258. *On Fractional Powers of the Stokes Operator*

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1. Introduction and summary. The present paper is concerned with the so-called Stokes operator described below. Our objective is to prove a theorem concerning domains of fractional powers of the Stokes operator. This theorem has some applications to the Navier-Stokes equation [4], as is expected from important roles played by the fractional powers of the Stokes operator in recent works on the Navier-Stokes equation. For instance, see Sobolevskii [11, 12], Kato-Fujita [7], Fujita-Kato [3], and Masuda [10]. Moreover, we hope that the theorem is of some interests also from the view point of theory of fractional powers of operators and theory of interpolation of spaces.

Let Ω be a bounded domain in R^m with smooth boundary $\partial\Omega$. By L we denote $L_2(\Omega)$ of real m-vector functions defined in Ω. $C_{0,\sigma}^\infty$ is the set of all vector functions $\varphi \in C^\infty(\Omega)$ with $\operatorname{div}\varphi=0$ and $\operatorname{supp}\varphi \subset \Omega$. We put

$$H_\sigma = \text{the closure of } C_{0,\sigma}^\infty \text{ in } L_2(\Omega),$$
$$H_\sigma^1 = \text{the closure of } C_{0,\sigma}^\infty \text{ in } W_2^1(\Omega).$$

Here, $W_2^l(\Omega)$ means the Sobolev space of order l. The orthogonal projection from L onto H_σ is denoted by P. The operator $A_0=-P\Delta$ with domain $C_{0,\sigma}^\infty$ is positive and symmetric in the Hilbert space H_σ. The Friedrichs extension A of A_0 is called the *Stokes operator* in Ω. A is positive and self-adjoint. It should be noted that $Au=Pf$ $(f \in L)$ implies that

$$(1.1)\qquad \begin{cases} \Delta u - \nabla p = -f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

with some scalar function p. Actually, it is known [2, 8] that

$$(1.2)\qquad \mathcal{D}(A) = W_2^2(\Omega) \cap H_\sigma^1,$$

where $\mathcal{D}(A)$ is the domain of the operator A. On the other hand, we put $B=-\Delta$ with

$$(1.3)\qquad \mathcal{D}(B) = W_2^2(\Omega) \cap H^1,$$

where H^1 is the set of all $u \in W_2^1(\Omega)$ satisfying $u|_{\partial\Omega}=0$. Obviously, B is a positive self-adjoint operator in L.

Our theorem now reads:

Theorem 1.1. *Let A and B be as above. Then for any α in $0<\alpha<1$, we have*

(1.4) $$\mathcal{D}(A^\alpha)=\mathcal{D}(B^\alpha)\cap H_\sigma.$$

Remark 1.2. For $u\in L=L_2(\Omega)$ the condition $u\in H_\sigma$ is equivalent to $(u,\nabla p)_L=0$ ($\forall\nabla p\in L$), and furthermore, equivalent to that $\operatorname{div} u=0$ and the normal component of u vanishes on $\partial\Omega$. On the other hand, concrete characterizations of domains of $B^\alpha=(-\Delta)^\alpha$ have been given by Fujiwara [5], Grisvard [6] and some others. Thus (1.4) enables us to deduce criterions for u to belong to $\mathcal{D}(A^\alpha)$ which, however, will not be stated here explicitly.

2. Proof of Theorem 1.1. We shall make use of the following lemma concerning the trace space which is a special case of a theorem due to Lions [9].

Lemma 2.1. *Let X be a Hilbert space and let S be a positive self-adjoint operator in X. $\mathcal{D}(S)$ is the domain of S regarded as a Hilbert space with the graph norm. Then for α in $0<\alpha<1$, we have*

(2.1) $$\mathcal{D}(S^\alpha)=T\left(2,\frac{1}{2}-\alpha\,;\mathcal{D}(S),X\right).$$

We recall that $a\in T(2,\theta\,;X_0,X_1)$ for $-\frac{1}{2}<\theta<\frac{1}{2}$ if and only if there exists a $u\colon[0,\infty)\to X_0\subset X_1$ such that

(2.2) $$t^\theta u\in L_2(0,\infty\,;X_0),$$

(2.3) $$t^\theta u'\in L_2(0,\infty\,;X_1),$$

and $u(0)=a$. Here $X_0\subset X_1$ are two Banach spaces such that X_0 is dense in X_1 and the injection is continuous.

The following lemma has been proved by Cattabriga [2] and Ladyzhenskaya [8], and also can be read off from the proof of general theorems in Agmon-Douglis-Nirenberg [1]. (Notice (1.1).)

Lemma 2.2. *There exist constants C_1 and C_2 such that*

(2.4) $$\|\Delta A^{-1}\psi\|_{L_2(\Omega)}\leqq C_1\|A^{-1}\psi\|_{W_2^2(\Omega)}\leqq C_2\|\psi\|_{L_2(\Omega)}$$

for all $\psi\in H_\sigma$.

Proof of Theorem 1.1. By Lemma 1.1 we have

$$\mathcal{D}(A^\alpha)=T\left(2,\frac{1}{2}-\alpha\,;\mathcal{D}(A),H_\sigma\right),$$

$$\mathcal{D}(B^\alpha)=T\left(2,\frac{1}{2}-\alpha\,;\mathcal{D}(B),L\right).$$

From this and in view of $H_\sigma\subset L$ and $\mathcal{D}(A)\subset\mathcal{D}(B)$, it is easy to see $\mathcal{D}(A^\alpha)\subset\mathcal{D}(B^\alpha)\cap H_\sigma$. Thus we have to show the other inclusion. To this end, we first introduce the operator $\mathring{K}\colon\mathcal{D}(B)\to\mathcal{D}(A)$ by setting

(2.5) $$\mathring{K}\varphi=-A^{-1}P\Delta\varphi=A^{-1}PB\varphi\qquad(\varphi\in\mathcal{D}(B)).$$

By virtue of (2.4) we can easily show that $\mathring{K}$ admits of a bounded extension from L to H_σ. It should be noted that $K\varphi=\varphi$ for $\varphi\in H_\sigma$. Now we take an a from $\mathcal{D}(B^\alpha)\cap H_\sigma$. Since $a\in\mathcal{D}(B^\alpha)$, there exists a $u\colon[0,\infty)\to\mathcal{D}(B)$ such that

(2.6) $$t^{1/2-\alpha}u\in L_2(0,\infty\,;\mathcal{D}(B)),$$

(2.7) $$t^{1/2-\alpha}u' \in L_2(0, \infty; L),$$

and $u(+0)=a$. We put $v(t)=Ku(t)$. Then we have $v(+0)=Ka=a$, for $a \in H_\sigma$. Obviously, $v(t) \in \mathscr{D}(A)$ and $v'(t) \in H_\sigma$ for almost every t. We have also

$$\|v(t)\|_{\mathscr{D}(A)} = \|Ku(t)\|_{\mathscr{D}(A)} = \|\mathring{K}u(t)\|_{\mathscr{D}(A)}$$
$$\leqq C\|A\mathring{K}u(t)\|_{H_\sigma} \leqq C\|Bu(t)\|_L \leqq C\|u(t)\|_{\mathscr{D}(B)}.$$

Here we use the symbol C for various positive constants indifferently and have made use of the fact that A^{-1} and B^{-1} are bounded. We have

$$\|v'(t)\|_{H_\sigma} = \|Ku'(t)\|_{H_\sigma} \leqq C\|u'(t)\|_L,$$

since K is bounded. Combining these estimates with (2.6) and (2.7), we notice

$$t^{1/2-\alpha}v \in L_2(0, \infty; \mathscr{D}(A))$$

and

$$t^{1/2-\alpha}v' \in L_2(0, \infty; H_\sigma).$$

Consequently, $a=v(0) \in T\left(2, \frac{1}{2}-\alpha; \mathscr{D}(A), H_\sigma\right) = \mathscr{D}(A^\alpha)$, which completes the proof.

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