A SCATTERING TRINITY: THE REPRODUCING KERNEL, NULL-FIELD EQUATIONS AND MODIFIED GREEN'S FUNCTIONS

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SUMMARY

The close relation between the null-field method or **T**-matrix method and the modified Green's-function method has previously been shown. In this paper we establish the connection between the reproducing kernel both with the null-field equations and with the modified Green's functions. Furthermore, we demonstrate the relationship between an infinite matrix derived from the reproducing kernel and known results.

1. Introduction

IN RECENT years the null-field equations (also known as the **T**-matrix, extended boundary condition or Waterman method) have received considerable attention. Waterman (**1**) first derived equations of this type for electromagnetic scattering and later for acoustic scattering (see (**2**)). One important feature of the null-field equations is that, for the exterior Dirichlet and Neumann problems, they are uniquely solvable for all wave numbers k satisfying $\operatorname{Im} k \geqslant 0$ (see (**3**; **4**, Theorem 3.45; **5**)). That is, no non-uniqueness difficulties occur at interior eigenvalues.

A different approach for obtaining uniquely-solvable integral equations for exterior scattering problems has been given by Jones (**6**) and Ursell (**7**). There they alter the standard integral equations for the exterior Dirichlet and Neumann problems by introducing a modified kernel function.

In (**8**, **9**) Kleinman and Roach give two different possibilities for choosing the coefficients of the modified kernel function of Jones (**6**) and Ursell (**7**) and appropriately call their kernel functions modified Green's functions.

In a recent paper Kleinman, Roach and Ström (**10**) establish an important relationship between the null-field method and the modified Green's-function method.

In this paper we extend the results in (**10**). Here we establish the connection of the reproducing kernel (also known as the Bergman–Schiffer kernel, see (**11**)) both with the modified Green's functions and with the null-field equations. In view of the intrinsic linkage of each method with the other two approaches, we shall hereafter refer to these three approaches as a *scattering trinity*. Furthermore, we establish the relation between the remarkable matrix Q given in (**10**) and the reproducing matrix defined in section 4 of this paper.

In the next section we give our notation, definitions and some preliminary results. For the most part we adopt the same notation here as was used in (**10**). In section 3 we state for the convenience of the reader some pertinent facts about the null-field equations, and give certain results contained in (**10**) which we shall need for our purposes. In section 4 we establish the connection between the reproducing kernel and the modified Green's functions. We demonstrate in the last section the relation between the reproducing kernel and the null-field equations.

2. Notation, definitions and preliminary results

Let B_- denote a bounded domain in $\mathbb{R}^3$ containing the origin with a closed, simply connected C^2-surface ∂B, and let B_+ denote the region exterior to $\bar{B}_-$. Let $\hat{\mathbf{n}}$ denote a unit normal on ∂B directed out of B_-. A typical point $\mathbf{x} \in \mathbb{R}^3$ has spherical coordinates $(r_\mathbf{x}, \theta_\mathbf{x}, \phi_\mathbf{x})$. Denote by $R = R(\mathbf{x}, \mathbf{y})$ the distance between the points $\mathbf{x}$ and $\mathbf{y}$.

By a radiating solution of the Helmholtz equation we mean a function $u(\mathbf{x})$ defined in B_+ which satisfies

$$(\Delta + k^2)u(\mathbf{x}) = 0, \qquad \mathbf{x} \in B_+,\ k \in \mathbb{R}, \tag{2.1}$$

and the radiation condition

$$\lim_{r_\mathbf{x} \to \infty} r_\mathbf{x}(\partial/\partial r_\mathbf{x} - ik)u(\mathbf{x}) = 0. \tag{2.2}$$

We say that $u(\mathbf{x})$ satisfies a Dirichlet boundary condition if

$$u(\mathbf{x}) = f_D(\mathbf{x}), \qquad \mathbf{x} \in \partial B, \tag{2.3}$$

or a Neumann boundary condition if

$$\partial u(\mathbf{x})/\partial n(\mathbf{x}) = f_N(\mathbf{x}), \qquad \mathbf{x} \in \partial B, \tag{2.4}$$

where f_D and f_N are known functions defined on ∂B. We define a fundamental solution to the Helmholtz equation by

$$\gamma_0(\mathbf{x}, \mathbf{y}) = -e^{ikR}/2\pi R. \tag{2.5}$$

Let $\mathfrak{R}(B_+)$ denote the linear space of all complex-valued functions $u \in C^2(B_+) \cap C(\bar{B}_+)$ for which the normal derivative on the boundary exists in the sense that the limit

$$\frac{\partial u(\mathbf{x})}{\partial n(\mathbf{x})} = \lim_{\substack{h \to 0 \\ h > 0}} \hat{\mathbf{n}}(\mathbf{x}) \cdot \operatorname{grad} u(\mathbf{x} + h\hat{\mathbf{n}}(\mathbf{x})), \qquad \mathbf{x} \in \partial B \tag{2.6}$$

exists uniformly on ∂B.

From Green's theorem we have the following integral representation for radiating solutions $u(\mathbf{x}) \in \mathfrak{R}(B_+)$ of the Helmholtz equation:

$$\int_{\partial B} \left\{ \gamma_0(\mathbf{x}, \mathbf{y}) \frac{\partial u(\mathbf{y})}{\partial n(\mathbf{y})} - u(\mathbf{y}) \frac{\partial \gamma_0(\mathbf{x}, \mathbf{y})}{\partial n(\mathbf{y})} \right\} dS_{\mathbf{y}} = \begin{cases} 2u(\mathbf{x}), & \mathbf{x} \in B_+, \\ u(\mathbf{x}), & \mathbf{x} \in \partial B, \\ 0, & \mathbf{x} \in B_-. \end{cases} \tag{2.7}$$

Expanding γ_0 in terms of spherical wave functions we have

$$\gamma_0(\mathbf{x}, \mathbf{y}) = \sum_{l=0}^{\infty} v_l^e(\mathbf{x}_>) v_l^i(\mathbf{x}_<), \tag{2.8}$$

where

$$\mathbf{x}_> = \begin{cases} \mathbf{x} & \text{if } r_{\mathbf{x}} > r_{\mathbf{y}}, \\ \mathbf{y} & \text{if } r_{\mathbf{y}} > r_{\mathbf{x}}, \end{cases} \qquad \mathbf{x}_< = \begin{cases} \mathbf{x} & \text{if } r_{\mathbf{x}} < r_{\mathbf{y}}, \\ \mathbf{y} & \text{if } r_{\mathbf{y}} < r_{\mathbf{x}}, \end{cases} \tag{2.9}$$

$$\begin{Bmatrix} v_{2l}^{e,i}(\mathbf{x}) \\ v_{2l+1}^{e,i}(\mathbf{x}) \end{Bmatrix} = \left[\frac{-ik}{2\pi} \epsilon_m (2n+1) \frac{(n-m)!}{(n+m)!} \right]^{\frac{1}{2}} z_n^{e,i}(kr) P_n^m(\cos\theta) \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix}, \tag{2.10}$$

where $\epsilon_0 = 1$ and $\epsilon_m = 2$ if $m > 0$; $z_n^{e,i}$ are the spherical Bessel functions $z_n^e = h_n^{(1)}$ and $z_n^i = j_n$, and P_n^m is an associated Legendre function. The single index l can be defined in terms of the pair (m, n) (see (**9**)) by

$$l = \tfrac{1}{2}(n^2 + n) + m, \qquad 0 \leq m \leq n \tag{2.11}$$

and the pair (m, n) can be determined uniquely from l by

$$n = \left[\!\left[\frac{-1 + (1 + 8l)^{\frac{1}{2}}}{2} \right]\!\right], \tag{2.12}$$

where $[\![\cdot]\!]$ denotes the greatest integer function and, once n is obtained,

$$m = l - \tfrac{1}{2}(n^2 + n). \tag{2.13}$$

The functions v_l^e are radiating solutions of the Helmholtz equation and lie in $\mathfrak{R}(B_+)$. Furthermore, it is known that both the sequences $\{v_l^e(\mathbf{x})\}$ and $\{\partial v_l^e(\mathbf{x})/\partial n(\mathbf{x})\}$ are linearly independent (see (**8**)) and complete (see (**12**, **13**, **14**)) on $L^2(\partial B)$. Here $L^2(\partial B)$ denotes the Hilbert space of complex-valued, square-integrable functions defined on ∂B.

3. The null-field equations

The null-field equations relate a radiating solution $u \in \mathfrak{R}(B_+)$ of the Helmholtz equation defined on ∂B to its normal derivative $\partial u/\partial n$ as follows:

$$\int_{\partial B} u(\mathbf{y}) \frac{\partial v_l^e(\mathbf{y})}{\partial n(\mathbf{y})} dS_{\mathbf{y}} = \int_{\partial B} \frac{\partial u(\mathbf{y})}{\partial n(\mathbf{y})} v_l^e(\mathbf{y})\, dS_{\mathbf{y}}, \qquad l = 0, 1, 2, \ldots. \tag{3.1}$$

There are two standard methods for obtaining the null-field equations. One method is based on the interior integral relation which is obtained from the integral representation (2.7) for $\mathbf{x} \in B_-$. This approach is discussed in (**5**).

The second method is based on Green's theorem by applying this theorem directly in the domain B_+ to the radiating solution u of the Helmholtz equation and to the radiating wave function v_l^e. This approach is given in (**10**, **15**, **16**).

For the Dirichlet problem (2.1), (2.2) and (2.3) we obtain from (3.1) that

$$\int_{\partial B} \frac{\partial u(\mathbf{y})}{\partial n(\mathbf{y})} v_l^e(\mathbf{y})\, dS_{\mathbf{y}} = \int_{\partial B} f_D(\mathbf{y}) \frac{\partial v_l^e(\mathbf{y})}{\partial n(\mathbf{y})}\, dS_{\mathbf{y}}, \tag{3.2}$$

and for the Neumann problem (2.1), (2.2) and (2.4) we obtain from (3.1) that

$$\int_{\partial B} u(\mathbf{y}) \frac{\partial v_l^e(\mathbf{y})}{\partial n(\mathbf{y})}\, dS_{\mathbf{y}} = \int_{\partial B} f_N(\mathbf{y}) v_l^e(\mathbf{y})\, dS_{\mathbf{y}}. \tag{3.3}$$

Once $\partial u/\partial n$ for the Dirichlet problem and u for the Neumann problem are found on ∂B, the solution u can be obtained in B_+ from (2.7).

For certain scattering surfaces ∂B both the sequences $\{v_l^e\}$ and $\{\partial v_l^e/\partial n\}$ are bases for $L^2(\partial B)$. This is known to be true, for example, when ∂B is sufficiently smooth so that the Rayleigh hypothesis is satisfied (see (**10**, **12**)). In this paper we assume that the geometry of ∂B is such that the Rayleigh hypothesis holds.

It is known (see (**10**)) that the coefficients of an expansion of u in terms of $\{v_l^e\}$ on ∂B are the same as the coefficients of an expansion of the normal derivative $\partial u/\partial n$ in terms of $\{\partial v_l^e/\partial n\}$ on ∂B. Consequently, the following result is obtained:

$$u(\mathbf{x}) = \sum_{l=0}^{\infty} c_l v_l^e(\mathbf{x}), \qquad \mathbf{x} \in B_+, \tag{3.4}$$

where the coefficients c_l are obtained from an appropriate expansion of the boundary data: that is, either

$$f_D(\mathbf{x}) = \sum_{l=0}^{\infty} c_l v_l^e(\mathbf{x}), \qquad \mathbf{x} \in \partial B \tag{3.5}$$

or

$$f_N(\mathbf{x}) = \sum_{l=0}^{\infty} c_l \frac{\partial v_l^e(\mathbf{x})}{\partial n(\mathbf{x})}, \qquad \mathbf{x} \in \partial B \tag{3.6}$$

in the case of the Dirichlet problem or Neumann problem, respectively. Furthermore, the infinite series in (3.4) converges uniformly with respect to x.

For numerous problems in acoustics we have either $f_D(\mathbf{x}) = -u^i(\mathbf{x})$ or $f_N(\mathbf{x}) = -\partial u^i(\mathbf{x})/\partial n(\mathbf{x})$, where $u^i(\mathbf{x})$ is a given incident field. Furthermore, the given incident field is generally expanded in terms of the standing waves $\{v_l^i\}$ rather than the outgoing waves $\{v_l^e\}$.

Following (**10**) we have for the Dirichlet problem

$$f_D(\mathbf{x}) = \sum_{l=0}^{\infty} a_l v_l^i(\mathbf{x}) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} a_l \alpha_{ml} v_m^e(\mathbf{x}) = \sum_{m=0}^{\infty} c_m v_m^e(\mathbf{x}), \tag{3.7}$$

where the coefficients a_l are known and the coefficients α_{ml} are defined by expanding each standing wave v_l^i in terms of the outgoing waves $\{v_m^e\}$:

$$v_l^i(\mathbf{x}) = \sum_{m=0}^{\infty} \alpha_{ml} v_m^e(\mathbf{x}). \tag{3.8}$$

Kleinman, Roach and Ström (**10**) refer to the infinite matrix

$$\mathbf{T}_D = (\alpha_{ml}) \tag{3.9}$$

as the transition matrix for the Dirichlet problem. From (3.7) and (3.9) it follows that the coefficients c_l in (3.4) are obtained from

$$\mathbf{c} = \mathbf{T}_D \mathbf{a}, \tag{3.10}$$

where $\mathbf{c}$ and $\mathbf{a}$ are infinite column vectors with entries $\{c_m\}$ and $\{a_l\}$, respectively.

In a similar fashion for the Neumann problem we have

$$f_N(\mathbf{x}) = \sum_{l=0}^{\infty} b_l \frac{\partial v_l^i(\mathbf{x})}{\partial n(\mathbf{x})} = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} b_l \beta_{ml} \frac{\partial v_m^e(\mathbf{x})}{\partial n(\mathbf{x})} = \sum_{m=0}^{\infty} c_m \frac{\partial v_m^e(\mathbf{x})}{\partial n(\mathbf{x})}, \tag{3.11}$$

where the coefficients b_l are known and the coefficients β_{ml} are obtained from

$$\frac{\partial v_l^i(\mathbf{x})}{\partial n(\mathbf{x})} = \sum_{m=0}^{\infty} \beta_{ml} \frac{\partial v_m^e(\mathbf{x})}{\partial n(\mathbf{x})}, \qquad \mathbf{x} \in \partial B. \tag{3.12}$$

The infinite matrix

$$\mathbf{T}_N = (\beta_{ml}) \tag{3.13}$$

is referred to as the transition matrix for the Neumann problem. From (3.11) it follows that the coefficients c_l in (3.4) are obtained from

$$\mathbf{c} = \mathbf{T}_D \mathbf{b}. \tag{3.14}$$

One important result established in (**10**) is that the matrices $\mathbf{T}_D$ and $\mathbf{T}_N$ can both be computed from the computation of the inverse of only one matrix. It can be shown that

$$\mathbf{T}_D = \mathbf{Q}^{-1}\mathbf{Q}^i, \tag{3.15}$$

$$\mathbf{T}_N = \mathbf{Q}^{-1}\mathbf{Q}^i - 2\mathbf{Q}^{-1}, \tag{3.16}$$

where $\mathbf{Q}^i$ has components

$$q_{nl}^i = \int_{\partial B} \frac{\partial v_n^e(\mathbf{y})}{\partial n(\mathbf{y})} v_l^i(\mathbf{y})\, dS_{\mathbf{y}} \tag{3.17}$$

and $\mathbf{Q}$ is an invertible matrix and has entries

$$q_{mn}=\int_{\partial B} v_m^e(\mathbf{y})\frac{\partial v_n^e(\mathbf{y})}{\partial n(\mathbf{y})}\,dS_{\mathbf{y}}. \tag{3.18}$$

4. The reproducing kernel and the modified Green's functions

Let $G(\mathbf{x},\mathbf{y})$ denote the Green's function for the Dirichlet problem. Specifically, $G(\mathbf{x},\mathbf{y})$ satisfies

$$(\Delta_{\mathbf{y}}+k^2)G(\mathbf{x},\mathbf{y})=2\delta(\mathbf{x}-\mathbf{y}), \tag{4.1}$$

$$\lim_{r_{\mathbf{y}}\to 0} r_{\mathbf{y}}(\partial/\partial r_{\mathbf{y}}-ik)G(\mathbf{x},\mathbf{y})=0, \tag{4.2}$$

$$G(\mathbf{x},\mathbf{y})=0, \qquad \mathbf{x}\in B_+, \mathbf{y}\in\partial B, \tag{4.3}$$

where δ denotes the Dirac delta function. Let $N(\mathbf{x},\mathbf{y})$ denote the Green's function for the Neumann problem. Then $N(\mathbf{x},\mathbf{y})$ satisfies (4.1), (4.2) and the Neumann boundary condition:

$$\frac{\partial N(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}=0, \qquad \mathbf{x}\in B_+, \mathbf{y}\in\partial B. \tag{4.4}$$

With the free-space Green's function γ_0 defined in (2.5), it can be shown that

$$G(\mathbf{x},\mathbf{y})=\gamma_0(\mathbf{x},\mathbf{y})-g(\mathbf{x},\mathbf{y}), \tag{4.5}$$

$$N(\mathbf{x},\mathbf{y})=\gamma_0(\mathbf{x},\mathbf{y})-n(\mathbf{x},\mathbf{y}); \tag{4.6}$$

where the functions g and n are radiating solutions of the Helmholtz equation and the minus sign is introduced for future convenience.

Applying Green's theorem to a radiating solution $u(\mathbf{y})$ of the Helmholtz equation and to the functions $G(\mathbf{x},\mathbf{y})$ and $N(\mathbf{x},\mathbf{y})$, respectively, we obtain

$$u(\mathbf{x})=-\frac{1}{2}\int_{\partial B}\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}u(\mathbf{y})\,dS_{\mathbf{y}}, \qquad \mathbf{x}\in B_+, \tag{4.7}$$

$$u(\mathbf{x})=\frac{1}{2}\int_{\partial B} N(\mathbf{x},\mathbf{y})\frac{\partial u(\mathbf{y})}{\partial n(\mathbf{y})}\,dS_{\mathbf{y}}, \qquad \mathbf{x}\in B_+. \tag{4.8}$$

In (**10**) the following definitions of the modified Green's functions for the Dirichlet and Neumann problems are given, respectively:

$$\gamma_1^D(\mathbf{x},\mathbf{y})=\gamma_0(\mathbf{x},\mathbf{y})-\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\alpha_{ml}v_l^e(\mathbf{x})v_m^e(\mathbf{y}), \qquad \mathbf{x},\mathbf{y}\in\bar{B}_+, \tag{4.9}$$

$$\gamma_1^N(\mathbf{x},\mathbf{y})=\gamma_0(\mathbf{x},\mathbf{y})-\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\beta_{ml}v_l^e(\mathbf{x})v_m^e(\mathbf{y}), \qquad \mathbf{x},\mathbf{y}\in\bar{B}_+, \tag{4.10}$$

where the coefficients α_{ml} and β_{ml} are defined in (3.8) and (3.12), respectively, and where it is assumed that the geometry of ∂B is such that the

Rayleigh hypothesis is satisfied. Furthermore, it is shown in (**10**) that

$$u(\mathbf{x}) = -\frac{1}{2}\int_{\partial B} \frac{\partial \gamma_1^D(\mathbf{x}, \mathbf{y})}{\partial n(\mathbf{y})} u(\mathbf{y})\, dS_{\mathbf{y}}, \qquad r_{\mathbf{x}} > \max_{\mathbf{y}\in\partial B} r_{\mathbf{y}}, \tag{4.11}$$

$$u(\mathbf{x}) = \frac{1}{2}\int_{\partial B} \gamma_1^N(\mathbf{x}, \mathbf{y}) \frac{\partial u(\mathbf{y})}{\partial n(\mathbf{y})}\, dS_{\mathbf{y}}, \qquad r_{\mathbf{x}} > \max_{\mathbf{y}\in\partial B} r_{\mathbf{y}}. \tag{4.12}$$

Results there depend upon the series expansion of γ_0 in (2.7) and it is essential that the field point $\mathbf{x}$ lies on some sphere centred at the origin which contains $\bar{B}_-$ in its interior.

We now define the reproducing kernel $K(\mathbf{x}, \mathbf{y})$ by

$$K(\mathbf{x}, \mathbf{y}) = N(\mathbf{x}, \mathbf{y}) - G(\mathbf{x}, \mathbf{y}). \tag{4.13}$$

Then from (4.3), (4.4), (4.7), (4.8) and (4.13) it follows that

$$u(\mathbf{x}) = \frac{1}{2}\int_{\partial B} \frac{\partial K(\mathbf{x}, \mathbf{y})}{\partial n(\mathbf{y})} u(\mathbf{y})\, dS_{\mathbf{y}}, \qquad \mathbf{x} \in B_+, \tag{4.14}$$

$$u(\mathbf{x}) = \frac{1}{2}\int_{\partial B} K(\mathbf{x}, \mathbf{y}) \frac{\partial u(\mathbf{y})}{\partial n(\mathbf{y})}\, dS_{\mathbf{y}}, \qquad \mathbf{x} \in B_+. \tag{4.15}$$

From (4.5), (4.6) and (4.13) it is seen that $K(\mathbf{x}, \mathbf{y}) = g(\mathbf{x}, \mathbf{y}) - n(\mathbf{x}, \mathbf{y})$, and consequently, as a function of $\mathbf{y}$, K is a radiating solution of the Helmholtz equation in B_+.

Now define $\gamma_1^K(\mathbf{x}, \mathbf{y})$ to denote the difference between the modified Green's functions for the Neumann and Dirichlet problems. That is

$$\gamma_1^K(\mathbf{x}, \mathbf{y}) = \gamma_1^N(\mathbf{x}, \mathbf{y}) - \gamma_1^D(\mathbf{x}, \mathbf{y}) \tag{4.16}$$

and we have from (4.9), (4.10) and (4.16) that

$$\gamma_1^K(\mathbf{x}, \mathbf{y}) = -\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \eta_{ml} v_l^e(\mathbf{x}) v_m^e(\mathbf{y}), \qquad \mathbf{x}, \mathbf{y} \in \bar{B}_+, \tag{4.17}$$

where

$$\eta_{ml} = \beta_{ml} - \alpha_{ml}. \tag{4.18}$$

Then from (3.8), (3.12), (4.9), (4.10), (4.11), (4.12) and (4.16) it follows that $\gamma_1^K(\mathbf{x}, \mathbf{y})$ satisfies the following reproducing properties:

$$u(\mathbf{x}) = \frac{1}{2}\int_{\partial B} \frac{\partial \gamma_1^K(\mathbf{x}, \mathbf{y})}{\partial n(\mathbf{y})} u(\mathbf{y})\, dS_{\mathbf{y}}, \qquad r_{\mathbf{x}} > \max_{\mathbf{y}\in\partial B} r_{\mathbf{y}}, \tag{4.19}$$

$$u(\mathbf{x}) = \frac{1}{2}\int_{\partial B} \gamma_1^K(\mathbf{x}, \mathbf{y}) \frac{\partial u(\mathbf{y})}{\partial n(\mathbf{y})}\, dS_{\mathbf{y}}, \qquad r_{\mathbf{x}} > \max_{\mathbf{y}\in\partial B} r_{\mathbf{y}}. \tag{4.20}$$

In the next theorem we show that the kernels $K(\mathbf{x}, \mathbf{y})$ and $\gamma_1^K(\mathbf{x}, \mathbf{y})$ are equal.

THEOREM 4.1. *$K(\mathbf{x}, \mathbf{y}) = \gamma_1^K(\mathbf{x}, \mathbf{y})$ for $\mathbf{x} \in B_+$, $\mathbf{y} \in \bar{B}_+$ or $\mathbf{x} \in \bar{B}_+$, $\mathbf{y} \in B_+$.*

Proof. For $\mathbf{x} \in B_+$ note that as a function of $\mathbf{y}$, $n(\mathbf{x}, \mathbf{y}) \in C^2(B_+) \cap C^1(\bar{B}_+)$ and (**4**, Theorem 3.27) that $g(\mathbf{x}, \mathbf{y}) \in C^2(B_+) \cap C^1(\bar{B}_+)$. Consequently, for $\mathbf{x} \in B_+$ as a function of $\mathbf{y}$, $K(\mathbf{x}, \mathbf{y})$ is a radiating solution of the Helmholtz equation and $K(\mathbf{x}, \mathbf{y}) \in C^2(B_+) \cap C^1(\bar{B}_+)$. It follows that for $\mathbf{x} \in B_+$ $K(\mathbf{x}, \mathbf{y})$ is an analytic function of $\mathbf{y}$ in B_+ (see (**4**, Theorem 3.5; **17**)).

It can be shown (see, for example, (**11**)) that both $G(\mathbf{x}, \mathbf{y})$ and $N(\mathbf{x}, \mathbf{y})$ satisfy the principle of reciprocity in B_+. Consequently, $K(\mathbf{x}, \mathbf{y}) = K(\mathbf{y}, \mathbf{x})$ for $\mathbf{x}, \mathbf{y} \in B_+$. From the above analysis, for $\mathbf{y} \in B_+$ as a function of $\mathbf{x}$, $K(\mathbf{x}, \mathbf{y})$ is a radiating solution of the Helmholtz equation; $K(\mathbf{x}, \mathbf{y}) \in C^2(B_+) \cap C^1(\bar{B}_+)$; and $K(\mathbf{x}, \mathbf{y})$ is analytic in B_+.

From the Rayleigh hypothesis it follows that there exist coefficients A_{ml} such that

$$K(\mathbf{x}, \mathbf{y}) = -\sum_{l=0}^{\infty} \sum_{m=0}^{\infty} A_{ml} v_l^e(\mathbf{x}) v_m^e(\mathbf{y}), \tag{4.21}$$

where $\mathbf{x} \in B_+$, $\mathbf{y} \in \bar{B}_+$ or $\mathbf{x} \in \bar{B}_+$, $\mathbf{y} \in B_+$. It is to be noted that we avoid the situation where $\mathbf{x} \in \partial B$ and $\mathbf{y} \in \partial B$ simultaneously. Furthermore, from the Rayleigh hypothesis, the infinite series in (4.21) is uniformly convergent in $\mathbf{x}$ and $\mathbf{y}$ when $\mathbf{x}, \mathbf{y} \in B_+$.

To prove the theorem we must show that the coefficients A_{ml} in equation (4.21) satisfy $A_{ml} = \eta_{ml}$ for all $m, l = 0, 1, 2, \ldots$.

Let D_ρ denote the ball centred at the origin defined by

$$D_\rho = \left\{\mathbf{x} : r_\mathbf{x} \leq \rho, \rho = \max_{\mathbf{y} \in \partial B} r_\mathbf{y}\right\}. \tag{4.22}$$

Let D_+ denote the region exterior to D_ρ. From (4.5), (4.6), (4.9) and (4.10) it follows from (**10**, section 4) that

$$g(\mathbf{x}, \mathbf{y}) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \alpha_{ml} v_l^e(\mathbf{x}) v_m^e(\mathbf{y}), \qquad \mathbf{x} \in D_+, \mathbf{y} \in \partial B, \tag{4.23}$$

$$\frac{\partial n(\mathbf{x}, \mathbf{y})}{\partial n(\mathbf{y})} = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \beta_{ml} v_l^e(\mathbf{x}) \frac{\partial v_m^e(\mathbf{y})}{\partial n(\mathbf{y})}, \qquad \mathbf{x} \in D_+, \mathbf{y} \in \partial B. \tag{4.24}$$

Since for $\mathbf{x} \in B_+$, $g(\mathbf{x}, \mathbf{y})$ and $n(\mathbf{x}, \mathbf{y})$ as functions of y are radiating solutions of the Helmholtz equation, it can be shown (the argument here follows a similar one used in (**10**, section 3)) that

$$g(\mathbf{x}, \mathbf{y}) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \alpha_{ml} v_l^e(\mathbf{x}) v_m^e(\mathbf{y}), \qquad \mathbf{x} \in D_+, \mathbf{y} \in \bar{B}_+, \tag{4.25}$$

$$n(\mathbf{x}, \mathbf{y}) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \beta_{ml} v_l^e(\mathbf{x}) v_m^e(\mathbf{y}), \qquad \mathbf{x} \in D_+, \mathbf{y} \in \bar{B}_+. \tag{4.26}$$

From (4.5), (4.6), (4.9), (4.10), (4.13), (4.16), (4.17), (4.18), (4.25) and

(4.26) we have

$$K(\mathbf{x}, \mathbf{y}) = \gamma_1^K(\mathbf{x}, \mathbf{y}), \qquad \mathbf{x} \in D_+, \mathbf{y} \in \bar{B}_+. \tag{4.27}$$

From (4.17), (4.21), (4.27) it follows that $A_{ml} = \eta_{ml}$ for $m, l = 0, 1, 2, \ldots$. Thus

$$K(\mathbf{x}, \mathbf{y}) = \gamma_1^K(\mathbf{x}, \mathbf{y}) \tag{4.28}$$

for $\mathbf{x} \in B_+, \mathbf{y} \in \bar{B}_+$ or for $\mathbf{x} \in \bar{B}_+, \mathbf{y} \in B_+$.

From (4.14), (4.15), (4.19) and (4.20) and the above theorem, it follows that the reproducing properties (4.19) and (4.20) can be extended to the entire exterior domain B_+.

THEOREM 4.2.

$$u(\mathbf{x}) = \frac{1}{2} \int_{\partial B} \frac{\partial \gamma_1^K(\mathbf{x}, \mathbf{y})}{\partial n(\mathbf{y})} u(\mathbf{y})\, dS_{\mathbf{y}}, \qquad \mathbf{x} \in B_+,$$

$$u(\mathbf{x}) = \frac{1}{2} \int_{\partial B} \gamma_1^K(\mathbf{x}, \mathbf{y}) \frac{\partial u(\mathbf{y})}{\partial n(\mathbf{y})}\, dS_{\mathbf{y}}, \qquad \mathbf{x} \in B_+.$$

We conclude this section by establishing the connection between the matrix $\mathbf{Q}$ whose components are given in (3.18) and the reproducing kernel. From (3.15) and (3.16) it follows that

$$\mathbf{T}_N - \mathbf{T}_D = 2\mathbf{Q}^{-1}. \tag{4.29}$$

Defining $\mathbf{T}_K$ as the *reproducing matrix* associated with the reproducing kernel, it follows from (3.9), (3.13), (4.17), (4.18) and (4.29) that

$$\mathbf{T}_K = (\eta_{ml}) = -2\mathbf{Q}^{-1}. \tag{4.30}$$

Consequently, the intriguing matrix $\mathbf{Q}$ introduced by Kleinman, Roach and Ström in (**10**) is related to the reproducing matrix $\mathbf{T}_K$ via (4.30). It is thus seen from the formulas (3.15) and (3.16) and the identity (4.30) that knowledge of $\mathbf{T}_K$ is sufficient to calculate both the transition matrices $\mathbf{T}_D$ and $\mathbf{T}_N$.

5. The reproducing kernels and the null-field equations

To complete the analysis of the scattering trinity, we demonstrate in this section the intrinsic relation between the reproducing kernel and the null-field equations.

Changing the index in (3.1) from l to m and then multiplying by $\eta_{ml} v_l^e(x)$ we obtain

$$\int_{\partial B} \eta_{ml} v_l^e(\mathbf{x}) \frac{\partial v_m^e(\mathbf{y})}{\partial n(\mathbf{y})} u(\mathbf{y})\, dS_{\mathbf{y}} = \int_{\partial B} \eta_{ml} v_l^e(\mathbf{x}) v_m^e(\mathbf{y}) \frac{\partial u(\mathbf{y})}{\partial n(\mathbf{y})}\, dS_{\mathbf{y}}, \tag{5.1}$$

where $\mathbf{x} \in B_+$. Summing over l and m in (5.1), then interchanging the order of summation and integration, we obtain from Theorem 4.1 the following exterior integral relation

$$\int_{\partial B} \frac{\partial K(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})} u(\mathbf{y})\, dS_{\mathbf{y}} = \int_{\partial B} K(\mathbf{x},\mathbf{y}) \frac{\partial u(\mathbf{y})}{\partial n(\mathbf{y})}\, dS_{\mathbf{y}}, \qquad \mathbf{x} \in B_+. \tag{5.2}$$

The justification for the interchange of summation and integration follows from the fact that the infinite series representing $K(\mathbf{x},\mathbf{y})$ and $\partial K(\mathbf{x},\mathbf{y})/\partial n(\mathbf{y})$ lie in $L^2(\partial B)$ for $\mathbf{x} \in B_+$ and since $u(\mathbf{y}) \in \mathfrak{R}(B_+)$, it follows that $u(\mathbf{y}), \partial u(\mathbf{y})/\partial n(\mathbf{y}) \in L^2(\partial B)$.

Conversely, starting with the integral representations in (4.14) and (4.15) one obtains the exterior integral relation (5.2). Then by reversing the above steps, the null-field equations are obtained.

Thus it is seen that the null-field equations are equivalent to the exterior integral relation (5.2) with reproducing kernel $K(\mathbf{x},\mathbf{y})$.

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