

6. Nonlinear problems

J. A. Nitsche
Lecture notes

Problem $F(x, u) = 0$

Aproximations $\varphi \in S_h : (F(\cdot, \varphi), \chi) = 0$ for $\chi \in S_h$

Regularity assumptions (roughly)

- i) There is a unique solution,
- ii) F, F_u are Lischitz continuous.

Error analysis

Put $f(x) := F_u(x, u(x))$ and $e := u - \varphi$, then

$$(fe, \chi) = (R, \chi) \text{ for } \chi \in S_h$$

with a remainder term

$$R = R(e) = F(\cdot, u - \varphi) + fe$$

resp.

$$(f\varphi, \chi) = (fu - R(e), \chi) \text{ for } \chi \in S_h.$$

Let P_h denote the L_2 – projection relative to $(f \cdot, \cdot)$. Then

$$\varphi = P_h(u - \frac{1}{f} R(e)),$$

resp.

$$e = (I - P_h)(u) + P_h(\frac{1}{f} R(e)) =: T(e).$$

This means, that the difference $e := u - \varphi$ is a fixpoint solution of the operator T .

Properties of the operator T

Lemma 1: There is a $\kappa > 0$ such that for sufficiently small

$$\bar{\varepsilon} := \inf_{\chi \in S_h} \|u - \chi\|_{L_\infty}$$

the operator T maps the ball

$$B_{\kappa\bar{\varepsilon}} := \{e \mid \|e\|_{L_\infty} \leq \kappa\bar{\varepsilon}\}$$

into itself.

Proof.

i) Because of P_h being bounded we have

$$\|(I - P_h)u\|_{L_\infty} \leq c_1\bar{\varepsilon}$$

ii) For the same reason ($f^{-1} < \infty$)

$$\left\| P_h \left(\frac{1}{f} R(e) \right) \right\|_{L_\infty} \leq c_2 \|Re\|_{L_\infty}$$

iii) It is

$$\|F(\cdot, u - e) + f \cdot e\|_{L_\infty} \leq c_3 \|e\|_{L_\infty}$$

with c_3 being the Lischitz constant of F_u .

From i)-iii) it follows

$$\|Te\|_{L_\infty} \leq c_1\bar{\varepsilon} + c_3c_2\kappa^2\bar{\varepsilon}^{-2} \leq \bar{\varepsilon}(c_1 + c_2c_3\kappa^2\bar{\varepsilon}) .$$

Now fix $\kappa > c_1$ and choose $\bar{\varepsilon}_0$ according to

$$\kappa = c_1 + c_2c_3\kappa^2\bar{\varepsilon}_0 .$$

Lemma 2: For $\bar{\varepsilon}$ small, the operator T is a contradiction in

$$B_{\kappa\bar{\varepsilon}} := \{e \mid \|e\|_{L_\infty} \leq \kappa\bar{\varepsilon}\}.$$

Proof:

$$\|T(e_1) - T(e_2)\|_{L_\infty} \leq \left\| P_h \left(\frac{1}{f} R(e_1) - R(e_2) \right) \right\|_{L_\infty} \leq c_2 \|R(e_1) - R(e_2)\|_{L_\infty}$$

Now

$$\begin{aligned} R(e_1) - R(e_2) &= F(\cdot, u - e_1) - F(\cdot, u - e_2) + f \cdot (e_1 - e_2) \\ &= (F_u(\cdot, \mathcal{G}) - F_u(\cdot, u))(e_1 - e_2) \end{aligned}$$

with

$$(F_u(\cdot, \mathcal{G}) := F_u(\cdot, u - \mathcal{G}e_1 - (1 - \mathcal{G})e_2)$$

and

$$\|F_u(\cdot, \mathcal{G}) - F_u(\cdot, u)\|_{L_\infty} \leq \kappa\bar{\varepsilon}c_3.$$

From this the assertion of the lemma follows for

$$\bar{\varepsilon} < \min \left\{ \bar{\varepsilon}_0, \frac{1}{c_2 c_3 \kappa} \right\}.$$

Consequence: The operator T has a unique fixpoint in $B_{\kappa\bar{\varepsilon}} := \{e \mid \|e\|_{L_\infty} \leq \kappa\bar{\varepsilon}\}$

Theorem 3: The FEM admits the error estimate

$$\|u - \varphi\|_{L_\infty} \leq c \inf_{\chi \in \mathcal{S}_h} \|u - \chi\|_{L_\infty}$$