# *On a Singular Variational Integral with Linear Growth, I: Existence and Regularity of Minimizers*

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## § 1. Introduction

In this paper we investigate the variational integral

$$\mathscr{I}(u) = \int_\Omega u\sqrt{1 + |Du|^2}\, dx, \tag{1}$$

which is to be minimized in the class of functions $u\colon \Omega \to \mathbb{R}$ under Dirichlet boundary conditions; $\Omega$ is a bounded domain in $\mathbb{R}^n$ whose boundary $\Sigma$ is Lipschitz continuous. Furthermore we impose the side condition $u(x) \geqq 0$ a.e. on $\Omega$ as so as to make $\mathscr{I}(u)$ bounded from below.

The essential properties of $\mathscr{I}(u)$ are
(i) the integrand

$$f(x, u, p) = u\sqrt{1 + |p|^2} \tag{2}$$

grows linearly in $p$ and depends on $u$ explicitly;
(ii) $f(x, u, p)$ degenerates if $u$ vanishes.
The most prominent example for an integral with linear growth is the area integral

$$\mathscr{A}(u) = \int_\Omega \sqrt{1 + |Du|^2}\, dx. \tag{3}$$

Minimum problems for $\mathscr{A}(u)$ can be solved by the theory of Cacciopoli sets; for details we refer to the recent books by GIUSTI [14] and MASSARI & MIRANDA [17]. In this approach $\mathscr{A}(u)$ is extended to $\mathrm{BV}(\Omega)$, the space of $L_1(\Omega)$-functions whose distributional derivatives are measures with finite total variation:

$$\int_\Omega \sqrt{1 + |Du|^2} = \sup\Big\{\int g_{n+1} + u \sum_{i=1}^{n} D_i g_i\, dx \colon g_i \in C_c^1(\Omega)\\ \forall i = 1, \ldots, n+1, \sum_{i=1}^{n+1} g_i^2(x) \leqq 1\Big\}. \tag{4}$$

For more general integrands $f(x, u, Du)$ such an extension may become rather complicated because $Du$ is no longer a function. GIAQUINTA, MODICA & SOUČEK [13] studied integrands of the form $f(x, p) + g(x, u)$, and DAL MASO [6] and ANZELLOTTI [2] treated the more general case $f(x, u, p)$. They define $\int_\Omega f(x, u, Du)$ on $\mathrm{BV}(\Omega)$ through the elliptic parametric integrand that is associated to $f(x, u, p)$. However, even for an integrand that unlike (2) does not degenerate and is close to the one in (4), e.g. $a(u)\sqrt{1+|p|^2}$ with $a(u) > 0$,[1] this approach does not lead to regularity of minima. In [6] and [2] the aim is to extend possibly all integrals with linear growth onto the space $\mathrm{BV}(\Omega)$. In the present paper however we proceed rather in the opposite way: we choose a particular function space that is adapted to the one variational integral $\mathscr{I}(u)$.

If we set $v = u^2$, we have

$$\begin{aligned} u\sqrt{1+|Du|^2} &= \sqrt{u^2 + u^2\,|Du|^2} \\ &= \sqrt{u^2 + \tfrac{1}{4}\,|D(u^2)|^2} \\ &= \sqrt{v + \tfrac{1}{4}\,|Dv|^2}; \end{aligned} \tag{6}$$

now

$$\mathscr{I}(v) = \int_\Omega \sqrt{v + \tfrac{1}{4}\,|Dv|^2}\,dx \tag{7}$$

can be extended onto $\mathrm{BV}(\Omega) \cap \{v(x) \geqq 0 \text{ a.e. in } \Omega\} \equiv \mathrm{BV}^+(\Omega)$ and we can apply the direct methods to (7) because the integrand

$$g(x, v, q) = \sqrt{v + \tfrac{1}{4}\,|q|^2} \tag{8}$$

is much simpler in structure than (2) is. The existence of a minimum for $\mathscr{I}(v)$ is shown in § 2 where we also cite a further result supporting our view that $\mathrm{BV}(\Omega)$ is possibly not the appropriate function space for all integrals with linear growth.

A further decisive property of $\mathscr{I}(u)$ is the fact that the integral degenerates if $u$ vanishes. For $\Omega = B_R(0)$, $R$ large enough, and $u|_{\partial\Omega} \equiv 1$ the minimum for our problem vanishes on a set $I \subset \Omega$ of positive measure; *cf.* Theorem 7. The analysis of the boundary $\partial I$ of the coincidence set $I$ is usually based on the regularity of the minimum, especially on the behavior of $Du$ on $\partial I$. Variation of the independent variables leads to

$$u\frac{|Du|^2}{\sqrt{1+|Du|^2}} - u\sqrt{1+|Du|^2} = 0 \qquad \text{on } \partial I; \tag{9}$$

as this relation is satisfied for any value of $Du$, it does not give any information about the angle at which $u$ leaves the coincidence set. This states the main differ-

[1] The contribution by TAUSCH [25], who obtaines existence and regularity for $\delta \int a(x, u)\sqrt{1+|Du|^2}\,dx = 0$ by completely different methods, indicates that the lack of regularity is due to the method rather than being a property of the solution. Inequalities as side conditions are not considered in [25].

ence from integrals of the form

$$\int_{\Omega\cap\{u>0\}} \{|Du|^2 + Q^2\}\, dx, \qquad Q(x) > 0 \text{ on } \Omega,$$

for which ALT & CAFFARELLI [1] developed a new method to investigate the free boundary $\partial I$; its regularity follows from the Lipschitz continuity of $u$. An observation by KEIPER [16] shows that in the 2-dimensional, axially symmetric case the extremals of (1) fail to be that regular. The procedure from [1] was extended by CAFFARELLI & FRIEDMAN to more general integrals

$$\int_{\Omega\cap\{u>0\}} f(x, u, Du)\, dx;$$

in particular they studied the capillary problem for the sessile drop; *cf.* [5]. Although this method does not apply to integrals like (1), the present paper and the investigation of the free boundary which will be published elsewhere owe very much to the work in [5].

In §§ 4, 5 we show that the minima of $\mathcal{J}(v)$ are continuous in $\Omega$ and analytic in the set where $u$ is positive. This implies in particular the equivalence of the variational problems for $\mathcal{I}(u)$ and $\mathcal{J}(v)$. These results are based on properties of the parametric analogues to $\mathcal{I}(u)$ and $\mathcal{J}(v)$. They allow us to interpret $\mathcal{I}$ as an area functional for surfaces in certain manifolds.

We close this introduction by referring to other works that are related to the present problem.

For $n = 1$ the integral (1) becomes

$$\mathcal{I}_1(u) := \int_a^b u \sqrt{1 + |u|^2}\, dt; \tag{10}$$

it describes the area of a surface that is generated by rotating about the $t$-axis the curve

$$\gamma := \{(t, u), u = u(t), t \in [a, b]\}. \tag{11}$$

Hence a minimum of (11) under Dirichlet boundary conditions gives a rotationally symmetric minimal surface that is bounded by two concentric circles. The solution of this problem is discussed in NITSCHE'S monograph [20] Chapter VI.3, especcially § 515. In view of our results on the coincidence set we remark that for certain Dirichlet data the absolute minimum of (10) consists in three line segments connecting the points $(a, u(a))$ and $(a, 0)$, then $(a, 0)$ and $(b, 0)$, and finally $(b, 0)$ and $(b, u(b))$. This solution was established by B. GOLDSCHMIDT in his celebrated paper [15] of 1831. The fact that besides regular extremals there may be piecewise smooth curves that give an even smaller value to the variational integral was known to EULER, *cf.* [9]; GOLDSCHMIDT and later on SINCLAIR [24] clarified the minimal properties of the various extremals. The minimum obtained in the present paper can be regarded as the $n$-dimensional analogue of GOLDSCHMIDT'S solution. It is remarkable that the simple geometric configuration consisting in three line segments can be minimal only in the one-dimensional problem; an elementary calculation shows that in higher dimensions such a configuration does not provide even a local minimum.

Another interpretation of $\mathscr{I}(u)$ is obtained if we regard the graph of $u$ as a material surface of constant mass density. In this setting the variational problem for $\mathscr{I}_1(u)$ characterizes the shape of a chain fixed at its endpoints and subject to its own weight. This interpretation gives the regular extremals of (11) the name "catenaries". For two-dimensional parametric surfaces $x\colon B \to \mathbb{R}^3$, $B = \{\xi + i\eta\colon \xi^2 + \eta^2 \leqq 1\}$ the functional now reads

$$\mathscr{I}(x) = \int_B z(\xi, \eta)\,|Dx(\xi, \eta)|^2\, d\xi\, d\eta \tag{12}$$

where $z$ denotes the third component of $x$. In this setting BÖHME, HILDEBRANDT & TAUSCH [4] provided several inclusion theorems; related results are proved by DIERKES [7], [8].

In [4] it is pointed out that the circular cone $u(x^1, x^2) = +\sqrt{(x^1)^2 + (x^2)^2}$ solves the Euler-Lagrange equations in $\{(x^1, x^2)\colon 0 < (x^1)^2 + (x^2)^2 < R\}$. KEIPER [16] observed that every extremal of the axially symmetric version of (1) which starts from the $u$-axis must-approach the cone $u(x^1, \dots, x^n) = \frac{1}{\sqrt{n-1}} \sqrt{(x^1)^2 + \dots + (x^n)^2}$ asymptotically. Again the $n$-cone satisfies the Euler-Lagrange equation in $R^n - \{0\}$. In view of this singular cone it is conceivable that minima of (1) will not necessarily be regular.

In variational problems for (11) one can also prescribe the length of the curve $u$ as side condition. This physically realistic constraint poses considerable difficulties in higher dimensions. Furthermore, for $n > 1$ the prescribed area of $u$ cannot be arbitrary large. This necessary condition, which is a genuinely $n$-dimensional property, was found in a recent investigation by J. C. C. NITSCHE [21]; there an upper bound in terms of the prescribed boundary curve is given. In the appendix to this paper as well as in [4] further historical notes may be found.

## § 2. The variational problem and the spaces $\mathrm{BV}_2$, $\mathrm{BV}_2^+$

In this part of our paper we show existence of a minimum to our variational problem. We remark that once the space $\mathrm{BV}_2^+(\Omega)$ is introduced the proofs follow rather closely the analogous arguments for non-parametric minimal surfaces; *cf. e.g.* [14] Chapter 14 or [17] § 3.5.

Suppose $\Omega \subset \mathbb{R}^n$, $n \geqq 2$, be a bounded domain with Lipschitz continuous boundary $\Sigma \equiv \partial\Omega$. We define

$$\mathrm{BV}_2(\Omega) := \{u \in L_2(\Omega)\colon u^2 \in \mathrm{BV}(\Omega)\}, \tag{13}$$

$$\mathrm{BV}_2^+(\Omega) := \{u \in \mathrm{BV}_2(\Omega)\colon u \geqq 0 \text{ a.e. in } \Omega\}. \tag{14}$$

In analogy to (4) we can now define (1) for more general functions $u \in \mathrm{BV}_2(\Omega)$:

$$\int_\Omega \sqrt{u^2 + \tfrac{1}{4}|D(u^2)|^2} := \sup\left\{\int_\Omega u g_{n+1} + \tfrac{1}{2} u^2 \operatorname{div} g \, dx : \right. \tag{15}$$

$$\left. g_i \in C_c^1(\Omega), \sum_{i=1}^{n+1} g_i^2(x) \leqq 1 \quad \forall x \in \Omega\right\}.$$

From (13) and (15) it is obvious that

$$\int_\Omega \sqrt{u^2 + \tfrac{1}{4}|D(u^2)|^2} < \infty$$

if and only if $u$ is in $\mathrm{BV}_2(\Omega)$. Moreover it is readily verified that (15) agrees with the integral (1) from which we began if only $u$ is regular enough:

$$\int_\Omega \sqrt{u^2 + \tfrac{1}{4}|D(u^2)|^2} = \int_\Omega u\sqrt{1 + |Du|^2}\, dx$$

for all $u \in H_1^1(\Omega) \cap L_\infty(\Omega)$ with $u \geqq 0$ a.e. on $\Omega$.

Setting $v := u^2$, $\psi := \varphi^2$ where $\varphi$ are the boundary data for $u$ we are thus led to the following variational problem:

$$\begin{aligned} &\text{minimize } \mathscr{J}(v) = \int_\Omega \sqrt{v + \tfrac{1}{4}|Dv|^2} \text{ in the class} \\ &\mathfrak{C}(\psi) := \{v \in \mathrm{BV}(\Omega) : v \geqq 0 \text{ a.e. in } \Omega,\ v = \psi \text{ on } \Sigma\}. \end{aligned} \tag{P}$$

As this set is not closed with respect to convergence in $L_1(\Omega)$, we consider the following problem:

$$\begin{aligned} &\text{minimize } \mathscr{J}^*(v) = \mathscr{J}(v) + \tfrac{1}{2} \oint_\Sigma |v - \psi| \, d\mathfrak{H}^{n-1} \\ &\text{in the class } \mathrm{BV}^+(\Omega) := \{v \in \mathrm{BV}(\Omega) : v \geqq 0 \text{ a.e. in } \Omega\}. \end{aligned} \tag{P*}$$

Here $\mathfrak{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. Problems (P) and (P*) are equivalent in the following sense:

**Lemma 1.** *For* $\Sigma \in C^1$, $\varphi \in L_2(\Sigma)$ *with* $\varphi \geqq 0$ a.e. *on* $\Sigma$ *and* $\psi := \varphi^2$,

$$\inf\{\mathscr{J}(v) : v \in \mathrm{BV}^+(\Omega), v = \psi \mid \text{on } \Sigma\} = \inf\{\mathscr{J}^*(v) : v \in \mathrm{BV}^+(\Omega)\}. \tag{16}$$

**Proof.** Fix $v \in \mathrm{BV}^+(\Omega)$ and $\varepsilon > 0$. A theorem of GAGLIARDO [11] provides an extension $w \in H_1^1(\Omega)$ of $\psi - v$ such that

$$w = \psi - v \quad \text{in } L_1(\Sigma), \tag{17}$$

$$\int_\Omega |Dw| \leqq (1 + \varepsilon) \oint_\Sigma |v - \psi| \, d\mathfrak{H}^{n-1}, \tag{18}$$

$$\int_\Omega |w| \, dx \leqq \varepsilon \oint_\Sigma |v - \psi| \, d\mathfrak{H}^{n-1}. \tag{19}$$

Now $(w+v)^+ := \max\{w+v, 0\}$ is in $\mathrm{BV}^+(\Omega)$, too, and has $\psi$ as its trace on $\Sigma$. Moreover

$$\int_\Omega \sqrt{(v+w)^+ + \tfrac14 |D(v+w)^+|^2} \leqq \int_\Omega \sqrt{v + \tfrac14 |Dv|^2} + \tfrac12 \int_\Omega |Dw| + \int_\Omega \sqrt{|w|}\, dx$$
$$\leqq \int_\Omega \sqrt{v + \tfrac14 |Dv|^2} + \tfrac12 (1+\varepsilon) \oint_\Omega |v-\psi|\, d\mathfrak{H}^{n-1}$$
$$+ (\operatorname{meas} \Omega)^{\frac12} \cdot \left(\varepsilon \cdot \oint_\Sigma |v-\psi|\, d\mathfrak{H}^{n-1}\right)^{\frac12}.$$

If we let $\varepsilon$ tend to zero and take the infimum over all $v \in \mathrm{BV}^+(\Omega)$, we obtain

$$\inf\{\mathscr{J}(v) : v \in \mathrm{BV}^+(\Omega), v = \psi\} \leqq \inf\{\mathscr{J}^*(v) : v \in \mathrm{BV}^+(\Omega)\}.$$

As the opposite relation is obvious, (16) is therefore proved. □

The functional $\mathscr{J}(v)$ is lower semicontinuous with respect to convergence in $L_{1,\mathrm{loc}}(\Omega)$.

**Lemma 2.** *Suppose* $0 \leqq v_k \to v$ *in* $L_{1,\mathrm{loc}}(\Omega)$. *Then*

$$\int_\Omega \sqrt{v + \tfrac14 |Dv|^2} \leqq \liminf_{k\to\infty} \int_\Omega \sqrt{v_k + \tfrac14 |Dv_k|^2}. \tag{20}$$

**Proof.** Set $u_k := \sqrt{v_k}$ and $u := \sqrt{v}$; then $|u_k - u| \leqq |v_k - v|^{\frac12}$, and consequently $u_k$ tends to $u$ in $L_{2,\mathrm{loc}}(\Omega)$. Lower semicontinuity then follows immediately. □

This lemma implies the following existence theorem because the class $\tilde{\mathfrak{C}}$ defined below is now closed with respect to $L_1$-convergence.

**Lemma 3.** *Let* $\Omega \subset\subset B \equiv B_R(0)$ *and* $\psi \in H_1^1(B - \bar{\Omega})$, $\psi \geqq 0$ a. e. *on* $B$ *be given. Then the variational problem*

$$\left.\begin{array}{l} \text{minimize } \int_B \sqrt{v + \tfrac14 |Dv|^2} \text{ in the class } \tilde{\mathfrak{C}} = \{v \in \mathrm{BV}^+(\Omega) : \\ v = \psi \quad \text{on} \quad B - \tilde{\Omega}\} \end{array}\right\} \tag{$\tilde{\mathrm{P}}$}$$

*has a solution.*

The main conclusion regarding existence is contained in

**Theorem 4.** *Suppose* $\Sigma$ *to be Lipschitz continuous, and let* $\psi$ *be in* $L_1(\Sigma)$. *Then the variational integral*

$$\mathscr{J}^*(v) = \int_\Omega \sqrt{v + \tfrac14 |Dv|^2} + \tfrac12 \oint_\Sigma |v-\psi|\, d\mathfrak{H}^{n-1}$$

*attains its minimum on* $\mathrm{BV}^+(\Omega)$.

**Proof.** Let $\tilde{\psi} \in H_1^1(B - \Omega)$ denote some non-negative extension of $\psi$; then we set for $v \in \mathrm{BV}^+(\Omega)$

$$\tilde{v}(x) = \begin{cases} v(x), & x \in \Omega \\ \tilde{\psi}(x), & x \in B - \Omega. \end{cases}$$

Because $\tilde{v} \in \mathrm{BV}^+(B)$, by use of the trace formula we obtain

$$\int_B \sqrt{\tilde{v} + \tfrac{1}{4}|D\tilde{v}|^2} = \int_\Omega \sqrt{v + \tfrac{1}{4}|Dv|^2} + \int_{B-\Omega} \sqrt{\psi + \tfrac{1}{4} D\psi^2} + \tfrac{1}{2} \oint_\Sigma |D\tilde{v}|$$
$$= \mathscr{J}(v) + c(\psi).$$

We now can apply Lemma 3, and the theorem is proved. □

As a further illustration of our procedure we refer to the work of Bemelmans [3] in which two-dimensional closed surfaces that are graphs over the unit sphere $S$ are studied. Their areas are given by

$$\mathscr{A}^*(u) = \oint_S \sqrt{u^4 + u^2|\mathscr{D}^*u|^2}\,\sqrt{g^*}\,d\xi$$

and hence constitutes an integral similar to (1)[2].

A variational problem for $\mathscr{A}^*(u)$ is considered where among other side conditions a volume constraint is imposed:

$$\mathscr{V}(u) = \oint_S u^3 \sqrt{g^*}\,d\xi = \text{const.} \tag{21}$$

Clearly, BV is not an appropriate function space for this problem, because the Sobolev embedding theorem yields $u \in L_p$, $p > 2$, which means that (21) is not a compact side condition. The space used in [3] instead is $\mathrm{BV}_2$, and the embedding $\mathrm{BV}_2 \to L_p$, $p < 4$, is compact. Therefore (21) which contains the $L_3$-norm of $u$ is a compact side condition as one expects from the geometric content of the integrals $\mathscr{A}^*(u)$ and $\mathscr{V}(u)$. The problem in this paper as well as the one in [3] indicates that the growth of the integrand alone does not determine the appropriate function space.

## § 3. A Maximum Principle

The maximum principle for the variational problem (P*) states that the $L_\infty$-norm of a minimizer $v$ can be bounded by the supremum of the boundary data; as a preparatory step we prove

**Lemma 5.** *Let* $v \in \mathrm{BV}^+(\Omega)$ *and* meas $A(k) > 0$, *where* $A(k) = \{x \in \Omega : v(x) > k\}$ *Then* $w = \min(v, k) \in \mathrm{BV}^+(\Omega)$, *and for almost all* $k$

$$\mathscr{J}(w) < \mathscr{J}(v). \tag{22}$$

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[2] $|\mathscr{D}^*u|^2 = g^{*ij} D_i u\, D_j u$, $g^{*ij}$ is the inverse of the metric tensor $g^*_{ij}$ on $S$; $g^* = \det(g^*_{ij})$.

**Proof.** The first assertion is well known; *cf.* [12] Lemma A.4. To prove (22) choose a sequence of functions $v_m \in C^1(\Omega)$ with the properties

$$v_m \to v \quad \text{in } L_1(\Omega), \tag{23}$$

$$\int_\Omega \sqrt{v_m + \tfrac{1}{4}\,|Dv_m|^2} \to \int_\Omega \sqrt{v + \tfrac{1}{4}\,|Dv|^2}. \tag{24}$$

For $w_m = \min(v_m, k)$ lower semicontinuity of $\mathscr{I}(w)$ and (24) yield

$$\begin{aligned}
&\int_\Omega \sqrt{v + \tfrac{1}{4}\,|Dv|^2} - \int_\Omega \sqrt{w + \tfrac{1}{4}\,|Dw|^2} \\
&\qquad \geqq \liminf_{m\to\infty} \left\{ \int_\Omega \sqrt{v_m + \tfrac{1}{4}\,|Dv_m|^2} - \int_\Omega \sqrt{w_m + \tfrac{1}{4}\,|Dw_m|^2} \right\} \\
&\qquad \geqq \lim_{m\to\infty} \left\{ \int_{\{v_m > k\}} \sqrt{v_m}\,dx - \int_{\{v_m > k\}} \sqrt{k}\,dx \right\} \\
&\qquad \geqq \int_{A(k)} \sqrt{v} - \sqrt{k}\,dx.
\end{aligned}$$

Here we used the facts that $\sqrt{v_m}$ tends to $\sqrt{v}$ in $L_1(\Omega)$ and that the characteristic function $\chi_{\{v_m > k\}}$ converges to $\chi_{\{v > k\}}$ for almost all $k$.

According to our hypothesis meas $A(k) > 0$ the last integral is positive, too.

**Theorem 6.** *Let* $v \in \mathrm{BV}^+(\Omega)$ *be a minimum of* (P*) *and let the boundary values* $\psi$ *satisfy* $0 \leqq \psi \leqq k < \infty$ *on* $\Sigma$. *Then*

$$\|v\|_{L_\infty(\Omega)} \leqq k. \tag{25}$$

**Proof.** Again we choose a sequence of $C^1(\Omega)$-functions $v_m$ that satisfy (23), (24) and

$$v_m \to v \quad \text{in } L_1(\Sigma). \tag{26}$$

Then a consequence of EMMER's lemma gives

$$w_m := \min(v_m, k) \to \min(v, k) =: w \quad \text{in } L_1(\Sigma);$$

*cf.* [12], Lemma A.2. Hence

$$\oint_\Sigma |w_m - \psi|\, d\mathfrak{H}^{n-1} \to \oint_\Sigma |w - \psi|\, d\mathfrak{H}^{n-1}$$

and

$$\oint_\Sigma |v_m - \psi|\, d\mathfrak{H}^{n-1} \to \oint_\Sigma |v - \psi|\, d\mathfrak{H}^{n-1}.$$

As $\psi$ is bounded by $k$ a.e. on $\Sigma$, we have

$$\begin{aligned}
\oint_\Sigma |w_m - \psi|\, d\mathfrak{H}^{n-1} &= \oint_\Sigma |\min(v_m, k) - \min(\psi, k)|\, d\mathfrak{H}^{n-1} \\
&\leqq \oint_\Sigma |v_m - \psi|\, d\mathfrak{H}^{n-1}.
\end{aligned}$$

Taking the limit, we obtain

$$\oint_{\Sigma} |w - \psi| \, d\mathfrak{H}^{n-1} \leqq \oint_{\Sigma} |v - \psi| \, d\mathfrak{H}^{n-1}. \tag{27}$$

If $A(k)$ were of positive measure this together with (22) would yield

$$\mathscr{I}(w) + \oint_{\Sigma} |w - \psi| \, d\mathfrak{H}^{n-1} < \mathscr{I}(v) + \oint_{\Sigma} |v - \psi| \, d\mathfrak{H}^{n-1},$$

i.e.

$$\mathscr{I}^*(w) < \mathscr{I}^*(v),$$

which contradicts the assumption that $v$ is a minimum of (P*). Hence meas $A(k)$ must vanish and the theorem is proved.

The results of [4] on inclusion suggest that the coincidence set $\{u = 0\}$ is empty provided the boundary values $\varphi$ lie above the $n$-cone $u(x) = \frac{1}{\sqrt{n-1}} |x|$. On the other hand it is very likely that meas $\{u(x) = 0\} > 0$ if $\varphi$ is small enough.

**Theorem 7.** *Suppose* $v \in \mathrm{BV}^+(\Omega)$ *is a minimum of* (P*) *when the boundary values satisfy* $0 \leqq \psi \leqq k < \infty$. *If* $k$ *satisfies*

$$\sqrt{k} < \frac{\text{meas}\,\Omega}{\text{meas}\,\Sigma}, \tag{28}$$

*then the coincidence set is of positive measure*:

$$\text{meas}\,\{u = 0\} > 0. \tag{29}$$

**Proof.** Set $u := \sqrt{v}$ and $[u - \varepsilon]^+ := \max(u - \varepsilon, 0)$. Then $[u - \varepsilon]^{+2}$ is an admissible comparison function in (P*):

$$\begin{aligned} 0 &\geqq \mathscr{I}^*(v) - \mathscr{I}^*([u - \varepsilon]^{+2}) \\ &= \int_{\Omega} \sqrt{u^2 + \tfrac{1}{4} |Du^2|^2} - \int_{\Omega} \sqrt{[u - \varepsilon]^{+2} + \tfrac{1}{4} |D[u - \varepsilon]^{+2}|^2} \\ &\quad + \tfrac{1}{2} \oint_{\Sigma} |u^2 - \psi| \, d\mathfrak{H}^{n-1} - \tfrac{1}{2} \oint_{\Sigma} |[u - \varepsilon]^{+2} - \psi| \, d\mathfrak{H}^{n-1}. \end{aligned}$$

As in the preceeding proofs there exists a sequence of $C^1(\Omega)$-functions $v_m$ that approximate $v$ in the sense of (23), (24), (26). If we set $u_m := \sqrt{v_m}$, then $u_m$ is continuous on all of $\Omega$ and continuously differentiable when it does not vanish: $u_m \in C^1(\{u_m > \varepsilon\})$, $\varepsilon > 0$. Therefore the inequality above leads to

$$\begin{aligned} 0 \geqq \liminf_{m \to \infty} \Big\{ &\int_{\Omega} \sqrt{u_m^2 + \tfrac{1}{4} |Du_m^2|^2} \, dx \\ &- \int_{\Omega} \sqrt{[u_m - \varepsilon]^{+2} + \tfrac{1}{4} |D[u_m - \varepsilon]^{+2}|^2} \, dx \Big\} \\ &+ \tfrac{1}{2} \oint_{\Sigma} \{ |u^2 - \psi| - |[u - \varepsilon]^{+2} - \psi| \} \, d\mathfrak{H}^{n-1}, \end{aligned}$$

where we used again the lower semicontinuity of $\mathscr{J}$. Next we obtain

$$0 \geqq \liminf_{m\to\infty} \Bigg\{ \int\limits_{\{u_m>\varepsilon\}} \sqrt{u_m^2 + \tfrac{1}{4}\,|Du_m^2|^2}\,dx - \int\limits_{\{u_m>\varepsilon\}} \sqrt{[u_m-\varepsilon]^{+2} + \tfrac{1}{4}\,|D[u_m-\varepsilon]^{+2}|^2}\,dx \Bigg\} + \tfrac{1}{2} \oint\limits_{\Sigma} \{|u^2-\psi| - |[u-\varepsilon]^{+2}-\psi|\}\,d\mathfrak{H}^{n-1}.$$

According to GERHARDT [12], Lemma A4,

$$[u-\varepsilon]^+ = u - \min(u,\varepsilon) \quad \text{in } L_1(\Sigma);$$

therefore

$$\begin{aligned} 0 &\geqq \liminf_{m\to\infty} \int\limits_{\{u_m>\varepsilon\}} \varepsilon\, \sqrt{1+|Du_m|^2}\,dx + \tfrac{1}{2}\oint\limits_{\Sigma} \{|u^2-\psi| - u^2 - \psi - 2u\min(u,\varepsilon) + \min^2(u,\varepsilon)\}\,d\mathfrak{H}^{n-1} \\ &\geqq \liminf_{m\to\infty} \varepsilon \int\limits_{\{u_m>\varepsilon\}} \sqrt{1+|Du_m|^2}\,dx - \tfrac{1}{2}\oint\limits_{\Sigma} |\min^2(u,\varepsilon) - 2u\min(u,\varepsilon)|\,d\mathfrak{H}^{n-1} \\ &\geqq \liminf_{m\to\infty} \varepsilon \int\limits_{\{u_m>\varepsilon\}} dx - \frac{\varepsilon}{2}\oint\limits_{\Sigma} 2u + \min(u,\varepsilon)\,d\mathfrak{H}^{n-1}. \end{aligned}$$

This gives, after dividing by $\varepsilon$:

$$0 \geqq \int\limits_{\{u>\varepsilon\}} dx - \oint\limits_{\Sigma} u\,d\mathfrak{H}^{n-1} - \tfrac{1}{2}\oint \min(u,\varepsilon)\,d\mathfrak{H}^{n-1}.$$

Now we let $\varepsilon$ tend to zero and obtain

$$\oint\limits_{\Sigma} u\,d\mathfrak{H}^{n-1} \geqq \int\limits_{\{u>0\}} dx = \operatorname{meas}\{u>0\}.$$

Theorem 6 implies that $0 \leqq u \leqq \sqrt{k}$; therefore

$$\sqrt{k}\,\operatorname{meas}\Sigma \geqq \operatorname{meas}\{u>0\}.$$

If $\{u>0\}$ were empty, we would have

$$\operatorname{meas}\{u>0\} = \operatorname{meas}\Omega,$$

hence

$$\sqrt{k}\,\operatorname{meas}\Sigma \geqq \operatorname{meas}\Omega.$$

This contradicts to (28); therefore $\{u>0\}$ must have positive measure, and the proof is complete. □

## § 4. Parametric versions of the variational problem

For Borel sets $U, V \subset Q \equiv \Omega \times [0, \infty)$ we define functinals $\mathscr{F}(U)$ and $\mathscr{G}(V)$ by

$$\begin{aligned}\mathscr{F}(U) &\equiv \int_Q \bar{x}^{n+1} |D\varphi_U| &(31)\\ &\equiv \sup\left\{\int_Q \varphi_U \sum_{i=1}^{n+1} D_i(\bar{x}^{n+1}\gamma_i)\,d\bar{x} : \gamma_i \in C_c^1(Q),\ \sum_{i=1}^{n+1} \gamma_i^2(\bar{x}) \leqq 1\right\}\end{aligned}$$

and

$$\begin{aligned}\mathscr{G}(V) &\equiv \int_Q \sqrt{\tfrac{1}{4}\sum_{i=1}^{n} v_i^2 + \bar{x}^{n+1} v_{n+1}^2}\,|D\varphi_V|\\ &\equiv \sup\left\{\int_Q \varphi_V \left[\tfrac{1}{2}\sum_{i=1}^{n} D_i\gamma_i + D_{n+1}\left(\sqrt{\bar{x}^{n+1}}\,\gamma_{n+1}\right)\right] d\bar{x} : \right. &(32)\\ &\qquad \left. \gamma_i \in C_c^1(Q),\ \sum_{i=1}^{n+1}\gamma_i^2(\bar{x}) \leqq 1 \quad \forall \bar{x} \in Q\right\}.\end{aligned}$$

As usual, $\varphi_E$ denotes the characteristic function of some set $E$; points in $Q \subset \mathbb{R}^{n+1}$ are denoted by $\bar{x} \equiv (\bar{x}^1, \ldots, \bar{x}^{n+1}) \equiv (x, \bar{x}^{n+1})$ with $x \in \mathbb{R}^n$. If $V$ is the subgraph of a function $v \in \mathrm{BV}^+(\Omega) \cap L_\infty(\Omega)$ and if $U$ is the subgraph of $u = \sqrt{v}$, both integrals $\mathscr{F}(U)$ and $\mathscr{G}(V)$ equal $\mathscr{I}(v) = \int_\Omega \sqrt{v + \frac{1}{4}|Dv|^2}$.

**Lemma 8.** *For* $v \in \mathrm{BV}^+(\Omega) \cap L_\infty(\Omega)$ *and* $u = {}_+\sqrt{v}$ *define*

$$U = \{\bar{x} \in Q : 0 \leqq \bar{x}^{n+1} < u(x)\} \quad \textit{and} \quad V = \{\bar{x} \in Q : 0 \leqq \bar{x}^{n+1} < v(x)\}.$$

*Then*

$$\int_\Omega \sqrt{v + \tfrac{1}{4}|Dv|^2} = \int_Q \bar{x}^{n+1}|D\varphi_U|, \tag{33}$$

$$\int_\Omega \sqrt{v + \tfrac{1}{4}|Dv|^2} = \int_Q \sqrt{g_{ij}v_i v_j}\,|D\varphi_V|, \tag{34}$$

*with*

$$\begin{aligned} g_{ij}(\bar{x}) &= \tfrac{1}{4}\delta_{ij} \quad \text{for } i, j = 1, \ldots, n,\\ g_{n+1,n+1} &= \bar{x}^{n+1}, \quad g_{i,n+1} = g_{n+1,i} = 0. \end{aligned} \tag{35}$$

*Remark.* $\int_Q \bar{x}^{n+1}|D\varphi_U|$ may serve as a definition for $\int u\sqrt{1 + |Du|^2}$, cf. [13], [6], [2]. If $\mathscr{I}(u)$ is defined in this way for $u = \sqrt{v}$ we obtain also

$$\int_\Omega u\sqrt{1 + |Du|^2} = \int_\Omega \sqrt{v + \tfrac{1}{4}|Dv|^2}.$$

**Proof.** The proof follows closely the one for the corresponding proposition for the area functional, *cf.* [14], chapter 14. Take $g_i \in C_c^1(\Omega)$ $\forall i = 1, \dots, n+1$, with $\sum_{i=1}^{n+1} g_i^2(x) \leqq 1$ and $\eta \in C_c^1([0, \sup u + 1])$ such that $\eta \equiv 1$ on $[0, \sup u]$ and $|\eta(t)| \leqq 1$ everywhere. Then $\gamma(x, t) = g(x)\eta(t)$ is an admissible test function in (31).

$$\begin{aligned}\int_Q \bar{x}^{n+1} |D\varphi_U| &\geqq \int_U \sum_{i=1}^{n+1} D_i(\bar{x}^{n+1}\gamma_i)\, dx\, d\bar{x}^{n+1} \\ &= \int_\Omega \left( \int_0^{u(x)} [D_{n+1}(\bar{x}^{n+1}\gamma_{n+1}) + \bar{x}^{n+1} \sum_{i=1}^{n} D_i\gamma_i]\, d\bar{x}^{n+1} \right) dx \\ &= \int_\Omega \left( u g_{n+1} + \int_0^{u(x)} \bar{x}^{n+1}\eta(\bar{x}^{n+1})\, d\bar{x}^{n+1} \sum_{i=1}^{n} D_i g_i \right) dx \\ &= \int_\Omega \left( \sqrt{v}\, g_{n+1} + \frac{v}{2} \sum_{i=1}^{n} D_i g_i \right) dx .\end{aligned}$$

If we take the supremum over all $g_i \in C_c^1(\Omega)$ with $\sum_{i=1}^{n+1} g_i^2(x) \leqq 1$ we get from the definition of $\mathscr{J}(v)$ in (16)

$$\int_Q \bar{x}^{n+1} |D\varphi_U| \geqq \int_\Omega \sqrt{v + \tfrac{1}{4} |Dv|^2}.$$

The reverse inequality is obvious for $v \in C^2(\Omega)$, $v \geqq 0$:

$$\begin{aligned}\int_\Omega \sqrt{v + \tfrac{1}{4} |Dv|^2}\, dx &= \int_{\Omega \cap \{v > 0\}} \sqrt{v + \tfrac{1}{4} |Dv|^2}\, dx \\ &= \int_{\Omega \cap \{u > 0\}} u \sqrt{1 + |Du|^2}\, dx \\ &= \int_Q \bar{x}^{n+1} |D\varphi_U|\end{aligned}$$

since $|D\varphi_U|$ coincides with the $n$-dimensional Hausdorff measure on $\partial U$. For $v \in BV^+(\Omega)$ there is a sequence $\{v_j\}_{j=1}^\infty$ of smooth functions such that

$$\int_\Omega \sqrt{v_j + \tfrac{1}{4} |Dv_j|^2} \to \int_\Omega \sqrt{v + \tfrac{1}{4} |Dv|^2},$$

as $j \to \infty$. This implies convergence in $L_{1,\mathrm{loc}}(Q)$ for the characteristic functions of the associated subgraphs $U_j$, and as $\mathscr{F}$ is lower semicontinuous we obtain

$$\begin{aligned}\int_Q \bar{x}^{n+1} |D\varphi_U| &\leqq \liminf_{j \to \infty} \int_Q \bar{x}^{n+1} |D\varphi_{U_j}| \\ &= \lim_{j \to \infty} \int_\Omega \sqrt{v_j + \tfrac{1}{4} |Dv_j|^2} \\ &= \int_\Omega \sqrt{v + \tfrac{1}{4} |Dv|^2}.\end{aligned}$$

To prove (34) we choose as before $g_i \in C_c^1(\Omega)$, $i = 1, \ldots, n+1$ and $\eta \in C_c^1$ $([0, \sup v + 1])$ with $\eta(t) \equiv 1$ on $[0, \sup v]$ and $|\eta(t)| \leqq 1$ otherwise. Then

$$\int_Q \sqrt{g_{ij}\nu_i\nu_j}\,|D\varphi_V| \geqq \int_Q \varphi_V \left[\tfrac{1}{2}\sum_{i=1}^n D_i(g_i\eta) + D_{n+1}\left(\sqrt{\bar{x}^{n+1}}\, g_{n+1}\eta\right)\right] dx\, d\bar{x}^{n+1}$$

$$= \int_\Omega \left\{\sqrt{v}\, g_{n+1} + \tfrac{1}{2} v \sum_{i=1}^n D_i g_i\right\} dx\,,$$

which implies that

$$\mathscr{G}(V) \geqq \int_\Omega \sqrt{v + \tfrac{1}{4}|Dv|^2}\,.$$

The reverse inequality follows again from the lower semicontinuity of $\mathscr{G}$ and the fact that for $v \in C^2(\Omega)$ with $v \geqq 0$

$$\sqrt{v + \tfrac{1}{4}|Dv|^2} = \left\{\tfrac{1}{4}\frac{|Dv|^2}{1+|Dv|^2} + \frac{v}{1+|Dv|^2}\right\}^{\frac{1}{2}} \sqrt{1+|Dv|^2}$$

$$= \sqrt{g_{ij}\nu_i\nu_j}\,|D\varphi_V|$$

because $\nu = (\nu_1, \ldots, \nu_{n+1}) = \left(-\dfrac{Dv}{\sqrt{1+|Dv|^2}}, \dfrac{1}{1+|Dv|^2}\right)$ is the unit normal.

Next we show that for a local minimizer $v$ of $\mathscr{J}(v)$ the subgraph $U$ of $\sqrt{v}$ minimizes $\mathscr{F}(U)$ locally. Moreover the subgraph $V$ of $v$ minimizes $\mathscr{G}(V)$ locally among all subgraphs. This implies in particular that $\sqrt{v}$ minimizes $\int_\Omega u\sqrt{1+|Du|^2}$ locally if the latter integral is defined by $\int_Q \bar{x}^{n+1}|D\varphi_U|$. The proof follows the one given by MIRANDA [18] for the corresponding proposition for the area integral.

**Lemma 9.** *Let $F \subset Q$ be a bounded Borel set, and put*

$$w(x) = \lim_{k\to\infty} \int_0^k \varphi_F(x,t)\,dt\,.$$

*Then $z = w^2$ satisfies the inequality*

$$\int_\Omega \sqrt{z + \tfrac{1}{4}|Dz|^2} \leqq \int_Q \bar{x}^{n+1}|D\varphi_F|\,. \tag{36}$$

**Proof.** We fix $T > 0$ such that $F \subset Q_T \equiv \Omega \times [0, T]$; then $\|w\|_{L_\infty(\Omega)} \leqq T$ and $\|z\|_{L_\infty(\Omega)} \leqq T^2$. As above we choose $g_i \in C_c^1(\Omega)$ with $\sum_{i=1}^{n+1} g_i^2(x) \leqq 1$ on $\Omega$ and $\eta \in C_c^1([0, T+1])$ with $\eta \equiv 1$ on $[0, T]$ and $0 \leqq \eta(t) \leqq 1$ on $[0, T+1]$.

Then by the definition of $\mathscr{F}$ we get

$$\int_Q \bar{x}^{n+1} |D\varphi_F| \geqq \int_Q \varphi_F \left[\sum_{i=1}^{n+1} D_i(\bar{x}^{n+1}\eta(\bar{x}^{n+1})\, g_i(x))\right] dx\, d\bar{x}^{n+1}$$

$$= \int_\Omega \int_0^T \varphi_F(x, \bar{x}^{n+1})\, \bar{x}^{n+1}\eta(\bar{x}^{n+1}) \sum_{i=1}^{n} D_i g_i(x)\, d\bar{x}^{n+1}\, dx$$

$$+ \int_\Omega g_{n+1}(x) \int_0^T \varphi_F(x, x^{n+1}) \frac{d}{d\bar{x}^{n+1}} (x^{n+1}\eta(x^{n+1}))\, d\bar{x}^{n+1}\, dx$$

$$\geqq \int_\Omega \left[\sum_{i=1}^{n} D_i g_i(x)\right] \int_0^{w(x)} \bar{x}^{n+1}\, d\bar{x}^{n+1} + g_{n+1}(x)\, w(x)\, dx$$

$$= \int_\Omega w g_{n+1} + \frac{z}{2} \sum_{i=1}^{n} D_i g_i\, dx;$$

in the last step we have used $\int_0^T \varphi_F(x, \bar{x}^{n+1})\, \bar{x}^{n+1}\, \eta(\bar{x}^{n+1})\, d\bar{x}^{n+1} \geqq \int_0^{w(x)} \bar{x}^{n+1}\, d\bar{x}^{n+1}$.

We get (35) by taking the supremum over all $g_i$ with $\sum_{i=1}^{n+1} g_i^2(x) \leqq 1$. □

**Theorem 10.** *Let $v \in \mathrm{BV}^+(\Omega)$ be a local minimum for $\mathscr{J}(v)$. Then the set $U = \{(x, \bar{x}^{n+1}) \in Q : 0 \leqq \bar{x}^{n+1} < \sqrt{v(x)}\}$ is a local minimum for $\mathscr{F}(E)$ among all measurable subsets $E$ of $Q$.*

*Furthermore, the subgraph $V = \{(x, \bar{x}^{n+1}) \in Q : 0 \leqq \bar{x}^{n+1} < v(x)\}$ is a local minimum for $\mathscr{G}(E)$ among all subsets $E$ of $Q$ that are subgraphs to some $L_1(\Omega)$-function.*

**Proof.** Let $F \subset Q_T$, $T < \infty$, be a Borel set that coincides with $U$ outside some compact set $K \subset A \times [0, \infty)$, $A \subset\subset \Omega$. By the minimum property of $v$ and Lemmas 8 and 9 we get with $w = \int_0^T \varphi_F(x, \bar{x}^{n+1})\, d\bar{x}^{n+1}$

$$\int_{A\times[0,\infty)} \bar{x}^{n+1} |D\varphi_U| = \int_A \sqrt{v + \tfrac{1}{4}|Dv|^2}$$

$$\leqq \int_A \sqrt{w + \tfrac{1}{4}|Dw|^2}$$

$$\leqq \int_{A\times[0,\infty)} \bar{x}^{n+1} |D\varphi_F.$$

To remove the restriction that $F$ be bounded we note first that $T$ will be finite if chosen to be $1 + \inf\{\tau : U \subset Q_\tau\}$ since $v$ is bounded according to Theorem 6. Furthermore if $F \subset Q$ coincides with $U$ outside some closed set $K \subset A \times [0, \infty)$ with $A \subset\subset \Omega$, we set $F_T = F \cap \Omega \times [0, T]$. By definition also $F_T$ coincides with

$U$ outside $K$ and moreover $\mathscr{F}(F_T) \leqq \mathscr{F}(F)$. This inequality follows from

$$\int_{A\times[0,\infty)} \bar{x}^{n+1}\,|D\varphi_{F_T}| = \int_{A\times(0,T)} \bar{x}^{n+1}\,|D\varphi_F| + T\cdot \mathfrak{H}^n(F\cap \Omega\times\{T\})$$
$$\leqq \int_{A\times(0,T)} \bar{x}^{n+1}\,|D\varphi_F| + T \int_{A\times[T,\infty)} |D\varphi_F|$$
$$\leqq \int_{A\times[0,\infty)} \bar{x}^{n+1}\,|D\varphi_F|\,.$$

The second assertion can be proved along the same lines. Lemma 9 is not needed because only subgraphs are taken into account rather than arbitrary measurable sets. □

## § 5. Regularity

The integrals $\mathscr{F}(U)$ and $\mathscr{G}(V)$ we introduced in the preceeding chapter can be interpreted as parametric integrals taken over the currents that are associated to the sets $U$ and $V$, respectively. $\mathscr{F}$ and $\mathscr{G}$ are parametric integrals over the $n$-rectifiable currents $S = \partial [\![U]\!]$ and $T = \partial[\![V]\!]$ where for some Borel set $A \subset \mathbb{R}^{n+1}$ the integral current $[\![A]\!]$ denotes integration over $A$ with the standard orientation and $\partial$ is the boundary operator $\partial[\![A]\!]\,(\omega) := [\![A]\!]\,(d\omega)$; for the background from geometric measure theory we refer to [10], [23]. In the notation of [22] the (singular) integrand $F\colon(\mathbb{R}^n\times\mathbb{R}^+)\times\mathbb{R}^{n+1}\to\mathbb{R}$ is given by

$$F(x, t, p) = t\,|p| \tag{37}$$

for $\mathscr{F}$ and by

$$F(x, t, p) = \left\{\tfrac{1}{4}\sum_{i=1}^{n} p_i^2 + tp_{n+1}^2\right\}^{\frac{1}{2}} \tag{38}$$

for the integral $\mathscr{G}$.

This apparatus allows us to apply regularity results for the minima of such parametric integrals which in turn yield regularity properties for the minima to $\mathscr{J}(v)$ according to Theorem 10. If we exploit the fact that the subgraph of $u$ can be regarded as a minimal surface of codimension 1 in a Riemannian manifold we can apply a much stronger regularity theorem from geometric measure theory which eventually leads to regular minima $u$ for $n < 7$. Let $F$ be the Riemannian manifold $\mathbb{R}^{n+1} \cap \{\bar{x}^{n+1} > 0\}$ endowed with the metric $f_{ij} = (x^{-n+1})^{2/n}\,\delta_{ij}$ $\forall$ $i,j = 1, \ldots, n+1$. Then $\mathscr{F}(U)$ coincides with the area functional $\int_Q |D\psi_U|_F$ on $F$, which is defined by

$$\int_Q |D\psi_U|_F := \sup\left\{\int_Q \psi_U \operatorname{div}_F g(x)\,\sqrt{\operatorname{discr}\{f_{ij}\}}\,\mathrm{d}\mathscr{L}^{n+1}(\bar{x})\,,\right.$$
$$\left. g\in C_c^1(Q;\mathbb{R}^{n+1}),\ |g|_F \leqq 1\right\}.$$

Here $\operatorname{div}_F g$ denotes the divergence, and $|g|_F$ the norm in the Riemannian space $F$. The functional $\mathscr{G}(V)$ does not admit a geometric interpretation of this type.

The regularity theorem from [23] requires $\mathscr{F}(U)$ to be formulated as follows. For $0 < a < b < \infty$ and some open ball $B \subset\subset \Omega$ let $E$ denote the open cylinder $B \times (a, b)$ in $F$. By the result of NASH [19] $E$ can be embedded isometrically into some $\mathbb{R}^{n+k}$ with $k$ large enough; let $I: E \to \mathbb{R}^{n+k}$ denote this isometry. The relation

$$*(g_1, \dots, g_{n+1}) = \sum_{j=1}^{n+1} (-1)^{j-1} g_j d\bar{x}^1 \wedge \dots \wedge d\bar{x}^{j-1} \wedge d\bar{x}^{j+1} \wedge \dots \wedge d\bar{x}^{n+1}$$

defines the canonical isomorphism $*: C_c^\infty(E, \mathbb{R}^{n+1}) \to \mathscr{D}^n(E)$ between vector fields $g$ and differential $n$-forms. For an $\mathscr{L}^{n+1}$-measurable set $U \subset E$ this allows us to consider the $n$-current $S = \partial[\![U]\!] \in \mathscr{D}_n(E)$ of integral multiplicity that is given by

$$\begin{aligned} S(*g) &= \partial[\![U]\!]\,(*g) \\ &= [\![U]\!]\,(d*g) \\ &= \int_E \varphi_U \operatorname{div} g \, dx. \end{aligned} \tag{39}$$

$U$ has locally finite perimeter in $E$ if and only if the mass $\mathbf{M}_W(\partial[\![U]\!])$ is finite for all $W \subset\subset E$. In this case

$$\begin{aligned} \mathbf{M}_W(S) &= \mathbf{M}_W(\partial[\![U]\!]) \\ &= \int_W |D\varphi_U| . \end{aligned} \tag{40}$$

Clearly the orientation $\vec{S}$ of the approximate tangent space is given by $\star\vec{S} = \overrightarrow{\partial[\![U]\!]} = \nu_U$, which holds $|D\varphi_U|$ almost everywhere; $\nu_U$ is the unit normal in the sense of measure theory; it is defined on the reduced boundary $\partial^* U$ and $\star$: $\Lambda_n(\mathbb{R}^{n+1}) \to \mathbb{R}^{n+1}$ is the canonical isometry. Consequently we can write

$$S(\omega) = \partial[\![U]\!]\,(\omega) = \int_{\partial^* U \cap E} \langle \omega(x), \vec{S} \rangle \, d\mathfrak{H}^n(x) \tag{41}$$

for all $\omega \in \mathscr{D}^n(E)$ where $\langle \, , \rangle$ denotes the dual pairing for covectors and vectors.

With $I$ as above the pushed forward current $I_\# S \in \mathscr{D}_n(V)$, $V \subset \mathbb{R}^{n+k}$ an open set with $I(E) \subset V$, is of the form

$$I_\# S(\omega) = \int_{I(\partial^* U)} \langle \omega(y), \tau(y) \rangle \, d\mathfrak{H}^n(y) \tag{42}$$

where $\omega \in \mathscr{D}^n(V)$ and $\tau$ is an orientation for the approximate tangent space $T_y(I(\partial^* U))$; *cf.* [23] § 27.2. Since $I$ is an isometry,

$$\mathbf{M}_W(I_\# S) = \mathfrak{H}^n(I(\partial^* U) \cap W) = \int_{I^{-1}(W \cap I(E))} x^{n+1} \, |D\varphi_U| \tag{43}$$

for $W \subset\subset V$.

If we now take $U$ to be the subgraph of $\sqrt{v}$, where $v$ is a solution of our variational problem (P), we can show that the current $\tilde{T} = I_\# \, \partial [\![ U \cap E ]\!]$ locally minimizes mass. This implies regularity, and as the function $v$ and the current $\tilde{T}$ describe the same geometric object, $v$ is regular, too.

**Lemma 11.** *Let $v$ be a minimum to the variational problem* (P), $U = \{(x, \bar{x}^{n+1}): 0 \leqq \bar{x}^{n+1} < \sqrt{v}\}$, *and let $T$ be the current $\partial [\![ U \cap E ]\!] \llcorner E$, $\tilde{T} = I_\# T$. Then*

$$\mathbf{M}_W(\tilde{T}) \leqq \mathbf{M}_W(R) \tag{44}$$

*for all $W \subset\subset V$, where $V \subset \mathbb{R}^{n+k}$ is an open set with $V \cap N \neq \emptyset$, $N = I(E)$ and $(\bar{N} - N) \cap V = \emptyset$, and for all $R \in \mathscr{D}_n(V)$ that are of integer multiplicity and satisfy $\partial R = \partial \tilde{T}$ in $V$ and* $\operatorname{supp}(R - \tilde{T}) \subset\subset N \cap W$.

**Proof.** The assumptions on $R$ imply

$$\operatorname{supp} R \subset N \cap V$$

and

$$R = \tilde{T} \quad \text{on } (N \cap V) \setminus (N \cap W).$$

Moreover there is a rectifiable current $E \in \mathscr{D}_{n+1}(V)$ of integral multiplicity $n+1$ such that

$$\partial E = R - \tilde{T} \quad \text{and } \operatorname{supp} \partial E \subset\subset N \cap W.$$

Now set

$$F = E + [\![ I(U \cap E) ]\!] \llcorner V.$$

Clearly $F \in \mathscr{D}_{n+1}(V)$ and moreover

$$\begin{aligned} \partial F &= R - \tilde{T} + \partial [\![ I(U \cap E) ]\!] \\ &= R - \tilde{T} + [\![ \partial I(U \cap E) ]\!] \\ &= R - \tilde{T} + I_\# [\![ \partial (U \cap E) ]\!] \\ &= R. \end{aligned}$$

According to the decomposition theorem for top-dimensional currents, *cf.* [23], § 27.8, there are $\mathfrak{H}^{n+1}$-measurable subsets $U_j \subset V \cap N$, $j \in \mathbb{Z}$, such that

$$\partial F = \sum_{j=-\infty}^{+\infty} \partial[\![U_j]\!]$$

and

$$\mu_{\partial F} = \sum_{j=-\infty}^{+\infty} \mu_{\partial[\![U_j]\!]}.$$

Therefore

$$\begin{aligned} \mathbf{M}_W(R) &= \mathbf{M}_W(\partial F) \\ &= \mu_{\partial F}(W) \\ &= \sum_{j=-\infty}^{+\infty} \mu_{\partial[\![U_j]\!]}(W) \\ &\geqq \mu_{\partial[\![U_1]\!]}(W) \\ &= \mathbf{M}_W(\partial[\![U_1]\!]) \\ &= \int_W |D\varphi_{U_1}| \\ &= \int_{I^{-1}(W\cap N)} \bar{x}^{n+1}\, |D\varphi_{\tilde{U}_1}| \end{aligned}$$

where $\tilde{U}_1 = I^{-1}(U_1)$.

As $\varphi_{\tilde{U}_1} = \varphi_U$ in $E - I^{-1}(W \cap N)$ (because $\operatorname{supp}(R - \tilde{T}) \subset\subset N \cap W$) we obtain from the minimal property of $U$

$$\begin{aligned} \int_{I^{-1}(W\cap N)} \bar{x}^{n+1}\, |D\varphi_{\tilde{U}_1}| &\geqq \int_{I^{-1}(W\cap N)} \bar{x}^{n+1}\, |D\varphi_U| \\ &= \int_W |D\varphi_{I(U\cap E)}| \\ &= \mathbf{M}_W(\partial I_\# [\![U \cap E]\!]) \\ &= \mathbf{M}_W(I_\# T) \\ &= \mathbf{M}_W(\tilde{T}). \end{aligned}$$

This completes the proof that $\tilde{T}$ is minimal in $N \cap V$. □

Lemma 11 allows to apply the regularity theorem for minimizing currents of codimension one; *cf.* [23], § 37.7. In this context this theorem reads

**Theorem 12.** *Let $v$ be a solution of problem* (P) *and let $U$ denote the subgraph of ${}_+\sqrt{v}$. Then the locally n-rectifiable integer multiplicity current $T = \partial[\![U]\!] \llcorner \mathring{Q}$ satisfies*

(i) $\operatorname{sing} T = \emptyset$ if $n \leqq 6$.
(ii) $\operatorname{sing} T$ is locally finite in $Q$ if $n = 7$.
(iii) $\mathfrak{H}^{n-7+\alpha}(\operatorname{sing} T) = 0$ $\forall \alpha > 0$ if $n > 7$.

*Remark.* We recall that the singular set sing $T$ is defined as supp $T$ — reg $T$, the set reg $T$ consisting in all points $x$ such that in a neighborhood of $x$ the current $T$ is of the form $[\![M]\!]$ for some $C^1$-submanifold $M \subset \mathbb{R}^{n+k}$.

**Theorem 13.** *Assume* $n \leqq 6$, *and let* $v$ *be a solution to the variational problem* (P). *Then* $v$ *is continuous in* $\Omega$.

**Proof.** Put $u = {}_+\sqrt{v}$ and let $x_0 \in \Omega$ be a point at which $u$ is positive. If $u$ were not continuous at $x_0$, the subgraph $\partial U$ would have to contain a vertical line segment $L$. Because of the analyticity of $\partial U$ this line $L$ must be unbounded. This clearly contradicts the assertion of Theorem 6 that $v$ is bounded. □

The same reasoning shows that $u(x)$ tends to zero for $x \to x_0$ and $u(x_0) = 0$. Hence $u$ and consequently $v$ are continuous in $\Omega$.

**Theorem 14.** *Assume* $n \leqq 6$, *and let* $v$ *be a solution of* (P). *Then* $v$ *and* $u = {}_+\sqrt{v}$ *are analytic in the* (*open*) *set* $\{x \in \Omega : u(x) > 0\}$.

**Proof.** According to theorem 10 the subgraph $U$ of $u$ minimizes $\mathscr{F}(E)$ locally, and $\partial U \cap Q$ is an analytic surface; its outer normal is $\nu = (\nu_1, \dots, \nu_{n+1})$. If $u$ were not analytic, then $\nu$ would become vertical at a point $y_0 = (x_0, u(x_0)) \in \partial U$, i.e. $\nu_{n+1}(y_0) = 0$. By performing a suitable rotation in $\mathbb{R}^n$ we can arrange that

$$\nu_1(y_0) = 1, \nu_2(y_0) = \dots = \nu_n(y_0) = 0.$$

Now let $y^1 = f(\hat{y})$, $\hat{y} = (y^2, \dots, y^{n+1})$ be a representation of $\partial U$ in a neighborhood $N_\varepsilon$ of $\hat{y}_0$; then $f$ is analytic and $Df(y_0^2, \dots, y_0^{n+1}) = 0$. Near $(y_0^2, \dots, y_0^{n+1})$

$$D_{n+1}f(y^2, \dots, y^{n+1}) \geqq 0 \quad \text{or} \quad D_{n+1}f(y^2, \dots, y^{n+1}) \leqq 0.$$

Furthermore $f$ solves the Euler equation for the functional $\int y^{n+1} \sqrt{1 + |Df|^2}\, dy^2 \dots dy^{n+1}$:

$$\sum_{i=2}^{n+1} D_i \left\{ \frac{y^{n+1} D_i F}{\sqrt{1 + |Df|^2}} \right\} = 0.$$

This is equivalent to

$$\sum_{i=2}^{n+1} D_i \frac{D_i f}{\sqrt{1 + |Df|^2}} = - \frac{D_{n+1} F}{y^{n+1}\sqrt{1 + |Df|^2}}. \tag{45}$$

Differentiating (45) with respect to $y^{n+1}$, we obtain a linear equation for $h = D_{n+1} f$:

$$\begin{aligned} \sum_{i,j=2}^{n+1} D_i &\left\{ \frac{(1 + |Df|^2)\, \delta_{ij} - D_i f\, D_j f}{(1 + |Df|^2)^{3/2}} D_j h \right\} \\ &= -D_{n+1} \left\{ \frac{h}{y^{n+1} \sqrt{1 + |Df|^2}} \right\} \qquad (46) \\ &= - \left\{ \frac{D_{n+1} h}{y^{n+1} \sqrt{1 + |Df|^2}} - \frac{h}{(y^{n+1})^2 \sqrt{1 + |Df|^2}} - \frac{h(Df \cdot D(D_{n+1} f))}{y^{n+1} (1 + |Df|^2)^{3/2}} \right\} \\ &= h \left\{ \frac{1 + |Df|^2 + y^{n+1}(Df \cdot D(D_{n+1} f))}{(y^{n+1})^2 (1 + |Df|^2)^{3/2}} \right\} - \frac{D_{n+1} h}{y^{n+1} \sqrt{1 + |Df|^2}}. \end{aligned}$$

If we set $\hat{y} = (y^2, \dots, y^{n+1})$, $b = \dfrac{1}{y^{n+1}\sqrt{1 + |Df|^2}}$,

$$a = -\left\{\frac{1 + |Df|^2 + y^{n+1} Df \cdot D(D_{n+1}f)}{(y^{n+1})^2 (1 + |Df|^2)^{3/2}}\right\},$$

and

$A_i(\hat{y}, p) = \sum_{j=2}^{n+1} \dfrac{(1 + |Df|^2)\,\delta_{ij} - D_i f D_j f}{(1 + |Df|^2)^{3/2}} p_j$, we can write (46) as

$$\int \sum_{i=2}^{n+1} A_i(\hat{y}, Dh)\, D_i\varphi \, d\hat{y} = \int \{ah - b\, D_{n+1}h\}\, \varphi \, d\hat{y}$$

for all $\varphi \in C_c(N_\varepsilon)$.

In view of $y_0^{n+1} > 0$ and $Df(\hat{y}_0) = 0$ there is a neighborhood $N_\delta$ of $\hat{y}_0$ in which $a(\hat{y}) \leqq 0$ holds. The strong maximum principle then implies in either case (i) or (ii) that $h(\hat{y}) = 0 \ \ \forall \hat{y} \in N_\delta$. Evidently this contradicts the continuity of $u$. □

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