6. Nonlinear problems

<u>Probl.</u> $F(x, u) = 0$

<u>Appr.</u> $\varphi \in S_h : (F(\cdot, \varphi), \chi) = 0$ for $\chi \in S_h$

Regularity - Ass. (roughly)

(i) there is a unique sol.,

(ii) F, F_u Lip-cont.

<u>Error Analysis</u>

Put $f(x) = F_u(x, u(x))$

and $e = u - \varphi$

$\varphi = u - e$

Then

$(fe, \chi) = (R, \chi)$

with a remainder term

$R = R(e) = F(\cdot, u - e) + fe$

resp.

$(f\varphi, \chi) = (fu - R(e), \chi).$

Let P_h denote the L_2-projection relative to $(f\cdot, \cdot)$. Then

$$\varphi = P_h(u - f^{-1}R(e))$$

resp.

$$e = (I - P_h)u + P_h\left(\frac{1}{f} R(e)\right) =: T(e)$$

i.e.: The difference

$$e = u - \varphi$$

is a fixpoint of T

Properties of T

Lemma 1: There is a $K > 0$ such that for

$$\bar{\varepsilon} := \inf_{X \in S_h} \|u - X\|_{L_\infty}$$

sufficiently sm~~ooth~~ all then

T maps the ball

$$B_{K\bar{\varepsilon}} := \{e \mid \|e\|_{L_\infty} \le K\bar{\varepsilon}\}$$

into itself.

Proof

(i) because of P_h being bdd. we have

$$\|(I - P_h)u\|_{L_\infty} \le c_1 \bar{\varepsilon}$$

(ii) for the same reason ($f^{-1} < \infty$)

$$\left\|P_h\left(\frac{1}{f} R(e)\right)\right\|_{L_\infty} \le c_2 \|Re\|_{L_\infty}$$

(iii) it is

$$\| F(\cdot, h-e) + f e \|_{L_\infty} \le c_3 \|e\|_{L_\infty}^2$$

(with c_3 being the Lip-constant of F_u).

$$\rightsquigarrow \|Te\|_{L_\infty} \le c_1 \bar{\varepsilon} + c_3 c_2 K^2 \bar{\varepsilon}^2$$

$$\le \bar{\varepsilon} \left(c_1 + c_2 c_3 K^2 \bar{\varepsilon} \right).$$

Now fix $K > c_1$ and choose $\bar{\varepsilon}_0$ according to

$$c_1 + c_2 c_3 K^2 \bar{\varepsilon}_0 = K \quad . \#$$

Lemma 2: For $\bar{\varepsilon}$ small, T is a contraction in $B_{K\bar{\varepsilon}}$.

Proof:

$$\| T(e_1) - T(e_2) \|_{L_\infty}$$

$$= \| P_h \tfrac{1}{f} \left(R(e_1) - R(e_2) \right) \|_{L_\infty}$$

$$\le c_2 \| R(e_1) - R(e_2) \|_{L_\infty} .$$

Now

$$R(e_1) - R(e_2) = F(\cdot, h-e_1) - F(\cdot, h-e_2) + f \cdot (e_1 - e_2)$$

$$= \left(F_u(\cdot, \vartheta) - F_u(\omega) \right) (e_1 - e_2) .$$

with

$$\bar{F}_h(\cdot,\vartheta) = \bar{F}_h(\cdot, u - \vartheta e_1 - (1-\vartheta)e_2)$$

$\rightsquigarrow$ $\| \bar{F}_h(\cdot,\vartheta) - \bar{F}_h(u) \| \leq K\bar{\varepsilon} \cdot C_3$

$\rightsquigarrow$ assertion of the lemma for

$$\bar{\varepsilon} \leq \min\left(\varepsilon_0, \frac{1}{C_2 C_3 K}\right). \#$$

Consequence: T has a unique fix-point in $B_{K\bar{\varepsilon}}$.

Th^m 3: The FEM admits the error estimate

$$\| u - \varphi \|_{L_\infty} \leq C \inf_{\chi \in S_h} \| u - \chi \|_{L_\infty}.$$