J.A. Nitsche's footprints related to Navier-Stokes equations problems

The 3 footprint domains

1st footprint

L_∞ – boundedness of the FEM Galerkin operator for parabolic problems

[JNi5] J.A. Nitsche, M.F. Wheeler, L_∞ – boundedness of the finite element method Galerkin operator for parabolic problems, Numer. Funct. Anal. Optim, 4, 325-353 (1981/1982)

In this paper Nitsche/Wheeler leveraged on the already in [JNi1] proven optimal convergence with respect to the convergence factor $O(h^r)$ but with non balanced norms (i.e. non balanced regularity assumptions to the heat equation solution). The basic idea is to estimate the solution of the heat equation with respect to the norm

$$\left\|w\right\|_k^2 := \int_0^T \left\|w\right\|_k^2 dt$$

using weight function in the form $\mu(x,t) := |x - x_0|^2 + |t - t_0|$, whereby $\left\|u\right\|_{L_\infty(L_\infty)} = u(x_0,t_0)$.

By estimating corresponding generalized Fourier coefficients $\int_0^T w_i^2(t)dt$ of the heat equation

$$\dot{u} - \Delta u = f \;,\; u(0) = u_0 \,,\; u\big|_{\partial\Omega} = 0$$

with

$$w_i(t) = e^{-\lambda_i t}u_0 + \int_0^t e^{-\lambda_i(t-\tau)}f_i(\tau)d\tau$$

the problem adequate shift theorem

$$\left\|w\right\|_{k+2}^2 \le c\left\|Aw\right\|_k^2$$

has been proven by changing the order of integration in the following form:

$$\int_0^T w_i^2(t)dt \le \int_0^T\left[\int_0^t e^{-\lambda_i(t-\tau)}d\tau\right]\left[\int_0^t e^{-\lambda_i(t-\tau)}f_i^2(\tau)d\tau\right]dt \le \lambda_i^{-1}\int_0^T f_i^2(\tau)\left[\int_\tau^T e^{-\lambda_i(t-\tau)}dt\right]d\tau \le \lambda_i^{-2}\int_0^T f_i^2(\tau)d\tau \,.$$

We recall the following spaces of divergence-free functions

$$C_{0,\sigma}^\infty(\Omega) = \left\{ v \in C_0^\infty(\Omega), div\,v = 0 \right\}$$

$$H = closure\_of\_C_{0,\sigma}^\infty(\Omega)\_in\_L_2(\Omega)$$

$$V = closure\_of\_C_{0,\sigma}^\infty(\Omega)\_in\_H_1(\Omega)$$

The space V is characterized by

$$V = \left\{ v \in H_0^1(\Omega), div\,v = 0 \right\}.$$

The space L_2 has the (Helmholtz) decomposition $L_2(\Omega) = H \oplus H^\perp$, where

$$H^\perp = \left\{ \phi \in L_2(\Omega), \exists p \in H_1(\Omega), \phi = \nabla v \right\}.$$

Let P denote the orthogonal projection from $L_2(\Omega)$ onto H. Then the operator $A : D(A) \to H$ given by $A = -P\Delta$ with domain $D(A) = H_2(\Omega) \cap V$ is called the Stokes operator. The operator is positive definite, self-adjoint and is characterized by the relation

$$(Aw, v) = (\nabla w, \nabla v) \quad \forall w \in D(A), v \in V.$$

The operator A^{-1} is linear continuous from H into $D(A)$, and since the injection of $D(A)$ in H is compact, A^{-1} is a compact operator in H. As an operator in H, A^{-1} is also self-adjoint. Therefore there exists a sequence of positive numbers $\mu_{j+1} \le \mu_j$ and an orthogonal basis of H, $\left\{ \varphi_j(x) \right\}_{j \in N}$ such that $A^{-1}\varphi_j = \lambda_j \varphi_j$. Let . since A^{-1} has range in $D(A)$ one obtains that

$$A\varphi_i = \lambda_i \varphi_i \quad \varphi_i \in D(A)$$

$0 < \lambda_1 < .... < \lambda_j < \lambda_{j+1}$ with $\lim\limits_{j \to \infty} \lambda_j = \infty$ and $\left\{ \varphi_j(x) \right\}_{j \in N}$ being an orthogonal basis of H.

The functional analytical approach to the N-S-E is built on the eigen pairs of the Stokes operator, i.e. in the linear case Hilbert scale approximation theory can be applied (e.g. Nitsche's lecture notes, including negative scaled Hilbert spaces); from a functional analytical point of view the Stokes operator A has the same "perfect" properties as the Laplacian, i.e.

    i) A is self-adjoint and positive

    ii) A^{-1} is compact.

This enables the definition of Hilbert scales with the inner products (J.A. Nitsche, Lecture Notes)

$$(x, y)_\alpha := \sum \lambda_i x_i y_i \quad \text{and} \quad (x, y)_{(-t)} := \sum e^{-\lambda_i t} x_i y_i \ , \quad x_i := (x, \varphi)_i \ .$$

Let A denote Stokes operator. We note the representation of the power of the operator in the form

$$A^\beta u = \sum \lambda_i^\beta x_i \varphi_i \ , \ u \in D(A^\beta), \beta \in R \ .$$

It holds

$$D(A^{\beta_1}) \subset D(A^{\beta_2}) \quad \text{for} \quad \beta_1 \leq \beta_2 \ .$$

An inner product resp. norm is given by

$$\left\| w \right\|_k^2 := \int_0^T \left\| w \right\|_k^2 dt \ .$$

For the non-stationary Stokes problem

$$\dot{u} + Au = f \quad u(0) = u_0$$

then it holds with the analogue arguments as above

$$\left\| u \right\|_{\alpha+2}^2 + \left\| \dot{u} \right\|_\alpha^2 \leq c \left\{ \left\| u_0 \right\|_\alpha^2 + \left\| f \right\|_\alpha^2 \right\} \quad \text{for} \quad \alpha \in R$$

The standard "energy" inequality is given in the form

$$\frac{1}{2} \left\| u \right\|^2 + \int_0^t \left\| \nabla u \right\|^2 ds \leq \frac{1}{2} \left\| u_0 \right\|^2$$

The non linear case

For the in-stationary Stokes solution it holds

$$u(t) = e^{t\Delta}u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta} \prod(u(s) \otimes u(s))ds$$

It is well known, that if $u_0 \in J_1(\Omega)$ then $u \in C([0,t), J_1(\Omega))$ and

$$\|\nabla u(t)\|^2 + \int_0^t \left\{ \|u(s)\|^2 + \|\nabla p(s)\|^2 + \|\dot{u}(s)\|^2 \right\}ds \le c\|\nabla u_0(t)\|^2 \quad , \quad 0 < t < T \ .$$

[JHe3] gave a counter example, that it is not possible to include the terms $\|p(t)\|_{\dot{L}_2}, \|\dot{u}(t)\|_{H_{-1}}$ on the left hand side, i.e. **it doesn't hold**

$$\|\nabla u(t)\| + \|p(t)\|_{\dot{L}_2} + \|\dot{u}(t)\|_{H_{-1}} + \int_0^t \left\{ \|u(s)\| + \|\nabla p(s)\| + \|\dot{u}(s)\| \right\}ds \le c\|\nabla u_0(t)\| \quad , \quad 0 < t < T \ .$$

It holds

$$\|p(t)\|_{\dot{L}_2} + \|\dot{u}(t)\|_{H_{-1}} \approx ct^{-1/2}\|\nabla u_0(t)\| \ .$$

Remark: The Oseen operator is the Fourier multiplier by the matrix

$$\Omega(t) = \Delta^{-1}(\nabla \otimes \nabla)e^{t\Delta} = (I - P)e^{t\Delta} = \Delta^{-1}(\partial_{x_j}\partial_{x_k})_{1 \le j,k \le n}$$

In [NLe] the convolution with the matrix $(F_{jk}(t,x))$ of the Oseen matrix operator $\Omega(t)$ is given.

The variational representation of the Navier-Stokes equations

$$(\dot{u}, u) + (Au, u) + (Bu, u) = 0 \ , \ u(x,0) = u_0$$

whereby $b(u, v, w) := (u\nabla \cdot)v, w)$ with $b(u, v, v) = (Bu, v) = 0$ in case $divu = 0$, leads to

$$\frac{d}{dt}\|u\|^2 + 2\|A^{1/2}u\|^2 = 0 \quad \text{resp.} \quad \|u(t)\|^2 + 2\int_0^t \|A^{1/2}u\|^2(\tau)d\tau = \|u_0\|^2 \ .$$

4

The functional analytical representation of the Navier-Stokes solution of

$$\dot{u} - \Delta u + \nabla p = f_0 + div[F - u \otimes u]$$

is given by

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{-(t-s)A}Pf_0 ds + \int_0^t A^{1/2}e^{-(t-s)A}A^{-1/2}Pdiv[F - uu]ds$$

In [HSo2] the Banach space

$$X := \left\{u : (0,t) \rightarrow L_\sigma^2(\Omega)\big|(A^{-1/2}u)_t, (A^{1/2}u) \in L_4(0,T; L_2(\Omega)), A^{-1/2}u(0) = 0\right\}$$

equipped with the norm

$$\|u\|_X^2 := \left\|(A^{-1/2}u)_t\right\|_{2.4.T}^2 + \left\|(A^{1/2}u)\right\|_{2.4.T}^2$$

has been defined to consider a fixed point problem for the strong solution. We further recall, that the space X is continuously embedded into

$$X \subset L_8((0,T), L_4(\Omega))$$

and it holds

$$\|u\|_X^2 := \|u\|_{4.8.T} \approx \left\|(A^{3/8}u)\right\|_{2.8.T}^2 \approx \|F\|_{2.4}^2 \quad.$$

for $f = divF \quad F = uu \in L_2((0,T); L_2(\Omega))$.

We note that $f \in W_2^{-1} \leftrightarrow F \in L_2$ i.e. $c\|u\|_{W_2^{-1}} \le \|F\|_{L_2} \le c\|f\|_{W_2^{-1}} \quad.$

Concerning the relation to Hilbert scales we note:

Lemma (J.A. Nitsche, lecture notes) Let $0 \le \alpha < \beta < \gamma$ and $u \in W_p^\gamma$ then for fixed p it holds

$$\|u\|_{W_p^\beta} \le c\|u\|_{W_p^\alpha}^\lambda \|u\|_{W_p^\gamma}^\mu$$

with

$$\lambda = \frac{\gamma - \beta}{\gamma - \alpha} \quad, \quad \mu = \frac{\beta - \beta}{\gamma - \alpha} \quad.$$

IDEE: appropriate Hilbert scale definition depending from non-linear problem, i.e. in case of $n = 3$ for the solution of the N-S-E problems this might be --> p=4 ?

„Saddle point problems and non linear minimization problems on convex manifolds".

From [WVe] we recall the minimization problem in the form:

$$(*) \quad J(u): a(u,u) - F(u) \rightarrow \min , \quad u - u_0 \in U .$$

Let $a(\cdot,\cdot): V \times V \rightarrow R$ a symmetric bilinear form with energy norm $\|u\|^2 := a(u,u)$. Let further $u_0 \in V$ and $F(\cdot): V \rightarrow R$ a functional with the following properties:

    i) $F(\cdot): V \rightarrow R$ is convex on the linear manifold $u_0 + U$,

    i.e. for every $u, v \in u_0 + U$ it holds $F((1-t)u + tv) \le (1-t)F(u) + tF(v)$ for every $t \in [0,1]$

    ii) $F(u) \ge \alpha$ for every $u \in u_0 + U$

    iii) $F(\cdot): V \rightarrow R$ is Gateaux differentiable, i.e. it exits a functional $F_u(\cdot): V \rightarrow R$ with

$$\lim_{t \to 0} \frac{F(u + tv) - F(v)}{t} = F_u(v) \cdot$$

Then the minimum problem (*) is equivalent to the variational equation

$$a(u,\varphi) + F_u(\varphi) = 0 \text{ for every } \varphi \in U$$

and admit only an unique solution. In case the sub space U and therefore also the manifold $u_0 + U$ is closed with respect to the energy norm and the functional $F(\cdot): V \rightarrow R$ is continuous with respect to convergence in the energy norm, then there exists a solution.

We note that the energy functional is even strongly convex in whole V .

From [YGi3] we recall the result of Calderón [ACa] for $0 < \alpha < \beta < \gamma$

$$\left\| A^{\beta} u \right\|_{L_p} \leq c \left\| A^{\alpha} u \right\|_{L_p}^{\lambda} \left\| A^{\gamma} u \right\|_{L_p}^{\mu}$$

whereby $\lambda + \mu = 1$, $\alpha\lambda + \gamma\mu = \beta$.

We denote the norms

$$\left\| w \right\|_{q.\rho.T} := \left(\int_0^T \left\| w \right\|_{L_q}^{\rho} \, dt \right)^{1/\rho}$$

The (scaled) Serrin's values are defined by

$$S(q,\rho) := \frac{n}{q} + \frac{2}{\rho}$$

The condition $S(q,\rho) \leq 1$ ensures convergent integrals, i.e. bounded norms $\left\| w \right\|_{4.8.T} < \infty$.
Uniqueness and regularity of N-S-E solutions are ensured, if

$$S(q,\rho) = n/2$$

In case of $n = 3$ ([HSo], if a weak solution of the full linear case fulfills the Serrin condition

$$\left\| w \right\|_{4.8.T} = \left[\int_0^{\infty} \left\| u \right\|_4^8 \right]^{1/8} < \infty$$

then u is uniquely determined by the data f and u_0.

In case of $n = 3$ there is gap of $1/2$ of the scale of Serrin's values, i.e.

$$S(q,\rho) := \frac{n}{q} + \frac{2}{\rho} = \frac{3}{4} + \frac{2}{8} = 1$$

fulfills $S(q,\rho) \leq 1$, i.e. one knows, that for $q = 4$ and $\rho = 8$ the norm $\left\| w \right\|_{q.\rho.T}$ is bounded. On the other side, what is required from the N-S-E energy inequality, is

$$1 < S(q,\rho) < \frac{n}{2} = \frac{3}{2} \qquad .$$

If $n = 3$, the energy equality holds even in the middle of the gap, namely $\left\| w \right\|_{q.\rho.T} < \infty$ is satisfied with $1 < S(4,4) < 1 + 1/4$.

From [YGi1] we note $\qquad \left\|\nabla u(t)\right\|_p \le ct^{-1/2}\left\|u_0\right\|_p$, $\quad \left\|A^{1/2}e^{-tA}u_0\right\|_{p=2} \le ct^{-1/2}\left\|u_0\right\|_{p=2}$

From [YGi3] we recall $\qquad \left\|A^\alpha e^{-tA}u\right\|_p \le ct^{-\alpha}\left\|u_0\right\|_p$ for $\alpha \in R^+, t > 0$.

Regarding the lowest possible initial value space requires an analysis of the condition in the form ([RFa])

$$\int_0^\infty \left\|e^{-tA}u_0\right\|_4 \le c$$

It holds

$$u \in L_8(0,T;L_4(\Omega)) \qquad \text{iff} \qquad \int_0^\infty \left\|e^{-tA}u_0\right\|_4 \le \infty \ .$$

It further holds

$$\int_0^\infty \left\|e^{-tA}u_0\right\|_4^8 \le c\left\|A^{1/8}u_0\right\|_2^8 \quad \text{if} \ u_o \in D(A^{1/8})$$

In [RFa] the estimate is derived

$$\left\|e^{-tA}u_0\right\|_4 \le ct^{-3/8}\left\|u_0\right\|$$

This simple means the integrability of the (continuous) function

$$t \to \left\|e^{-tA}u_0\right\|_4 \qquad \text{near} \qquad t = 0 \ .$$

This and the below might indicate, that the norm $\|w\|_{q.\rho.T}$ is not adequate to handle the N-S-E properly.

$$[w]_{q.\rho.\beta.T} := \left(\int_0^T t^\beta \left\|w\right\|_{L_q}^\rho dt\right)^{1/\rho} \quad \text{with} \quad \beta = \beta(n)$$

2nd footprint

Non-linear parabolic equation (free boundary problem, 1D Stefan and 2D Stokes' flow)

[JNi2] J.A. Nitsche, Finite Element Approximation to the One dimensional Stefan Problem, Proceedings on Recent Advances in Numerical analysis (C. de Boor and G. Golub, eds.) Academic Press, New York, 1978, pp. 119-142

[JNi3] J.A. Nitsche, A Finite Element Method For Parabolic Free Boundary Problems, Intensive seminar on free boundary problems, Pavia, Italy, September 4-21, 1979

[JNi6] J.A. Nitsche, Free boundary problems for the Stokes' flows and finite element methods, in J. Vosmansky and M. Zlamal: Equadiff 6, Proceedings of the International Conference on differential Equations and Their Applications held in Brno, Czechoslovakia, 1985, pp. 327-332

J.A. Nitsche, Stokes equations and mixed FE approximations with piecewise constant FE spaces $S_{2h}^{0,1} \subset \dot{L}_2$, lecture notes

Then the free boundary Stefan problem with its solution $U(y, \tau)$ can be transformed into the non-linear parabolic equation looking for a solution $u(x,t) = U(y,\tau)$ fulfilling

$$u_t(y,\tau) - u_{xx}(x,t) = -xu_x(1,t)u_x \text{ in } Q$$

with the boundary conditions

$$(*) \qquad u_x(0,t) = 0 \text{ for } t > 0$$

$$(*) \qquad u(1,t) = 0 \text{ for } t > 0$$

$$u(x,0) = f(x) \text{ for } x \in (0,1) \ .$$

Let

$$\dot{H}_1 := \left\{ w \middle| w \in H_1(0,1), > 0, w(0) = 0 \right\} = \left\{ w \middle| w' \in L_2(0,1), > 0, w(0) = 0 \right\}$$

Then $v := u_x$ belongs to $\dot{H}_1$ and, for any $v \in \dot{H}_1$ the function defined by

$$u(x,t) = -\int_x^1 v(z,t)dz$$

satisfies the boundary condition (*). Multiplying the differential equation above with w_x ($w \in \dot{H}_1$) and integration gives the variational equation

$$\int_0^1 u_{xx}w_x + u_{xt}wdx = u_x(1,t)\int_0^1 xu_xw_xdx \ .$$

In [JNi2] the non optimal FE error estimate of order $h^\alpha / \sqrt{t}$ with $0 < \alpha < 1$ has been proven in case of non regular initial value function.

A proof of optimal FE approximation convergence of order $h/\sqrt{t}$ with non-regular initial value function seems to be still missing.

The non-optimal convergence order $h^\alpha/\sqrt{t}$ has been proven built on the inequalities

$$\|z\|^2 + \int_0^t \|z'\|^2 d\tau \le 2\|g\|^2 \quad , \quad t\|z'\|^2 + \int_0^t \tau\|\dot{z}\|^2 d\tau \le c = c(\|g\|)$$

And the following a priori estimates

$$\sup_{0\le t\le T}\left\{ t^{2k}\left\|\partial_t^k v_h\right\|^2 + \int_0^t \tau^{2k}\left\|\partial_t^k v_h'\right\|^2 d\tau \right\} \le c_{2k}^2 \quad , \quad \sup_{0\le t\le T}\left\{ t^{2k+1}\left\|\partial_t^k v_h'\right\|^2 + \int_0^t \tau^{2k+1}\left\|\partial_t^{k+1} v_h\right\|^2 d\tau \right\} \le c_{2k+1}^2$$

The proof applies Young inequality and uses the Gronwall lemma for inequalities of the form

$$\lambda := \lambda(t) \le \|g\|^2 + c\int_0^t \lambda^3(\tau)d\tau \quad , \quad \lambda := \lambda(t) \le k_1 + k_2\int_0^t \tau^{-1/}\lambda^{3/2}(\tau)d\tau \quad \cdot$$

The learning from point 1 is, that applying the lemma of Gronwall leads to an unbalance (with respect to the norms) inequality. "Optimal" convergence order could be proven by using a parabolic (heat) equation duality "Ansatz" in combination with cut-off function with respect to the time variable (as has been successfully applied for linear parabolic equations with non regular initial value function, J.A. Nitsche, Lecture Notes).

In general parabolic Hölder norms define the appropriate function spaces in case of non-linear parabolic problems; the paper of K. Höllig gives a modified heat integral kernel to estimate non-linear parabolic equation with singular coefficient function (in the same order, than the initial singular initial value function of the Stefan problem ($t^{-1/2}$).

Remark (Hölder spaces): In [JNi6] appropriate Hölder spaces for the 2D case are defined.

Remark (Gronwall lemma): Putting

$$y(t) := \left\|A^{1/2}u\right\|^2$$

In the context of the inequality

$$\frac{1}{2}\frac{d}{dt}\left\|A^{1/2}u\right\|^2 + \|Au\|^2 \le \frac{1}{2}\|Au\|^2 + \left\|A^{1/2}u\right\|^6$$

leads to $\quad y'(t) \le cy^3(t)$. From this it follows, that every positive solution blows up, i.e. there is no global boundedness.

Bem: --> applying Lemma von Gronwall läuft immer auf sowas hinaus (oder zwingt zu Formen wie Gronwall version 4 (see appendix), i.e. $y'(t) \le cy^\delta(t)$ $0 < \delta < 1$.

In the framework of Hilbert scales (see appendix) there are also "negative" norm.

$$(x, y)_\alpha := \sum \lambda_i x_i y_i \qquad \text{and} \qquad (x, y)_{(-t)} := \sum e^{-\lambda_i t} x_i y_i$$

Because of

$$\lambda^{-\alpha} \le \delta^{2\alpha} + e^{t(\delta^{-1} - \sqrt{\lambda})}$$

It holds

$$\|x\|_{-\alpha}^2 \le \delta^{2\alpha} \|x\|_0^2 + e^{t\delta^{-1}} \|x\|_{(-t)}^2 \ .$$

From this one e.g. can conclude for every $\alpha > 0$

$$\|x\|_{-\alpha}^2 \le t^{2\alpha} \|x\|_0^2 + e\|x\|_{(-t)}^2 \xrightarrow[t \to 0]{} e\|x\|_0^2$$

Remark: (R. Rannacher, lecture notes): For the purposes of numerical analysis one needs regularity of the solution uniformly down to $t = 0$, which turns out to be a delicate requirement. To illustrate this, assuming that the solution is uniformly smooth as $t = 0$. Then, applying the divergence operator to the Navier-Stokes equations and letting $t = 0$ one obtains an over determined Neumann problem for the initial pressure including a compatibility condition (tangent direction along $\partial\Omega$).

We define the following two singular integral operators (see appendix)

(A) $\quad (Nu)(x) := -\oint \log 2 \sin \dfrac{x-y}{2} u(y) dy =: \oint k(x-y) u(y) dy \quad$ and $\quad D(N) = H = L_2^*(\Gamma)$

(H) $\quad (Hu)(x) := [u](x) := \dfrac{1}{2\pi} \oint \cot \dfrac{x-y}{2} u(y) dt = -\lim_{\varepsilon \to 0} \dfrac{1}{2\pi} \int_\varepsilon^\pi [u(x+y) - u(x-y)] \cot \dfrac{y}{2} dy \quad \cdot$

__Idee__: first solve the open problem of the not proven quais-optimal convergence for the Stefan problem with non regular ($v_0 \in L_2$ in case of putting $v := u_x \in H_1$) initial value function (Helsinki paper, J.A. Nitsche)

The following ideas might be helpful:

1. instead of putting $v := u_x \in H_1$ put

$$v := N u_x = -Hu \quad resp. \quad u = Hv \quad , \text{ see also } ([JNi8)]$$

It would require less regularity assumption on the weak solution $u \in H_1$ than currently $u \in H_2$

At the same time the variational equation

$$(u_{xx}, w_x) + (u_{xt}, w) = u_x(1, t)(x u_x, w_x)$$

would be transformed into

$$(N u_{xx}, w_x) + (N u_{xt}, w) = N(u_x(1, t))(N(x u_x), w_x) \quad .$$

We note

i) $\quad (N(x u_x), w_x) = ((Nx - xN)u_x, w_x) + (x N u_x, w_x) \approx (H u_x, w_x) + (x H u, w_x)$

ii) $\quad u_x(1, t) \approx t^{-1/4}$

iii) $\quad \left\| A^1 u \right\| \approx \left\| N u \right\|_1 \approx \left\| Hu \right\| \approx \left\| u \right\| \quad , \quad \left\| A^{-1} u \right\| \approx \left\| N u \right\|$

An analogue projection operator from

$$H_1 \to H_1^\sigma := \left\{ w \in H_1 \,\middle|\, div w = 0 \right\}$$

could project onto

$$\to \left\{ w \in H_1 \,\middle|\, w_x = 0 \right\} \quad or \ (?) \quad \to \left\{ w \in H_1 \,\middle|\, x w_x = 0 \right\}$$

Remark: concerning an appropriate projection space: a condition like $(x u_x)_x = 0$
would mean, that $\quad x u_x = const \quad i.e. \quad u \approx c \log(x)$!!!!

Der Ansatz $v := N u_x = -Hu$ entspricht in etwas (?) der reduzierten Regularität der externen Kräfte beim „unusual proof of the„Nitsche-Stokes-flow" shift theorem.

Evtl. wäre dies auch ein Hilfsschritt, um in „negative" Hilbert scales „runterzufahren" (für $p = 2$), die dann in geeigneterem Zusammenhang stehen mit Hilbert scales bzgl. $p = 4$? Zusammenhang über t-Potenz unter dem Integral?

2. $u_x(1) := u_x(1,t) \approx t^{-1/4}$ „behandeln" mit Höllig-approach (Hölder spaces) with kernel function

$$k(x,t,s) := \frac{1}{\sqrt{4\pi}} \frac{1}{\sqrt{t-s}} e^{-\frac{[x+2(\sqrt{t}-\sqrt{s})]^2}{4(t-s)}} =: \frac{1}{\sqrt{4\pi}} \frac{1}{\sqrt{t-s}} e^{-A^2(x,t,s)}$$

fulfilling the relation $\quad k_s - k_{xx} - 2s^{-1/2}k_x = 0 \quad$ and $\quad e^{-A^2(x,t,s)} \le c e^{-\frac{x^2}{t-s}}$

Eigentlich würde ich lieber das Modellproblem

$$u_t - u_{xx} - \frac{2x}{\sqrt{t}} u_x = 0$$

in gleicher Weise behandelt wissen, wie das K. Höllig-Problem

3. cut-off functions bzgl. T-Variable für den Dualitätsansatz

4. the estimates $\quad z^k e^{-z} \le c e^{-cz} \quad$ and $\quad \int_0^t s^{-k/2}(t-s)^{-1/2} e^{-z/s} ds \le c e^{(1-k)/2}$.

3rd footprint

Stokes Operator shift theorem and Cauchy-Riemann differential equations

[JNi7] J.A. Nitsche, Direct Proofs of Some Unusual Shift-Theorems, Anal. Math. Appl., Gauthier-Villars, Montrouge, 1988, pp. 383-400

C-R differential equation being used to de-couple Stokes equations In a potential and a bi-potential PDE; corresponding boundary conditions transformation is no issue, due to appropriate rotation properties of C-R-differential equations; shift theorem uses negative scale Hilbert space and appropriate Hölder spaces; Hilbert transform and 2D Newton potential integral operator being analyzed to enable transform into negative scaled Hilbert spaces; n>1 counterpart of Hilbert transform are Riesz transformations, Conjugate harmonic functions, ([ESt2], S. 120), which are the $n > 2$ extensions of the C-R-PDE and characterization von Hardy spaces und BMO ([ESt1] III, [HAb]).

Sowohl die Cauchy-Riemann Dgl. als auch die verallgemeinerten C-R-Dgl. sind rotationsinvariant, bzw. die Riesz transforms sind rotationsinvariant [ESt2], S58):

$$\rho R_j \rho^{-1} f = \sum_k \rho_{jk} R_k f$$

From [JKa] we recall:

If $u_0 \in L_\infty(R^n)$ the non-linear Navier-Stokes equations admits a unique time-local (regular) solution u with

$$p = \sum_{i,k=1,n} R_i R_k u_i u_k$$

Theorem ([HAb] p.117):

i) The dual space of the Hardy space is BMO:

$$(H^1)^\bullet \cong BMO$$

ii) Let $R_j f = F^{-1}\left[\dfrac{i\xi_j}{|\xi|}\hat{f}(\xi)\right]$ be the Riesz operators. Then

$$f \in H^1 \quad \text{iff} \quad f \in L_1 \text{ and } R_j f \in L_1 \text{ for all } \quad j = 1,2,...n$$

$$f \in BMO \quad \text{iff} \quad \text{there are some } g_j \in L_\infty \text{ for } j = 0,1,2,...n$$

$$\text{such that} \quad f = g_0 + \sum_1^n R_j g_j$$

Lemma ([JKa]):Let $1 \le i,j,k \le n$ and $R_{ij}^\varepsilon f := \partial_i \partial_j k_\varepsilon * f$ with related fundamental solution $k(x)$ of the Laplacian $-\Delta$.

i)   for $f \in L_\infty$ we have $\lim_{\varepsilon \to 0}(R_{ij}^\varepsilon f, \varphi) = (R_i R_j f, \varphi)$ for all $\varphi \in S$ with $\int \varphi = 0$.

     Moreover, we have $\lim_{\varepsilon \to 0} R_{ij}^\varepsilon \partial_l f = \partial_l R_i R_j f$ in S'

ii)   for $f \in S'$ with $div f = 0$ and $0 < \varepsilon < 1/4$ we have $\sum_{j=1}^n R_{ij}^\varepsilon f_j = 0$ in S'

iii)   for $f \in BMO$ we have $\lim_{\varepsilon \to 0} \sum_{j=1}^n R_{ij}^\varepsilon \partial_j f = -\partial_i f$ in S'

Die Verallgemeinerung der C-R-Differentialgleichungen auf drei Dimensionen ([CRu]) ist gegeben durch

$$\nabla \cdot v = 0 \ , \ \nabla \times v = 0$$

Diese Verallgemeinerung lässt nicht nur Potentialströmungen zu, sondern auch Wirbel. Dabei verlangt sie von den Wirbellinien, dass $w \times v = 0$, d.h. dass die Richtung der Wirbelline der Richtrung der Geschwindigkeit gleich oder entgegengesetzt ist. Die Wirbellinien müssen daher mit den Stromlinien zusammenfallen und jedes Wirbelelement muss in Richtung seiner Achse oder der entgegengesetzten Richtung der Stromlinie entlang fliessen.

Die Bedingung ist mit den Eulerschen Gleichungen einer inkompressiblen stationären Strömung vereinbar, wegen

$$(\nabla \times v) \times v = (v \cdot \nabla)v - \frac{1}{2}\nabla(v \cdot v)$$

Damit ergibt sich aus (*):

$$\nabla \cdot v = 0 \ , \ (v \cdot \nabla)v = \frac{1}{2}\nabla(v \cdot v)$$

Was nichts anderes ist, als die Eulerschen Gleichungen für das stationäre Strömung einer inkompressiblen Flüssigkeit unter der Voraussetzung, dass die aeusseren Kräfte, falls vorhanden, ein Potential haben und der folgenden zweiten Voraussetzung:

In der Bernoullischen Gleichung

$$\frac{\rho}{2}(v \cdot v) = -p - \Omega + C$$

hat die Grösse C auf den verschiedenen Stromlinien denselben Wert. In this case gibt es eine skalare Funktion , so dass an irgend einer Stelle der Flüssigkeit die auf das Volumenteilchen ausgeübte Kraft gleich

ist, womit die Eulerschen Gleichungen der stationären inkompressiblen Strömung lauten:

Die linke Seite der zweiten Gleichungen ist gleich ρ -mal der Beschleunigung der Flüssigkeit an der betrachteten Stelle. Durch Multiplikation mit v und Integration nach der Zeit erhält man die bekannte Bernoullische Gleichung

$$\frac{\rho}{2}(v \cdot v) = -p - \Omega + C \ \cdot$$

The Stokes equations and the Heywood counter example

The Stokes equations are the simplest model to describe a flow. In this case it describes an extreme viscous flow (like honey), which does not allow small scaled rotations. Therefore it should provide no problems, to rotate the external forces, without any effect to the Stokes solution??

Even in this simplest version of a viscous flow there arise numerical difficulties, especially in the context of the Babuska-Brizzi (inf-sup-) condition. This is a general problem in the context of saddle point problem on finite-dimensional approximation spaces.

Heywood-Rannacher [JHe2] gave certain FE convergence estimates, which requires a singular term $t^{-1/2}$ in the pressure error estimate. In [JHe3] a counter example is given, that

Based on the function space decomposition

$$L_2(\Omega) = J(\Omega) \oplus G(\Omega)$$

it follows from the Navier-Stokes equations the representation

$$\left\| \Delta u - u \cdot \nabla u \right\|^2 = \left\| \dot{u} \right\|^2 + \left\| \nabla p \right\|^2$$

which is in line with (*). Based on the function space decomposition

$$\overset{\circ}{W}{}_2^1(\Omega) = J_1(\Omega) \oplus R(A)$$

it follows from the Navier-Stokes equations the representation

$$\left\| \dot{u} - u \cdot \nabla u \right\|_{-1}^2 = \left\| \nabla u \right\|^2 + \left\| \nabla Ap \right\|^2$$

whereby $\left\| \nabla Ap \right\|^2 \approx \left\| p \right\|^2$. This means that there is $\left\| \dot{u} \right\|_{-1}^2$ and $\left\| p \right\|^2$ on the opposite

sides of this equation. So it appears that both could be large, even when $\left\| \nabla u \right\|^2$ is small. According to the theorem in [JHe3] that actual happens. One of the overall propositions is the assumption of an initial value for the pressure, when the initial velocity belongs to

$$J_1(\Omega) \oplus W_2^2(\Omega) \ .$$

The pressure is uniquely (possible up to a constant) determined by the velocity field. There holds the stability estimate ("inf-sup" stability)

$$\inf_{q \in L_2} \sup_{\varphi \in H_0^1} \frac{(q, \nabla \cdot \varphi)}{\left\| q \right\| \left\| \nabla \varphi \right\|} \geq \gamma_0 > 0 \ .$$

For $n = 2$ the Stokes equations are given by

$$-\Delta u - p_x = f \quad \text{in } \Omega$$
$$-\Delta v - p_y = g \quad \text{in } \Omega$$
$$u_x + v_y = r \quad \text{in } \Omega$$

with the compatibility condition: $(r, 1) = 0$.

Der Ansatz mittels der Cauchy-Riemann'schen Differentialgleichungen

$$u = w_x - z_y$$
$$v = w_y + z_x$$

führt auf die beiden entkoppelten Dgln.

$$\Delta w = r \quad \text{in } \Omega$$
$$w = 0 \quad \text{auf } \partial\Omega$$

und

$$\Delta^2 z = -(f_y - g_x) \quad \text{in } \Omega \quad \text{(Rotation der Kräfte)}$$
$$z_s = -w_n \quad \text{auf } \partial\Omega$$
$$z_n = 0 \quad \text{auf } \partial\Omega \quad .$$

Das ist eine hübsche Sache, allerdings sind die beiden Funktionen w, z nicht mehr „in balance", was die Regularität betrifft ($f = -\nabla F = -\nabla\sigma$). Mittels reduzierten Regularitäts-Voraussetzungen an die äusseren Kräfte löst Nitsche ([JNi7]) das Problem eleganter mit dem Ergebnis:

$$\|u\|_{H_1} + \|p\|_{\dot{L}_2} \le c\|\sigma\|_{H_0} + \|h\|_{\dot{L}_2}$$

$$\|u\|_{C_{1.\lambda}} + \|p\|_{\dot{C}_{0.\lambda}} \le c\|\sigma\|_{C_{0.\lambda}} + \|h\|_{\dot{C}_{0.\lambda}}$$

Linear parabolic equations

We consider the two parabolic equations

$$\dot{w} - w'' = f \qquad\qquad \dot{z} - z'' = 0 \qquad\qquad \text{in } (0,1) \times [0,T]$$

$$w(0,t) = w(1,t) = 0 \qquad z(0,t) = z(1,t) = 0 \qquad \text{for } t \in (0,T]$$

$$w(x,o) = 0 \qquad\qquad z(x,o) = g(x) \qquad \text{for } x \in (0,1) \ .$$

The following compatibility relations for the initial value function have to be fulfilled in order to ensure corresponding regularity of the solution z :

$$g(1) = 0 \ , \ g'(0) = 0 \ , \ g''(1) = g'^2(1) \ , \text{ etc.}$$

Let $w_i := (w, \varphi_i)$ resp. $f_i := (f, \varphi_i)$ being the generalized Fourier coefficient related to the eigen pairs $-v_i'' = \lambda_i v_i$. Then it holds

$$\dot{w}_i(t) + \lambda_i w_i(t) = f_i(t) \ \text{ and } \ w_i(0) = 0 \ .$$

with the solution

$$w_i(t) = \int_0^t e^{-\lambda_i(t-\tau)} f_i(\tau) d\tau \ \cdot$$

The following shift theorems hold true:

Lemma: i) $\displaystyle \int_0^T \|w\|_{k+2}^2 dt \leq c \int_0^T \|f\|_k^2 dt$

                  ii) $\displaystyle \int_0^T t^{-1/2} \|w\|_{k+2}^2 dt \leq c \int_0^T t^{-1/2} \|f\|_k^2 dt$

                  iii) $\displaystyle \|z(t)\|_k^2 \leq c t^{-(k-l)} \|g\|_l^2 \quad , \quad \int_0^T t^{-1/2} \|z'\|_{-1/2}^2 dt \leq c\|g\| \ \cdot$

Proof: is given in the appendix, but we recall the basic idea, which is about change the order of integration:

$$\int_0^T \|w\|_{k+2}^2 dt = \sum \lambda_i^{k+2} \int_0^T w_i^2(t) dt \leq \sum \lambda_i^{k+2} \int_0^T \left[\int_0^t e^{-\lambda_i(t-\tau)} d\tau \right] \left[\int_0^t e^{-\lambda_i(t-\tau)} f_i^2(\tau) d\tau \right] dt$$

$$\leq \sum \lambda_i^{k+2} \int_0^T \lambda_i^{-1} \left[\int_0^t e^{-\lambda_i(t-\tau)} f_i^2(\tau) d\tau \right] dt = \sum \lambda_i^{k+1} \int_0^T f_i^2(\tau) \left[\int_\tau^T e^{-\lambda_i(t-\tau)} dt \right] d\tau \ \cdot$$

Linear parabolic equations with a singular lower order coefficient
(Klaus Höllig)

For the solution of the problem (with a $t^{-1/2}$ singularity)

$$u_t - u_{xx} - 2t^{-1/2}u_x = t^{-1/2}f \qquad (x,t) \in [0,1] \times [0,T]$$

$$u(*,0) = \Phi$$

$$u(0,*) = u(1,*) = 0$$

the estimate

$$\left\| u, u_x, \sqrt{t}\, u_{xx} \right\|_{\alpha, \alpha/2} \le c \left\{ \|f\|_{\alpha, \alpha/2} + \|\Phi, \Phi'\|_{\alpha, \alpha/2} \right\}$$

holds if the data f, Φ satisfy the appropriate compatibility condition.

Proof: The proof is built on a series of estimates, based on the kernel function

$$k(x,t,s) := \frac{1}{\sqrt{4\pi}} \frac{1}{\sqrt{t-s}} e^{-\frac{\left[x + 2(\sqrt{t} - \sqrt{s})\right]^2}{4(t-s)}}$$

which fulfills the relation

$$k_s - k_{xx} - 2s^{-1/2}k_x = 0 \quad .$$

We recall the definition of the Hölder norms:

$$\rho^\alpha(z_1, z_2) := \sqrt{|x_1 - x_2|^2 + |t_1 - t_2|}$$

$$[u]_{\alpha/2.\alpha} := \sup_{z_1 \ne z_2} \frac{\left(u(z_1) - u(z_2)\right)}{\rho^{\alpha/2}(z_1 - z_2)}$$

$$[u]_{1+\alpha/2.2+\alpha} := [\dot{u}]_{\alpha/2.\alpha} + \sum_{i,j}^{n} \left[u_{x_i x_j}\right]_{\alpha/2.\alpha} < \infty$$

and note for the heat equation the following shift theorem

$$[u]_{1+\alpha/2.2+\alpha} \le c[f]_{\alpha/2.\alpha} \quad .$$

Remark: In Nitsche/Wheeler Nitsche's weight function (to prove L_∞ estimate for elliptic problems) has been modified to capture specific parabolic weighting:

$$\mu_{x_0, t_0}(x,t) := |x - x_o|^2 + |t - t_0| \quad , \quad u(x_0, t_0) := \|u\|_{L_\infty(L_\infty)} \quad .$$

Appendix

§ 1 An unusual proof of the shift theorem for the Stokes problem (([JNi7]))

The proof is restricted to $n=2$, as the argument is based on Cauchy-Riemann Differentialgleichungen, um die solenoid condition und die Stokes Equations zu entkoppeln.

Stationary Stokes problem ([JNi7]): let n=2; consider the boundary value problem

$$-\Delta \underline{v} - \nabla p = \underline{f} \quad \text{in } \Omega$$

$$div \, \underline{u} = h \quad \text{in } \Omega$$

$$\underline{u} = 0 \quad \text{on } \partial\Omega$$

Let $\dot{L}_2$ denote the factor space L_2 / R equipped with the corresponding factor norm and let the right hand sides $\underline{f}$ with a reduced regularity assumptions in the form

$$\underline{f} = -div(\underline{\sigma}) \text{ i.e.} \quad f_i = \sum_{j=1}^{n} \sigma_{ij|j}$$

In this case the weak solution of the Stokes boundary value problem is characterized by

$$(\nabla \underline{v}, \nabla \underline{w}) - (p, div \, \underline{w}) = (\underline{\sigma}, \nabla \underline{w}) \quad \text{for all } \underline{w} \in D(\Omega)$$

$$(q, \nabla \underline{v},) = (q, h) \quad \text{for all } q \in \dot{L}_2 \ .$$

The following two shift theorems hold true ([JNi7], [HSo] p. 107)

Theorem:
1. Assume the regularity $\underline{f} = -div(\underline{\sigma})$ with $\underline{\sigma} \in H_{-0}$ and $h \in \dot{L}_2$. Then the unique (weak) solution $\{\underline{v}, p\}$ of the boundary value problem has the regularity $\underline{v} \in H_{-1}$ and $p \in \dot{L}_2$ and the a priori estimate holds true:

$$\left\| \underline{v} \right\|_{H_1} + \left\| p \right\|_{\dot{L}_2} \leq c \left\| \underline{\sigma} \right\|_{H_0} + \left\| h \right\|_{\dot{L}_2}$$

2. Assume the regularity $\underline{f} = -div(\underline{\sigma})$ with $\underline{\sigma} \in C_{-0.\lambda}$ and $h \in \dot{C}_{0.\lambda} := C_{0.\lambda} \cap \dot{L}_2$. Then the unique (weak) solution $\{\underline{v}, p\}$ of the boundary value problem has the regularity $\underline{v} \in C_{-1.\lambda}$ and $p \in \dot{C}_{0.\lambda}$ and the a priori estimate holds true:

$$\left\| \underline{v} \right\|_{C_{-1.\lambda}} + \left\| p \right\|_{\dot{C}_{0.\lambda}} \leq c \left\| \underline{\sigma} \right\|_{C_{-0.\lambda}} + \left\| h \right\|_{\dot{C}_{0.\lambda}}$$

§2 Parabolic shift theorems

We consider the two parabolic equations

$$\dot{w} - w'' = f \qquad\qquad \dot{z} - z'' = 0 \qquad\qquad \text{in } (0,1) \times [0,T]$$

$$w(0,t) = w(1,t) = 0 \qquad z(0,t) = z(1,t) = 0 \qquad \text{for } t \in (0,T]$$

$$w(x,o) = 0 \qquad\qquad z(x,o) = g(x) \qquad\qquad \text{for } x \in (0,1) \;.$$

The following compatibility relations for the initial value function have to be fulfilled in order to ensure corresponding regularity of the solution z:

$$g(1) = 0 \;,\; g'(0) = 0 \;,\; g''(1) = g'^2(1) \;, \text{ etc.}$$

Let $w_i := (w, \varphi_i)$ resp. $f_i := (f, \varphi_i)$ being the generalized Fourier coefficient related to the eigen pairs $-v_i'' = \lambda_i v_i$. Then it holds

$$\dot{w}_i(t) + \lambda_i w_i(t) = f_i(t) \;\text{ and }\; w_i(0) = 0 \;.$$

with the solution

$$w_i(t) = \int_0^t e^{-\lambda_i(t-\tau)} f_i(\tau) d\tau \;\cdot$$

The following shift theorem holds true:

Lemma:

i) $$\int_0^T t^{-1/2} \|w\|_{k+2}^2 \, dt \le c \int_0^T t^{-1/2} \|f\|_k^2 \, dt$$

ii) $$\|z(t)\|_k^2 \le ct^{-(k-l)} \|g\|_l^2 \;,\; \int_0^T t^{-1/2} \|z'\|_{-1/2}^2 \, dt \le c\|g\| \;\cdot$$

Proof: i) It holds for $\tau \le t$

$$\int_0^T t^{-1/2} \|w\|_{k+2}^2 \, dt = \sum \lambda_i^{k+2} \int_0^T t^{-1/2} w_i^2(t) dt \le \sum \lambda_i^{k+2} \int_0^T \left[\int_0^t e^{-\lambda_i(t-\tau)} d\tau \right] \left[\int_0^t \tau^{-1/2} e^{-\lambda_i(t-\tau)} f_i^2(\tau) d\tau \right] dt$$

$$\le \sum \lambda_i^{k+2} \int_0^T \lambda_i^{-1} \left[\int_0^t \tau^{-1/2} e^{-\lambda_i(t-\tau)} f_i^2(\tau) d\tau \right] dt \;\cdot$$

Exchanging the order of integration gives

$$\int_0^T \int_0^t \tau^{-1/2} e^{-\lambda_i(t-\tau)} F_i^2(\tau) d\tau dt = \int_0^T \int_t^T \tau^{-1/2} e^{-\lambda_i(t-\tau)} F_i^2(\tau) dt d\tau = \int_0^T F_i^2(\tau) dt \left[\int_t^T e^{-\lambda_i(t-\tau)} d\tau \right] \le \lambda_i^{-1} \int_0^T \tau^{-1/2} F_i^2(\tau) dt$$

and therefore

$$\int_0^T t^{-1/2}\|w\|_{k+2}^2\,dt \le c\int_0^T t^{-1/2}\|F\|_k^2\,dt \ .$$

ii) From $z(x,t)=\sum z_\nu(t)\varphi_\nu(x)$ it follows

$$\dot z - z'' = \sum(\dot z_\nu(t)+\lambda_\nu z_\nu(t))\varphi_\nu(x) = 0 \ .$$

Therefore

$$z_\nu(t)=z_\nu(0)e^{-\lambda_\nu t} \text{ and } z_\nu(0)=g_\nu=(g,\varphi_\nu) \ .$$

Putting
$$C_{k.l}(t):=\sup_{\lambda_\nu\ge m>0}\lambda_\nu^{k-l}e^{-2\lambda_\nu t}$$

it follows

$$\|z(t)\|_k^2 = \sum\lambda_\nu^k z_\nu^2(t) = \sum\lambda_\nu^k e^{-2\lambda_\nu t}g_\nu \le C_{k.l}(t)\sum\lambda_\nu^l e^{-2\lambda_\nu t}$$

The conditions

$$(k-l)\lambda^{k-l-1}e^{-2\lambda_\nu t}+\lambda^{k-l}(-2t)e^{-2\lambda_\nu t}=0 \quad \text{resp.} \quad (k-l)\lambda^{k-l-1}e^{-2\lambda_\nu t}=2t\lambda^{k-l}e^{-2\lambda_\nu t}$$

leads to (for the critical case $k>l$) $\lambda\approx t^{-1}$ **q.e.d.**

§3 Generalized Lemmas of Gronwall

Generalized Lemma of Gronwall (version 1): Let $\psi(t) \in C^0[0, a]$ be a real valued function and $h(t) \in L_1(0, a)$ be non-negative function with

$$\psi(t) \le \alpha + \int_0^t h(\tau)\psi(\tau)d\tau \quad , \quad \alpha \in R \ .$$

Then

$$\psi(t) \le \alpha * e^{\int_0^t h(\tau)d\tau} \ .$$

Generalized Lemma of Gronwall (version 2): Let $\psi(t) \in C^0[0, a]$ be a real valued function and $h(t) \in L_1(0, a)$ be non-negative function with

$$\psi(t) \le \alpha(t) + \int_0^t h(\tau)\psi(\tau)d\tau \quad , \quad \alpha \in R \ .$$

Then

$$\psi(t) \le \alpha(t) + \int_0^t \alpha(\tau)h(\tau)e^{H(t)-H(\tau)}d\tau$$

with $H(\tau) := \int_0^\tau h(s)ds$.

Generalized Lemma of Gronwall (version 3: log type) ([YGi1]):**:** Let a, β be non-negative constants. Assume that a non negative function $a(t, s)$ satisfies $a(*,*) \in C(0 \le s < t \le T)$, $a(t,*) \in L_1(0, t)$ for all $t \in ((0, T)]$. Furthermore, we assume that there exists a positive constant ε_0 such that

$$\sup_{0 \le t \le T} \int_{t-\varepsilon_0}^t a(t, s)ds \le 1/2 \quad \cdot$$

If a non negative function $f \in C([0, T])$ satisfies

$$f(t) \le \alpha + \int_0^t a(t,s)f(s)ds + \beta \int_0^t \{1 + \log(1 + f(s))\}f(s)ds$$

for all $t \in [0, T]$. Then we have

$$f(t) \le e^{\left\{1 + \frac{\gamma}{\beta} + \log(1 + 2\alpha)\right\}e^{2\beta t}}$$

for all $t \in [0, T]$. Here we put

$$\gamma := \sup_{0 \le t \le T}\left\{ \sup_{0 \le s \le t - \varepsilon_0} a(t, s)\right\}$$

Lemma of Gronwall (version 4): Let $a(t)$ and $b(t)$ nonnegative functions in $[0, A)$ and $0 < \delta < 1$. Suppose a nonnegative function $y(t)$ satisfies the differential inequality

$$y'(t) + b(t) \leq \alpha(t) y^{\delta}(t) \qquad \text{on} \qquad [0, A)$$

$$y(0) = y_0.$$

Then for $0 \leq t < A$

$$y(t) + \int_0^t b(\tau) d\tau \leq (2^{\delta/(1-\delta)} + 1) y_0 + 2^{\delta/(1-\delta)} \left[\int_0^t \alpha(\tau) d\tau \right]^{\delta/(1-\delta)}$$

Proof: solving

$$y'(t) \leq \alpha(t) y^{\delta}(t)$$

leads to

$$\cdot \ y(t) \leq y_0 + \left[\int_0^t \alpha(\tau) d\tau \right]^{\delta/(1-\delta)}$$

A standard formula: For

$$F(x) := \int_{\varphi(t)}^{\psi(t)} u(x,t) dx$$

It holds

$$F'(x) := \int_{\varphi(t)}^{\psi(t)} \dot{u}(x,t) dx + \dot{\psi}(t) u(\psi(t),t) - \dot{\varphi}(t) u(\varphi(t),t)$$

From this it follows for

$$s(t) = s - \int_0^{s(t)} u(x,t) dx$$

the relation

$$\dot{s}(t) = 1 - \int_0^{s(t)} \dot{u}(x,t) dx + \dot{s}(t) u(s(t),t) - 0 = 1 - \int_0^{s(t)} u_{xx}(x,t) dx = 1 - u_x(s(t),t) + u_x(0),t) = -u_x(s(t),t) \ \cdot$$

§4 The one dimensional Stefan problem

We recall the relation of the free boundary Stefan problem with our model problem ([JNi3]): Let

$$\Omega := \{(y,\tau)|\tau > 0, 0 < y < s(\tau)\} \quad \text{with } s(0) = 1 .$$

The free boundary Stefan problem is looking for a solution $U(y,\tau)$ fulfilling

$$U_\tau(y,\tau) - U_{yy}(y,\tau) = 0 \qquad \text{in } \Omega$$

$$U_y(0,\tau) = 0 \qquad \text{for } \tau > 0 .$$

Along the free boundary $y = s(\tau)$ the function $U(y,\tau)$ vanishes, i.e.

$$U(s(\tau),\tau) = 0 \qquad \text{for } \tau > 0$$

and the function $s(\tau)$ fulfills the additional condition

$$s_\tau(\tau) + U_y(s(\tau),\tau) = 0 \quad \text{for } \tau > 0 .$$

The Stefan problem can be transformed to a non-linear parabolic differential equation with fixed boundary of the area

$$Q := \{(x,\tau)|\tau > 0, 0 < x < 1\}$$

by the transformation

$$x = s^{-1}(\tau)y$$

and the variable change $\tau \rightarrow t$ defined by

$$\frac{d\tau}{dt} = s^2(\tau) , \quad \tau(0) = 0 .$$

Then the free boundary problem is looking for a solution $u(x,t) = U(y,\tau)$ fulfilling

$$u_t(y,\tau) - u_{xx}(x,t) = -xu_x(1,t)u_x \quad \text{in} \quad Q$$

with the boundary conditions

$$(*) \quad u_x(0,t) = 0 \text{ for } t > 0$$

$$(*) \quad u(1,t) = 0 \text{ for } t > 0$$

$$u(x,0) = f(x) \text{ for } x \in (0,1) \ .$$

The free boundary can then be derived from the differential equation

$$\frac{ds}{dt} = -u_x(1,t)s(t) \ , \quad s(0) = 1 \ .$$

Let

$$\dot{H}_1 := \{ w \mid w \in H_1(0,1), > 0, w(0) = 0 \} = \{ w \mid w' \in L_2(0,1), > 0, w(0) = 0 \}$$

Then $v := u_x$ belongs to $\dot{H}_1$ and, for any $v \in \dot{H}_1$ the function defined by

$$u(x,t) = -\int_x^1 v(z,t)dz$$

satisfies the boundary condition (*). Multiplying the differential equation above with w_x ($w \in \dot{H}_1$) and integration gives

$$\int_0^1 u_{xx}w_x + u_{xt}wdx = u_x(1,t)\int_0^1 xu_x w_x dx \ .$$

From [JNixxx] we recall the Model Problem:

P_v : Find v such that $v(*,x) \in \dot{H}_1$ and

$$(\dot{v}, w) + (v', w') = v(1)(xv, w') \text{ for } w \in \dot{H}_1 \text{ and } t > 0$$

$$v(*,0) = f'$$

with its related Finite Element approximation problem:

Problem P_h : let $S_h \subset \dot{H}_1$ be an approximation space. Find a function v_h with $v_h(*,t) \in S_h \subset \dot{H}_1$ fulfilling

$$(\dot{v}_h, \chi) + (v'_h, \chi') = v_h(1)(xv_h, \chi') \text{ for } \chi(*,t) \in S_h \subset \dot{H}_1 \text{ and } t > 0$$

$$v_h(*,0) = f'_h =: g_h := P_h g \ .$$

In case of a regular initial value function $v(*,0) = f'$ **quasi-optimal** order of convergence has been proven ([JNi1]) in the form

$$\|v - v_h\|_{L_\infty(0,\bar{t};L_\infty)} = O(h^r) .$$

In case of a reduced regularity assumption in the form

$$v(x,0) = u_x(x,0) = f'(x) =: g(x) \in L_2(0,1)$$

a **non-quasi-optimal** convergence of the FEM has been proven ([JNi3]) in the form

$$\|v - v_h\| = c_\alpha h^\alpha t^{-1/2} , \quad \alpha < 1 .$$

The key estimate to prove this result is given by

$$\frac{d}{dt}\left\{\|\Phi\|^2 + c\|\varphi'\|^2\right\} + \left\{\|\Phi'\|^2 + \|\Phi\|^2\right\} \le ct^\alpha \left\{\|\varepsilon'\|^2 + t^{-1/2}\|\varepsilon\|^2\right\}$$

with $c := \kappa e^{-\eta^{1-\alpha}}$ after using a duality argument with

$$-w'' = \Phi \quad x \in (0,1)$$

$$w(0) = w'(1) = 0$$

to estimate

$$\|\Phi\|^2 = -(\dot{\Phi}, \varphi) + a(\varepsilon, \varphi) + v(1)(x\Phi, \varphi') + \Phi(1)(xv_h, \varphi')$$

$$= -\frac{1}{2}\frac{d}{dt}\|\varphi'\|^2 + a(\varepsilon, \varphi) + v(1)(x\Phi, \varphi') + \Phi(1)(xv_h, \varphi')$$

and applying the lemma of Gronwall.

In this case the initial value function $g \in L_2(0,1)$ is approximated by the $L_2 - projection$ $g_h = P_h g \in S_h$, i.e.

$$(g_h, \vartheta) = (g, \vartheta) \quad \text{for} \quad \vartheta \in S_h \subset \dot{H}_1 .$$

28

We recall the core elements of the proof, in order to motivate our alternative proposal of an adequate Hilbert space (which is $H_{1/2}$) to prove **quasi-optimal** convergence simulatiously for both cases. The corresponding proposed "energy" norms are:

i) $$\frac{d}{dt}\|v\|_{1/2}^2 + \|v'\|_{1/2}^2$$

ii) $$t^{1/2}\|v\|_{1/2}^2(t) + \int_0^T t^{1/2}\|v'\|_{1/2}^2\,dt$$

For $w, z \in \dot{H}_1$ by

$$\tilde{a}(w,z) := (w', z') - v(1)(xw, z') - w(1)(xv, z')$$

a bilinear form is defined, fulfilling the following relation:

i) $\tilde{a}(*,*)$ is bounded in $\dot{H}_1$, i.e. $|\tilde{a}(w,z)| \le M\|w'\|\|z'\|$

i) $\tilde{a}(*,*)$ is coercive in $\dot{H}_1$, i.e. $\tilde{a}(w,w) \ge m\|w'\|^2 - \Lambda\|w\|^2$

for $m, M, \Lambda > 0$ depending only on $\|v\|_{L_\infty(L_\infty)}$. Therefore by

$$\tilde{a}_\Lambda(w,z) := \tilde{a}(w,z) + \Lambda(w,z)$$

a bounded, positive definite, not symmetric bilinear form is defined, which is applied to define the corresponding Ritz-Galerkin approximation:

Let $\tilde{v}_h := R_h(v) \in \dot{S}_h$ with $\dot{S}_h := S_h \cap \{\chi | \chi(0) = 0\}$ be the Ritz-Galerkin-approximation to v with respect to this form, i.e. let

$$\tilde{a}_\Lambda(v - \tilde{v}_h, \chi) = 0 \text{ for } \chi(*,t) \in S_h \subset \dot{H}_1 \text{ and } t > 0 \ .$$

The Finite Element space $S_h^{k,t} \subset H_k(I)$ with $k < t$ consists of functions $\chi \in S_h^{k,t}$ with the properties:

i) the restriction of χ to any triangle Δ of the triangulation $\Delta \in \Gamma_h$ is a polynomial of degree less than t

ii) χ is $(k-1)-$ time continuously differentiable in $I := (0,1)$.

The following properties are valid:

i) $S_h^{k,t} \subset H_k(I)$

ii) $\inf_{\chi \in S_h} \|v - \chi\|_k \le ch^{l-k}\|v\|_l$ for $v \in H_l$

iii) $\|\chi\|_k \le ch^{-(k-l)}\|v\|_l$ for $\chi \in S_h$.

The error $e = v - v_h$ of the problem P_h is split in the form

$$e-= v - \tilde{v}_h - (v_h - \tilde{v}_h) =: \varepsilon - \phi \quad \text{with} \quad \phi \in S_h = S_h^{k,t} \subset \dot{H}_1 \;.$$

The error $\varepsilon = v - \tilde{v}_h$ of the approximation according to the BLF $a_\Lambda(w,z)$ is estimated by

$$\|\varepsilon\|_\beta + \|\dot{\varepsilon}\|_\beta \le ch^{r-\beta} \quad \text{for} \quad \beta = -1,0,1 \;.$$

The case $\beta = 1$ follows directly from the fact, that by $a_\Lambda(w,w)$ a norm is defined, which is equivalent to $\|w'\|^2$. With the additional regularity assumption that the approximation spaces are at least quadratic splines, i.e. $t > 2$ the duality argument of Nitsche-Aubin can be applied to prove the case $\beta = 0,-1$.

In order to estimate the correction term $\phi \in S_h \subset \dot{H}_1$ and therefore e itself the norm equivalence of $\tilde{a}_\Lambda(\phi,\phi)$ and (ϕ',ϕ') in combination with the defining approximation equation

$$(\dot{\phi},\chi) + \tilde{a}_\Lambda(\phi,\chi) = \Lambda(\phi,\chi) + (\dot{\varepsilon},\chi) - \Lambda(\varepsilon,\chi) - e(1)(xe,\chi')$$

is applied.

In a **first step** the linear problem

$$(\dot{\phi},\chi) + \tilde{a}_\Lambda(\phi,\chi) = \Lambda(\phi,\chi) + E(1)(x\phi,\chi') + (\dot{\varepsilon},\chi) - \Lambda(\varepsilon,\chi) - E(1)(x\varepsilon,\chi')$$

is analyzed, where an arbitrarily function $E(1)$ is chosen, replacing the quadratic term $e(1)$. This leads to the estimations of the type

$$\|e\|_{L_\infty(L_2)} \le ch^r(1 + \|E\|_{L_\infty(L_\infty)}) \quad \text{and} \quad \|e\|_{L_\infty(L_\infty)} \le c(h^r + h^{r-1/2})\|E\|_{L_\infty(L_\infty)} \;.$$

In a **second step**, because the image of e of any E with

$$E \in B_1 := \left\{ w \big| \|w\|_{L_\infty(L_\infty)} \le 1 \right\}$$

is contained in for $h \le \bar{h} := c^*$, the Schauder's fix point theorem guarantees the existence of an E with $e = E$ for $h \le \bar{h}$. This then proves

$$\|v - v_h\|_{L_\infty(0,\bar{t};L_\infty)} = O(h^k)$$

for sufficiently smooth v in $(0,1) \times [o,\bar{t}]$, properly chosen $\bar{t} > 0$ and at least quadratic splines.

Remark: In [JNi3] it shown the a priori estimate

Theorem: Consider the problem P_h with the assumed regularity $g \in L_2(0,1)$ of the initial data. Further let $S_h \subset \dot{H}_1$ be a finite dimensional approximation space. There is a $T > 0$ depending only on $\|g\|$ such that the problem P_h has a unique solution for $t \leq T$. For the semi-discrete Galerkin-approximation v_h it holds $v_h \in C^\infty((0,T); \dot{H}(0,1))$ and the a priori bounds

i)
$$\sup_{0 \leq t \leq T} \left\{ t^{2k} \left\| \partial_t^k v_h \right\|^2 + \int_0^t \tau^{2k} \left\| \partial_t^k v_h' \right\|^2 d\tau \right\} \leq c_{2k}^2$$

ii)
$$\sup_{0 \leq t \leq T} \left\{ t^{2k+1} \left\| \partial_t^k v_h' \right\|^2 + \int_0^t \tau^{2k+1} \left\| \partial_t^{k+1} v_h \right\|^2 d\tau \right\} \leq c_{2k+1}^2$$

are valid. The constants c_v are independent of S_h and

$$\overline{c}_k := \max\{c_{2k}, c_{2k+1}\}$$

is bounded by

$$\overline{c}_k \leq C^k (k!)^2$$

with C depending only on $\|g\|$. One especially gets

$$\left| v_h(*,t) \right| \leq \sqrt{2} \left\| v_h \right\|^{1/2} \left\| v_h' \right\|^{1/2} \leq \sqrt{2} \overline{c}_0 t^{-1/4}$$

Since the approximation s_h on the free boundary is defined by

$$\dot{s}_h = -v_h s_h \quad , \quad s_h(0) = 1$$

a uniform Hölder-continuity of s_h with an exponent up to $3/4$ is the consequence.

§ 5 (Hyper) Singular Integral Operator

Let Γ denote the boundary of the unit sphere and $\oint$ the integral from 0 to 2π in the Cauchy-sense. Then for $u \in H = \overset{*}{L_2}(\Gamma)$ and for real β the Fourier coefficients are defined by

$$u_\nu := \frac{1}{2\pi} \oint u(x) e^{-i\nu x} dx \ .$$

This enables the definitions of the norms (e.g. [KBr])

$$\|u\|_\beta^2 := \sum_{-\infty}^{\infty} |\nu|^{2\beta} |u_\nu|^2$$

with corresponding Hilbert scales

$$H_\beta := \left\{ u \mid \|u\|_\beta^2 < \infty \right\} \ .$$

For the two singular integral operators

(A) $\quad (Nu)(x) := -\oint \log 2\sin\frac{x-y}{2} u(y) dy =: \oint k(x-y)u(y)dy \quad$ and $\quad D(N) = H = \overset{*}{L_2}(\Gamma)$

(H) $\quad (Hu)(x) := [u](x) := \frac{1}{2\pi} \oint \cot\frac{x-y}{2} u(y) dt = -\lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_\varepsilon^\pi [u(x+y) - u(x-y)]\cot\frac{y}{2} dy \quad \cdot$

we note the following properties:

Lemma: The operator H is skew-symmetric in the space $L_2(0,2\pi)$ ([DGa],[BPe], chapter 2, §9) and maps the space $H := L_2(0,2\pi) - R$ isometric onto itself. It holds [KBr1]

$$\|Hu\| = \|u\| \quad \text{and} \quad H^2 = -I \ , \ (Hu,v) = -(u,Hv) \quad , \quad [u'](x) = [u]'(x)$$

$$[xH - Hx](u(x)) = \frac{1}{\pi} \int_{-\infty}^{\infty} u(y) dy \quad \cdot$$

We note

$$(Hu)(x) = i \sum_1^{\infty} \left[u_{-\nu} e^{-i\nu x} - u_\nu e^{i\nu x} \right] \in L_2 \text{ for } u \in L_2 \ .$$

Lemma: Let $c > 0$ denote different constants, then it holds

    i) $\|Nu\|_\beta = c\|u\|_{\beta-1}$, $(Nu, v)_\beta = (u, Nv)_\beta = c(u, u)_{\beta-1/2}$;

    ii) $c\|u\|_\beta = \|Hu\|_\beta$, $-(Hu, v)_\beta = (u, Hv)_\beta$;

    iii)   if $u \in L_2$, then $Hu \in L_2$;

    iv) $(Nu')(x) = (Hu)(x)$, $(Nu)'(x) = -(Hu)(x)$, $H(u')(x) = (H(u)'(x)$;

    v)   if $u \in L_2$, then $v = Hu \in H_1$;

    vi)   For a Hilbert-transformed function $u^H(x) := [Hu](x)$ it holds
            $[xH - Hx](u^H(x)) = 0$, $[xN - Nx](u) \approx Hu$

    vii) $\|u\|_{L_\infty} \leq c\|u'\|_{-1/2} c\|u\|_{1/2}$.

Proof: The Fourier coefficients of the convolutions (A), (H) are given by ([DGa] pp.63, appendix, [SGr] 1.441)

$$(Nu)_\nu = k_\nu^A u_\nu = \frac{1}{2|\nu|} u_\nu$$

$$(Hu)_\nu = k_\nu^H u_\nu = -i\,sign(\nu)u_\nu \qquad .$$

This leads to the propositions i)-v) ([DGa],[BPe],chapter 2, §9, [NMu], §18, 19)

vi) follows by the inequality

$$\left|u^2(x)\right| = 2\left|\int_0^x u(\xi)u'(\xi)d\xi\right| \leq c|(u, u')| \leq c|(Nu', u')| = c\|u'\|_{-1/2}^2 . \qquad \textbf{q.e.d.}$$

Remark: The operator H defines also a bijective mapping from the Hölder space $C^{0,\lambda}$ onto itself [NMu], §18, 19.

For the general case we mention: let

$$(H^* u)(x) := \frac{1}{\pi} \oint \log \frac{1}{x-y} du(y) \qquad ,$$

then the integral equation $v = Bu$ has the more general structure [JNi5]

$$v = H^* u + \tilde{H}u \qquad \text{resp.} \qquad u = -H^* v + H\tilde{H}v$$

with $\tilde{H}$ being an operator compact in $H = L_2(0, 2\pi) - R$.

§6 Hilbert Scales

Nitsche lecture notes:

There are certain relations between the spaces $\{H_\alpha | \alpha \geq 0\}$ for different indices:

Lemma: Let $\alpha < \beta$. Then

$$\|x\|_\alpha \leq \|x\|_\beta$$

and the embedding $H_\beta \to H_\alpha$ is compact.

Lemma: Let $\alpha < \beta < \chi$. Then

$$\|x\|_\beta \leq \|x\|_\alpha^\mu \|x\|_\gamma^\nu \text{ for } x \in H_\gamma$$

with $\mu = \dfrac{\gamma - \beta}{\gamma - \alpha}$ and $\nu = \dfrac{\beta - \alpha}{\gamma - \alpha}$.

Lemma: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

i) $\qquad \|x - y\|_\alpha \leq t^{\beta - \alpha} \|x\|_\beta$

ii) $\qquad \|x - y\|_\beta \leq \|x\|_\beta$, $\|y\|_\beta \leq \|x\|_\beta$

iii) $\qquad \|y\|_\gamma \leq t^{-(\gamma - \beta)} \|x\|_\beta$.

Corollary: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

i) $\qquad \|x - y\|_\rho \leq t^{\beta - \rho} \|x\|_\beta \qquad$ for $\alpha \leq \rho \leq \beta$

ii) $\qquad \|y\|_\sigma \leq t^{-(\sigma - \beta)} \|x\|_\beta \qquad$ for $\beta \leq \sigma \leq \gamma$.

Remark: Our construction of the Hilbert scale is based on the operator A with the two properties i) and ii). The domain $D(A)$ of A equipped with the norm

$$\|Ax\|^2 = \sum_{i=1}^{} \lambda_i^2 (x, \varphi_i)^2$$

turned out to be the space H_2 which is densely and compactly embedded in $H = H_0$. It can be shown that on the contrary to any such pair of Hilbert spaces there is an operator A with the properties i) and ii) such that

$$D(A) = H_2 \quad R(A) = H_0 \text{ and } \|x\|_2 = \|Ax\|.$$

For $t > 0$ we introduce an additional inner product resp. norm by

$$(x, y)^2_{(t)} = \sum_{i=1} e^{-\sqrt{\lambda_i} t} (x, \varphi_i)(y, \varphi_i)$$

$$\|x\|^2_{(t)} = (x, x)^2_{(t)} \ .$$

Now the factor have exponential decay $e^{-\sqrt{\lambda_i} t}$ instead of a polynomial decay in case of λ_i^α. Obviously we have

$$\|x\|_{(t)} \leq c(\alpha, t) \|x\|_\alpha \quad \text{for } x \in H_\alpha$$

with $c(\alpha, t)$ depending only from α and $t > 0$. Thus the $(t) - norm$ is weaker than any $\alpha - norm$. On the other hand any negative norm, i.e. $\|x\|_\alpha$ with $\alpha < 0$, is bounded by the $0 - norm$ and the newly introduced $(t) - norm$. It holds:

Lemma 5: Let $\alpha > 0$ be fixed. The $\alpha - norm$ of any $x \in H_0$ is bounded by

$$\|x\|^2_{-\alpha} \leq \delta^{2\alpha} \|x\|^2_0 + e^{t/\delta} \|x\|^2_{(t)}$$

with $\delta > 0$ being arbitrary.

Remark 2: This inequality is in a certain sense the counterpart of the logarithmic convexity of the $\alpha - norm$, which can be reformulated in the form ($\mu, \nu > 0, \mu + \nu > 1$)

$$\|x\|^2_\beta \leq \nu \varepsilon \|x\|^2_\gamma + \mu e^{-\nu/\mu} \|x\|^2_\alpha$$

applying Young's inequality to

$$\|x\|^2_\beta \leq (\|x\|^2_\alpha)^\mu (\|x\|^2_\gamma)^\nu \ .$$

The counterpart of lemma 4 above is

Lemma 6: Let $t, \delta > 0$ be fixed. To any $x \in H_0$ there is a $y = y_t(x)$ according to

   i) $\|x - y\| \leq \|x\|$

   ii) $\|y\|_1 \leq \delta^{-1} \|x\|$

   iii) $\|x - y\|_{(t)} \leq e^{-t/\delta} \|x\| \ .$

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