h^{\tau-\kappa} \|f'\|_{L_2(H_\tau)} \qquad , \; -(t+1) \le \kappa \le -1/2 \quad , \; \tau \le t \; .$$

$$\|\varepsilon\|_{L_\infty((0,t),L_\infty(0,2\pi))} \le c \cdot \inf_{\chi \in S_h} \|v - \chi\|_{L_\infty((0,t),L_\infty(0,2\pi))} \quad .$$

Applying same techniques as in [JNi5] the analog error estimates can be derived for the non-stationary Stokes problem. This is "*when Navier meets* (*generalized*) *Fourier* (*based Hilbert scale (approximation) theory*)" ([*CaM*], [*BrK1*]). The (energy) Hilbert space $H_{1/2}$ plays also a key role answering the question "*can one hear the degree of continuous maps (i.e. the winding number)*?" ([BrH]).

The parabolic Galerkin approximation behaviors will be used to apply the Schauder fixed point theorem in order to govern the critical quadratic non-linear term $b(e,e,\chi) = e(1)(e,\chi)$ below.

The defining error ($e := v - v_h$) variation equation for our model problem is given by

$$(\dot{e}, \chi)_{-1/2} + (e, \chi)_{1/2} - 2b(v, e, \chi) = -b(e, e, \chi) \quad \text{for} \quad \chi \in S_h \subset \dot{H}_{1/2}$$

resp.

$$(\dot{e}, \chi)_{-1/2} + a_v(e, \chi) = -b(e, e, \chi) \quad \text{for} \quad \chi \in S_h \subset \dot{H}_{1/2}$$

whereby $b(e,e,\chi) = e(1)(e,\chi)_0$.

Putting $e := (v - \tilde{v}_h) - (\tilde{v}_h - v_h) =: \varepsilon - \Phi$ with now $\Phi \in S_h \subset \dot{H}_{1/2}$ leads to

$$(\dot{\Phi}, \chi)_{-1/2} + a_v(\Phi, \chi) = e(1)(e, \chi)_0 \, .$$

In order to show the existence of a finite element solution in the neighborhood of v the quadratic term $e(1) = e(1,t)$ is replaced by $E(1)$ for some function $E = T(e)$. Then

$$(\dot{\Phi}, \chi)_{-1/2} + a_v(\Phi, \chi) + E(1)(\Phi, \chi)_0 = E(1)(\varepsilon, \chi)_0$$

is a linear problem. Therefore, for any $E(1)=E(1,t)$ there exists a solution Φ with $\Phi(0)=0$. Therefore the same is true for $e=\varepsilon-\Phi$, but now e depends on E .

Following the same arguments as in ([NiJ1]) by applying the Schauder fixed point theorem (appendix) then it follows that there is a E with $E=T(e)=e$.

From [NiJ7] we recall that for the regularity of piecewise defined function with respect to scales of Sobolev spaces it holds

$$S_\Gamma \subset H_\delta \text{ with } \delta < 1/2 .$$

Therefore the problem P_{v_h} requires (just not piecewise, but) linear splines only which is different from the requirements in ([NiJ1].

We summaries the above in

Theorem: The problem P_v has an unique bounded solution $v \in H_{1/2}$ and an unique (linear spline) finite element approximation v_h of problem P_{v_h} in the neighborhood of v with optimal order of convergence

$$\|v-v_h\|_{L_\infty(0,t);L_\infty)} \le O(h) .$$

Due to the properties of the Hilbert transform operator the same estimate is true for the original solution $u \in H_{1/2}$, i.e. it holds

$$\|u-u_h\|_{L_\infty(0,t);L_\infty)} \le O(h) .$$

An approximation $u_h \in S_h \subset H_{1/2}$ for the solution $u \in H_{1/2}$ is given by $u_h = Hv_h$ i.e. the spline order of u_h is the same as the spline order of v_h .

Let $v_h = Au_h$ the corresponding approximation of the Ritz approximation u_h of $Au=f$. Then the error $v-v_h$ can be represented in the form $(v-v_h)=(\ln|\xi-\circ|,e)$ whereby

$$v_h(\xi)=\oint \ln|\xi-\eta| u_h(y)dy$$

and $\xi=\xi_1+i\xi_2$, $\eta=\eta_1+i\eta_2=e^{iy}$. For $\xi=\xi_1+i\xi_2$ with $|\xi|<1$ fixed the function $g(y):=\ln|\xi-e^{iy}|$ is analytical with respect to y . From the Schwarz inequality it therefore follows

$$|v(\xi)-v_h(\xi)| \le \|g\|_\beta \|e\|_{-\beta} \le ch^{2t+1}\|u\|_t \|g\|_{t+1} \text{ in case } S_h = S_h^{k.t} .$$

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Appendix

Related topics

The Ritz-Galerkin Approximation Theory

A well-defined Ritz-Galerkin approximation method in a Hilbert space H_α and corresponding approximation $u_h := Ru$ of the linear operator equation $Bu = f$ requires certain properties to the linear operator, as well as adequate properties of the operator domain $D(A)$ embedded into a Hilbert space H_α and appropriate related properties of the finite dimensional approximation spaces $S_h \subset H_\alpha$. In the one-dimensional case the parameter h corresponds to space dimension in the form $h \approx n^{-1}$.

In order to apply the generalized Lax-Milgram lemma the $H_\beta -$coerciveness can be weakened to the Garding type inequality which is given by one of the following forms

$$(Bu,v) \geq c_1\|u\|_\alpha^2 - c_2\|v\|_\beta^2 \quad \text{or} \quad (Bu,v) \geq c\|u\|_\alpha \cdot \|v\|_\alpha - (Ku,v)$$

whereby H_α is compactly embedded in H_β resp. K describes a corresponding compact operator. This enables the so-called "weak inf-sup condition" ([AzA], [BrK], [NiJ7], [NiJ8]) which we summaries in the following

Lemma: Given three Hilbert spaces H, H_1, H_2 with $H_1 \subset H$ compactly embedded and the bilinear form $b(u,v) := (Bu,v) : H_1 \times H_2 \to R$ with

i) $|b(u,v)| \leq c \cdot \|u\|_{H_1} \|v\|_{H_2}$ for all $u \in H_1, v \in H_2$
ii) For $u \in H_1$ with $b(u,v)$ for all $v \in H_2$ it follows $u = 0$
iii) For all $n \in N$ the approximation spaces fulfill $S_n \subset H_1, T_n \subset H_2$ and $\dim S_n = \dim T_n$
iv) $T_n \subset T_{n+1}$ and $\bigcup_{n\in N} T_n$ is dense in H_2
v) $\forall \varphi \in S_n \exists \psi \in T_n : |b(\varphi,\psi)| \geq (c_1\|\varphi\|_{H_1} - c_2\|\psi\|_H) - \|\psi\|_{H_2}$ (the weak inf-sup condition).

Then there is a $N > 0$ and a constant $c_3 > 0$ that for all $n \geq N$ it holds

$$\forall \varphi \in S_n \exists \psi \in T_n \text{ with } |b(\varphi,\psi)| \geq c_3\|\varphi\|_{H_1} \cdot \|\psi\|_{H_2}.$$

The required properties the approximation spaces are e.g. given by finite elements. The standard notation of finite element spaces is given by $S_h^{k.t} \subset H_k$ with $k < t$ which is about $(k-1)-$time continuously differentiable functions $\chi \in S_h^{k.t}$ with the property that the restriction of χ to any triangle Δ of the triangulation $\Delta \in \Gamma_h$ is a polynomial of degree less than t.

The corresponding approximation properties are usual described in the following way:

i) $S_h^{k.t} \subset H_k$

ii) $\inf_{\chi \in S_h} \|v-\chi\|_k \le ch^{l-k} \|v\|_l$ for $v \in H_l$

iii) $\|\chi\|_k \le ch^{-(k-l)} \|v\|_l$ for $\chi \in S_h$.

The Hölder resp. Lipschitz spaces are the adequate ones to derive approximation estimates for non-linear problems. A proof of the boundedness of the Ritz operator in Hölder resp. Lipschitz spaces is a consequence of the below AI-condition and lemma (by the choice $X_1 = C^k$ and $X_2 = C^{k.\lambda}$) in combination with the boundedness of the Ritz operator in in $L_\infty -$ norm ([JNi4]).

The boundedness of the Ritz operator as mapping of C^0 into itself is a consequence of the boundedness with respect to the $L_\infty -$ norm. As the Hölder spaces $C^{k.\lambda}$ are compactly embedded into C^0 it follows the boundedness of the Ritz operator in Hölder spaces.

The so-called approximation (A) and inverse (I)-condition for the collection of approximation spaces $\{S_h | 0 < h \le 1\}$ are given by

AI-condition: let $\{S_h | 0 < h \le 1\}$ a collection of subspaces of X_2 with **a**pproximation and **i**nverse-quantities σ_h and τ_h according to

i) $\forall y \in X_2 \exists \eta \in S_h \;\; \|y-\eta\|_1 \le \sigma_h \|y\|_2$, $\|\eta\|_2 \le c_1 \|y\|_2$ with c_1 independently of h .

ii) $\forall \eta \in S_h$ a Bernstein type inequality holds $\|\chi\|_2 \le \tau_h \|\chi\|_1$.

The collection $\{S_h | 0 < h \le 1\}$ fulfills the AI condition if $K := \sup \sigma_h \cdot \tau_h < \infty$.

Lemma: let $\{P_h | X_1 \to S_h\}$ be a collection of linear projection operators of X_1 onto S_h which is uniformly bounded as mappings of X_1 into itself, i.e.

$$\|P_h\|_1 = \sup \frac{\|P_h y\|_1}{\|y\|_1} \le p_1 \text{ with } p_1 \text{ independently of } h.$$

If $\{S_h\}$ fulfills the AI-condition then $\{P_h\}$ as mapping of X_2 into itself is uniformly bounded with

$$\|P_h\|_2 = \sup \frac{\|P_h y\|_2}{\|y\|_2} \le p_2 := (c_1 + 3K) p_1 .$$

The boundedness of the Ritz operator in in $L_\infty -$ norm can be proven by the weighted norm technique of J. A. Nitsche ([JNi4], next section). In [BrK] interior ‘optimal’ error estimates of the Ritz method for Pseudo-differential operators are derived.

Approximation Theory in Hilbert scales

Lecture Notes

J. A. Nitsche

Let H_α denotes a Hilbert scale and let for simplicity reasons α be restricted to $\alpha \in [0,1]$. Further, let S denote an appropriate approximation space, also for simplicity reasons with the regularity assumption $S \subset H_1$. Given a x not in S one may ask for

$$d(x,S) = dist(x,S) = \inf_{\xi \in S} \|x - \xi\|_\alpha$$

for any $\alpha -$ norm. In applications the smallest constant is sought such that for all $x \in H_\beta$ it holds

$$\inf_{\xi \in S} \|x - \xi\|_\alpha \le \kappa_{\alpha\beta} \|x\|_\beta .$$

This constant is given by

$$\kappa_{\alpha\beta} := \sup \left\{ \inf_{\xi \in S} \|x - \xi\|_\alpha \, \middle| \, x \in H_\beta \wedge \|x\|_\beta = 1 \right\} .$$

The simultaneous approximation property is given by

Theorem 1: Let $\gamma \in (0,1)$ be fixed and $\kappa := \kappa_{\gamma 1}^{1/(1-\gamma)}$. Then

$$\inf_{\xi \in S} \left\{ \|x - \xi\|_0 + \kappa^\gamma \|x - \xi\|_\gamma \right\} \le c \cdot \kappa \|x\|_1$$

with $c := 2 \cdot \left\{ 1 + 2^{\gamma/(1-\gamma)} \right\}$ and there is a $\xi \in S$ such that simultaneously

$$\|x - \xi\|_\beta \le c \cdot \kappa^{1-\beta} \|x\|_1$$

holds true for $0 \le \beta \le \gamma$.

A generalization is given by

Theorem 2: Let κ be defined according to

$$\inf_{\xi \in S} \|x - \xi\|_0 \le \kappa \|x\|_1 .$$

To $x \in H_0$ there is a $\xi \in S$ such that

$$\|x - \xi\|_b \le c(b) \cdot \kappa^{-b} \|x\|_0$$

holds with $c(b)$ depending only on b for $b \le 0$.

The central concept to prove this theorem is built on an additional inner product resp. norm with a not polynomial decay as the given by the $\alpha -$ norms. For $t > 0$ this is about

$$(x,y)_{(t)} := \sum e^{-\sqrt{\lambda_i}t}(x,\varphi_i)(y,\varphi_i)$$

resp.

$$\|x\|_{(t)}^2 := (x,x)_{(t)} \ .$$

The proof of the theorem is a consequence of the following lemma. Obviously it holds $\|x\|_{(t)} \le c(\alpha,t)\cdot\|x\|_{\alpha}$. On the other side any negative norm, i.e. $\|x\|_{\alpha}$ with $\alpha < 0$, is bounded by the $0-$norm and the new $(t)-$norm, i.e. it holds

Lemma: i) Let $\alpha > 0$ be fixed. The $(-\alpha)-$norm of any $x \in H_0$ is bounded by

$$\|x\|_{-\alpha}^2 \le \delta^{2\alpha}\|x\|_0^2 + e^{t/\delta}\|x\|_{(t)}^2$$

with $\delta > 0$ being arbitrary. Let $t,\delta > 0$ be fixed. To any $x \in H_0$ there is an $y \in H_1$ according to

$$\|x-y\| \le \|x\| \ , \ \|y\|_1 \le \delta^{-1}\|x\| \ , \ \|x-y\|_{(t)} \le e^{-t/\delta}\|x\| \ .$$

and therefore

$$E_t(x) := \inf_{\xi\in S}\left\{e^{-t/2\kappa}\|x-\xi\|_0 + \|x-\xi\|_{(t)}\right\} \le 4e^{-t/2\kappa}\|x\| \ .$$

$L_\infty -$ Boundedness of the Ritz Operator for Singular Integral Equation Problems

The (Hilbert transform) singular integral equation problem is given by

$$Hu = f$$

with

$$Hu(s) = \frac{1}{2\pi}\oint \cot\frac{s-t}{2}u(st)dt = f(s)\cdot$$

In [JNi6] the $L_\infty -$ boundedness of the Ritz-Galerkin approximation operator R_h onto finite element subspaces S_h is defined by

$$(\chi, Hu) = (\chi, H(R_h u)) =: (\chi, H\varphi) \text{ for } \chi \in S_h .$$

As a consequence the boundedness holds true also in the norm of $C^{0,\lambda}$.

Analogue to the analysis in case of boundary value problems the proof deals with weighted norms in the form

$$\mu := \rho^2 + \sin^2(s - s_0) \text{ , with } s_0 \in [0,2\pi) \text{ appropriately chosen.}$$

For $\alpha \in R$ the weighted scalar products resp. norms are defined by

$$((v,w))_\alpha := \oint \mu^{-\alpha} v \cdot w ds \text{ , } |||v|||_\alpha^2 := ((v,v))_\alpha .$$

The estimates require for space dimension $n=1$ a value e.g. $\alpha = 1 > n/2$, for the space dimension $n=2$ a value e.g. $\alpha = 2 > n/2$. The connection of weighted norms and the $L_\infty -$ norm is given by the inequality (for $\alpha = 1$)

$$|||v|||_\alpha \le ch^{-1/2} |||v|||_\infty \text{ for } v \in C^0$$

and

$$\|\chi\|_{L_\infty} \le ch^{1/2} \sup\{|||\chi|||_\alpha | s_0 \in [0,2\pi)\} \text{ for } \chi \in \dot{S}_h .$$

An analogue proof shows the $L_\infty -$ boundedness of the Ritz-Galerkin approximation operator R_h of the singular integral equation problem $Au = f$ with

$$Au(x) = -\frac{1}{\pi}\oint \log 2\left|\sin\frac{x-y}{2}\right| u(y)dy \quad .$$

The BEM is given by

$$(\chi, Au) = (\chi, A(R_h u)) \text{ for } \chi \in S_h .$$

For the error $e := u - R_h u$ we use the split $e := u - R_h u = (u - \eta) - (R_h u - \eta) =: \varepsilon - \Phi$, $\eta, \Phi \in S_h$. Then the error is defined by

$$(\chi, A\varepsilon) = (\chi, A\Phi) \text{ for } \chi \in S_h .$$

The objective is to derive an estimate in the form $\|\Phi\|_\alpha^2 \le c\|\varepsilon\|_\alpha^2$ which then gives

$$\|e\|_\alpha^2 \le c\|\varepsilon\|_\alpha^2$$

from which the boundedness follows (by appropriately chosen $\eta \in S_h$) according to the relationship of weighted and $L\infty -$norm of finite elements.

Analog to the approach as in [JNi6] the baseline to start from is the identity

$$\|\Phi\|_\alpha^2 = \left(\Phi, \mu^{-\alpha}\Phi\right) = \left(\Phi, \mu^{-\alpha}\Phi - A\lambda\right) + \left(\Phi, \varepsilon\right)_\alpha - \left(\varepsilon, \mu^{-\alpha}\Phi - A\lambda\right)$$

whereby

$$\chi := R_h\left(\mu^{-\alpha}\lambda\right): \quad \left(\mu^{-\alpha}\lambda, \psi\right) = (A, \psi) \text{ for } \psi \in \dot{S}_h .$$

It follows from the above (for $n = 1,2$)

$$\begin{aligned} &\le c_\delta\|\varepsilon\|_\alpha^2 + \delta\left\|\mu^{-\alpha}\Phi - A\lambda\right\|_{-\alpha}^2 \\ &\le c_\delta\|\varepsilon\|_\alpha^2 + \delta\cdot c\cdot \inf_{\psi\in S_h}\left\|\mu^\alpha A\lambda - \psi\right\|_\alpha^2 \\ &\le c_\delta\|\varepsilon\|_\alpha^2 + \delta\cdot c\cdot h^{2n}\left\|\nabla^n(\mu^\alpha A\lambda)\right\|_\alpha^2 . \end{aligned}$$

In the following we consider only the case $n = 1 = \alpha$. Then the second term gives

$$\nabla(\mu A\lambda) \le \rho^2|\nabla(A\lambda)| + \left|\sin^2\frac{s-s_0}{2}\right||A\lambda| + \mu|\nabla(A\lambda)|$$

and therefore (because of $\sin^2((s-s_0)/2)/\mu \le 1$)

$$\begin{aligned} \left\|\nabla(\mu^\alpha A\lambda)\right\|_\alpha^2 &\le \rho^4\|\nabla(A\lambda)\|_\alpha^2 + \|(A\lambda)\|^2 + \|\nabla(A\lambda)\|_{-\alpha}^2 \\ \left\|\nabla(\mu^\alpha A\lambda)\right\|_\alpha^2 &\le \rho^2\|\lambda\|^2 + \|(A\lambda)\|^2 + \left\|\sin\frac{s-s_0}{2}\nabla(A\lambda)\right\|^2 . \end{aligned}$$

The following estimates then enable the proof:

$$\left\|\sin\frac{s-s_0}{2}\nabla A\chi\right\|_0^2 \le \left\|\nabla\sin\frac{s-s_0}{2}A\chi\right\|_0^2 + c\|A\chi\|_0^2 \;, \quad \left\|\nabla\sin\frac{s-s_0}{2}A\chi\right\|_0^2 \le \|\nabla(sA-As)\chi\|_0^2 + \|\nabla A(s\chi)\|_0^2$$

$$\|s\chi\|_0^2 \le \|s\chi-\psi\|_0^2 + \|\psi\|_0^2 \le ch^2\|\chi\|_0^2 + \sup_{\xi\in S_h}\frac{(\psi, A\xi)}{\|A\xi\|_0^2} \le c\|A\chi\|_0^2 + \sup_{\xi\in S_h}\frac{(s\chi, A\xi) - (s\chi-\psi, A\xi)}{\|A\xi\|_0^2}$$

$$(s\chi, A\xi) = (\chi, (sA-As)\xi) + (\chi, A(s\xi))$$

$$(\chi, (sA-As)\xi) \le \|\chi\|_{-1}\|(sA-As)\xi\|_1 \le \|A\chi\|\cdot\|\xi\|_{-1} \le \|A\chi\|\cdot\|A\xi\|$$

$$(s\chi-\psi, A\xi) \le ch\|\chi\|\cdot\|A\xi\| \le c\|\chi\|_{-1}\cdot\|A\xi\| \le c\|A\chi\|\cdot\|A\xi\|$$

$L_\infty -$ **Boundedness of the Ritz-Galerkin Operator for Linear and Non-linear Elliptic and Parabolic Problems**

In [JNi5] $L_\infty -$ boundedness of the finite element method Ritz-Galerkin operator for parabolic problems and related optimal FEM approximation estimates are derived. In [NiJ3]) $L_\infty -$ error estimates are derived for a nonlinear boundary value problem. $L_\infty -$ boundedness in Hölder resp. Lipschitz norms of the Ritz-Galerkin operator for the Laplace equation are analyzed in [JNi4].

Non-linear functional analysis is well established in Banach and Hilbert space frameworks (e.g. [RuM], Banach & Schauder fixed point theorem, theory of (pseudo) monotone operators, variation inequalities, theorem of Brouwder). It is applied also for the stationary and non-stationary NSE when "*Navier meets Fourier*" [BrK1], [CaM]).

Hölder/Litschitz spaces are the adequate ones in treating nonlinear elliptic and parabolic problems. The boundedness of the Ritz operator in the corresponding norms at least simplifies the analysis of finite element procedures in some cases it is essential.

The two central aspects of [JNi5] are

i) an 'optimal' "parabolic type" shift theorem for the solution of the heat equation in the form

$$|||w|||^2_{H_{k+2}} \le c|||Aw|||^2_{H_k} \qquad \text{with} \qquad |||w|||^2_{H_k} := \int_0^T \|w\|^2_{H_k} dt$$

ii) Nitsche's proposed weighted norm concept which is about norms

$$\|\nabla v\|^2_{\alpha.\Omega'} := \sum_{|\xi|=k} \iint_{\Omega'} \mu^{-\alpha} \left|D^\xi v\right|^2 dx$$

with

$\mu(x,t) := |x-x_0|^2 + |t-t_0|$, and x_0, t_0 chosen that $\|u\|_{L_\infty(L_\infty)} = u(x_0,t_0)$.

The proof of the shift theorem is based on appropriate estimates of the generalized Fourier coefficients $w_i\ (t)$ of the heat equation

$$\dot{u} - \Delta u = f \ , \ u(0) = u_0 \, , \ u|_{\partial\Omega} = 0$$

with
$$w_i(t) = e^{-\lambda_i t} u_0 + \int_0^t e^{-\lambda_i (t-\tau)} f_i(\tau) d\tau \ .$$

The "trick" to go there is about changing the order of integration in the following form:

$$\int_0^T w_i^2(t) dt \le \int_0^T \left[\int_0^t e^{-\lambda_i (t-\tau)} d\tau \right] \left[\int_0^t e^{-\lambda_i (t-\tau)} f_i^2(\tau) d\tau \right] dt \ .$$

$$\le \lambda_i^{-1} \int_0^T f_i^2(\tau) \left[\int_\tau^T e^{-\lambda_i (t-\tau)} dt \right] d\tau \le \lambda_i^{-2} \int_0^T f_i^2(\tau) d\tau \ .$$

Approximation Theory for Non-linear Problems

Lecture Notes

J. A. Nitsche

In [NiJ3]) $L_\infty -$ error estimates are derived for the nonlinear boundary value problem

$$div(a(\circ,u)\nabla u)=f$$

As for $u\in C^{1.1}$

$$Tu(x):=a(x,u)\Delta u+\frac{\partial a}{\partial u}\sum(\frac{\partial u}{\partial x_i})^2+\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i}\in C^{2.1}$$

the Schauder fixed point theorem can be applied ($M:=C^{1.1}$) to prove the existence of a solution u.

Schauder fixed point theorem

> Let X be a Banach space and $M\subseteq X$ closed, bounded, convex and $T:M\to M$ compact. Then there exists a $\bar{x}\in M$ with $\bar{x}=T\bar{x}$.

In the following we give a FEM 'optimal' error estimate in $L_\infty -$ norm for non-linear problems.

Let the problem be given by

$$F(x,u)=0$$

with the (roughly) regularity assumptions:

i) there is a unique solution

ii) F,F_u are Lipschitz continuous.

The approximation problem is given by:

$$\text{find}\quad \varphi\in S_h \quad (F(\cdot,\varphi),\chi)=0 \quad \text{for}\ \chi\in S_h \ .$$

Theorem: The FEM admits the error estimate

$$\|u-\varphi\|_{L_\infty}\le c\inf_{\chi\in S_h}\|u-\chi\|_{L_\infty} \ .$$

Approximation error analysis

Put

$$f(x) = F_u(x, u(x)) \quad \text{and} \quad \varphi = u - e$$

then

$$(fe, \chi) = (R, \chi)$$

with a remainder term

$$R := R(e) := F(\cdot, u - e) + fe$$

resp.

$$(fe, \chi) = (fu - R(e), \chi).$$

Let P_h denote the L_2 – projection related to $(f\cdot,\cdot) = (R, \chi)$, then

$$\varphi = P_h(u - \frac{1}{f} R(e))$$

resp.

$$e = (I - P_h)u + P_h \frac{1}{f} R(e)) =: T(e) \cdot$$

Therefore the difference $e = u - e$ is a fixed point of T.

Let

$$B_{\kappa\bar{\varepsilon}} := \left\{ e \middle| \|e\|_{L_\infty} \leq \kappa\bar{\varepsilon} \right\} \text{ and } \bar{\varepsilon} := \inf_{\chi \in S_h} \|u - \chi\|_{L_\infty}.$$

The application of the Schauder fixed point theorem is enabled by the following properties of T:

Lemma:

i) There is a $\kappa > 0$ such that for $\bar{\varepsilon}$ sufficiently small, then T maps the ball $B_{\kappa\bar{\varepsilon}}$ into itself.

ii) for $\bar{\varepsilon}$ sufficiently small, T is a contradiction in $B_{\kappa\bar{\varepsilon}}$.

Proof: i) Because of P_h and f^{-1} are being bounded it holds

$$\|I - P_h\|_{L_\infty} \leq c_1 \inf_{\chi \in S_h} \|u - \chi\|_{L_\infty} = \bar{\varepsilon}$$

and

$$\left\| P_h(\frac{1}{f} R(e)) \right\|_{L_\infty} \leq c_2 \|R(e)\|_{L_\infty}.$$

It is

$$\|F(\cdot,u-e)+fe\|_{L_\infty} \le c_3\|e\|_{L_\infty}^2 = c_3\kappa^2\bar{\varepsilon}^2$$

with c_3 being the Lipschitz constant of F_u and therefore

$$\|T(e)\|_{L_\infty} \le c_1\bar{\varepsilon} + c_3c_2\kappa^2\bar{\varepsilon}^2 \ .$$

Now fixing $\kappa > c_1$ and choosing $\bar{\varepsilon}_0$ according to $\kappa = c_1 + c_3c_2\kappa^2\bar{\varepsilon}_0$ gives i)

ii) it holds $\|T(e_1)-T(e_2)\|_{L_\infty} = \left\|P_h(\frac{1}{f}(R(e_1)-R(e_2))\right\|_{L_\infty} \le c_2\|R(e_1)-R(e_2)\|_{L_\infty}$

and

$$R(e_1)-R(e_2) = F(\cdot,u-e_1)-F(\cdot,u-e_2) = (F_u(\cdot,\vartheta)-F_u(u)(e_1-e_2) \ .$$

With

$$F_u(\cdot,\vartheta) = F_u(\cdot,u-\vartheta e_1-(1-\vartheta)e_2)$$

one gets

$$\|F_u(\cdot,\vartheta)-F_u(\cdot,u)\| \le \kappa\bar{\varepsilon}c_3 \ .$$

Choosing $\bar{\varepsilon} < Min(\varepsilon_0,(c_2c_3\kappa)^{-1})$ then proves ii).

Consequence: The operator T has a unique fixed point in the ball $B_{\kappa\bar{\varepsilon}}$.

From this it follows the theorem above.

Linear Parabolic Problems with Singular Coefficients

Klaus Braun

For $v=u'$ we consider the model problem

P_v^* : find $v\in\dot H_{1/2}$ with

$$(\dot v,w)_{-1/2}+(v',w')_{-1/2}+t^{-1/4}(v,w)_0=0 \qquad \text{for } w\in\dot H_{1/2} \text{ and } t>0$$

$$(v(\cdot,0),w)_{-1/2}=(f,w)_0 \quad \text{for } w\in\dot H_{1/2} \text{ and } t=0$$

and its related Ritz-Galerkin approximation problem

$P_{v_h}^*$: find $v_h\in\dot H_{1/2}$ with

$$(\dot v_h,\chi)_{-1/2}+(v_h,\chi)_{1/2}+t^{-1/4}(v_h,\chi)_0=0 \qquad \text{for } \chi\in S_h\subset\dot H_{1/2} \text{ and } t>0$$

$$(v_h(\cdot,0),w)_{-1/2}=(f,w)_0 \quad \text{for } \chi\in S_h\subset\dot H_{1/2} \text{ and } t=0 .$$

For the approximation error $e=v-v_h\in H_{1/2}$ we prove the following

Theorem: For linear splines (finite element $S_h^{1,t}$) approximation spaces the Ritz-Galerkin approximation is 'optimal' in the following sense

$$\|u-u_h\|_0\approx\|e\|_{-1/2}=O(h)$$

Proof: We define with $\varepsilon:=v-\tilde v_h\in H_{1/2}$ the error of the Ritz-Gakerkin approximation of the corresponding heat equation with respect to the $\dot H_{-1/2}$ –Hilbert space inner product, i.e.

$$(\dot\varepsilon,\chi)_{-1/2}+(\varepsilon,\chi)_{1/2}=0 \quad \text{for } \chi\in S_h\subset\dot H_{1/2} .$$

From [JNi5] it follows for $\beta>-1$, $\chi\in S_h=S_h^{k,t}\subset\dot H_{1/2}$

$$\int_0^t\tau^\beta\|\varepsilon\|_{-1/2}^2d\tau\le ch^{2t+1}\int_0^t\tau^\beta\|v\|_t^2d\tau$$

i.e. especially for $v\in H_{1/2}$ only

$$\int_0^t\tau^{-1/2}\|\varepsilon\|_{-1/2}^2d\tau\le ch^2\int_0^t\tau^{-1/2}\|v\|_{1/2}^2d\tau \ .$$

Putting $e=\varepsilon-\Phi:=(v-\tilde v_h)-(v_h-\tilde v_h)\in H_{1/2}$ one gets

$$(\dot\Phi,\chi)_{-1/2}+(\Phi,\chi)_{1/2}+t^{-1/4}(\Phi,\chi)_0=t^{-1/4}(\varepsilon,\chi)_0 \ \text{ for } \chi\in S_h\subset\dot H_{1/2}$$

and therefore

$$\frac{1}{2}\frac{d}{dt}\|\Phi\|_{-1/2}^2 + \|\Phi\|_{1/2}^2 + t^{-1/4}\|\Phi\|_0^2 = t^{-1/4}(\varepsilon,\Phi) \le \delta\|\Phi\|_{1/2}^2 + c_\delta t^{-1/2}\|\varepsilon\|_{-1/2}^2$$

resp.

$$\frac{1}{2}\frac{d}{dt}\|\Phi\|_{-1/2}^2 + \|\Phi\|_{1/2}^2 + t^{-1/4}\|\Phi\|_0^2 \le c_\delta t^{-1/2}\|\varepsilon\|_{-1/2}^2 .$$

From this it follows

$$\|e\|_{-1/2}^2(t) \le \|\varepsilon\|_{-1/2}^2(t) + \|\Phi\|_{-1/2}^2(t) \le h^2\|v\|_{1/2}^2(t) + c\int_0^t \tau^{-1/2}\|\varepsilon\|_{-1/2}^2 d\tau = O(h^2) \ .$$