

Conjugate flow action functionals

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Abstract. We present a new method to construct an action functional for a field theory described in terms of nonlinear partial differential equations (PDEs). The key idea relies on an intrinsic representation of the PDEs governing the physical system relatively to a diffeomorphic flow of coordinates which is assumed to be a functional of their solution. This flow, which will be called the conjugate flow of the theory, evolves in space and time similarly to a physical fluid flow of classical mechanics and it can be selected in order to symmetrize the Gâteaux derivative of the field equations relatively to a suitable (advective) bilinear form. This is equivalent to require that the equations of motion of the field theory can be derived from a principle of stationary action on a Lie group manifold. By using a general operator framework, we obtain the determining equations of such symmetrizing manifold for a second-order nonlinear scalar field theory. The generalization to vectorial and tensorial theories is straightforward.

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1. Introduction

The problem of finding an action functional whose Euler-Lagrange equations correspond to a prescribed set of partial differential equations (PDEs) is known as the inverse problem of the calculus of variations and it has attracted the attention of researches for more than one century. Perhaps, one the main reasons is that the formulation of a field theory in terms of an action functional is very elegant and, more importantly, it allows us to establish an immediate connection between symmetry principles and conservation laws [23, 32]. As is well known, the existence conditions of an action functional can be put in a correspondence with the theory of irrotational vector fields [26, 39, 43]. Essentially, if the path integral of the nonlinear operator representing the field equations is independent on the trajectory of functions connecting two specified points in a function space (i.e. in the domain of the nonlinear operator), then there exist a scalar field, *the action*, whose variational derivative is stationary in correspondence of the field equations of the theory. The path integral of an operator along a trajectory of functions is defined in terms of a bilinear form [25, 41]. Thus, the solution to the inverse problem of the calculus of variations, i.e. how to determine the action functional of a given set of PDEs, can be reduced to looking for a bilinear form that makes a given nonlinear operator “irrotational”, if any. It has been shown by Tonti [42] that there exist not just one but an infinite number of bilinear forms, often depending on the field equations of the theory, that satisfy this requirement. Therefore, an infinite number of action functionals can be constructed for a given set of PDEs. However, the physical meaning of the corresponding action principle sometimes may be obscured by the generalized bilinear forms that have to be adjusted on the specific problem. An alternative way

to proceed is to select a specific bilinear form having a physical meaning, and then look for ways of modifying the given field equations as to obtain a new problem which is potential with respect to the chosen bilinear form. Among known methods devised to do so, we recall the method of adding the adjoint equations [30, 15, 27, 20] and the integrating operator method of Tonti [42].

The purpose of this paper is to introduce a new approach to construct an action functional for an arbitrary field described in terms of nonlinear partial differential equations. The key idea relies on an intrinsic representation of the field equations relatively to a flow which is assumed to be a functional of their solution. This flow will be called the *conjugate flow* of the theory. Let us briefly describe the main ideas that led us to introduce this conjugate flow and, more importantly, their relevance in the context of known physical systems. To this end, we first notice that flows of coordinates being functionals of the solution to a field equation or a system of field equations arise naturally in many areas of mathematical physics. Perhaps, the most relevant example is in the context of classical fluid mechanics, where the set of fluid element trajectories in space is related to the velocity field that solves, e.g., the Navier-Stokes equations [1, 2]. This flow of curvilinear coordinates is known as *Lagrangian coordinate system* and it is obtained by integrating the definition of the velocity field [46]. In this sense, physical fluid flow of classical mechanics can be considered as a *very particular type* of conjugate flow. Another example is the free-falling coordinate system [49] in the Einstein's theory of gravitation. The conjugate flow here appears as a geodesic flow [23] in a four-dimensional Riemannian manifold whose metric is determined by a particular distribution of energy and momentum through the Einstein's field equations.

In both examples above, the relation between the solution to the field equations of the physical system and the flow of curvilinear coordinates is *known* and it reduces to the definition of the velocity field in the case of Navier-Stokes equations and to the definition of geodesics in the case of Einstein's theory of gravitation. In a broader framework, however, such functional relation may be left *unspecified*. This key observation provides us with an infinite number of functional degrees of freedom (those associated with the conjugate flow) that can be selected, e.g., by requiring that the field theory, expressed in conjugate flow intrinsic coordinates, is derivable from a principle of stationary action. In other words, we are posing the following fundamental question: *does it exist a functional flow of curvilinear coordinates such that if we represent the given set of field equations relatively to that flow then an action functional can be constructed?* This is equivalent to look for a representation of the field equations on a Riemannian manifold [7, 34] that depends on their solution, in such a way that the Gâteaux derivative of the nonlinear operator associated with the field equations (in intrinsic coordinates) is symmetric. This formulation of the inverse problem of the calculus of variations brings together concepts of differential geometry and nonlinear functional analysis and, as we will see, it results in the formulation of *new types of action principles* generalizing those ones based on specific functional flows of coordinates, such as the Herivel-Lin principle [19, 11, 6].

This paper is organized as follows. In section 2 we introduce the theory of the conjugate flow and we characterize the group of infinitesimal perturbations by using methods of nonlinear functional analysis. The representation of arbitrary nonlinear field equations in terms of conjugate flow intrinsic coordinates is discussed in section 3. Section 4 deals with the existence of a principle of stationary action in the context of conjugate flow variations. Formal symmetry conditions and corresponding determining equations for the conjugate flow are finally obtained and discussed in section 5 for second-order nonlinear scalar field equations.

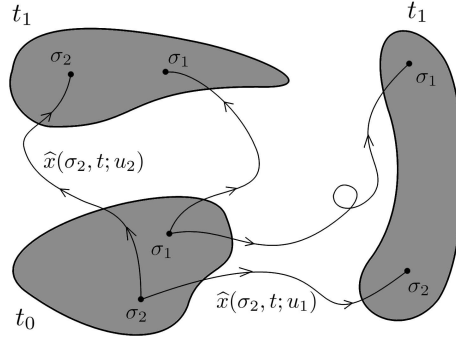


Figure 1. Sketch of the transport phenomenon of a volume of particles labeled by σ^μ according to two different realizations of the conjugate flow $\widehat{x}^\mu(\sigma^\nu; u^j)$, namely, $\widehat{x}^\mu(\sigma^\nu; u_1^j)$ and $\widehat{x}^\mu(\sigma^\nu; u_2^j)$. Shown are the trajectories of two different particles labeled as σ_1 and σ_2 . In this sketch, the time coordinate is not transformed, i.e. we have $\widehat{x}^0 = \sigma^0 = t$.

2. The conjugate flow

Conjugate flow is an intuitive physical notion which is represented mathematically by a continuous point transformation of $(n + 1)$ -dimensional (n denotes the number of spatial dimensions) Euclidean or Riemannian space into itself. In order to set up this transformation, let us consider a particle labeled by σ^ν ($\nu = 0, \dots, n$, 0 being the temporal component) and represent its trajectory in a fixed space-time Cartesian system as

$$x^\mu = \widehat{x}^\mu(\sigma^\nu; u^j), \quad \mu, \nu = 0, \dots, n, \quad (1)$$

where u^j ($j = 1, \dots, N$) is a vector field that solves a prescribed system of field equations. The transformation (1) is assumed to be invertible (with differentiable inverse) and eventually to possess even continuous derivatives up to a prescribed order except possibly at certain singular surfaces, curves or points. These requirements make (1) a *diffeomorphism*, i.e. a time-dependent flow of curvilinear coordinates [46, 28, 37, 38, 24] whose motion in space resembles *in toto* a physical fluid flow of classical mechanics. In figure 1 we sketch two realizations of the conjugate flow (1) for two different solutions fields corresponding, e.g., to different boundary or initial conditions.

Coordinate flows being functionals of the solution to a field equation are obviously not new in the scientific literature. For instance, in the context of symmetry analysis of partial differential equations the so-called non-classical symmetries [5] are remarkable examples of solution-dependent transformations. Similarly, in classical Lagrangian fluid dynamics the trajectories of the fluid elements in space are obtained as local functionals of the velocity field $U^j(x^k, t)$ that solves, e.g., the Navier-Stokes equations (x^k is a fixed Cartesian coordinate system)

$$\frac{\partial U^j}{\partial t} + U^k \frac{\partial U^j}{\partial x^k} = -\frac{\partial P}{\partial x^k} + \frac{1}{Re} \frac{\partial^2 U^j}{\partial x^k \partial x^k}, \quad \frac{\partial U^k}{\partial x^k} = 0, \quad k, j = 1, \dots, n. \quad (2)$$

Such functional relation is defined by the solution to the well-known problem

$$\frac{\partial \widehat{X}^j(\sigma^i, t)}{\partial t} = U^j(\widehat{X}^k(\sigma^i, t), t), \quad \widehat{X}^j(\sigma^i, t_0) = \sigma^j. \quad (3)$$

Note that the physical fluid flow \widehat{X}^j can be considered as a very particular type of conjugate flow since it is functionally dependent on the solution to Eq. (2) by means of Eq. (3). Another conjugate flow which is different from the physical fluid flow may be defined, e.g., by solving

$$\frac{\partial \widehat{X}^j(\sigma^i, t)}{\partial t} = U^j \left(\widehat{X}^k(\sigma^i, t) + B^\omega(\sigma^i, t), t \right), \quad \widehat{X}^j(\sigma^i, t_0) = \sigma^j. \quad (4)$$

where $B^\omega(\sigma^i, t)$ is a realization of a space-time Brownian motion. A remarkable result by Gomes [17] shows that ensemble averaging - over random B^ω - of diffeomorphisms of type (4) allows to construct a variational principle for the Navier-Stokes equations.

2.1. Infinitesimal flow perturbations

The components of the vector field u^j appearing in Eq. (1) are, by definition, Cartesian components expressed in terms of conjugate flow intrinsic coordinates σ^ν . In other words, if we denote by $U^j(x^\mu)$ the Cartesian components of the solution to a prescribed set of field equations expressed in Cartesian coordinates (e.g. Eqs. (2)), then u^j are defined as

$$u^j(\sigma^\nu) \stackrel{\text{def}}{=} U^j(\widehat{x}^\mu(\sigma^\nu)). \quad (5)$$

We remark that these are *not* the tensorial components [1, 44] of the vector field. Let us consider an infinitesimal perturbation of u^j in the form

$$u^j(\sigma^\nu) + \epsilon \varphi^j(\sigma^\nu), \quad \text{for } \epsilon \rightarrow 0. \quad (6)$$

Then, to the first-order in ϵ , we obtain the following perturbation in the conjugate flow trajectories (1)

$$\widehat{x}^\mu(\sigma^\nu; u^j + \epsilon \varphi^j) \simeq \widehat{x}^\mu(\sigma^\nu; u^j) + \epsilon \frac{\delta \widehat{x}^\mu}{\delta u^j} \varphi^j, \quad (7)$$

where, by definition

$$\frac{\delta \widehat{x}^\mu}{\delta u^j} \varphi^j \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow 0} \frac{\widehat{x}^\mu(\sigma^\nu; u^j + \epsilon \varphi^j) - \widehat{x}^\mu(\sigma^\nu; u^j)}{\epsilon}. \quad (8)$$

The quantity $\delta \widehat{x}^\mu / \delta u^j$ is known as Gâteaux derivative [45] of the functional \widehat{x}^μ with respect to u^j and under rather weak requirements [31] it is a continuous linear operator. The perturbed flow $\widehat{x}^\mu(\sigma^\nu; u^j + \epsilon \varphi^j)$ is schematically depicted in figure 2 and it is assumed to have the same regularity properties of the unperturbed one, i.e. invertibility and continuous derivatives up to prescribed order in all variables.

Let us now *postulate* that the solution field u^j is also functionally connected to the conjugate flow \widehat{x}^μ and let us denote this functional relation by $u^j(\sigma^\nu; \widehat{x}^\mu)$. This fundamental assumption implies that an infinitesimal flow perturbation $\widehat{x}^\mu + \epsilon \widetilde{\phi}^\mu$ induces the following variation in the solution field u^j

$$u^j(\sigma^\nu; \widehat{x}^\mu + \epsilon \widetilde{\phi}^\mu) \simeq u^j(\sigma^\nu; \widehat{x}^\mu) + \epsilon \frac{\delta u^j}{\delta \widehat{x}^\mu} \widetilde{\phi}^\mu, \quad (9)$$

where, in analogy with Eq. (8), we have defined the Gâteaux differential as

$$\frac{\delta u^j}{\delta \widehat{x}^\mu} \widetilde{\phi}^\mu \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow 0} \frac{u^j(\sigma^\nu; \widehat{x}^\mu + \epsilon \widetilde{\phi}^\mu) - u^j(\sigma^\nu; \widehat{x}^\mu)}{\epsilon}. \quad (10)$$

In the context of the Navier-Stokes equations, this means that a perturbation in the conjugate flow \widehat{x}^j determines - by assumption - a perturbation in the velocity field U^j that solves Eq. (2). This ultimately results in a perturbation of the physical fluid flow \widehat{X}^j by means of Eq. (3). In

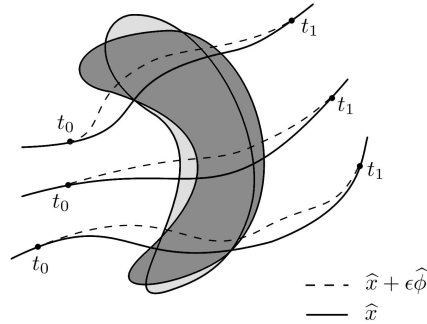


Figure 2. Conjugate flow perturbation $\hat{x}^\mu + \epsilon \hat{\phi}^\mu$ induced by a field perturbation $u^j + \epsilon \varphi^j$ and corresponding deformation of the volume of particles σ^ν transported by the flow.

other words, by perturbing the conjugate flow in this case we are actually perturbing the physical fluid flow. At this point it is convenient to set

$$\tilde{\varphi}^j = \frac{\delta u^j}{\delta \hat{x}^\mu} \tilde{\phi}^\mu, \quad (11)$$

$$\hat{\phi}^\mu = \frac{\delta \hat{x}^\mu}{\delta u^j} \varphi^j \quad (12)$$

and write Eq. (7) and Eq. (9) as

$$u^j(\sigma^\mu; \hat{x}^\nu + \epsilon \tilde{\phi}^\nu) \simeq u^j(\sigma^\mu; \hat{x}^\nu) + \epsilon \tilde{\varphi}^j(\sigma^\mu; \hat{x}^\nu), \quad (13)$$

$$\hat{x}^\mu(\sigma^\nu; u^j + \epsilon \varphi^j) \simeq \hat{x}^\mu(\sigma^\nu; u^j) + \epsilon \hat{\phi}^\mu(\sigma^\nu; u^j). \quad (14)$$

Note that in these equations we have $\hat{\phi}^\mu \neq \tilde{\phi}^\mu$ and $\varphi^j \neq \tilde{\varphi}^j$. In fact, if we arbitrarily perform a simultaneous perturbation of u^j and \hat{x}^μ we cannot obviously expect that, in general, the functional disturbances arising from the Gâteaux differentials (11) and (12) coincide with the perturbations at the left hand side of Eqs. (13) and (14). This immediately leads us to the question of which variable between u^j and \hat{x}^μ should be chosen as *independent* when performing perturbations. In the sequel we will be mostly concerned with perturbations induced in the conjugate flow \hat{x}^μ through a variation of the solution field u^j , i.e. we will mostly employ Eq. (14), although the other approach, i.e. the one based Eq. (13), can be equivalently considered.

3. Conjugate flow representation of field equations

Several field equations of mathematical physics, remarkably the fluid mechanics equations, include naturally the concept of a conjugate flow within their formulation. Such a flow usually has a direct physical interpretation, e.g., trajectories of fluid elements in space, and it often constitutes the ground work on which dynamical results are constructed [1, 2]. Many other equations, however, do not include explicitly any term having a direct reference to a conjugate flow. This is the case, for example, of the classical heat equation, the Maxwell's equations of electrodynamics, the laws of elasticity and, undoubtedly, many others. The fundamental question at this point is whether it is possible to formulate a law that include *both* the field equations and the conjugate flow and allows to study their interaction, e.g., in the context of the principle of stationary action. The answer is affirmative and the simplest way to achieve this result is to represent the field equations relatively to a coordinate system which is advected by the conjugate

flow, namely, coordinates σ^ν . In other words, we represent the partial differential equations governing the physical system on a curvilinear manifold which is assumed to be functionally dependent on their solution.

This procedure obviously introduces a functional dependence in the form of the field equations from their solution. As a consequence, the equations look completely different in conjugate flow intrinsic coordinates and, in general, they are *highly nonlinear*. For example, by using the mathematical tools of Appendix A, it can be shown that the classical one-dimensional heat equation

$$\frac{\partial U}{\partial t} - \alpha \frac{\partial^2 U}{\partial x^2} = 0, \quad (15)$$

where $U(x, t)$ denotes the temperature in fixed Cartesian coordinates, can be written in terms of conjugate flow intrinsic coordinates as

$$\frac{\partial u}{\partial t} - \frac{1}{\partial \hat{x} / \partial \sigma} \frac{\partial u}{\partial \sigma} \frac{\partial \hat{x}}{\partial t} - \frac{\alpha}{(\partial \hat{x} / \partial \sigma)^3} \left(\frac{\partial \hat{x}}{\partial \sigma} \frac{\partial^2 u}{\partial \sigma^2} - \frac{\partial^2 \hat{x}}{\partial \sigma^2} \frac{\partial u}{\partial \sigma} \right) = 0, \quad (16)$$

where the flow \hat{x} is a functional of u . For illustration purposes, here we have assumed that the time variable is not transformed, i.e. we have set $\hat{x}^0 = \sigma^0 = t$. By examining Eq. (16) under the conjugate flow perspective we see that a perturbation in the field $u(\sigma, t)$ induces also a perturbation in conjugate flow $\hat{x}(\sigma, t; u)$ (see Eq. (14)) and therefore the perturbed equation in conjugate flow intrinsic coordinates includes many terms arising from the perturbations of both u and \hat{x} . Clearly, if the conjugate flow is in rest with respect to the fixed Cartesian coordinate system then Eq. (16) coincides with Eq. (15), although the effects of the aforementioned functional perturbations are still present.

The conjugate flow representation of a field equation is obviously much more complex than a standard formulation in fixed Cartesian coordinates. This has been observed, e.g., by Temam [36], in the context of the Lagrangian representation of the Navier-Stokes equations. He pointed out that “the Lagrangian representation is not used too often because the Navier-Stokes equations in Lagrangian coordinates are highly nonlinear”. Indeed, by using the results of Appendix A, it can be shown that these equations can be written in general conjugate flow intrinsic coordinates as

$$\frac{\partial u^j}{\partial \sigma^\nu} A_0^\nu + u^k \frac{\partial u^j}{\partial \sigma^\nu} A_k^\nu = - \frac{\partial p}{\partial \sigma^\nu} A_j^\nu + \frac{1}{Re} \left(\frac{\partial^2 u^j}{\partial \sigma^\lambda \partial \sigma^\rho} A_k^\rho A_k^\lambda + \frac{\partial u^j}{\partial \sigma^\lambda} \frac{\partial A_k^\lambda}{\partial \sigma^\rho} A_k^\rho \right), \quad (17)$$

where the quantities A_ν^μ , defined in Eq. (A.6), are rather complex functionals of \hat{x}^μ . Clearly, when the coordinate system σ^μ is advected exactly by the physical fluid flow, i.e. when the functional link between u^j and \hat{x}^j is defined by Eq. (3), then Eq. (17) coincides with the Lagrangian representation of the Navier-Stokes equations.

3.1. Functional setting

Let us associate with the physical system the linear function space \mathcal{U} , whose elements are the N -tuples $u = (u^1, \dots, u^N)$. Similarly, let us also consider the *configuration space* \mathcal{X} , whose elements, denoted as $\hat{x} = (\hat{x}^0, \dots, \hat{x}^n)$, represent $(n+1)$ -dimensional conjugate flows, n being the number of spatial dimensions. In general, the configuration space is not a linear space because the summation of two conjugate flows is not a conjugate flow. This is due to the fact the superimposition of two invertible flows may not be invertible (the summation of two invertible Jacobian matrices is not necessarily invertible). However, the requirement that the perturbed conjugate flow has the same properties of the unperturbed one, i.e. it is still a diffeomorphism, is equivalent to state that locally, i.e. in the neighborhood of a particular flow \hat{x} , the configuration

space \mathcal{X} is linear or can be linearized. In this sense we can say that the configuration space \mathcal{X} is *locally linear*. Given this, an arbitrary field equation (or a system of field equations) written in terms of conjugate flow intrinsic coordinates can be synthesized as

$$\mathbf{N}_{\hat{x}}(u) = \emptyset_{\mathcal{V}}, \quad (18)$$

where $\mathbf{N}_{\hat{x}}$ is, in general, a nonlinear operator while $\emptyset_{\mathcal{V}}$ denotes the null element of a third topological linear space \mathcal{V} . The subscript \hat{x} in $\mathbf{N}_{\hat{x}}$ reminds us that the operator is defined in terms of conjugate flow intrinsic coordinates σ^ν , i.e. on the manifold \hat{x} . For example, Eq. (16) can be written in the form (18) by defying $\mathbf{N}_{\hat{x}}$ as

$$\mathbf{N}_{\hat{x}}(u) \stackrel{\text{def}}{=} \frac{\partial u}{\partial t} - \frac{1}{\partial \hat{x} / \partial \sigma} \frac{\partial u}{\partial \sigma} \frac{\partial \hat{x}}{\partial t} - \frac{\alpha}{(\partial \hat{x} / \partial \sigma)^3} \left(\frac{\partial \hat{x}}{\partial \sigma} \frac{\partial^2 u}{\partial \sigma^2} - \frac{\partial^2 \hat{x}}{\partial \sigma^2} \frac{\partial u}{\partial \sigma} \right). \quad (19)$$

The domain of the nonlinear operator $\mathbf{N}_{\hat{x}}$, is a space of functions satisfying the initial or the boundary conditions of the problem. In the conjugate flow theory, however, the operator $\mathbf{N}_{\hat{x}}(u)$ acts on both u and \hat{x} and therefore it implicitly identifies two different domains, one within the space of fields \mathcal{U} and the other one within the configuration space \mathcal{X} . These two domains will be denoted by $D_{\mathcal{U}}(\mathbf{N}_{\hat{x}}) \subseteq \mathcal{U}$ and $D_{\mathcal{X}}(\mathbf{N}_{\hat{x}}) \subseteq \mathcal{X}$, respectively (see figure 3). The range of the operator $\mathbf{N}_{\hat{x}}$ will be denoted by $R(\mathbf{N}_{\hat{x}}) \subseteq \mathcal{V}$. The next fundamental step in the functional setting of the conjugate flow theory of field equations is to introduce duality pairings between the linear spaces \mathcal{U} , \mathcal{V} and the locally linear one \mathcal{X} through non-degenerate *local bilinear forms* [47, 26]. To this end, let us define

$$\langle \cdot, \cdot \rangle_u : \mathcal{V} \times \mathcal{U} \rightarrow \mathbb{R}, \quad (20)$$

$$\langle \cdot, \cdot \rangle_{\hat{x}} : \mathcal{V} \times \mathcal{X} \rightarrow \mathbb{R}. \quad (21)$$

The subscripts u and \hat{x} in Eqs. (20) and (21) emphasize the fact that such forms depend also on u and \hat{x} , respectively (in a possibly nonlinear way). An explicit expression of (20) will be given in section 4.2. For the moment we simply observe that, locally, the forms (20) and (21) can be put in a correspondence through the linear transformations defined by Eqs. (11) and (12). In fact, as shown in figure 3, the elements of $D_{\mathcal{U}}(\mathbf{N}_{\hat{x}})$ in the neighborhood of a certain u are in correspondence with the elements of $D_{\mathcal{X}}(\mathbf{N}_{\hat{x}})$ in the neighborhood of a certain \hat{x} . In practice, such a correspondence can be established locally through the linear operators $\delta u / \delta \hat{x}$ and $\delta \hat{x} / \delta u$. For instance, by using Eq. (11) we obtain

$$\langle v, \tilde{\varphi} \rangle_u = \langle v, \frac{\delta u}{\delta \hat{x}} \tilde{\phi} \rangle_u = \langle v, \tilde{\phi} \rangle_{\hat{x}}. \quad (22)$$

We shall conclude this section by explaining why we have chosen the definition “conjugate flow” for the transformation (1). To this end, we recall that the Gâteaux differential of \hat{x} with respect to u defines a linear functional from the space \mathcal{U} to the space $\mathcal{X} \equiv \mathcal{U}^\dagger$, which is the *conjugate* space of \mathcal{U} . Thus, for every admissible $u \in \mathcal{U}$, the flow \hat{x} belongs to the conjugate space of \mathcal{U} , hence the definition “conjugate flow”. In a broader sense, the adjective “conjugate” simply emphasizes that there exist a functional relation between the flow \hat{x} , the dynamic equations \mathbf{N} of the field theory and their solution u . We also remark that a definition of conjugate flow already appeared in the scientific literature [3, 4], as “a flow uniform in the direction of streaming which separately satisfy the hydrodynamical equations”. This definition, however, is clearly different from ours.

3.2. Perturbation expansions

By employing the operatorial approach developed in the previous section we can easily synthesize in a single operator equation the perturbative form of an arbitrary nonlinear field equation

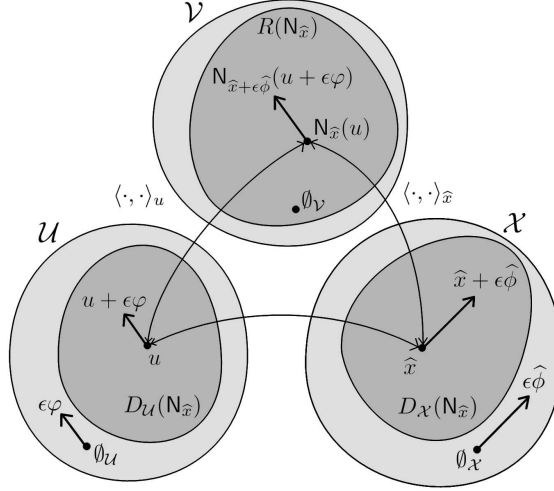


Figure 3. Sketch of the function spaces employed for the functional setting of the conjugate flow theory. Shown are the domains $D_{\mathcal{U}}(\mathbf{N}_{\hat{x}}) \subseteq \mathcal{U}$ and $D_{\mathcal{X}}(\mathbf{N}_{\hat{x}}) \subseteq \mathcal{X}$ of the nonlinear operator $\mathbf{N}_{\hat{x}}$ representing the field equations. The range of $\mathbf{N}_{\hat{x}}$ is denoted by $R(\mathbf{N}_{\hat{x}}) \subseteq \mathcal{V}$. We also show the correspondence between field perturbations $(u + \epsilon\varphi)$, conjugate flow perturbations $(\hat{x} + \epsilon\hat{\phi})$ and corresponding perturbations induced in the field equations $(\mathbf{N}_{\hat{x} + \epsilon\hat{\phi}}(u + \epsilon\varphi))$ relatively to a specific representation $(u, \hat{x}, \mathbf{N}_{\hat{x}})$. The local bilinear forms that put the various spaces in duality are indicated in between the sets.

(or a system of nonlinear equations) in the presence of a conjugate flow perturbation, i.e. a simultaneous perturbation of both the solution field and conjugate flow. To the first-order in ϵ we have

$$\mathbf{N}_{\hat{x} + \epsilon\hat{\phi}}(u + \epsilon\varphi) = \mathbf{N}_{\hat{x}}(u) + \epsilon \left[\frac{\delta \mathbf{N}_{\hat{x}}}{\delta u} \varphi + \frac{\delta \mathbf{N}_{\hat{x}}}{\delta \hat{x}} \hat{\phi} \right] + \dots, \quad (23)$$

where the Gâteaux differentials appearing in Eq. (23) are defined as

$$\frac{\delta \mathbf{N}_{\hat{x}}}{\delta u} \varphi \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow 0} \frac{\mathbf{N}_{\hat{x}}(u + \epsilon\varphi) - \mathbf{N}_{\hat{x}}(u)}{\epsilon}, \quad (24)$$

$$\frac{\delta \mathbf{N}_{\hat{x}}}{\delta \hat{x}} \hat{\phi} \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow 0} \frac{\mathbf{N}_{\hat{x} + \epsilon\hat{\phi}}(u) - \mathbf{N}_{\hat{x}}(u)}{\epsilon}, \quad (25)$$

provided that such limits exist. A function space representation of the conjugate flow perturbation is sketched in figure 3.

4. Conjugate flow action functionals

Let us consider a field $u \in D_{\mathcal{U}}(\mathbf{N}_{\hat{x}})$ and a conjugate flow $\hat{x} \in D_{\mathcal{X}}(\mathbf{N}_{\hat{x}})$. The couple (u, \hat{x}) does not necessarily have to be a solution to the field equation, i.e. $\mathbf{N}_{\hat{x}}(u) \neq \emptyset_{\mathcal{V}}$. Disregarding the particular form of the operator $\mathbf{N}_{\hat{x}}$, it is useful to consider

$$v = \mathbf{N}_{\hat{x}}(u) \in R(\mathbf{N}_{\hat{x}}) \quad (26)$$

as a definition two vector fields [31]: one in $D_{\mathcal{U}}(\mathbf{N}_{\hat{x}})$ and the other one $D_{\mathcal{X}}(\mathbf{N}_{\hat{x}})$, respectively. This allows us to introduce in a conceptually simple way the notion of a line integral of an operator according to a geometric standpoint which seems originally due to Volterra [48]. To this end, let us consider a one-parameter family of fields in the domain $D_{\mathcal{U}}(\mathbf{N}_{\hat{x}})$

$$u = u_{\lambda} \quad (0 \leq \lambda \leq 1). \quad (27)$$

This can be regarded as a line in the function space $D_{\mathcal{U}}(\mathbf{N}_{\hat{x}})$. With such line we can associate the number

$$\ell_u = \int_0^1 \langle \mathbf{N}_{\hat{x}}(u_\lambda), \frac{\partial u_\lambda}{\partial \lambda} \rangle_{u_\lambda} d\lambda, \quad (28)$$

i.e. the path integral of the operator $\mathbf{N}_{\hat{x}}$ along the trajectory of functions $u_\lambda \in D_{\mathcal{U}}(\mathbf{N}_{\hat{x}})$. We recall that $\langle \cdot, \cdot \rangle_u$ in (28) denotes the local bilinear form (20). In the context of the conjugate flow theory, we can also define the path integral of the operator $\mathbf{N}_{\hat{x}}$ along a trajectory of flows \hat{x}_λ in the space $D_{\mathcal{X}}(\mathbf{N}_{\hat{x}})$, i.e.

$$\ell_{\hat{x}} = \int_0^1 \langle \mathbf{N}_{\hat{x}_\lambda}(u), \frac{\partial \hat{x}_\lambda}{\partial \lambda} \rangle_{\hat{x}_\lambda} d\lambda, \quad (29)$$

where $\langle \cdot, \cdot \rangle_{\hat{x}}$ denotes the local bilinear form (21). Therefore, we can define an infinitesimal circulation of an operator around a certain field u as well as around a certain conjugate flow \hat{x} , these circulations being of course related by Eqs. (11) and (12). If the line integrals (28) and (29) are independent of the path of integration then the operator $\mathbf{N}_{\hat{x}}$ is said to be *potential* with respect to the chosen local bilinear form. In this case the line integral from a prefixed element u_0 to any element u in the domain of $\mathbf{N}_{\hat{x}}$ along an arbitrarily chosen path defines the *action functional*

$$A_u[u] = A_u[u_0] + \int_0^1 \langle \mathbf{N}_{\hat{x}}(u_\lambda), \frac{\partial u}{\partial \lambda} \rangle_{u_\lambda} d\lambda. \quad (30)$$

Similarly, the line integral from a prefixed conjugate flow \hat{x}_0 to another flow \hat{x} along an arbitrarily chosen line \hat{x}_λ defines another (*dual*) action functional

$$A_{\hat{x}}[\hat{x}] = A_{\hat{x}}[\hat{x}_0] + \int_0^1 \langle \mathbf{N}_{\hat{x}_\lambda}(u), \frac{\partial \hat{x}_\lambda}{\partial \lambda} \rangle_{\hat{x}_\lambda} d\lambda. \quad (31)$$

In turn, the operator $\mathbf{N}_{\hat{x}}$ is said to be the *gradient* of the functionals $A_u[u]$ or $A_{\hat{x}}[\hat{x}]$. This definition relies on the fact that if we calculate an infinitesimal line integral by using u or \hat{x} as independent variable then we obtain, respectively,

$$\delta A_u[u] = \langle \mathbf{N}_{\hat{x}}(u), \delta u \rangle_u, \quad \delta A_{\hat{x}}[\hat{x}] = \langle \mathbf{N}_{\hat{x}}(u), \delta \hat{x} \rangle_{\hat{x}}. \quad (32)$$

The relations (32) show that the equations of motion of the system, i.e. $\mathbf{N}_{\hat{x}}(u) = \emptyset_{\mathcal{V}}$, can be obtained as a stationary point of either $A_u[u]$ or $A_{\hat{x}}[\hat{x}]$, for arbitrary variations δu and $\delta \hat{x}$, respectively. Thus, the theory of conjugate flows allow us to look for action functionals associated with field equations in two different ways, depending on which variable between u or \hat{x} is assumed as *independent*. Clearly, if we consider u as independent then we are looking for the set of conjugate flows such that the field equation is potential. On the contrary, if we consider the conjugate flow \hat{x} as independent then we are looking for the set of fields u such that the field equation is potential. In the sequel we will be mostly concerned with conjugate flows action functionals where the field u is considered as independent variable.

4.1. Existence conditions

In order to formulate the existence conditions of conjugate flow action functionals we follow the approach of Magri [26]. To this end, we consider two infinitesimal trajectories (two infinitesimal straight lines) of the field u in the function space $D_{\mathcal{U}}(\mathbf{N}_{\hat{x}})$

$$\begin{aligned} \text{I} & : u \rightarrow u + \epsilon\varphi \\ \text{II} & : u \rightarrow u + \nu\psi \end{aligned}$$

Correspondingly, we have the following infinitesimal conjugate flow perturbations

$$\begin{aligned} \text{I} &: \hat{x} \rightarrow \hat{x} + \epsilon \hat{\phi} \\ \text{II} &: \hat{x} \rightarrow \hat{x} + \nu \hat{\eta} \end{aligned}$$

where $\hat{\phi}$ and $\hat{\eta}$ are related to the field perturbations φ and ψ in the sense of Eq. (12). Due to this fundamental relation, an infinitesimal circulation of the operator $\mathbf{N}_{\hat{x}}$ around the element $u \in D_{\mathcal{U}}(\mathbf{N}_{\hat{x}})$ is associated with an infinitesimal circulation of $\mathbf{N}_{\hat{x}}$ around a flow $\hat{x} \in D_{\mathcal{X}}(\mathbf{N}_{\hat{x}})$. The vanishing of these simultaneous circulations with respect to the local bilinear form (20) is synthesized by the condition

$$\begin{aligned} \langle \mathbf{N}_{\hat{x}}(u), \epsilon \varphi \rangle_u + \langle \mathbf{N}_{\hat{x} + \epsilon \hat{\phi}}(u + \epsilon \varphi), \nu \psi \rangle_{u + \epsilon \varphi} = \\ \langle \mathbf{N}_{\hat{x}}(u), \nu \psi \rangle_u + \langle \mathbf{N}_{\hat{x} + \nu \hat{\eta}}(u + \nu \psi), \epsilon \varphi \rangle_{u + \nu \psi}. \end{aligned} \quad (33)$$

To the second-order in ϵ and ν we have

$$\begin{aligned} \langle \mathbf{N}_{\hat{x} + \epsilon \hat{\phi}}(u + \epsilon \varphi), \nu \psi \rangle_{u + \epsilon \varphi} = \langle \mathbf{N}_{\hat{x}}(u), \psi \rangle_u + \\ \epsilon \nu \left\langle \frac{\delta \mathbf{N}_{\hat{x}}}{\delta u} \varphi + \frac{\delta \mathbf{N}_{\hat{x}}}{\delta \hat{x}} \hat{\phi}, \psi \right\rangle_u + \epsilon \nu \langle \varphi; \mathbf{N}_{\hat{x}}(u), \psi \rangle_u, \end{aligned} \quad (34)$$

where

$$\langle \varphi; v, \psi \rangle_u \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow 0} \frac{\langle v, \psi \rangle_{u + \epsilon \varphi} - \langle v, \psi \rangle_u}{\epsilon} \quad (35)$$

denotes the Gâteaux differential of the local bilinear form (20), considered as a particular type of nonlinear operator on u . A substitution of Eq. (34) into Eq. (33) gives

$$\begin{aligned} \left\langle \frac{\delta \mathbf{N}_{\hat{x}}}{\delta u} \varphi + \frac{\delta \mathbf{N}_{\hat{x}}}{\delta \hat{x}} \hat{\phi}, \psi \right\rangle_u + \langle \varphi; \mathbf{N}_{\hat{x}}(u), \psi \rangle_u = \\ \left\langle \frac{\delta \mathbf{N}_{\hat{x}}}{\delta u} \psi + \frac{\delta \mathbf{N}_{\hat{x}}}{\delta \hat{x}} \hat{\eta}, \varphi \right\rangle_u + \langle \psi; \mathbf{N}_{\hat{x}}(u), \varphi \rangle_u. \end{aligned} \quad (36)$$

Finally, by using Eq. (12) we can write the vanishing condition of the infinitesimal circulation entirely in terms of field perturbations ψ and φ as

$$\langle \mathbf{G}_{\hat{x}} \varphi, \psi \rangle_u + \langle \varphi; \mathbf{N}_{\hat{x}}(u), \psi \rangle_u = \langle \mathbf{G}_{\hat{x}} \psi, \varphi \rangle_u + \langle \psi; \mathbf{N}_{\hat{x}}(u), \varphi \rangle_u, \quad (37)$$

where the linear operator $\mathbf{G}_{\hat{x}}$ is defined as

$$\mathbf{G}_{\hat{x}} \stackrel{\text{def}}{=} \frac{\delta \mathbf{N}_{\hat{x}}}{\delta u} + \frac{\delta \mathbf{N}_{\hat{x}}}{\delta \hat{x}} \frac{\delta \hat{x}}{\delta u}. \quad (38)$$

Thus, if the circulation vanishes along any infinitesimal closed line in $D_{\mathcal{U}}(\mathbf{N}_{\hat{x}})$ then Eq. (37) must hold for every φ , ψ and for all admissible u . This is the necessary condition for operators to be potential with respect to the local bilinear form (20). If the domain of the operator $\mathbf{N}_{\hat{x}}$ is simply connected, then this condition is also sufficient. This happens, for example, when $D_{\mathcal{U}}(\mathbf{N}_{\hat{x}})$ is defined by linear homogeneous initial or boundary conditions (in this case $D_{\mathcal{U}}(\mathbf{N}_{\hat{x}})$ is a convex set). We notice that Eq. (37) includes also interesting subcases. For example, if the flow \hat{x} is not a functional of u then we have

$$\left\langle \frac{\delta \mathbf{N}_{\hat{x}}}{\delta u} \varphi, \psi \right\rangle_u + \langle \varphi; \mathbf{N}_{\hat{x}}, \psi \rangle_u = \left\langle \frac{\delta \mathbf{N}_{\hat{x}}}{\delta u} \psi, \varphi \right\rangle_u + \langle \psi; \mathbf{N}_{\hat{x}}, \varphi \rangle_u. \quad (39)$$

In addition, if the bilinear form $\langle \cdot, \cdot \rangle_u$ does not depend on u , i.e. if we are dealing with a global bilinear form, then the potential theory coincides with the classical theory of Vainberg [45]. In this case the condition (37) reduces to

$$\left\langle \frac{\delta \mathbf{N}_{\hat{x}}}{\delta u} \varphi, \psi \right\rangle = \left\langle \frac{\delta \mathbf{N}_{\hat{x}}}{\delta u} \psi, \varphi \right\rangle, \quad (40)$$

namely, the Gâteaux derivative of the operator $\mathbf{N}_{\hat{x}}$ must be symmetric with respect to the bilinear form $\langle \cdot, \cdot \rangle$.

4.2. The local bilinear form defining the conjugate flow action functional

In classical, relativistic and quantum field theories the action functional has the standard form [22, 23, 49]

$$A = \int \mathcal{L} \sqrt{g} d\sigma. \quad (41)$$

where \mathcal{L} denotes a Lagrangian density, g is the determinant of the metric tensor and $\sqrt{g}d\sigma$ is the invariant space-time volume element ($d\sigma$ being a shorthand notation for $d\sigma^0 d\sigma^1 \cdots d\sigma^n$). We recall that the square root of the metric tensor determinant is equal to the Jacobian determinant J of the transformation from fixed Cartesian to conjugate flow intrinsic coordinates (see Appendix A for further details). A comparison between Eq. (41) and Eq. (30) suggests that the local bilinear form to be considered for the conjugate flow formulation of the inverse problem of the calculus of variations is

$$\langle a, b \rangle_u \stackrel{\text{def}}{=} \int_{\Sigma} ab J d\sigma, \quad a \in \mathcal{U}, \quad b \in \mathcal{V} \quad (42)$$

where the Jacobian determinant J is a rather complex functional of \hat{x} (see Eq. (A.5)). The domain Σ appearing in the integral (42) is a four-dimensional volume of particles σ^ν advected by the conjugate flow (see figure 1 and figure 2). The form (42) generalizes the bilinear form appearing in the Herivel-Lin variational principle [19, 6, 11, 35], where the volume of particles is advected precisely by the physical fluid flow. Note also that (42) is symmetric, non-degenerate and non-negative, i.e., it satisfies all the properties of an inner product. By using Eq. (A.21) we obtain the following Gâteaux derivative

$$\begin{aligned} \langle \varphi; a, b \rangle_u &\stackrel{\text{def}}{=} \frac{d}{d\epsilon} [\langle a, b \rangle_{u+\epsilon\varphi}]_{\epsilon=0} \\ &= \int_{\Sigma} ab J \nabla \cdot \hat{\phi} d\sigma \end{aligned} \quad (43)$$

where, according to Eq. (12), $\hat{\phi}$ is a linear functional of φ , i.e. (43) is a trilinear form in a , b and φ .

4.3. Incompressible flow perturbations

Let us assume that the divergence of the perturbation field $\hat{\phi}$ appearing in Eq. (43) vanishes. Under this assumption the symmetry condition (37) simplifies to

$$\langle \mathbf{G}_{\hat{x}} \varphi, \psi \rangle_u = \langle \mathbf{G}_{\hat{x}} \psi, \varphi \rangle_u. \quad (44)$$

Equation (44) basically requires the symmetry of $\mathbf{G}_{\hat{x}}$ relatively to the local inner product (42). Thus, the application of the conjugate flow theory to the inverse problem of the calculus of variations is now reduced to look for an incompressible four-dimensional flow of curvilinear coordinates that symmetrizes the operator $\mathbf{G}_{\hat{x}}$.

5. Symmetrizing flows

A field equation is said to be formally symmetric when the operator symmetry condition, e.g. Eq. (44), is satisfied disregarding the particular form of the boundary or the initial conditions associated with the problem. Clearly, when the domain of the operator $\mathbf{N}_{\hat{x}}$ is formed by a set of functions satisfying local homogeneous boundary and initial conditions then formal symmetry is a necessary condition for symmetry. Such a condition, however, is not sufficient even in the case of homogeneous boundaries [13]. In any case, it is useful to establish formal symmetry conditions

for particular classes of field equations. This has been done by Tonti [39, 40] using classical inner products in fixed Cartesian coordinates. In this section we obtain similar conditions in the context of conjugate flow variations. To this end, let us consider the following second-order nonlinear scalar field equation

$$\mathbf{N}_{\hat{x}}(u) = f(u; u_{,\mu}; u_{,\mu\nu}; \hat{x}_{,\nu}^{\mu}; \hat{x}_{,\nu\lambda}^{\mu}) = 0, \quad (45)$$

where the comma denotes partial differentiation with respect to σ^{μ} ($\mu = 0, \dots, 3$), i.e. $u_{,\mu} \stackrel{\text{def}}{=} \partial u / \partial \sigma^{\mu}$. The Gâteaux differential of Eq. (45) is obtained as

$$\frac{\delta \mathbf{N}_{\hat{x}}}{\delta u} \varphi + \frac{\delta \mathbf{N}_{\hat{x}}}{\delta \hat{x}} \hat{\phi} = \frac{\partial f}{\partial u} \varphi + \frac{\partial f}{\partial u_{,\mu}} \varphi_{,\mu} + \frac{\partial f}{\partial u_{,\mu\nu}} \varphi_{,\mu\nu} + \frac{\partial f}{\partial \hat{x}_{,\nu}^{\mu}} \hat{\phi}_{,\nu}^{\mu} + \frac{\partial f}{\partial \hat{x}_{,\nu\lambda}^{\mu}} \hat{\phi}_{,\nu\lambda}^{\mu}. \quad (46)$$

The conjugate flow perturbation $\hat{\phi}^{\mu}$ is related to the field perturbation φ by Eq. (12). We remark that, in general, such transformation could involve both derivatives and integrals. For example, it could be in the form

$$\hat{\phi}^{\mu} = \int_{\Sigma} K^{\mu}(\sigma^{\nu}; u) \varphi d^A \sigma + A^{\mu}(u; \sigma^{\nu}) \varphi + Q_{\lambda}^{\mu}(u; \sigma^{\nu}) \varphi_{,\lambda}. \quad (47)$$

The choice of the functional dependence between $\hat{\phi}^{\mu}$ and φ is actually a matter of investigation when looking for a conjugate flow variational principles. In this section we shall limit ourselves to *algebraic* flows, i.e. flows that can be expressed in the form

$$x^{\mu} = \hat{x}^{\mu}(\sigma^{\nu}; u), \quad (48)$$

where the functions \hat{x}^{μ} (to be determined) are *local functionals* of u , i.e. they do not involve integrals of the field u . Under these assumptions, the flow perturbation $\hat{\phi}^{\mu}$ is easily obtained as

$$\hat{\phi}^{\mu} = \frac{\partial \hat{x}^{\mu}}{\partial u} \varphi. \quad (49)$$

At this point we set

$$a^{\mu} \stackrel{\text{def}}{=} \frac{\partial \hat{x}^{\mu}}{\partial u}, \quad (50)$$

$$b_{\nu}^{\mu} \stackrel{\text{def}}{=} a_{,\nu}^{\mu} + \frac{\partial a^{\mu}}{\partial u} u_{,\nu}, \quad (51)$$

$$c_{\nu\rho}^{\mu} \stackrel{\text{def}}{=} b_{\nu,\rho}^{\mu} + \frac{\partial b_{\nu}^{\mu}}{\partial u} u_{,\rho}. \quad (52)$$

This allows us to write the conjugate flow perturbation and its partial derivatives as

$$\hat{\phi}^{\mu} = a^{\mu} \varphi, \quad (53)$$

$$\hat{\phi}_{,\nu}^{\mu} = b_{\nu}^{\mu} \varphi + a^{\mu} \varphi_{,\nu}, \quad (54)$$

$$\hat{\phi}_{,\nu\rho}^{\mu} = c_{\nu\rho}^{\mu} \varphi + b_{\nu}^{\mu} \varphi_{,\rho} + b_{\rho}^{\mu} \varphi_{,\nu} + a^{\mu} \varphi_{,\nu\rho}. \quad (55)$$

Remarkably, the derivative order of $\hat{\phi}^{\mu}$ is the same as that of φ . In other words, the k th-order derivative of u with respect to σ^{ν} involves the k th-order derivative of \hat{x}^{μ} (see Eqs. (A.14) and (A.15)). This is why we have included the second-order derivative of the conjugate flow in the second-order scalar field equation (45). The symmetry condition (44) can be now explicitly written as

$$\begin{aligned} & \int_{\Sigma} \psi H \varphi J d\sigma + \int_{\Sigma} \psi B^{\mu} \varphi_{,\mu} J d\sigma + \int_{\Sigma} \psi F^{\rho\nu} \varphi_{,\rho\nu} J d\sigma = \\ & \int_{\Sigma} \varphi H \psi J d\sigma + \int_{\Sigma} \varphi B^{\mu} \psi_{,\mu} J d\sigma + \int_{\Sigma} \varphi F^{\rho\nu} \psi_{,\rho\nu} J d\sigma, \end{aligned} \quad (56)$$

where H , B^μ and $F^{\rho\nu}$ are obtained as

$$H = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial \hat{x}^\mu_{,\nu}} b^\mu_\nu + \frac{\partial f}{\partial \hat{x}^\mu_{,\nu\rho}} c^\mu_{\nu\rho}, \quad (57)$$

$$B^\nu = \frac{\partial f}{\partial u_{,\nu}} + \frac{\partial f}{\partial \hat{x}^\mu_{,\nu}} a^\mu + \left(\frac{\partial f}{\partial \hat{x}^\mu_{,\nu\rho}} + \frac{\partial f}{\partial \hat{x}^\mu_{,\rho\nu}} \right) b^\mu_\rho, \quad (58)$$

$$F^{\rho\nu} = \frac{\partial f}{\partial u_{,\nu\rho}} + \frac{\partial f}{\partial \hat{x}^\mu_{,\rho\nu}} a^\mu. \quad (59)$$

Next, we integrate by parts the integrals at right hand side of Eq. (56) and we neglect all the boundary terms (we are looking for formal symmetry). By using the identity (A.22) we obtain, for instance

$$\int_\Sigma \varphi B^\mu \psi_{,\mu} J d\sigma = - \int_\Sigma \psi B^\nu \varphi_{,\nu} J d\sigma - \int_\Sigma \psi B^\nu_{,\nu} \varphi J d\sigma - \int_\Sigma \psi B^\nu \Gamma^\mu_{\mu\nu} \varphi J d\sigma, \quad (60)$$

where $\Gamma^\alpha_{\mu\nu}$ denotes the affine connection of the conjugate flow. Proceeding similarly with the other terms at the right hand of Eq. (56), we conclude that the formal symmetry requirement is satisfied if and only if

$$F^{\nu\rho} = F^{\rho\nu}, \quad (61)$$

$$B^\mu = F^\mu_{,\rho} + \Gamma^\nu_{\nu\rho} F^{\mu\rho}, \quad (62)$$

$$B^\mu_{,\mu} = F^\nu_{,\rho\nu} + 2\Gamma^\mu_{\mu\nu} F^{\nu\rho} + \Gamma^\beta_{\beta\nu} \Gamma^\mu_{\mu\rho} F^{\nu\rho} + \Gamma^\mu_{\mu\rho,\nu} F^{\nu\rho} - \Gamma^\mu_{\mu\nu} B^\nu. \quad (63)$$

A differentiation of Eq. (62) with respect to σ^μ and subsequent substitution in Eq. (63) yields the single relation

$$\left(F^\nu_{,\rho} + \Gamma^\beta_{\beta\rho} F^{\rho\nu} - B^\nu \right) \Gamma^\mu_{\mu\nu} = 0, \quad (64)$$

which is equivalent to the following system of *determining equations* for the conjugate flow

$$F^\nu_{,\rho} + \Gamma^\beta_{\beta\rho} F^{\rho\nu} = B^\nu. \quad (65)$$

The system (65) defines the relation $\hat{x}^\mu(\sigma^\nu; u)$, i.e. the *Lie group manifold* that guarantees the existence of a principle of stationary action for the field equation (45). Once $\hat{x}^\mu(\sigma^\nu; u)$ is available, the action functional of the field theory can be explicitly constructed by calculating the integral (30) along an arbitrary trajectory of admissible functions u_λ . This yields an action of type (41).

We notice that if we remove the functional link between \hat{x}^μ and u , then the conditions (65) consistently reduce to those of Tonti [39, 40]. In order to see this, we simply set a^μ , b^μ_ν equal to zero in Eq. (58) and Eq. (59) and then substitute them into Eq. (65). The result in fixed Cartesian coordinates ($\Gamma^\beta_{\beta\rho} = 0$) is

$$\frac{\partial}{\partial x^\mu} \left(\frac{\partial f}{\partial u_{,\mu\nu}} \right) - \frac{\partial f}{\partial u_{,\nu}} = 0. \quad (66)$$

This is the classical condition arising from the symmetry requirement of a second-order nonlinear scalar equation [13, 15].

6. Summary

We have developed a new approach to construct an action functional for a field theory described in terms of nonlinear partial differential equations (PDEs). The key idea relies on an intrinsic representation of the PDEs governing the physical system relatively to a diffeomorphic flow of coordinates (the conjugate flow) which is assumed to be a functional of their solution. This flow can be selected in order to symmetrize the Gâteaux derivative of the field equations relatively to

a suitable (advective) bilinear form. This is equivalent to require that the equations of motion of the field theory can be derived from a principle of stationary action on a *Lie group manifold*. By using a general operator framework, we have obtained the determining equations of such symmetrizing manifold for a second-order nonlinear scalar field theory and shown that they are consistent with classical results in fixed Cartesian coordinates. Once the symmetrizing manifold is available, then the conjugate flow action functional can be constructed explicitly through path integration. In particular, the duality principle between the conjugate flow and the solution field discussed in section 4 allows us to perform integrations either in terms of flows or in terms of fields.

The proposed new methodology can be generalized to vectorial and tensorial field theories. In particular, it can be applied to the Navier-Stokes equations, for which a great research effort has focused in obtaining a physically meaningful principle of stationary action [8, 21, 29, 14]. Recent results of Gomes [17, 18], Eyink [12] and Constantin [10, 9] indeed have shown that an action principle can be constructed for the Navier-Stokes equations on *random diffeomorphisms* [16, 50, 33]. These random flows are usually defined in terms of perturbations of a Lagrangian base flow. In the proposed new framework, the variational principle for the Navier-Stokes equations may be constructed on a generalized diffeomorphism (Lie group manifold) satisfying a set of PDEs similar to (65). These equations, however, are strongly nonlinear and their study will be the objective of a future work.

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Appendix A. Representation of field equations in conjugate flow intrinsic coordinates

We recall some fundamental identities from differential geometry [23, 1, 44, 22] that allow us to write the dynamic equations of a physical system in terms of conjugate flow intrinsic coordinates. To this end, let us first consider the Jacobian of the conjugate flow transformation (1)

$$J_\nu^\mu \stackrel{\text{def}}{=} \frac{\partial \hat{x}^\mu}{\partial \sigma^\nu}. \quad (\text{A.1})$$

It is easy to verify that the transpose of the algebraic complement of J_ν^μ has tensorial expression (repeated indices are summed)

$$C_\rho^\lambda \stackrel{\text{def}}{=} \frac{1}{n!} \epsilon^{\lambda\nu\alpha\cdots} \epsilon_{\rho\mu\lambda\cdots} \frac{\partial \hat{x}^\mu}{\partial \sigma^\nu} \frac{\partial \hat{x}^\lambda}{\partial \sigma^\alpha} \cdots, \quad (\text{A.2})$$

where n denotes the total number of spatial dimensions while $\epsilon^{\lambda\nu\alpha\cdots}$ and $\epsilon_{\rho\mu\lambda\cdots}$ are multi-dimensional permutation symbols, i.e. Levi-Civita tensorial densities. In particular, if we consider only two dimensions (e.g., one spatial and one temporal dimension) then we obtain the simple expression (all indices are from 0 to 1)

$$C_\rho^\lambda = \epsilon^{\lambda\nu} \epsilon_{\rho\mu} \frac{\partial \hat{x}^\mu}{\partial \sigma^\nu}. \quad (\text{A.3})$$

Similarly, in 1 + 3 dimensions, i.e. one temporal and three spatial dimensions (all indices are from 0 to 3)

$$C_\rho^\lambda = \frac{1}{6} \epsilon^{\lambda\nu\alpha\beta} \epsilon_{\rho\mu\lambda\delta} \frac{\partial \hat{x}^\mu}{\partial \sigma^\nu} \frac{\partial \hat{x}^\lambda}{\partial \sigma^\alpha} \frac{\partial \hat{x}^\delta}{\partial \sigma^\beta}. \quad (\text{A.4})$$

By using Eqs. (A.1) and (A.2) we obtain the Jacobian determinant

$$J \stackrel{\text{def}}{=} \det(J_\nu^\mu) = \frac{1}{n+1} J_\nu^\mu C_\mu^\nu. \quad (\text{A.5})$$

This allows us to write the the inverse of the Jacobian matrix (A.1) as

$$A_\rho^\lambda \stackrel{\text{def}}{=} \frac{C_\rho^\lambda}{J}. \quad (\text{A.6})$$

We denote by

$$\sigma^\nu = \widehat{\sigma}^\nu(x^\mu; u^j), \quad \mu, \nu = 0, \dots, n \quad (\text{A.7})$$

the inverse transformation of (1). Such inverse transformation exists and is differentiable by definition of conjugate flow. From the well known identity

$$\frac{\partial \widehat{\sigma}^\nu}{\partial x^\mu} \frac{\partial \widehat{x}^\mu}{\partial \sigma^\lambda} = \delta_\lambda^\nu \quad (\text{A.8})$$

we obtain the following *fundamental expression* of the partial derivatives $\partial \widehat{\sigma}^\nu / \partial x^\mu$ as a function of σ^λ

$$\frac{\partial \widehat{\sigma}^\nu}{\partial x^\mu} = A_\mu^\nu(\sigma^\lambda; u^j), \quad (\text{A.9})$$

where A_μ^ν is defined in Eq. (A.6). It is useful to write down A_μ^ν explicitly for the two-dimensional case

$$\begin{bmatrix} A_0^0 & A_1^0 \\ A_0^1 & A_1^1 \end{bmatrix} = \frac{1}{J} \begin{bmatrix} \partial \widehat{x}^1 / \partial \sigma^1 & -\partial \widehat{x}^0 / \partial \sigma^1 \\ -\partial \widehat{x}^1 / \partial \sigma^0 & \partial \widehat{x}^0 / \partial \sigma^0 \end{bmatrix}, \quad (\text{A.10})$$

where

$$J = \frac{\partial \widehat{x}^1}{\partial \sigma^1} \frac{\partial \widehat{x}^0}{\partial \sigma^0} - \frac{\partial \widehat{x}^0}{\partial \sigma^1} \frac{\partial \widehat{x}^1}{\partial \sigma^0}. \quad (\text{A.11})$$

If time is not transformed, i.e. if $x^0 = \sigma^0 = t$, then Eq. (A.10) reduces to

$$\begin{bmatrix} A_0^0 & A_1^0 \\ A_0^1 & A_1^1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -(\partial \widehat{x} / \partial t) / (\partial \widehat{x} / \partial \sigma) & 1 / (\partial \widehat{x} / \partial \sigma) \end{bmatrix}, \quad (\text{A.12})$$

where we have denoted by $\sigma \equiv \sigma^1$ and $x \equiv x^1$.

Partial differentiation in intrinsic coordinates

We consider a vector field $U^j(x^\mu)$ expressed relatively to a fixed Cartesian coordinate system x^μ . If we express the x^μ -dependence of U^j in terms of the trajectories of the particles σ^ν (advected by the conjugate flow) we obtain the following equivalent representations

$$U^j(x^\mu) = U^j(\widehat{x}^\mu(\sigma^\nu)) = u^j(\sigma^\nu) = u^j(\widehat{\sigma}^\nu(x^\mu)). \quad (\text{A.13})$$

The transformation law for partial derivatives of U^j is obtained by differentiating Eq. (A.13)

$$\frac{\partial U^j}{\partial x^\mu} = \frac{\partial u^j}{\partial \sigma^\nu} \frac{\partial \widehat{\sigma}^\nu}{\partial x^\mu} = \frac{\partial u^j}{\partial \sigma^\nu} A_\mu^\nu, \quad (\text{A.14})$$

where the quantities $\partial \widehat{\sigma}^\nu / \partial x^\mu$ are expressed in coordinates σ^ν through the fundamental relation (A.9). Let us now evaluate the second derivative with respect to x^ν and express the result in conjugate flow intrinsic coordinates. To this end let us perform an additional differentiation of (A.14) with respect to x^ν . This yields

$$\frac{\partial^2 U^j}{\partial x^\mu \partial x^\nu} = \frac{\partial^2 u^j}{\partial \sigma^\lambda \partial \sigma^\rho} A_\nu^\rho A_\mu^\lambda + \frac{\partial u^j}{\partial \sigma^\lambda} \frac{\partial A_\mu^\lambda}{\partial \sigma^\rho} A_\nu^\rho. \quad (\text{A.15})$$

By using the expressions of J and C_μ^ν obtained in (A.5) and (A.2) it is possible to manipulate Eq. (A.15) further. However, it is more convenient to obtain first A_μ^ν explicitly as a function of σ^μ and then perform the differentiation appearing in (A.15).

Perturbations in the metric tensor, affine connection and Jacobian determinant

When the conjugate flow (1) undergoes an infinitesimal disturbance of type (14) then all the quantities related to its intrinsic geometry are subject to small variations. For instance, the metric tensor

$$g_{\mu\nu} \stackrel{\text{def}}{=} \frac{\partial \hat{x}^\beta}{\partial \sigma^\mu} \frac{\partial \hat{x}^\beta}{\partial \sigma^\nu}. \quad (\text{A.16})$$

becomes, to the first-order in ϵ , $g_{\mu\nu} + \epsilon h_{\mu\nu}$ where

$$h_{\mu\nu} \stackrel{\text{def}}{=} \frac{\partial \hat{\phi}^\beta}{\partial \sigma^\mu} \frac{\partial \hat{x}^\beta}{\partial \sigma^\nu} + \frac{\partial \hat{x}^\beta}{\partial \sigma^\mu} \frac{\partial \hat{\phi}^\beta}{\partial \sigma^\nu}. \quad (\text{A.17})$$

The corresponding perturbation in the affine connection (Christoffel symbols of the second kind)

$$\Gamma_{\mu\nu}^\alpha \stackrel{\text{def}}{=} \frac{1}{2} g^{\alpha\rho} \left(\frac{\partial g_{\rho\mu}}{\partial \sigma^\nu} + \frac{\partial g_{\rho\nu}}{\partial \sigma^\mu} - \frac{\partial g_{\mu\nu}}{\partial \sigma^\rho} \right) \quad (\text{A.18})$$

is

$$\delta \Gamma_{\mu\nu}^\alpha = -\epsilon g^{\alpha\rho} h_{\rho\beta} \Gamma_{\mu\nu}^\beta + \frac{\epsilon}{2} g^{\alpha\rho} \left(\frac{\partial h_{\rho\mu}}{\partial \sigma^\nu} + \frac{\partial h_{\rho\nu}}{\partial \sigma^\mu} - \frac{\partial h_{\mu\nu}}{\partial \sigma^\rho} \right). \quad (\text{A.19})$$

This can be also expressed in a covariant form as [49]

$$\delta \Gamma_{\mu\nu}^\alpha = \frac{\epsilon}{2} g^{\alpha\rho} (h_{\rho\mu;\nu} + h_{\rho\nu;\mu} - h_{\mu\nu;\rho}) \quad (\text{A.20})$$

the covariant derivatives “;” being of course constructed by using the unperturbed affine connection $\Gamma_{\mu\nu}^\alpha$. These results allow to compute the conjugate flow perturbation of other fundamental geometric quantities such as the Riemann-Christoffel curvature tensor. Next, we determine the perturbation of the Jacobian determinant (A.5) induced by a small disturbance in the conjugate flow. To this end, we substitute Eq. (14) into Eq. (A.5) and we keep only the terms that are linear in ϵ . This yields

$$J + \epsilon C_\mu^\nu \frac{\partial \hat{\phi}^\mu}{\partial \sigma^\nu} = J \left(1 + \epsilon \frac{\partial \hat{\phi}^\mu}{\partial x^\mu} \right), \quad (\text{A.21})$$

where $\partial \hat{\phi}^\mu / \partial x^\mu$ denotes the divergence of the flow perturbation (remember that $\hat{\phi}^\mu$ are Cartesian components). Another useful formula involving the Jacobian determinant is

$$\frac{\partial J}{\partial \sigma^\mu} = J \Gamma_{\nu\mu}^\nu. \quad (\text{A.22})$$

This can be easily proved by directly differentiating Eq. (A.5) with respect to σ^μ

$$\frac{\partial J}{\partial \sigma^\mu} = C_\rho^\nu \frac{\partial^2 \hat{x}^\rho}{\partial \sigma^\nu \partial \sigma^\mu} \stackrel{(\text{A.6})}{=} J \underbrace{\frac{\partial \hat{\sigma}^\nu}{\partial x^\rho} \frac{\partial^2 \hat{x}^\rho}{\partial \sigma^\nu \partial \sigma^\mu}}_{\Gamma_{\nu\mu}^\nu}. \quad (\text{A.23})$$

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