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Spectral Properties of Operators of the Theory of Harmonic Potential

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ABSTRACT. We classify the points of the spectrum of the operators B and B^* of the theory of harmonic potential on a smooth closed surface $S \subset \mathbb{R}^3$. These operators give the direct value on S of the normal derivative of the simple layer potential and the double layer potential. We show that zero can belong to the point spectrum of both operators in $L_2(S)$. We prove that the half-interval $[-2\pi, 2\pi)$ is densely filled by spectrum points of the operators for a varying surface; this is a generalization of the classical result of Plemelj. We obtain a series of new spectral properties of the operators B and B^* on ellipsoidal surfaces.

§1. Introduction

Let Ω be a simply connected finite domain in $\mathbb{R}^3$ bounded by a Lyapunov surface S. The aim of this paper is the study of spectral properties of the following operators of the classical potential theory on the space $L_2(S)$:

$$\begin{aligned}(T\sigma)(x) &:= \int_S \sigma(y)|x-y|^{-1}\,dS_y,\\ (B\varphi)(x) &:= \int_S \varphi(y)\frac{\partial|x-y|^{-1}}{\partial n_x}\,dS_y,\\ (B^*\psi)(x) &:= \int_S \psi(y)\frac{\partial|x-y|^{-1}}{\partial n_y}\,dS_y.\end{aligned} \tag{1}$$

The operator T is the value of the simple layer potential on S, B is the operator of the direct value on S of the normal derivative of the simple layer potential, and the adjoint operator B^* is the operator of the direct value on S of the double layer potential. It is known (see [1, p. 160]) that nonzero points of the spectrum of the compact operators B and B^* are real simple poles of their resolvents; they belong to the half-interval $[-2\pi, 2\pi)$ and the corresponding eigenfunctions are continuous. To the simple eigenvalue $\mu_0 = -2\pi$ of the operators B and B^* there correspond eigenfunctions σ_0 and c_0 such that

$$\int_S \sigma_0\,dS \neq 0 \qquad \text{and} \qquad c_0 = \text{const}$$

(see [2, p. 276] and [3, p. 334]). Moreover, the operator B is quasi-Hermitian (see [4, p. 394 of the Russian edition]), i.e., there exists a positive self-adjoint operator T (for instance, we can take the compact operator T in (1)) such that $TB = B^*T$ (see [5]). Let $\{\mu_n\}$ be the sequence of nonzero eigenvalues of B (and, therefore, of B^*), and $E(\mu, B)$ denote the finite-dimensional eigensubspace of B that corresponds to an eigenvalue $\mu \in \{\mu_n\}$. Since B is quasi-Hermitian, we see that the system of eigenfunctions of the operator B^* is total, the eigenfunctions of B corresponding to different eigenvalues are orthogonal with respect to the energy operator T (i.e., with respect to the inner product $(T\cdot,\cdot)$), and the following relations hold (see [5]):

$$E(\mu, B^*) = T\big(E(\mu, B)\big) \qquad \text{and} \qquad \varphi \in \operatorname{Ker} B \Leftrightarrow T\varphi \in \operatorname{Ker} B^*.$$

If $S \in C^\infty$, then all operators in (1) are pseudodifferential operators (PDO) of order (-1), that is, are bounded from the Sobolev space $H^r(S)$ into $H^{r+1}(S)$ for all $r \in R$, and the operator T is an elliptic PDO whose inverse extends to be a continuous operator from $L_2(S)$ onto $H^{-1}(S)$ (if $S \in C^\infty$, then T^{-1} maps $H^r(S)$ onto $H^{r-1}(S)$ for all $r \in \mathbb{R}$ (see [6] and [7, p. 151])). Thus, the spectrum of the compact self-adjoint positive operator T consists of positive eigenvalues of finite multiplicity and of zero,

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which belongs to the continuous spectrum. The cited properties of the operators B and B^* imply that their spectrum consists of a system of eigenvalues of finite multiplicity from the half-interval $[-2\pi, 2\pi)$, to which there correspond infinitely smooth eigenfunctions, and zero.

In potential theory, the following problem concerning the classification of zero in the spectrum of the operators B and B^* has been open since a while ago (see [8]): to which part of the spectrum (to the point part σ_p, to the continuous part σ_c, or to the residual part σ_r) of these operators can zero belong? In this paper, we obtain a partial answer to this question. Moreover, we give a generalization of Plemelj's classical result (1911) concerning the fact that the spectra of the operators B and B^* belong to the half-interval $[-2\pi, 2\pi)$; specifically, we prove that every point of the half-interval is an eigenvalue of these operators for a suitable choice of the surface. For completeness, in §3 we include a short exposition of some results obtained by the authors in [9]. In closing, we prove a series of new spectral properties of the operators B and B^* for ellipsoidal surfaces.

§2. Preliminary results

Since the operator B is quasi-Hermitian, we have the following assertion.

Theorem 1. $0 \notin \sigma_r(B^*)$.

Proof. Assume the contrary, that is, $0 \in \sigma_r(B^*) \Rightarrow \dim \operatorname{Ker} B \neq 0$. Let $\varphi \in \operatorname{Ker} B$; then

$$\varphi \neq 0 \Rightarrow TB\varphi = 0 \Rightarrow B^*T\varphi = 0 \Rightarrow T\varphi \in \operatorname{Ker} B^* \Rightarrow 0 \in \sigma_p(B^*) \Rightarrow 0 \notin \sigma_r(B^*),$$

which proves the assertion of the theorem. □

Taking into account this result and the relationship between the spectra of adjoint operators [7, p. 34], we can readily establish that only three possibilities can occur *a priori*:

$$0 \in \sigma_c(B), \qquad 0 \in \sigma_c(B^*), \tag{2}$$

$$0 \in \sigma_p(B), \qquad 0 \in \sigma_p(B^*), \tag{3}$$

$$0 \in \sigma_r(B), \qquad 0 \in \sigma_p(B^*). \tag{4}$$

It is known (see [8] and [10]) that possibility (2) occurs for the case in which S is a sphere or a prolate spheroid. In the paper [8], the following problem was posed: Does zero always belong to the continuous spectrum of the operators B and B^* for sufficiently smooth surfaces S? In the next section, we give a negative answer to this question by presenting an example for which possibility (3) occurs. As to (4), the problem remains open. We obtain the following result in this direction.

Theorem 2. $0 \in \sigma_r(B) \Leftrightarrow \varnothing \neq \operatorname{Ker} B^* \setminus \{0\} \subset L_2(S) \setminus H^1(S)$.

Proof. 1. Let $0 \in \sigma_r(B) \Rightarrow 0 \in \sigma_p(B^*) \Rightarrow \operatorname{Ker} B^* \neq \{0\}$. Assume that there is a ψ such that $\psi \neq 0$, $\psi \in H^1(S)$, and $B^*(\psi) = 0$. Since T is a bijection of $L_2(S)$ onto $H^1(S)$ (see [11, p. 87] and [12]), we see that there is a $\varphi \in L_2(S)$, $\varphi \neq 0$, such that

$$T\varphi = \psi \Rightarrow B^*T\varphi = 0 \Rightarrow TB\varphi = 0 \Rightarrow B\varphi = 0 \Rightarrow 0 \in \sigma_p(B);$$

this is a contradiction.

2. Let $\varnothing \neq \operatorname{Ker} B^* \setminus \{0\} \subset L_2(S) \setminus H^1(S) \Rightarrow 0 \in \sigma_p(B^*) \Rightarrow 0 \in \sigma_r(B)$, or $0 \in \sigma_p(B)$. Let us exclude the second possibility. We have

$$0 \in \sigma_p(B) \Rightarrow \exists \varphi \neq 0: \; B\varphi = 0 \Rightarrow \psi := T\varphi \in H^1(S), \; B^*\psi = TB\varphi = 0$$
$$\Rightarrow \operatorname{Ker} B^* \setminus \{0\} \not\subset L_2(S) \setminus H^1(S);$$

this is a contradiction, which completes the proof. □

Can $\operatorname{Ker} B^* \setminus \{0\}$ consist solely of "nonsmooth" functions? Note that an attempt to prove that (4) is impossible must be based on specific properties of the operators B and B^*, because for general quasi-Hermitian operators, possibility (4) can occur [13].

§3. Spectral properties of the operators B and B^* for an oblate spheroid

In this section we prove that possibility (3) can occur for the case in which S is an oblate spheroid (an ellipsoid of rotation). The Cartesian coordinates (x, y, z) are related to the coordinates (ξ, η, φ) of the oblate spheroid by the formulas [14, p. 24]

$$x = p\cos\varphi, \qquad y = p\sin\varphi, \qquad z = d\xi\eta/2,$$

where $p^2 = d^2(1+\xi^2)(1-\eta^2)/4$, d is the interfocal distance, $0 \le \xi < \infty$, $-1 \le \eta \le 1$, and $0 \le \varphi < 2\pi$. The coordinate surfaces $\xi = \text{const}$ are oblate spheroids of rotation about the z-axis. If we take such a surface for S, then the eigenvalues of the operators B^* and B acquire the form [15]

$$\lambda_n^m(\xi) = -2\pi\left\{1 + 2(-1)^m \frac{(n-m)!}{(n+m)!} Q_n^m(i\xi)\left[(n+1)i\xi P_n^m(i\xi) - (n-m+1)P_{n+1}^m(i\xi)\right]\right\}, \tag{5}$$

and the corresponding eigenfunctions of these operators are

$$\begin{cases} C_n^m(\eta,\varphi) := P_n^m(\eta)\cos m\varphi, \\ S_n^m(\eta,\varphi) := P_n^m(\eta)\sin m\varphi, \end{cases} \qquad \begin{cases} C_n^m(\xi,\eta,\varphi) := C_n^m(\eta,\varphi)(\xi^2+\eta^2)^{-0.5}, \\ S_n^m(\xi,\eta,\varphi) := S_n^m(\eta,\varphi)(\xi^2+\eta^2)^{-0.5}, \end{cases}$$

where $n = 0, 1, 2, \ldots$, $0 \le m \le n$, and P_n^m and Q_n^m denote the associated Legendre functions. Let us study the asymptotic behavior of eigenvalues (5) as $\xi \to \infty$ and $\xi \to 0$.

Lemma 1. *$\lambda_n^m(\xi) \to -2\pi/(2n+1)$ as $\xi \to \infty$.*

Proof. Using the asymptotic formulas [16, p. 165 of the Russian edition]

$$P_n^m(i\xi) = \frac{(2i\xi)^n\Gamma(n+1/2)}{\sqrt{\pi}\Gamma(n-m+1)}(1+O(\xi^{-2})),$$

$$Q_n^m(i\xi) = (-1)^m \frac{\sqrt{\pi}\Gamma(n+m+1)}{(2i\xi)^{n+1}\Gamma(n+3/2)}(1+O(\xi^{-2})),$$

where Γ denotes the gamma function, after some algebraic manipulations with (5), we obtain

$$\lambda_n^m(\xi) = -\frac{2\pi}{2n+1} + O(\xi^{-2}).$$

To complete the proof, we note that, as expected, as $\xi \to \infty$ we obtain the eigenfunctions of the operators B and B^* for the sphere. □

Lemma 2. *$\lambda_n^m(\xi) \to 2\pi(-1)^{n+m+1}$ as $\xi \to 0$.*

Proof. By substituting $z = i\xi$ into the formula expressing $Q_n^m(z)$ via the hypergeometric function of the argument z^2 [17, p. 155 of the Russian edition], after some algebraic manipulations we obtain

$$Q_n^m(i\xi) = i\exp\left(\frac{i\pi(3m-n)}{2}\right)\sqrt{\pi}2^{m-1}(-\xi^2-1)^{m/2}\left\{-\frac{\Gamma((n+m+1)/2)}{\Gamma((n-m+2)/2)} + O(\xi)\right\}. \tag{6}$$

Starting from a similar formula for $P_n^m(z)$ [17, p. 154 of the Russian edition], we obtain

$$P_n^m(i\xi) = 2^m(-\xi^2-1)^{-m/2}\pi^{-1/2}\left\{\frac{\cos(\pi\frac{n+m}{2})\Gamma(\frac{n+m+1}{2})}{\Gamma(\frac{n-m+2}{2})} + O(\xi)\right\}. \tag{7}$$

Substituting (6) and (7) into (5), we obtain the formula

$$\lambda_n^m(\xi) = -2\pi\Big\{1 + 2^m\alpha_n^m \exp\left(-i\pi\frac{n+m}{2}\right)i(n-m+1)\Gamma\left(\frac{n+m+1}{2}\right)\Gamma\left(\frac{n+m+2}{2}\right)$$
$$\times\cos\left(\pi\frac{n+m+1}{2}\right)\Big/\left[\Gamma\left(\frac{n-m+2}{2}\right)\Gamma\left(\frac{n-m+3}{2}\right)\right]\Big\} + O(\xi),$$

where we write $\alpha_n^m := (n-m)!/(n+m)!$ for convenience.

Now we can readily establish that

$$\lambda_n^m(\xi) = \begin{cases} -2\pi + O(\xi) & \text{for even } n+m, \\ 2\pi + O(\xi) & \text{for odd } n+m; \end{cases}$$

this completes the proof of the lemma. □

The following theorem states that possibility (3) can occur.

Theorem 3. *There is a surface S, which is an oblate spheroid, for which $0 \in \sigma_p(B)$ and $0 \in \sigma_p(B^*)$.*

Proof. Let the sum $n+m$ be odd. Then it follows from Lemmas 1 and 2 and from the continuity of $\lambda_n^m(\xi)$ that there exists an $\xi_n^m \in (0,\infty)$ such that $\lambda_n^m(\xi_n^m) = 0$. The assertion of the theorem holds for all oblate spheroids with $\xi = \xi_n^m$. □

This theorem answers the question raised by Ahner in [8]. In the general case, we cannot state that the operators B and B^* are injective for all smooth surfaces; the spectral nature of zero depends on the form of S.

The following theorem is a generalization of Plemelj's classical result [8] that states that the half-interval $[-2\pi, 2\pi)$ for the eigenvalues of B and B^* cannot be shrinked.

Theorem 4. *For each $b \in [-2\pi, 2\pi)$ there exists a smooth surface S such that b is an eigenvalue of the operators B and B^* defined on S.*

Proof. For $b = -2\pi$, the assertion holds for any smooth surface, and it follows from Lemmas 1 and 2 that the union of the ranges of the functions $\lambda_1^0(\xi)$ and $\lambda_2^0(\xi)$ coincides with the interval $(-2\pi, 2\pi)$; this completes the proof of the theorem. □

§4. Certain spectral properties of the operators B and B^* for ellipsoidal surfaces

Formula (5), as well as a similar formula for the eigenvalues of B and B^* for the case of a prolate spheroid [10], can be written in the form

$$\lambda_n^m(\xi) = 2\pi i(-1)^m(1+\xi^2)\alpha_n^m \frac{\partial\{P_n^m(i\xi)Q_n^m(i\xi)\}}{\partial\xi}, \qquad \xi \in (0,\infty), \tag{8}$$

for an oblate spheroid and

$$\lambda_n^m(\xi) = 2\pi(\xi^2-1)\alpha_n^m(-1)^m \frac{\partial\{P_n^m(\xi)Q_n^m(\xi)\}}{\partial\xi}, \qquad \xi \in (1,\infty), \tag{9}$$

for a prolate spheroid, $n = 0, 1, 2, \dots$, $0 \le m \le n$. The following assertion proves an interesting property of these eigenvalues, which is obvious for the sphere.

Theorem 5. *The eigenvalues for the oblate spheroid (8) and the prolate spheroid (9) satisfy the relation*

$$S_n(\xi) := \lambda_n^0(\xi) + 2\sum_{m=1}^{n} \lambda_n^m(\xi) = -2\pi, \qquad n = 0, 1, \dots. \tag{10}$$

First, we obtain some useful formulas.

Lemma 3. *The following relations hold:*

$$\int_{-1}^{1} \frac{A_k(x)(1-x^2)^{m/2}P_n^m(x)\,dx}{z-x} = 2(-1)^m A_k(z)(z^2-1)^{m/2}Q_n^m(z), \tag{11}$$

$$\int_{-1}^{1} \frac{[P_n^m(x)]^2\,dx}{z-x} = 2(-1)^m P_n^m(z)Q_n^m(z), \tag{12}$$

$$P_n(z)Q_n(z) + 2\sum_{m=1}^{n}(-1)^m\alpha_n^m P_n^m(z)Q_n^m(z) = Q_0(z) := 0.5\ln\left[\frac{z+1}{z-1}\right], \tag{13}$$

where $n = 0, 1, \dots$, $0 \le m \le n$, A_k is an arbitrary polynomial of degree $k \le n-m$, and $z \in \mathbb{C}\setminus(-1,1)$ ($\mathbb{C}$ is the complex plane).

Proof. We were able to find formula (11) for $A_k(x) = x^k$ only in the handbook [19, p. 200, formula 8], where it is presented with a mistake. Therefore, we had to derive it independently, starting from the Neumann integral [20, p. 232 of the Russian edition]

$$Q_n(z) = 0.5 \int_{-1}^{1} P_n(t) \frac{dt}{z-t}, \qquad z \in \mathbb{C} \setminus (-1, 1). \tag{14}$$

Substituting $A_k(x) = d^m P_n(x)/dx^m$ into (11) and taking into account the known expressions for the associated Legendre functions via the derivatives of the polynomials inside and outside the interval $(-1, 1)$, we obtain (12). Furthermore, by substituting $y = x$ and $a = 0$ into the formula expressing the addition theorem for Legendre polynomials [20, p. 233 of the Russian edition],

$$P_n\{xy + [(1-x^2)(1-y^2)]^{1/2} \cos a\} = P_n(x)P_n(y) + 2\sum_{m=1}^{n} \alpha_n^m \cos(ma) P_n^m(x) P_n^m(y),$$

and taking the relation $P_n(1) = 1$ into account, we obtain

$$P_n^2(x) + 2\sum_{m=1}^{n} \alpha_n^m [P_n^m(x)]^2 = 1.$$

By multiplying both parts of this identity by $(z-x)^{-1}$ and by integrating with respect to x from (-1) to 1, we obtain the main formula (13), in view of (12) and (14). The proof of the lemma is complete. □

The assertions of Lemma 3 yield the proof of Theorem 5.

Proof of Theorem 5. By substituting the values $\lambda_n^m(\xi)$ from (8) into (10) and by taking into account (13), we obtain

$$S_n(\xi) = 2\pi i(1+\xi^2) \frac{d[0.5 \ln((i\xi+1)/(i\xi-1))]}{d\xi} = -2\pi$$

for an oblate spheroid. Similarly, it follows from formulas (10), (9), and (13) that for a prolate spheroid we have

$$S_n(\xi) = 2\pi(\xi^2-1) \frac{d[0.5 \ln((\xi+1)/(\xi-1))]}{d\xi} = -2\pi,$$

which completes the proof of the theorem. □

Numerical calculations show that, apparently, a similar formula holds not only for a spheroid, but also for an arbitrary ellipsoid.

Consider the problem of the signs of the eigenvalues of the operators B and B^* defined on the surface of the spheroid. Formulas (8) and (9) can be written in a unified form

$$\lambda_n^m(z) = 2\pi(z^2-1)\alpha_n^m(-1)^m \frac{d\{P_n^m(z)Q_n^m(z)\}}{dz},$$

where $z = \xi > 1$ for a prolate spheroid and $z = i\xi$, $\xi > 0$, for an oblate spheroid. Taking into account (12), we have

$$\lambda_n^m(z) = \pi\alpha_n^m(z^2-1)\frac{d}{dz}\int_{-1}^{1} \frac{[P_n^m(x)]^2\,dx}{z-x}, \qquad z \in \mathbb{C} \setminus (-1, 1). \tag{15}$$

For a prolate spheroid, this formula gives

$$\lambda_n^m(\xi) = -\pi\alpha_n^m(\xi^2-1)\int_{-1}^{1} \left\{\frac{P_n^m(x)}{\xi - x}\right\}^2 dx, \qquad \xi > 1,$$

and we can repeat the conclusion, made in [10], that all eigenvalues of the operators B and B^* are negative. It follows from (15) that for an oblate spheroid we have

$$\lambda_n^m(\xi) = 2\pi(1+\xi^2)\alpha_n^m \int_0^1 \frac{(x^2-\xi^2)[P_n^m(x)]^2\,dx}{(x^2+\xi^2)^2}, \qquad \xi > 0.$$

Hence, for $\xi \geq 1$ all eigenvalues $\lambda_n^m(\xi)$ are negative. Studying the graphs of the functions $\lambda_n^m(\xi)$ for various n and m, we can conjecture that this property holds for all $\xi > \xi_1^0 \simeq 0.660068$, but this conjecture needs a rigorous proof.

In conclusion, let us find an explicit expression for the solution of the Roben problem for an ellipsoid. As is known [21, p. 218], this problem can be reduced to the determination of the eigenfunction $\sigma_0(x)$ that corresponds to the eigenvalue $\mu_0 = -2\pi$ of the operator B on the surface S or, which is the same, is reduced to the solution of the equation $T\sigma_0 = \text{const}$ on S. Our result is based upon an unexpected theorem proved by Ritter in [22] and asserting the following. Let S be the surface of the ellipsoid with semiaxes $a \geq b \geq c > 0$. Then for $p(x) := \big(n(x), x\big)$, $x \in S$, and for any function $\varphi \in C(S)$, we have the following relation on S:

$$B(p\varphi) = pB^*(\varphi). \tag{16}$$

Here $n(x)$ is the unit vector of the outward normal to S at the point $x \in S$ and $\big(n(x), x\big)$ is the corresponding inner product in $\mathbb{R}^3$. By setting $\varphi(x) \equiv 1$ in (16), we readily see that if for some surface S relation (16) holds, then the function $p(x)$ is necessarily a solution of the Roben problem for this surface. Thus, the following assertion holds.

Theorem 6. *For an ellipsoid, the solution of the Roben problem has the form*

$$\sigma_0(x) = \big(n(x), x\big), \qquad x \in S.$$

We can easily find the explicit form of this function:

$$\sigma_0(x) = \left(\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4} + \frac{x_3^2}{c^4}\right)^{-1/2},$$

where $x = (x_1, x_2, x_3) \in S$. Regretfully, the assertion of the theorem cannot be generalized to all smooth surfaces (in particular, this assertion fails for the torus).

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References

1. N. M. Gyunter, *Potential Theory and its Applications to Main Problems of Mathematical Physics* [in Russian], Gostekhizdat, Moscow (1953).
2. S. G. Mikhlin, *Linear Partial Differential Equations* [in Russian], Vysshaya Shkola, Moscow (1977).
3. V. I. Smirnov, *A Course of Higher Mathematics* [in Russian], Vol. 4, Pt. 2, Nauka, Moscow (1981).
4. J. Dieudonné, *Eléments d'analyse*, Gauthier-Villars, Paris (1975).
5. V. V. Dyakin and V. Ya. Raevskii, "Properties of operators of the classical potential theory," *Mat. Zametki* [*Math. Notes*, **45**, No. 2, 138–140 (1989).
6. M. S. Agranovich, *Addendum to the book: N. N. Voitovich, B. Z. Katsenelenbaum, and A. N. Sivov, Generalized Method of Eigenoscillations in Diffraction Theory* [in Russian], Nauka, Moscow (1977).
7. V. G. Maz'ya, in: *Contemporary Problems in Mathematics. Fundamental Directions* [in Russian], Vol. 27, Itogi Nauki i Tekhniki, VINITI, Moscow (1988), pp. 131–228.
8. J. F. Ahner, "Some spectral properties of an integral operator in potential theory," *Proc. Edinburgh Math. Soc.*, **29**, No. 3, 405–411 (1986).
9. J. F. Ahner, V. V. Dyakin, and V. Ya. Raevskii, "New spectral results for the electrostatic integral operator," *J. Math. Anal. Appl.*, **185**, No. 2, , 391–402 (1994).
10. J. F. Ahner and R. F. Arenstorf, "On the eigenvalues of the electrostatic integral operator," *J. Math. Anal. Appl.*, **117**, No. 1, 187–197 (1986).
11. G. Verchota, "Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domain," *J. Funct. Anal.*, **59**, No. 3, 572–611 (1984).

12. V. N. Belykh, "On the problem of numerical solution of the Dirichlet problem by a harmonic simple layer potential," *Dokl. Ross. Akad. Nauk* [*Russian Math. Dokl.*], **329**, No. 4, 392–395 (1993).
13. J. Dieudonné, "Quasi-hermitian operators," in: *Proc. of the International Symposium on Linear Spaces*, Oxford, London, New York, and Jerusalem (1961), pp. 115–122.
14. V. T. Erofeenko, *Addition Theorems* [in Russian], Nauka i Tekhnika, Minsk (1989).
15. J. F. Ahner , "On the eigenvalues of the electrostatic integral operator, II," *J. Math. Anal. Appl.*, **181**, No. 2, 328–334 (1994).
16. H. Bateman and A. Erdélyi, *Higher Transcendental Functions*, Vol. 2, McGraw–Hill, New York (1953).
17. M. Abramowitz and I. A. Stegun (editors), *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Nat. Bur. Standards, Washington, D.C. (1964).
18. J. Plemelj, "Potentialtheoretische Untersuchungen," in: *Preisschriften der Fürstlich Jablonowskischen Gesellschaft zu Leipzig*, Teubner Verlag, Leipzig (1911).
19. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series. Supplementary Chapters* [in Russian], Nauka, Moscow (1986).
20. F. W. J. Olver, *Introduction to Asymptotics and Special Functions*, Academic Press, New York–London (1974).
21. P. P. Zabreiko, A. I. Koshelev, M. A. Krasnosel'skii, S. G. Mikhlin, L. S. Rakovshchik, and V. Ya. Stetseňko, *Integral Equations* [in Russian], Nauka, Moscow (1968).
22. St. Ritter, "The spectrum of the electrostatic integral operator for an ellipsoid," in: *Methoden und Verfahren der Mathematischen Physik*. Proceeding-Band der gleichnamigen Konferenz vom 12.–18.12.93 in Oberwolfach, Preprint.

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