ON A VARIATIONAL APPROACH TO THE NAVIER-STOKES EQUATIONS

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be a domain. The initial-boundary value problem for the incompressible Navier-Stokes Equations is the following one,

$$
\begin{cases} (i) \ \ \frac{\partial v}{\partial t} + div_x \,(v \otimes v) + \nabla_x p = \nu \Delta_x v + f \qquad \forall (x,t) \in \Omega \times (0,T)\,, \\ (ii) \ \ div_x \, v = 0 \qquad \forall (x,t) \in \Omega \times (0,T)\,, \\ (iii) \ \ v = 0 \qquad \forall (x,t) \in \partial\Omega \times (0,T)\,, \\ (iv) \ \ v(x,0) = v_0(x) \qquad \forall x \in \Omega\,. \end{cases} \tag{1.1}
$$

Here $v = v(x,t) : \Omega \times (0,T) \to \mathbb{R}^N$ is an unknown velocity, $p = p(x,t) : \Omega \times (0,T) \to \mathbb{R}$ is an unknown pressure, associated with v, $\nu > 0$ is a given constant viscosity, $f : \Omega \times (0,T) \to \mathbb{R}^N$ is a given force field and $v_0 : \Omega \to \mathbb{R}^N$ is a given initial velocity. The existence of weak solution to (1.1) satisfying the Energy Inequality was first proved in the celebrating works of Leray (1934). There are many different procedures for constructing weak solutions (see Leray [3],[4] (1934); Kiselev and Ladyzhenskaya [2] (1957); Shinbrot [5] (1973)). These methods are all based on the so called "Faedo-Galerkin" aproximation process. In this paper we give a new variational method to investigate the Navier-Stokes Equations. As an application of this method we give a new relatively simple proof of the existence of weak solutions to the problem (1.1).

Let us briefly describe our method. Consider for simplicity $f = 0$ in (1.1). For every smooth $u : \bar{\Omega} \times [0,T] \to \mathbb{R}^N$ satisfying conditions $(ii) - (iv)$ of (1.1) define the energy functional

$$
E(u) := \frac{1}{2} \int_0^T \int_\Omega \Big(\nu |\nabla_x u|^2 + \frac{1}{\nu} |\nabla_x \bar{H}_u|^2 \Big) \, dxdt + \frac{1}{2} \int_\Omega |u(x,T)|^2 \, dx\,, \tag{1.2}
$$

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where $\bar{H}_u(x,t)$ solves the following Stokes system for every $t \in (0,T)$,

$$\begin{cases} \Delta_x \bar{H}_u = \left(\frac{\partial v}{\partial t} + div_x\,(v\otimes v)\right) + \nabla_x p \qquad x \in \Omega\,, \\ div_x\, \bar{H}_u = 0 \qquad x\in\Omega\,, \\ \bar{H}_u = 0 \qquad \forall x \in \partial\Omega\,. \end{cases} \tag{1.3}$$

A simple integration by parts gives

$$E(u) = \frac{1}{2\nu}\int_0^T\int_\Omega \left(|\nu\nabla_x u - \nabla_x \bar{H}_u|^2\right) dxdt + \frac{1}{2}\int_\Omega |v_0(x)|^2\,dx\,. \tag{1.4}$$

Therefore if there exists at least a smooth solution to (1.1) (with $f = 0$) then a smooth function $u : \Omega\times(0,T)\to\mathbb{R}^N$ will be a solution to (1.1) (with $f = 0$) if and only if it is a minimizer of the functional in (1.2) among all smooth divergence free vector fields satisfying the boundary and the initial value conditions of (1.1). For the rigorous formulations and statements see Section 5. This remark relates the problem of existence of solutions of the Navier-Stokes equations to that of the problem of minimizing the energy $E(u)$.

Unfortunately, when applying this method to the Navier-Stokes Equation one meets certain difficulties, for example in proving the existence of minimizers to E. But we can apply this method to a suitable approximation of problem (1.1). We approximate (1.1) by replacing the nonlinear term $div_x(v\otimes v)$ with the terms $div_x\big\{f_n(|v|^2)(v\otimes v)\big\}$, where $f_n : \mathbb{R}^+ \to \mathbb{R}^+$ are regular cutoff functions satisfying $f_n(s) = 1$ for $s \le n$ and $f_n(s) = 0$ for $s > 2n$. The approximating problems are simpler than (1.1), since the nonlinear term has higher integrability. Next we consider the energies E_n corresponding to the approximating problems and investigate the Euler-Lagrange equations of E_n and the existence of minimizers. In this way we get solutions to the approximating problems which satisfy the energy equality (in fact these solutions will be regular if the initial data and the domain are). Next we pass to the limit for $n\to\infty$ and obtain a weak solution to (1.1). For the details see Section 3.

2. Preliminaries

For two matrices $A, B \in \mathbb{R}^{p\times q}$ with ij-th entries a_{ij} and b_{ij} respectively, we write $A : B := \sum\limits_{i=1}^{p}\sum\limits_{j=1}^{q} a_{ij}b_{ij}$.

Given a vector valued function $f(x) = \big(f_1(x),\dots,f_k(x)\big) : \Omega \to \mathbb{R}^k$ ($\Omega \subset \mathbb{R}^N$) we denote by $\nabla_x f$ the $k\times N$ matrix with ij-th entry $\frac{\partial f_i}{\partial x_j}$.

For a matrix valued function $F(x) := \{F_{ij}(x)\} : \mathbb{R}^N \to \mathbb{R}^{k\times N}$ we denote by $div\, F$ the

$\mathbb{R}^k$-valued vector field defined by $div\, F := (l_1, \ldots, l_k)$ where $l_i = \sum_{j=1}^{N} \frac{\partial F_{ij}}{\partial x_j}$. Throughout the rest of the paper we assume that Ω is domain in $\mathbb{R}^N$.

Definition 2.1. We denote:

- By $\mathcal{V}_N$ the space $\{\varphi \in C_c^\infty(\Omega, \mathbb{R}^N) : div\, \varphi = 0\}$ and by L_N the space, which is the closure of $\mathcal{V}_N$ in the space $L^2(\Omega, \mathbb{R}^N)$, endowed with the norm $\|\varphi\| := \left(\int_\Omega |\varphi|^2 dx\right)^{1/2}$.
- By $\bar{H}_0^1(\Omega, \mathbb{R}^N)$ the closure of $C_c^\infty(\Omega, \mathbb{R}^N)$ with respect to the norm $|||\varphi||| := \left(\int_\Omega |\nabla\varphi|^2 dx\right)^{1/2}$. This space differ from $H_0^1(\Omega, \mathbb{R}^N)$ only in the case of unbounded domain.
- By V_N the closure of $\mathcal{V}_N$ in $\bar{H}_0^1(\Omega, \mathbb{R}^N)$.
- By V_N^{-1} the space dual to V_N.
- By $\mathcal{Y}$ the space

$$\mathcal{Y} := \{\varphi(x,t) \in C_c^\infty(\Omega \times [0,T], \mathbb{R}^N) : div_x\, \varphi = 0\}.$$

Remark 2.1. It is obvious that $u \in \mathcal{D}'(\Omega, \mathbb{R}^N)$ (rigorously the equivalency class of u, up to gradients) belongs to V_N^{-1} if and only if there exists $w \in V_N$ such that

$$\int_\Omega \nabla w : \nabla \delta \, dx = - < u, \delta > \qquad \forall \delta \in V_N .$$

In particular $\Delta w = u + \nabla p$ as a distribution and

$$|||w||| = \sup_{\delta \in V_N,\, |||\delta||| \leq 1} < u, \delta > = |||u|||_{-1} .$$

Definition 2.2. We will say that the distribution $l \in \mathcal{D}'(\Omega \times (0,T), \mathbb{R}^N)$ belongs to $L^2(0,T; V_N^{-1})$, if there exists $v(\cdot,t) \in L^2(0,T; V_N^{-1})$, such that for every $\psi(x,t) \in C_c^\infty(\Omega \times (0,T), \mathbb{R}^N)$, satisfying $div_x\, \psi = 0$, we have

$$< l(\cdot,\cdot), \psi(\cdot,\cdot) > = \int_0^T < v(\cdot,t), \psi(\cdot,t) > dt .$$

Remark 2.2. Let $v(\cdot,t) \in L^2(0,T; V_N^{-1})$. For a.e. $t \in [0,T]$ consider $V_v(\cdot,t)$ as in Remark 2.1, corresponding to $v(\cdot,t)$, i.e.

$$\int_\Omega \nabla_x V_v(x,t) : \nabla_x \delta(x) \, dx = - < v(\cdot,t), \delta(\cdot) > \qquad \forall \delta \in V_N .$$

Then it is clear that $V_v(\cdot,t) \in L^2(0,T; V_N)$ and

$$\|V_v\|_{L^2(0,T;V_N)} = \|v\|_{L^2(0,T;V_N^{-1})} .$$

In the sequel we will use the following compactness result which is a particular case of Theorem 2.3 in the book of Temam [6].

Theorem 2.1. *Let Ω be a bounded domain. Consider the sequence $\{u_n\} \subset L^2(0,T;V_N)$ such that $\partial_t u_n \in L^1(0,T,V_N^{-1})$. Assume that the sets $\{u_n\}$ and $\{\partial_t u_n\}$ are bounded in $L^2(0,T;V_N)$ and $L^1(0,T,V_N^{-1})$ respectively. Then $\{u_n\}$ is pre-compact in $L^2(0,T;L_N)$.*

3. Existence of the weak solution to the Navier-Stokes Equations

Throughout this section we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain.

Definition 3.1. Let $F(v) = \{F_{ij}(v)\} \in C^1(\mathbb{R}^N, \mathbb{R}^{N\times N}) \cap Lip$ satisfy $F(0) = 0$ and $\frac{\partial F_{ij}}{\partial v_m}(v) = \frac{\partial F_{mj}}{\partial v_i}(v)$ for all $v \in \mathbb{R}^N$ and $m, i, j \in \{1, \ldots, N\}$. Denote the class of all such F by $\mathfrak{F}$.

Remark 3.1. Let $F \in \mathfrak{F}$. Then it is clear that there exists $G(v) = (G_1(v), \ldots, G_N(v)) \in C^2(\mathbb{R}^N, \mathbb{R}^N)$, such that $\frac{\partial G_j}{\partial v_i}(v) = F_{ij}(v)$ i.e. $\nabla_v G(v) = (F(v))^T$.

Using our variational approach, we will prove in the sequel the existence of a solution of the following problem

$$\begin{cases} \frac{\partial v}{\partial t} + div_x\, F(v) + \nabla_x p = \Delta_x v \qquad \forall (x,t) \in \Omega \times (0,T)\,, \\ div_x\, v = 0 \qquad \forall (x,t) \in \Omega \times (0,T)\,, \\ v = 0 \qquad \forall (x,t) \in \partial\Omega \times (0,T)\,, \\ v(x,0) = v_0(x) \qquad \forall x \in \Omega\,, \end{cases} \tag{3.1}$$

for every $F \in \mathfrak{F}$, which in addition satisfies the Energy Equality (see Theorem 4.1). But first of all, in the proof of the following theorem we would like to explain how this fact implies the existence of weak solution to the Navier-Stokes Equation.

Theorem 3.1. *Assume $N \leq 4$. Let $v_0(x) \in L_N$. Then there exists $u \in L^2(0,T;V_N) \cap L^\infty(0,T;L_N)$ satisfying*

$$\int_\Omega v_0(x)\cdot\psi(x,0)\,dx + \int_0^T\int_\Omega \big(u\cdot\partial_t\psi + (u\otimes u) : \nabla_x\psi\big) = \int_0^T\int_\Omega \nabla_x u : \nabla_x\psi\,, \tag{3.2}$$

for every $\psi(x,t) \in C_c^\infty(\Omega\times[0,T), \mathbb{R}^N)$ such that $div_x\,\psi = 0$, i.e.

$$\Delta_x u = \partial_t u + div_x\,(u\otimes u) + \nabla_x p\,, \quad \textit{and } u(x,0) = v_0(x)\,.$$

Moreover, for a.e. $t \in [0,T]$ we have

$$\int_0^t \int_\Omega |\nabla_x u|^2\, dxdt \le \frac{1}{2}\Big(\int_\Omega v_0^2(x)dx - \int_\Omega u^2(x,t)dx\Big). \tag{3.3}$$

Proof. Fix some $h(s) \in C^\infty(\mathbb{R},[0,1])$, satisfying $h(s)=1\ \forall s \le 1$ and $h(s)=0\ \forall s \ge 2$. For every $n \in \mathbb{N}$ define $f_n(s) := h(s/n)$. Consider

$$F_n(v) := f_n(|v|^2)(v \otimes v) + g_n(|v|^2) I_N\,, \tag{3.4}$$

where I_N is a $N \times N$-unit matrix and $g_n(r) := \frac{1}{2}\int_0^r f_n(s)ds$. Then for every n we have $F_n \in \mathfrak{F}$ and there exists $A>0$ such that $|F_n(v)| \le A|v|^2$ for every v and n. Fix also some sequence $\{v_0^{(n)}\}_{n=1}^\infty \subset \mathcal{V}_N$ such that $v_0^{(n)} \to v_0$ strongly in L_N as $n \to \infty$. By Theorem 4.1, bellow, for every n there exist a function $u_n \in L^2(0,T;V_N)\cap L^\infty(0,T;L_N)$, such that $\partial_t u_n \in L^2(0,T;V_N^{-1})$ and $u_n(\cdot,t)$ is L_N-weakly continuous in t on $[0,T]$, which satisfy

$$\int_\Omega v_0^{(n)}(x)\cdot\psi(x,0) + \int_0^T\int_\Omega \big(u_n\cdot\partial_t\psi + F_n(u_n):\nabla_x\psi\big) = \int_0^T\int_\Omega \nabla_x u_n : \nabla_x\psi\,, \tag{3.5}$$

for every $\psi(x,t) \in C_c^\infty(\Omega\times[0,T),\mathbb{R}^N)$, such that $div_x\,\psi = 0$. Moreover, by the same Theorem, for every $t \in [0,T]$ we obtain

$$\frac{1}{2}\int_\Omega u_n^2(x,t)dx + \int_0^t\int_\Omega |\nabla_x u_n|^2\,dxdt = \frac{1}{2}\int_\Omega (v_0^{(n)})^2(x)dx\,. \tag{3.6}$$

Consider $V_n(\cdot,t), W_n(\cdot,t) \in L^2(0,T;V_N)$ as in Remark 2.2, corresponding to $(\partial_t u_n + div_x\,F_n(u_n))$ and $div_x\,F_n(u_n)$ respectively. But $V_n = u_n$. Therefore, by (3.6) we obtain

$$\int_0^T\int_\Omega |\nabla_x u_n|^2\,dxdt + \int_0^T\int_\Omega |\nabla_x V_n|^2\,dxdt \le C_0\,.$$

From the other hand, since $N \le 4$, we have

$$\int_0^T\Big(\int_\Omega|\nabla W_n|^2\Big)^{1/2} \le \int_0^T\Big(\int_\Omega|F_n(u_n)|^2\Big)^{1/2} \le A\int_0^T\Big(\int_\Omega|u_n|^4\Big)^{1/2} \le C\int_0^T\int_\Omega|\nabla_x u_n|^2$$

Therefore $\{\partial_t u_n\}$ is bounded in $L^1(0,T;V_N^{-1})$. Then we can use Theorem 2.1 and (3.6), to deduce that there exists $u \in L^2(0,T;V_N)\cap L^\infty(0,T;L_N)$ satisfying that, up to a subsequence, $u_n \to u$ strongly in $L^2(0,T;L_N)$ and $u_n \rightharpoonup u$ weakly in $L^2(0,T;V_N)$. Then, up to a further subsequence, we have $u_n(x,t) \to u(x,t)$ almost everywhere in

$\Omega \times (0,T)$. In particular $f_n\big(|u_n(x,t)|^2\big) \to 1$ almost everywhere in $\Omega \times (0,T)$. Then,

$$\overline{\lim_{n\to\infty}} \int_0^T \int_\Omega \big|f_n(|u_n|^2)(u_n \otimes u_n) - (u \otimes u)\big|\, dxdt \leq$$

$$\overline{\lim_{n\to\infty}} \int_0^T \int_\Omega |f_n(|u_n|^2)| \cdot |(u_n \otimes u_n) - (u \otimes u)|, dxdt + \overline{\lim_{n\to\infty}} \int_0^T \int_\Omega u^2 \big|f_n(|u_n|^2) - 1\big|\, dxdt = 0\,.$$

Therefore, letting n tend to ∞ in (3.5), we obtain (3.2). Moreover, by (3.6), for a.e. $t \in [0,T]$ we obtain (3.3). This completes the proof. □

4. Proof of the existence of solutions to (3.1)

Throughout this section we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain. In the sequel we will need several lemmas. The following Lemma can be proved in the same way as Lemmas 2.1 and 2.2 in [1].

Lemma 4.1. *Let $u \in L^2(0,T;V_N) \cap L^\infty(0,T;L_N)$ be such that $\partial_t u \in L^2(0,T;V_N^{-1})$. Consider $V_0(\cdot,t) \in L^2(0,T;V_N)$ as in Remark 2.2, corresponding to $\partial_t u$. Then we can redefine u on a subset of $[0,T]$ of Lebegue measure zero, so that $u(\cdot,t)$ will be L_N-weakly continuous in t on $[0,T]$. Moreover, for every $0 \leq a < b \leq T$ and for every $\psi(x,t) \in \mathcal{Y}$ (see Definition 2.1) we will have*

$$\int_a^b \int_\Omega \nabla_x V_0 : \nabla_x \psi\, dxdt - \int_a^b \int_\Omega u \cdot \partial_t \psi\, dxdt = \int_\Omega u(x,a) \cdot \psi(x,a)dx - \int_\Omega u(x,b) \cdot \psi(x,b)dx\,. \tag{4.1}$$

Remark 4.1. Let $F \in Lip\,(\mathbb{R}^N, \mathbb{R}^{N\times N})$ satisfying $F(0) = 0$. Then for every $u \in L^\infty(0,T;L_N)$ we have $F(u) \in L^\infty\big(0,T;L^2(\Omega,\mathbb{R}^{N\times N})\big)$ and therefore $div_x\, F(u) \in L^2(0,T;V_N^{-1})$. If in addition $\partial_t u \in L^2(0,T;V_N^{-1})$ then we obtain $\partial_t u + div_x\, F(u) \in L^2(0,T;V_N^{-1})$.

We have the following Corollary to Lemma 4.1.

Corollary 4.1. *Let u be as in Lemma 4.1 and let $F \in Lip\,(\mathbb{R}^N, \mathbb{R}^{N\times N})$ satisfying $F(0) = 0$. Assume, in addition, that $u(\cdot,t)$ is L_N-weakly continuous in t on $[0,T]$ (see Lemma 4.1). Consider $V(\cdot,t) \in L^2(0,T;V_N)$ as in Remark 2.2, corresponding to $\partial_t u + div_x\, F(u)$. Then for every $0 \leq a < b \leq T$ and for every $\psi(x,t) \in \mathcal{Y}$ we have*

$$\int_a^b \int_\Omega \nabla_x V : \nabla_x \psi\, dxdt - \int_a^b \int_\Omega \big(u \cdot \partial_t \psi + F(u) : \nabla_x \psi\big)\, dxdt = \int_\Omega u(x,a) \cdot \psi(x,a)dx - \int_\Omega u(x,b) \cdot \psi(x,b)dx\,. \tag{4.2}$$

The following Lemma can be proved in the same way as Theorem 4.1 in [1].

Lemma 4.2. *Let $u \in L^2(0,T;V_N) \cap L^\infty(0,T;L_N)$ be such that $\partial_t u \in L^2(0,T;V_N^{-1})$ and let $u(\cdot,t)$ be L_N-weakly continuous in t on $[0,T]$ (see Lemma 4.1). Consider $V_0(\cdot,t) \in L^2(0,T;V_N)$ as in Remark 2.2, corresponding to $\partial_t u$. Then for every $t \in [0,T]$ we have*

$$\int_0^t \int_\Omega \nabla_x u : \nabla_x V_0 \, dxdt = \frac{1}{2}\left(\int_\Omega u^2(x,0)dx - \int_\Omega u^2(x,t)dx \right).$$

Corollary 4.2. *Let $u \in L^2(0,T;V_N)$ be such that $\partial_t u \in L^2(0,T;V_N^{-1})$. Then $u \in L^\infty(0,T;L_N)$.*

We will give the proof of this Corollary in the Appendix.

Next we have the second Corollary to Lemma 4.2.

Corollary 4.3. *Let $F \in \mathfrak{F}$ and let $u \in L^2(0,T;V_N) \cap L^\infty(0,T;L_N)$ be such that $\partial_t u \in L^2(0,T;V_N^{-1})$ and let $u(\cdot,t)$ be L_N-weakly continuous in t on $[0,T]$ (see Lemma 4.1). Consider $V(\cdot,t) \in L^2(0,T;V_N)$ as in Remark 2.2, corresponding to $\partial_t u + div_x F(u)$ (see Remark 4.1). Then for every $t \in [0,T]$ we have*

$$\int_0^t \int_\Omega \nabla_x u : \nabla_x V \, dxdt = \frac{1}{2}\left(\int_\Omega u^2(x,0)dx - \int_\Omega u^2(x,t)dx \right). \tag{4.3}$$

Proof. By Lemma 4.2, for every $t \in [0,T]$ we obtain

$$\int_0^t \int_\Omega \nabla_x V : \nabla_x u \, dxdt - \int_0^t \int_\Omega F(u) : \nabla_x u \, dxdt = \frac{1}{2}\left(\int_\Omega u^2(x,0)dx - \int_\Omega u^2(x,t)dx \right). \tag{4.4}$$

But for almost every $t \in [0,T]$ $u(\cdot,t) \in V_N$, therefore, for every such fixed t there exists a sequence $\{\delta_n(\cdot)\}_{n=1}^\infty \in \mathcal{V}_N$, such that $\delta_n(\cdot) \to u(\cdot,t)$ in V_N. But for every $\delta \in \mathcal{V}_N$ we obtain

$$\int_\Omega F(\delta) : \nabla_x \delta = \int_\Omega \sum_{i=1}^N \sum_{j=1}^N : F_{ij}(\delta) \frac{\partial \delta_i}{\partial x_j} = \int_\Omega \sum_{i=1}^N \sum_{j=1}^N \frac{\partial G_j}{\partial v_i}(\delta) \frac{\partial \delta_i}{\partial x_j} = \int_\Omega div_x \, G(\delta) = 0,$$

where G is as in Remark 3.1. Therefore, since F is Lipshitz function, we obtain

$$\int_\Omega F(u(x,t)) : \nabla_x u(x,t) \, dx = \lim_{n\to\infty} \int_\Omega F(\delta_n(x)) : \nabla_x \delta_n(x) \, dx = 0.$$

Therefore, using (4.4), we obtain (4.3) and the result follows. ☐

Definition 4.1. Let $u \in L^2(0,T;V_N)\cap L^\infty(0,T;L_N)$ be such that $\partial_t u \in L^2(0,T;V_N^{-1})$ and such that $u(\cdot,t)$ is L_N-weakly continuous in t on $[0,T]$. Denote the set of all such functions u by $\mathcal{R}$. For a fixed $F \in \mathfrak{F}$ and for every $u \in \mathcal{R}$ let $H_u(\cdot,t) \in L^2(0,T;V_N)$ be as in Remark 2.2, corresponding to $\partial_t u + div_x\, F(u)$. That is for every $\psi(x,t) \in C_c^\infty(\Omega\times(0,T),\mathbb{R}^N)$ such that $div_x\,\psi = 0$ we have

$$\int_0^T\int_\Omega \big(u\cdot\partial_t\psi + F(u):\nabla_x\psi\big)dxdt = \int_0^T\int_\Omega \nabla_x H_u : \nabla_x\psi\, dxdt\,.$$

Define a functional $I_F(u):\mathcal{R}\to\mathbb{R}$ by

$$I_F(u) := \frac{1}{2}\bigg(\int_0^T\int_\Omega \Big(|\nabla_x u|^2 + |\nabla_x H_u|^2\Big)dxdt + \int_\Omega |u(x,T)|^2 dx\bigg)\,, \tag{4.5}$$

and for every $v_0 \in V_N$ consider the minimization problem

$$\inf\{I_F(u):\ u\in\mathcal{R}, u(\cdot,0) = v_0(\cdot)\}\,. \tag{4.6}$$

Lemma 4.3. *For every $u \in \mathcal{R}$ and every $\delta(x,t)\in\mathcal{Y}$, such that $\delta(x,0)=0$, we have*

$$\lim_{s\to 0}\frac{I_F(u+s\delta)-I_F(u)}{s} = \int_0^T\int_\Omega \nabla_x u:\nabla_x\delta + \int_\Omega u(x,T)\cdot\delta(x,T)\,dx$$
$$- \int_0^T\int_\Omega \partial_t\delta\cdot H_u\,dxdt + \int_0^T\int_\Omega\Big(\sum_{j=1}^N \delta_j\frac{\partial F}{\partial u_j}(u)\Big):\nabla_x H_u\,dxdt\,. \tag{4.7}$$

Proof. It is clear that

$$\lim_{s\to 0}\frac{1}{2s}\bigg(\int_0^T\int_\Omega|\nabla_x(u+s\delta)|^2 - \int_0^T\int_\Omega|\nabla_x u|^2\bigg) = \int_0^T\int_\Omega \nabla_x u:\nabla_x\delta\,. \tag{4.8}$$

Moreover,

$$\lim_{s\to 0}\frac{1}{2s}\bigg(\int_\Omega |u(x,T)+s\delta(x,T)|^2\,dx - \int_\Omega|u(x,T)|^2\,dx\bigg) = \int_\Omega u(x,T)\cdot\delta(x,T)\,dx\,. \tag{4.9}$$

Next we have

$$\frac{1}{2s}\int_0^T\int_\Omega\Big(|\nabla_x H_{(u+s\delta)}|^2 - |\nabla_x H_u|^2\Big) = \tag{4.10}$$
$$\frac{1}{2s}\int_0^T\int_\Omega\big(\nabla_x H_{(u+s\delta)} - \nabla_x H_u\big):\big(\nabla_x H_{(u+s\delta)} + \nabla_x H_u\big) =$$
$$-\frac{1}{2s}\int_0^T\Big\langle\big(s\cdot\partial_t\delta + div_x\,F(u+s\delta) - div_x\,F(u)\big)(\cdot,t), \big(H_{(u+s\delta)}+H_u\big)(\cdot,t)\Big\rangle\,dt$$
$$= -\int_0^T\int_\Omega \partial_t\delta(x,t)\cdot\frac{1}{2}\big(H_{(u+s\delta)}(x,t) + H_u(x,t)\big)\,dxdt$$
$$+\int_0^T\int_\Omega\frac{1}{s}\big(F(u+s\delta)-F(u)\big):\frac{1}{2}\big(\nabla_x H_{(u+s\delta)} + \nabla_x H_u\big)\,dxdt\,.$$

Since F is Lipschitz and C^1, we obtain

(4.11)
$$\frac{1}{s}\big(F(u+s\delta)-F(u)\big)\to\sum_{j=1}^{N}\delta_j\frac{\partial F}{\partial u_j}(u)\quad\text{as } s\to 0\quad\text{strongly in } L^2(\Omega\times(0,T),\mathbb{R}^{N\times N})\,.$$

On the other hand, for every $h(x,t)\in L^2(0,T;V_N)$ we obtain

$$(4.12)\quad \lim_{s\to 0}\int_0^T\int_\Omega\big(\nabla_x H_{(u+s\delta)}-\nabla_x H_u\big):\nabla_x h(x,t)=$$
$$\lim_{s\to 0}\left(-s\int_0^T\int_\Omega\partial_t\delta\cdot h\,dxdt+\int_0^T\int_\Omega\big(F(u+s\delta)-F(u)\big):\nabla_x h\,dxdt\right)=0\,.$$

Therefore

$$(4.13)\qquad H_{(u+s\delta)}\rightharpoonup H_u\quad\text{weakly in } L^2(0,T;V_N)\,.$$

In particular $H_{(u+s\delta)}$ remains bounded in $L^2(0,T;V_N)$ as $s\to 0$. Therefore, by (4.10), we obtain

$$\lim_{s\to 0}\int_0^T\int_\Omega\left(|\nabla_x H_{(u+s\delta)}|^2-|\nabla H_u|^2\right)=0\,.$$

So

$$(4.14)\qquad H_{(u+s\delta)}\to H_u\quad\text{strongly in } L^2(0,T;V_N)\,.$$

Therefore, using (4.14) and (4.11) in (4.10), we infer

$$(4.15)\quad \lim_{s\to 0}\frac{1}{2s}\int_0^T\int_\Omega\left(|\nabla_x H_{(u+s\delta)}|^2-|\nabla_x H_u|^2\right)=$$
$$-\int_0^T\int_\Omega\partial_t\delta\cdot H_u\,dxdt+\int_0^T\int_\Omega\Big(\sum_{j=1}^{N}\delta_j\frac{\partial F}{\partial u_j}(u)\Big):\nabla_x H_u\,dxdt\,.$$

Plugging (4.8), (4.9) and (4.15), we obtain that for every $\delta(x,t)\in\mathcal{Y}$, such that $\delta(x,0)=0$, we must have (4.7). □

Lemma 4.4. *Let $u\in\mathcal{R}$ be a minimizer to (4.6). Then $H_u=u$, i.e.*

$$\Delta_x u=\partial_t u+div_x\,F(u)+\nabla_x p\,.$$

Proof. Fix some $\delta(x,t)\in\mathcal{Y}$, such that $\delta(x,0)=0$. Then for every $s\in\mathbb{R}$ $(u+s\delta)\in\mathcal{R}$ and $(u+s\delta)(\cdot,0)=v_0(\cdot)$. Therefore,

$$(4.16)\qquad \lim_{s\to 0}\frac{I_F(u+s\delta)-I_F(u)}{s}=0\,.$$

So, by (4.7) in Lemma 4.3, we must have

$$\begin{aligned}(4.17)\quad &\int_0^T\int_\Omega \nabla_x u : \nabla_x \delta + \int_\Omega u(x,T)\cdot\delta(x,T)\,dx \\ &- \int_0^T\int_\Omega \partial_t\delta\cdot H_u\,dxdt + \int_0^T\int_\Omega \Big(\sum_{j=1}^N \delta_j \frac{\partial F}{\partial u_j}(u)\Big) : \nabla_x H_u\,dxdt = 0\,.\end{aligned}$$

Using Lemma 4.1 (see (4.2)), for every $\delta(x,t)\in\mathcal{Y}$, such that $\delta(x,0)=0$ we obtain

$$\begin{aligned}(4.18)\quad &\int_0^T\int_\Omega \nabla_x H_u : \nabla_x\delta\,dxdt - \int_0^T\int_\Omega \partial_t\delta(x,t)\cdot u(x,t)\,dxdt \\ &- \int_0^T\int_\Omega F(u) : \nabla_x\delta\,dxdt + \int_\Omega u(x,T)\cdot\delta(x,T)dx = 0\,.\end{aligned}$$

Since F is Lipschitz function, for a.e. t we have $\nabla_x F(u)\in L^2$. Then for a.e. $t\in(0,T)$,

$$\begin{aligned}(4.19)\quad &\int_\Omega F\big(u(x,t)\big) : \nabla_x\delta(x,t)\,dx = -\int_\Omega div_x\, F(u)\cdot\delta\,dx = \\ &-\int_\Omega \sum_{1\le i,j,m\le N} \frac{\partial F_{jm}}{\partial u_i}(u)\,\frac{\partial u_i}{\partial x_m}\,\delta_j\,dx = -\int_\Omega \sum_{1\le i,j,m\le N} \frac{\partial F_{im}}{\partial u_j}(u)\,\frac{\partial u_i}{\partial x_m}\,\delta_j\,dx \\ &= -\int_\Omega \sum_{j=1}^N \Big(\delta_j \frac{\partial F}{\partial u_j}(u)\Big) : \nabla_x u\,dx\,.\end{aligned}$$

Inserting (4.19) into (4.18), we deduce

$$\begin{aligned}(4.20)\quad &\int_0^T\int_\Omega \nabla_x H_u : \nabla_x\delta\,dxdt - \int_0^T\int_\Omega \partial_t\delta(x,t)\cdot u(x,t)\,dxdt \\ &+ \int_0^T\int_\Omega \sum_{j=1}^N \Big(\delta_j \frac{\partial F}{\partial u_j}(u)\Big) : \nabla_x u\,dxdt + \int_\Omega u(x,T)\cdot\delta(x,T)dx = 0\,.\end{aligned}$$

Next define $W_u := u - H_u$. Then $W_u\in L^2(0,T;V_N)$ and subtracting (4.20) from (4.17), for every $\delta(x,t)\in\mathcal{Y}$, such that $\delta(x,0)=0$, we obtain

$$\begin{aligned}(4.21)\quad &\int_0^T\int_\Omega \nabla_x W_u : \nabla_x\delta + \int_0^T\int_\Omega \partial_t\delta(x,t)\cdot W_u(x,t)\,dxdt \\ &- \int_0^T\int_\Omega \Big(\sum_{j=1}^N \delta_j \frac{\partial F}{\partial u_j}(u)\Big) : \nabla_x W_u\,dxdt = 0\,.\end{aligned}$$

Since $\frac{\partial F}{\partial u_j}\in L^\infty$, we obtain that the functional $L(\phi): V_N\to\mathbb{R}$ defined by

$$L(\phi) := \int_\Omega \Big(\sum_{j=1}^N \phi_j \frac{\partial F}{\partial u_j}(u)\Big) : \nabla_x W_u\,dx$$

belongs to V_N^{-1} for a.e. $t \in (0,T)$. Moreover there exists $Q(x,t) \in L^2(0,T;V_N)$ such that for a.e. $t \in (0,T)$ we have

$$L(\phi) := \int_\Omega \Big(\sum_{j=1}^N \phi_j \frac{\partial F}{\partial u_j}(u) \Big) : \nabla_x W_u \, dx = \int_\Omega \nabla_x Q(x,t) : \nabla_x \phi(x) \, dx \quad \forall \phi \in V_N \, .$$

Then from (4.21) we obtain that $\partial_t W_u \in L^2(0,T;V_N^{-1})$ and we have

$$(4.22) \quad < \partial_t W_u(\cdot,\cdot), \psi(\cdot,\cdot) >= - \int_0^T \int_\Omega \nabla_x (Q - W_u) : \nabla_x \psi \, dxdt$$
$$\forall \psi \in C_c^\infty(\Omega \times (0,T), \mathbb{R}^N) \text{ s.t. } div_x \, \psi = 0 \, .$$

Therefore, by Corollary 4.2 and Lemma 4.1, we can redefine $W_u(\cdot,t)$ on a set of Lebegue measure zero on $[0,T]$ so that $W_u(\cdot,t)$ be L_N-weakly continuous in t on $[0,T]$. From now we consider such W_u. Moreover, by (4.2) and (4.22), for every $\delta \in \mathcal{Y}$, such that $\delta(x,0) = 0$, we obtain

$$\int_0^T \int_\Omega \nabla_x (Q - W_u) : \nabla_x \delta \, dxdt - \int_0^T \int_\Omega W_u \cdot \partial_t \delta \, dxdt = - \int_\Omega W_u(x,T) \cdot \delta(x,T) dx \, ,$$

or in the another form

$$(4.23) \quad \int_0^T \int_\Omega \nabla_x W_u : \nabla_x \delta \, dxdt - \int_0^T \int_\Omega \Big(\sum_{j=1}^N \delta_j \frac{\partial F}{\partial u_j}(u) \Big) : \nabla_x W_u \, dxdt$$
$$+ \int_0^T \int_\Omega W_u \cdot \partial_t \delta \, dxdt - \int_\Omega W_u(x,T) \cdot \delta(x,T) dx = 0 \, .$$

Comparing (4.23) with (4.21), we obtain that $W_u(\cdot,T) = 0$. Therefore, by Corollary 4.2 and Lemma 4.2, for every $t \in [0,T]$ we obtain

$$\int_t^T \int_\Omega \nabla_x W_u : \nabla_x (Q - W_u) \, dxds = \frac{1}{2} \int_\Omega W_u^2(x,t) dx \, ,$$

or in the equivalent form

(4.24)
$$\int_t^T \int_\Omega |\nabla_x W_u|^2 \, dxds + \frac{1}{2} \int_\Omega W_u^2(x,t) dx = \int_t^T \int_\Omega \Big(\sum_{j=1}^N (W_u)_j \, \frac{\partial F}{\partial u_j}(u) \Big) : \nabla_x W_u \, dxds \, .$$

In particular there exists $C > 0$, independent of t, such that

$$\int_t^T \int_\Omega |\nabla_x W_u|^2 \, dxds \leq \int_t^T \int_\Omega \Big(\sum_{j=1}^N (W_u)_j \, \frac{\partial F}{\partial u_j}(u) \Big) : \nabla_x W_u \, dxds$$
$$\leq C \Big(\int_t^T \int_\Omega |\nabla_x W_u|^2 \, dxds \cdot \int_t^T \int_\Omega |W_u|^2 \, dxds \Big)^{1/2} .$$

So

$$\int_t^T \int_\Omega |\nabla_x W_u|^2 \, dxds \le C^2 \int_t^T \int_\Omega |W_u|^2 \, dxds \,. \tag{4.25}$$

Then, using (4.24) and (4.25) we obtain

$$\frac{1}{2}\int_\Omega W_u^2(x,t)dx \le \int_t^T \int_\Omega \Big(\sum_{j=1}^N (W_u)_j \frac{\partial F}{\partial u_j}(u) \Big) : \nabla_x W_u \, dxds \tag{4.26}$$

$$\le C \Big(\int_t^T \int_\Omega |\nabla_x W_u|^2 \, dxds \cdot \int_t^T \int_\Omega |W_u|^2 \, dxds \Big)^{1/2} \le C^2 \int_t^T \int_\Omega |W_u|^2 \, dxds \,.$$

Then by Gronwall's Lemma $\int_\Omega W_u^2(x,t)dx = 0$. So, by definition of W_u we obtain $H_u = u$. This completes the proof. □

Theorem 4.1. *For every $v_0(\cdot) \in V_N$ there exists a minimizer u to* (4.6)*. It satisfies $H_u = u$, i.e.*

$$\Delta_x u = \partial_t u + div_x \, F(u) + \nabla_x p \,,$$

$u(x,0) = v_0(x)$ and

$$\frac{1}{2}\int_\Omega u^2(x,t)dx + \int_0^t \int_\Omega |\nabla_x u|^2 \, dxdt = \frac{1}{2}\int_\Omega v_0^2(x)dx \qquad \forall t \in [0,T] \,. \tag{4.27}$$

Moreover if $v \in \mathcal{R}$ satisfy $v(\cdot,0) = v_0(\cdot)$ and $H_v = v$, i.e. $\Delta_x v = \partial_t v + div_x \, F(v) + \nabla_x p$, then v is a minimizer to (4.6)*.*

Proof. First of all we want to note that the set $A_{v_0} := \{u \in \mathcal{R} : \; u(\cdot,0) = v_0(\cdot)\}$ is not empty. In particular the function $u_0(\cdot,t) := v_0(\cdot)$ belongs to A_{v_0}. Let

$$K := \inf_{u \in A_{v_0}} I_F(u) \,.$$

Then $K \ge 0$. Consider the minimizing sequence $\{u_n\} \subset A_{v_0}$, i.e. the sequence such that $\lim_{n\to\infty} I_F(u_n) = K$. Then, by the definition of I_F in (4.5), we obtain that there exists $C > 0$, independent of n, such that

$$\int_0^T \int_\Omega \big(|\nabla_x u_n|^2 + |\nabla_x H_{u_n}|^2 \big) \, dxdt \le C \,. \tag{4.28}$$

Then using Theorem 2.1 we also obtain that, up to a subsequence,

$$u_n \to u_0 \quad \text{strongly in } L^2(0,T;L_N) \,, \tag{4.29}$$

$$u_n \rightharpoonup u_0 \quad \text{weakly in } L^2(0,T;V_N) \quad \text{and } H_{u_n} \rightharpoonup \bar{H} \quad \text{weakly in } L^2(0,T;V_N) \,.$$

From the other hand, by Corollary 4.3, for every $t \in [0,T]$ we have

$$\int_\Omega u_n^2(x,t)dx = \int_\Omega u_n^2(x,0)dx - 2\int_0^t \int_\Omega \nabla_x u_n : \nabla_x H_{u_n}\,.$$

Therefore, since, u_n and H_{u_n} are bounded in $L^2(0,T;V_N)$ by (4.28) and $u_n(\cdot,0)$ is bounded in L_N we obtain that there exists $C>0$ independent of n and t such that

$$\|u_n(\cdot,t)\|_{L_N} \le C \quad \forall n\in\mathbb{N}, t\in[0,T]. \tag{4.30}$$

But by (4.29) and (4.2), for every $t\in[0,T]$ and for every $\phi\in\mathcal{V}_N$, we have

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega u_n(x,t)\cdot\phi(x)dx = \\ \lim_{n\to\infty}\left(\int_\Omega u_n(x,0)\cdot\phi(x)dx - \int_0^t\int_\Omega \nabla_x H_{u_n}:\nabla_x\phi + \int_0^t\int_\Omega F(u_n):\nabla_x\phi\right) \\ = \int_\Omega v_0(x)\cdot\phi(x)dx - \int_0^t\int_\Omega \nabla_x\bar{H}:\nabla_x\phi + \int_0^t\int_\Omega F(u_0):\nabla_x\phi\,.\end{aligned} \tag{4.31}$$

Since $\mathcal{V}_N$ is dense in L_N, by (4.30), and (4.31), for every $t\in[0,T]$ there exists $u(\cdot,t)\in L_N$ such that

$$u_n(\cdot,t)\rightharpoonup u(\cdot,t) \quad \text{weakly in } L_N \quad \forall t\in[0,T]\,. \tag{4.32}$$

Moreover there exists $\hat{C}>0$, independent of t, such that $\|u(\cdot,t)\|_{L_N}\le\hat{C}$. But we have $u_n\to u_0$ in $L^2(0,T;L_N)$, therefore $u=u_0$ a.e. and so $u\in L^2(0,T;V_N)\cap L^\infty(0,T;L_N)$. Moreover, by (4.31) we obtain that $u(\cdot,t)$ is L_N-weakly continuous in t on $[0,T]$. Therefore, by (4.32) and (4.29),

$$\int_0^T\int_\Omega|\nabla_x u|^2\,dxdt+\int_\Omega|u(x,T)|^2dx \le \varliminf_{n\to\infty}\left(\int_0^T\int_\Omega|\nabla_x u_n|^2\,dxdt+\int_\Omega|u_n(x,T)|^2dx\right). \tag{4.33}$$

Next for every $\psi(x,t)\in C_c^\infty(\Omega\times(0,T),\mathbb{R}^N)$ such that $div_x\,\psi=0$ we obtain

$$\begin{aligned}\lim_{n\to\infty}\int_0^T\int_\Omega\big(u_n\cdot\partial_t\psi+F(u_n):\nabla_x\psi\big)dxdt = \\ \lim_{n\to\infty}\int_0^T\int_\Omega\nabla_x H_{u_n}:\nabla_x\psi\,dxdt = \int_0^T\int_\Omega\nabla_x\bar{H}:\nabla_x\psi\,dxdt\,.\end{aligned} \tag{4.34}$$

But we obtained that $u_n\to u$ strongly in $L^2(0,T;L_N)$. Therefore, since F is a Lipschitz function we obtain

$$\lim_{n\to\infty}\int_0^T\int_\Omega\big(u_n\cdot\partial_t\psi+F(u_n):\nabla_x\psi\big)dxdt = \int_0^T\int_\Omega\big(u\cdot\partial_t\psi+F(u):\nabla_x\psi\big)dxdt$$

So, by (4.34), for every $\psi(x,t) \in C_c^\infty(\Omega \times (0,T), \mathbb{R}^N)$ such that $div_x\, \psi = 0$ we obtain

$$\int_0^T \int_\Omega \big(u \cdot \partial_t \psi + F(u) : \nabla_x \psi\big) dxdt = \int_0^T \int_\Omega \nabla_x \bar{H} : \nabla_x \psi \, dxdt\,. \tag{4.35}$$

In particular $\partial_t u + div_x\, F(u) \in L^2(0,T; V_N^{-1})$. Therefore $\partial_t u \in L^2(0,T; V_N^{-1})$ and then $u \in A_{v_0} = \{u \in \mathcal{R} : \; u(\cdot, 0) = v_0(\cdot)\}$. Moreover, by (4.35), we obtain that $H_u = \bar{H}$. So, as before,

$$\int_0^T \int_\Omega |\nabla_x H_u|^2 \, dxdt \le \varliminf_{n\to\infty} \int_0^T \int_\Omega |\nabla_x H_{u_n}|^2 \, dxdt\,. \tag{4.36}$$

Combining (4.36) with (4.33), we infer

$$I_F(u) \le \varliminf_{n\to\infty} I_F(u_n) = K\,.$$

Therefore, u is a minimizer to (4.6). By Lemma 4.4 it satisfies $H_u = u$, i.e.

$$\Delta_x u = \partial_t u + div_x\, F(u) + \nabla_x p\,.$$

Moreover, by Lemma 4.2, for every $t \in [0,T]$ we have

$$\int_0^t \int_\Omega \nabla_x u : \nabla_x H_u \, dxdt = \frac{1}{2}\Big(\int_\Omega v_0^2(x)dx - \int_\Omega u^2(x,t)dx\Big)\,.$$

Therefore we obtain (4.27). Moreover, $I_F(u) = \frac{1}{2}\int_\Omega v_0^2(x)dx$. Finally if $v \in \mathcal{R}$ satisfy $v(\cdot,0) = v_0(\cdot)$ and $H_v = v$ then since

$$\int_0^t \int_\Omega \nabla_x v : \nabla_x H_v \, dxdt = \frac{1}{2}\Big(\int_\Omega v_0^2(x)dx - \int_\Omega v^2(x,t)dx\Big)\,,$$

we have $I_F(v) = \frac{1}{2}\int_\Omega v_0^2(x)dx = I_F(u)$. So v is a minimizer to (4.6). □

Remark 4.2. For a fixed $r(x,t) \in L^2(0,T; V_N)$ we can define a functional $\bar{I}_{\{F,r\}}(u) : \mathcal{R} \to \mathbb{R}$ by

(4.37)
$$\bar{I}_{\{F,r\}}(u) := \frac{1}{2}\Big(\int_0^T \int_\Omega \Big(|\nabla_x u + \nabla_x r|^2 + |\nabla_x H_u - \nabla_x r|^2\Big) dxdt + \int_\Omega |u(x,T)|^2 dx\Big)\,,$$

and for every $v_0 \in V_N$ we can consider the minimization problem

$$\inf\{\bar{I}_{\{F,r\}}(u) : \; u \in \mathcal{R}, u(\cdot,0) = v_0(\cdot)\}\,. \tag{4.38}$$

Then similarly to the proof of Theorem 4.1 we can prove that there exists a minimizer u to (4.38) and it satisfies $H_u = u + r$, i.e.

$$\Delta_x u + \Delta_x r = \partial_t u + div_x\, F(u) + \nabla_x p\,.$$

Then, using this fact, as in the proof of Theorem 3.1 we can deduce the existence of a weak solution to (1.1) with $f \in L^2(0,T; V_N^{-1})$.

Remark 4.3. Similar method as in the proof of Theorem 3.1 we can apply to the unbounded domain Ω. In this case we consider a sequence of smooth bounded domains $\{\Omega_n\}$, such that $\Omega_n \subset \Omega_{n+1}$ and $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$, and a sequence $v_0^{(n)} \to v_0$ in L_N, such that $\operatorname{supp} v_0^{(n)} \subset \Omega_n$. Consider $u_n(x,t) \in \mathcal{R}(\Omega_n)$, such that $u_n(\cdot,0) = v_0^{(n)}(\cdot)$ and for every $\psi(x,t) \in C_c^\infty(\Omega_n \times (0,T), \mathbb{R}^N)$, satisfying $div_x\, \psi = 0$, we have (3.5), where F_n is defined by (3.4). Then we can deduce that there exists $u \in L^2(0,T;V_N) \cap L^\infty(0,T;L_N)$ such that, up to a subsequence, $u_n \to u$ strongly in $L^2_{loc}(\Omega \times (0,T), \mathbb{R}^N)$. Then u will satisfy conditions (i)-(ii) of Theorem 3.1.

5. Variational principle for more regular solutions of the Navier-Stokes Equations

Let $\Omega \subset \mathbb{R}^N$ be a domain with Lipschitz boundary (not necessarily bounded). We denote by H_N the closure of $\mathcal{V}_N$ in $H_0^1(\Omega, \mathbb{R}^N)$ (the spaces H_N and V_N differ only in the case of unbounded domain). For every $u \in L^4(\Omega \times (0,T), \mathbb{R}^N)$ we have $(u \otimes u) \in L^2\big(0,T; L^2(\Omega, \mathbb{R}^{N\times N})\big)$ and therefore $div_x\,(u \otimes u) \in L^2(0,T;V_N^{-1})$. If in addition $\partial_t u \in L^2(0,T;V_N^{-1})$ then we obtain $\partial_t u + div_x\,(u\otimes u) \in L^2(0,T;V_N^{-1})$.

Definition 5.1. Let $u \in L^2(0,T;H_N) \cap L^\infty(0,T;L_N)$ be such that $\partial_t u \in L^2(0,T;V_N^{-1})$ and such that $u(\cdot,t)$ is L_N-weakly continuous in t on $[0,T]$. Denote the set of all such functions u by $\mathcal{R}'$. Denote the set $\mathcal{R}' \cap L^4(\Omega \times (0,T), \mathbb{R}^N)$ by $\mathcal{P}$. For every $u \in \mathcal{P}$ let $\bar{H}_u(\cdot,t) \in L^2(0,T;V_N)$ be as in Remark 2.2, corresponding to $\partial_t u + div_x\,(u \otimes u)$. That is for every $\psi(x,t) \in C_c^\infty(\Omega \times (0,T), \mathbb{R}^N)$ such that $div_x\,\psi = 0$ we have

$$\int_0^T \int_\Omega \big(u \cdot \partial_t \psi + (u \otimes u) : \nabla_x \psi\big) dxdt = \int_0^T \int_\Omega \nabla_x \bar{H}_u : \nabla_x \psi \, dxdt \,.$$

For a fixed $r(x,t) \in L^2(0,T;V_N)$ define a functional $J_{\{\varphi,r\}}(u) : \mathcal{P} \to \mathbb{R}$ by

(5.1)

$$J_{\{\varphi,r\}}(u) := \frac{1}{2}\bigg(\int_0^T \int_\Omega \Big(|\nabla_x u + \nabla_x r|^2 + |\nabla_x \bar{H}_u - \nabla_x r|^2\Big) dxdt + \int_\Omega |u(x,T)|^2 dx \bigg) .$$

Theorem 5.1. *Let $v_0 \in L_N$ and $r(x,t) \in L^2(0,T;V_N)$. Assume that there exists $u \in \mathcal{P}$ which satisfies $u(x,0) = v_0(x)$, and*

$$\int_0^T \int_\Omega \big(u \cdot \partial_t \psi + (u \otimes u) : \nabla_x \psi\big) dxdt = \int_0^T \int_\Omega (\nabla_x u + \nabla_x r) : \nabla_x \psi \, dxdt \tag{5.2}$$

for every $\psi(x,t) \in C_c^\infty(\Omega \times (0,T), \mathbb{R}^N)$, such that $div_x\,\psi = 0$, i.e.

$$\Delta_x u = \partial_t u + div_x\,(u \otimes u) + \nabla_x p - \Delta_x r \,.$$

Then u is a minimizer of the following problem

$$\inf\{J_{\{\varphi,r\}}(u):\ u\in\mathcal{P}, u(\cdot,0)=v_0(\cdot)\}\,. \tag{5.3}$$

Moreover if $\bar u$ is a minimizer to (5.3)*, then $\bar u$ is a solution to* (5.2)*.*

Proof. In the same way as in the proof of Theorem 4.1 in [1] we obtain that for every $\bar u\in\mathcal{P}$ we must have

$$\int_0^T\int_\Omega \nabla_x\bar u:\nabla_x H_{\bar u}\,dxdt=\frac{1}{2}\Big(\int_\Omega \bar u^2(x,0)dx-\int_\Omega \bar u^2(x,T)dx\Big)\,. \tag{5.4}$$

Therefore,

$$J_{\{\varphi,r\}}(\bar u)=\frac{1}{2}\int_0^T\int_\Omega \big(|\nabla_x\bar u+\nabla_x r-\nabla_x H_{\bar u}|^2+|\nabla_x r|^2\big)\,dxdt+\frac{1}{2}\int_\Omega \bar u^2(x,0)dx\,. \tag{5.5}$$

Therefore, $u\in\mathcal{P}$ which satisfy $u(x,0)=v_0(x)$ and $\nabla_x u+\nabla_x r=\nabla_x H_u$ will be the minimizer to (5.3). Then also every minimizer $\bar u$ will satisfy $\nabla_x\bar u+\nabla_x r=\nabla_x H_{\bar u}$, i.e. will satisfy (5.2). □

Appendix A

Proof of Corollary 4.2. Let $\eta\in C_c^\infty(\mathbb{R},\mathbb{R})$ be a mollifying kernel, satisfying $\eta\geq 0$, $\int_{\mathbb{R}}\eta(t)dt=1$, $\operatorname{supp}\eta\subset[-1,1]$ and $\eta(-t)=\eta(t)\ \forall t$. Given small $\varepsilon>0$ and $\psi(x,t)\in C_c^\infty(\Omega\times(2\varepsilon,T-2\varepsilon),\mathbb{R}^N)$ such that $div_x\,\psi=0$, define

$$\psi_\varepsilon(x,t):=\frac{1}{\varepsilon}\int_0^T \eta\Big(\frac{s-t}{\varepsilon}\Big)\psi(x,s)ds\,. \tag{A.1}$$

Then $\psi_\varepsilon(x,t)\in C_c^\infty(\Omega\times(0,T),\mathbb{R}^N)$ and satisfies $div_x\,\psi_\varepsilon=0$. Therefore we obtain

$$\int_0^T\int_\Omega u\cdot\partial_t\psi_\varepsilon\,dxdt=\int_0^T\int_\Omega \nabla_x V_u:\nabla_x\psi_\varepsilon\,dxdt\,, \tag{A.2}$$

where $V_u(\cdot,t)\in L^2(0,T;V_N)$ is as in Remark 2.2, corresponding to $\partial_t u$. But

$$\int_0^T\int_\Omega u\cdot\partial_t\psi_\varepsilon\,dxdt=\int_0^T\int_\Omega u(x,t)\cdot\Big(\frac{1}{\varepsilon}\int_0^T\eta\Big(\frac{s-t}{\varepsilon}\Big)\partial_s\psi(x,s)ds\Big)\,dxdt=$$
$$\int_0^T\int_\Omega \partial_t\psi(x,t)\cdot\Big(\frac{1}{\varepsilon}\int_0^T\eta\Big(\frac{s-t}{\varepsilon}\Big)u(x,s)ds\Big)\,dxdt=\int_0^T\int_\Omega \partial_t\psi(x,t)\cdot u_\varepsilon(x,t)\,dxdt\,,$$

where $u_\varepsilon(x,t) = \frac{1}{\varepsilon}\int_0^T \eta((s-t)/\varepsilon)u(x,s)ds$. By the other hand

$$\int_0^T\int_\Omega \nabla_x V_u : \nabla_x \psi_\varepsilon \, dxdt = \int_0^T\int_\Omega \nabla_x V_u(x,t) : \Big(\frac{1}{\varepsilon}\int_0^T \eta\Big(\frac{s-t}{\varepsilon}\Big)\nabla_x\psi(x,s)ds\Big)\, dxdt$$
$$= \int_0^T\int_\Omega \nabla_x\psi(x,t) : \nabla_x\Big(\frac{1}{\varepsilon}\int_0^T \eta\Big(\frac{s-t}{\varepsilon}\Big)V_u(x,s)ds\Big)\, dxdt$$
$$= \int_0^T\int_\Omega \nabla_x\psi(x,t) : \nabla_x(V_u)_\varepsilon(x,t)\, dxdt\,,$$

where $(V_u)_\varepsilon(x,t) = \frac{1}{\varepsilon}\int_0^T \eta((s-t)/\varepsilon)V_u(x,s)ds$. Therefore, by (A.2), we infer

$$\text{(A.3)}\qquad \int_0^T\int_\Omega u_\varepsilon\cdot\partial_t\psi\, dxdt = \int_0^T\int_\Omega \nabla_x(V_u)_\varepsilon : \nabla_x\psi\, dxdt\,.$$

So $\partial_t u_\varepsilon \in L^2(2\varepsilon, T-2\varepsilon; V_N^{-1})$. Moreover $u_\varepsilon \in L^2(0,T;V_N)\cap L^\infty(0,T;L_N)$. We have $u_\varepsilon \to u$ and $(V_u)_\varepsilon \to V_u$ strongly in $L^2(0,T;V_N)$ as $\varepsilon\to 0$. Moreover, up to a subsequence $\varepsilon_n \to 0$, we have $u_{\varepsilon_n}(\cdot,t)\to u(\cdot,t)$ strongly in L_N a.e. in $[0,T]$. In addition, by Lemma 4.2, for every $a,b\in[2\varepsilon, T-2\varepsilon]$ we have

$$\text{(A.4)}\qquad \int_a^b\int_\Omega \nabla_x u_\varepsilon : \nabla_x(V_u)_\varepsilon\, dxdt = \frac{1}{2}\Big(\int_\Omega u_\varepsilon^2(x,a)dx - \int_\Omega u_\varepsilon^2(x,b)dx\Big)\,.$$

Then letting $\varepsilon\to 0$ in (A.4), we obtain that for almost every a and b in $(0,T)$ we have

$$\int_a^b\int_\Omega \nabla_x u : \nabla_x V_u\, dxdt = \frac{1}{2}\Big(\int_\Omega u^2(x,a)dx - \int_\Omega u^2(x,b)dx\Big)\,.$$

So $u\in L^\infty(0,T;L_N)$. □

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