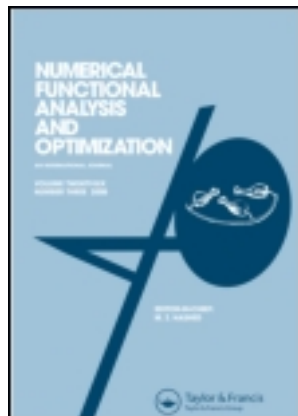


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### $L_\infty$ -boundedness of the finite element galerkin operator for parabolic problems

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$L_\infty$ -BOUNDEDNESS OF THE FINITE ELEMENT GALERKIN OPERATOR  
FOR PARABOLIC PROBLEMS

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ABSTRACT

In this paper the heat equation with Dirichlet boundary conditions in  $N \leq 3$  space dimensions - serving as model problem of second order parabolic initial boundary value problems - is considered. We prove: The standard finite element method is uniformly bounded in  $L_\infty$  with respect to space and time if the underlying finite elements are at least cubics.

0. INTRODUCTION

Let the model problem

$$(0.1) \quad \begin{aligned} \dot{u} - \Delta u &= f & \text{in } \Omega \times (0, T] &, \\ u &= 0 & \text{on } \partial\Omega \times (0, T] &, \\ u_{t=0} &= u_0 & \text{in } \Omega & \end{aligned}$$

be given. With  $S_h^0 \subseteq W_2^1$  being a finite dimensional space - we will consider only finite elements - the

standard Galerkin approximation  $u_h = u_h(t) \in \overset{\circ}{S}_h$  is defined by

$$(0.2) \quad (\dot{u}_h, \chi) + D(u_h, \chi) = (f, \chi) \\ \text{for } \chi \in S_h \text{ and } t \in (0, T]$$

with

$$(0.3) \quad u_h(0) = P_h u_0 .$$

Here  $(\cdot, \cdot)$  is the  $L_2(\Omega)$  - inner product and  $D(\cdot, \cdot)$  the Dirichlet integral.  $P_h$  may be any linear projection onto  $\overset{\circ}{S}_h$ . We will restrict ourselves in the present paper to  $P_h$  being the  $L_2$ -projection defined by - for  $v$  arbitrary:  $\varphi = P_h v \in \overset{\circ}{S}_h$  and

$$(0.4) \quad (\varphi, \chi) = (v, \chi) \quad \text{for } \chi \in \overset{\circ}{S}_h .$$

The result of our paper is

Theorem 1: In  $N = 1, 2, 3$  space dimensions and for finite elements of order 4 or higher, i.e. at least cubics, the mapping  $u \rightarrow u_h$  is bounded in  $L_\infty$  :

$$(0.5) \quad \|u_h\|_{L_\infty(0, T; L_\infty(\Omega))} \leq C \|u\|_{L_\infty(0, T; L_\infty(\Omega))} .$$

A direct consequence is

Corollary 1: Under the above assumptions the almost best error estimate in  $L_\infty$

$$(0.6) \quad \|u - u_h\|_{L_\infty(0, T; L_\infty(\Omega))} \\ \leq (1+C) \inf_{\chi \in S_h} \|u - \chi\|_{L_\infty(0, T; L_\infty(\Omega))}$$

is valid.

Without aiming for completeness we refer to earlier work on maximum norm error estimates:

Bramble-Schatz-Thomée-Wahlbin [ 1 ], Dobrowolski [ 3 ], [ 4 ], Nitsche [ 6 ], Schatz-Thomée-Wahlbin [ 8 ], Thomée [ 9 ], [ 10 ], Wahlbin [ 11 ] and Wheeler [ 12 ]

### 1. NOTATIONS, FINITE ELEMENTS

In the following  $\Omega \subseteq \mathbb{R}^N$  ( $N \leq 3$ ) denotes a bounded domain with boundary  $\partial\Omega$  sufficiently smooth. For any  $\Omega' \subseteq \Omega$  let  $W_p^k(\Omega')$  be the Sobolev space of functions having  $L_p$ -integrable generalized derivatives of order up to  $k$ . The norms are indicated by the corresponding subscripts. Moreover  $\Omega'$  is skipped in case of  $\Omega' = \Omega$ .

The use of weighted norms resp. semi-norms will be essential. They are defined by

$$(1.1) \quad \|\nabla^k v\|_{\alpha, \Omega'} = \left\{ \sum_{|\xi|=k} \iint_{\Omega'} \mu^{-\alpha} |D^\xi v|^2 dx \right\}^{1/2}$$

with  $\mu$  given by

$$(1.2) \quad \mu = \mu(x) = |x - x_0|^2 + \rho^2$$

( $x_0 \in \bar{\Omega}$ ,  $\rho > 0$ ). The boundary semi-norms  $|\nabla^k(\cdot)|_{\alpha, \Omega'}$  are defined in the corresponding way.

By  $\Gamma_h$  a subdivision of  $\Omega$  into generalized simplices is meant, i.e.  $\Delta \in \Gamma_h$  is a simplex if  $\Delta$  intersects  $\partial\Omega$  in at most a finite number of points and otherwise one of the faces of  $\Delta$  may be curved.  $\Gamma_h$  is called  $\kappa$ -regular if to any  $\Delta \in \Gamma_h$  there are two spheres of diameter  $\kappa^{-1}h$  and  $h$  such that  $\Delta$  contains the one and is contained in the other.

The finite element spaces  $S_h = S(\Gamma_h)$  we will work with have the following structure: Let  $m$  be an integer fixed. Any element  $\chi \in S_h$  is continuous in  $\Omega$  and the restriction to  $\Delta \in \Gamma_h$  is a polynomial of degree less than  $m$ . In curved elements we use isoparametric modifications as discussed by Ciarlet-Raviart [2], Zlamal [13].  $S_h$  is the intersection of  $S_h$  and  $W_2^1$ , the closure in  $W_2^1$  of the functions with compact support.

By construction we have  $S_h \subset W_2^1$  but in general  $S_h \not\subset W_2^k$  for  $k \geq 2$ . It will be useful to introduce the spaces  $W_k' = W_k'(\Gamma_h)$  consisting of functions the restriction of which to any  $\Delta \in \Gamma_h$  is in  $W_2^k(\Delta)$ . Obviously  $S_h \subset W_k'$  for all  $k$ .

Parallel to above we introduce 'broken' semi-norms

$$(1.3) \quad \|\nabla^k v\|_{\alpha}' = \left\{ \sum_{\Delta \in \Gamma_h} \|\nabla^k v\|_{\alpha \cdot \Delta}^2 \right\}^{1/2},$$

$$(1.4) \quad |\nabla^k v|_{\alpha}' = \left\{ \sum_{\Delta \in \Gamma_h} |\nabla^k v|_{\alpha \cdot \Delta}^2 \right\}^{1/2}.$$

The (semi)-scalar-products corresponding to (1.3) resp. (1.4) are denoted by  $(\dots)_{\alpha}'$  resp.  $\langle \dots \rangle_{\alpha}'$ . Schwarz' inequality in the form

$$(1.5) \quad |(v, w)_{\alpha}| \leq \|v\|_{\alpha+\beta} \|w\|_{\alpha-\beta}$$

will be used quite often. We remark also the obvious inequality for  $\beta > 0$

$$(1.6) \quad \|v\|_{\alpha+\beta} \leq \rho^{-\beta} \|v\|_{\alpha} .$$

## 2. APPROXIMATION THEORY OF FINITE ELEMENTS IN WEIGHTED NORMS

In this section we put together some direct and inverse theorems of constructive function theory for finite elements in weighted norms. The proofs are given in [ 6 ] resp. [ 7 ]. In the formulation we restrict ourselves to the one which is needed in the subsequent sections.  $c$  will denote a numerical constant which may differ at different places. If it is desirable we use the numbering  $c_1, c_2, \dots$ . Of course the constants will depend on (i) the dimension  $N$ , (ii) the domain  $\Omega$ , (iii) the degree  $m$  of the finite elements, (iv) the regularity parameter  $\kappa$  of the subdivisions  $\Gamma_h$ , (v) the index - we will use in this section the letter  $\beta$  - of the weighted norms.

$N$  is restricted to  $N \leq 3$ ,  $\Omega$  is a domain fixed as well as  $m$  is an integer fixed. We assume the subdivisions  $\Gamma_h$  to be  $\kappa$ -regular with  $\kappa$  fixed. Finally the weight-indices will be in a fixed interval - see the next sections. With respect to these restrictions the  $c$ 's are 'numerical' constants.

Lemma 2.1: Let  $\beta > N/2$ . Then for  $v \in L_{\infty}$

$$(2.1) \quad \|v\|_{\beta}^2 \leq c \rho^{-2\beta+N} \|v\|_{L_{\infty}}^2 .$$

Lemma 2.2: Let  $\beta > N/2$  and  $h \leq \rho$ . Then for  $\chi \in S_h$

$$(2.2) \quad \|\chi\|_{L^\infty}^2 \leq c \rho^{2\beta} h^{-N} \sup_{x_0 \in \Omega} \|\chi\|_\beta^2 .$$

Lemma 2.3: Let  $h \leq \rho$ . To any  $v \in W_1^1$  resp.  $v \in \overset{0}{W}_2^1 \cap W_1^1$  there is a  $\chi \in S_h$  resp.  $\chi \in \overset{0}{S}_h$  according to

$$(2.3) \quad \|\nabla^k(v-\chi)\|'_\beta \leq c h^{1-k} \|\nabla^1 v\|'_\beta \text{ for } 0 \leq k < 1 \leq m .$$

Corollary 2.3: The approximating  $\chi \in S_h$  also fulfills

$$(2.4) \quad |\nabla^k(v-\chi)|'_\beta \leq c h^{1-k-1/2} \|\nabla^1 v\|'_\beta .$$

Lemma 2.4: For  $\chi \in S_h$  and  $0 \leq k \leq 1 < m$  the inverse relations hold true

$$(2.5) \quad \|\nabla^1 \chi\|'_\beta \leq c h^{-(1-k)} \|\nabla^k \chi\|'_\beta .$$

Corollary 2.4: In addition to (2.5) also the inequalities are valid

$$(2.6) \quad |\nabla^1 \chi|'_\beta \leq c h^{-(1-k+1/2)} \|\nabla^k \chi\|'_\beta .$$

Lemma 2.5: Let  $\varphi \in \overset{0}{S}_h$  be given. The function  $\mu^{-\beta} \varphi$  can be approximated by an element  $\chi \in \overset{0}{S}_h$  according to

$$\begin{aligned}
 (2.7) \quad & \|\nabla(\mu^{-\beta}\varphi-\chi)\|_{-\beta} + h^{1/2} \|\nabla(\mu^{-\beta}\varphi-\chi)\|'_{-\beta} + \\
 & + h\|\nabla^2(\mu^{-\beta}\varphi-\chi)\|'_{-\beta} \leq \\
 & \leq c \frac{h}{\rho} \left\{ \|\varphi\|_{\beta+1} + \|\nabla\varphi\|_{\beta} \right\} .
 \end{aligned}$$

Finally we recall (Corollary 1 in [6]):

Lemma 2.6: There is a constant  $\underline{c}$  such that for  $\underline{c}h \leq \rho$  the  $L_2$ -projection admits:

$$\begin{aligned}
 (2.8) \quad & \|v - P_h v\|_{\beta} + h\|\nabla(v - P_h v)\|_{\beta} \leq \\
 & \leq c \inf \left\{ \|v - \chi\|_{\beta} + h\|\nabla(v - \chi)\|_{\beta} \mid \chi \in \overset{\circ}{S}_h \right\} .
 \end{aligned}$$

### 3. SHIFT-THEOREMS

In Section 5 we will need a certain shift theorem in weighted norms. In order not to interrupt the context there this is discussed in the present section.

We introduce the Hilbert-scale  $\{H_k \mid k \geq 0\}$  in the following way: Let  $\{v_i, \lambda_i\}$  be the orthonormal set of eigen-pairs of the Laplacian, i.e.

$$\begin{aligned}
 (3.1) \quad & -\Delta v_i = \lambda_i v_i \quad \text{in } \Omega , \\
 & v_i = 0 \quad \text{on } \partial\Omega .
 \end{aligned}$$

Any  $z \in L_2(\Omega)$  admits the representation

$$(3.2) \quad z = \sum z_i v_i$$

with

$$(3.3) \quad z_i = (z, v_i) .$$



In addition Parseval's equation holds:

$$(3.4) \quad \|z\|^2 = \sum z_1^2 .$$

Now  $H_k$  is the subspace of functions such that

$$(3.5) \quad \|z\|_k^2 = \sum \lambda_1^k z_1^2$$

is finite.

Remark: Since we have accepted only  $z \in L_2$  the index  $k$  has to be non-negative.

For integer  $k \leq 4$  - only these values will be relevant - the spaces  $H_k$  are connected with the usual Sobolev-spaces  $W_2^m$  by:

$$(3.6) \quad \begin{aligned} H_0 &= L_2, \quad H_1 = \overset{0}{W}_2^1, \quad H_2 = \overset{0}{W}_2^1 \cap W_2^2, \\ H_3 &= \{z \mid z \in H_2 \wedge \Delta z \in H_1\}, \\ H_4 &= \{z \mid z \in H_2 \wedge \Delta z \in H_2\}. \end{aligned}$$

The  $H_k$ -norms are equivalent in these spaces to the corresponding  $W_2^k$ -norms.

If  $z = z(t)$  is an element of  $H_k$  for almost every  $0 < t < T$  and the  $H_k$ -norm is  $L_2$ -integrable with respect to  $t$  we will use the notation

$$(3.7) \quad \|z\|_{L_2(H_k)}^2 = \|z\|_{L_2(0,T;H_k)}^2 = \int_0^T \|z(t)\|_{H_k}^2 dt .$$

For the sake of completeness we will give the proof of the standard shift-theorem:

**Theorem 3.1:** Let the operator  $A$  be defined by

$$(3.8) \quad \begin{aligned} Az &= \dot{z} - \Delta z && \text{in } \Omega \times (0, T) , \\ z &= 0 && \text{on } \partial\Omega \times (0, T) , \\ z_{t=0} &= 0 && \text{in } \Omega . \end{aligned}$$

Then  $A$  is a bijective mapping of  $L_2(H_{k+2}) \cap \{z \mid \dot{z} \in L_2(H_k)\}$  to  $L_2(H_k)$  and

$$(3.9) \quad \|z\|_{L_2(H_{k+2})} \leq \|Az\|_{L_2(H_k)} .$$

**Proof:** Let  $z_i$  resp.  $f_i$  denote the 'Fourier'-coefficients of  $z$  resp.  $Az$  with respect to  $\{v_i\}$ . Multiplication of (3.8.1) with  $v_i$  and integration over  $\Omega$  leads to the uncoupled first order system

$$(3.10) \quad \begin{aligned} \dot{z}_i + \lambda_i z_i &= f_i && \text{for } 0 < t < T , \\ z_i(0) &= 0 \end{aligned}$$

the solution of which is

$$(3.11) \quad z_i(t) = \int_0^t e^{-\lambda_i(t-s)} f_i(s) ds .$$

Application of Schwarz' inequality in the proper way gives

$$(3.12) \quad \begin{aligned} |z_i|^2 &\leq \left\{ \int_0^t e^{-\lambda_i(t-s)} f_i^2 ds \right\} \left\{ \int_0^t e^{-\lambda_i(t-s)} ds \right\} \\ &\leq \lambda_i^{-1} \int_0^t e^{-\lambda_i(t-s)} f_i^2 ds \end{aligned}$$

and further by interchanging the order of integration

$$\int_0^T z_1^2 dt \leq \lambda_1^{-1} \int_0^T f_1^2(s) ds \int_s^T e^{-\lambda_1(t-s)} dt$$

(3.13)

$$\leq \lambda_1^{-2} \int_0^T f_1^2 ds .$$

Because of the definition of the  $H_k$  - resp.  $L_2(H_k)$ -norms (3.9) is proven.

In Section 2 we discussed weights depending only on  $x \in \Omega$ . It will be advantageous to work also with time-dependent weights (for a different approach see Dobrowolski [5]): Let  $t_0 > 0$  be fixed. For  $0 < t \leq t_0$  we define - see (1.2)

$$(3.14) \quad \mu = \mu(x, t) = |x - x_0|^2 + \rho^2 + t_0 - t .$$

Since there will be no confusion we use the same letter as in Section 2. Obviously all the lemmata and corollaries of Section 2 remain valid with the new weight  $\mu$  except Lemma 2.2. For  $t < t_0$  then  $\rho^2$  in (2.2) has to be replaced by  $\rho^2 + t_0 - t$ . But we will apply this lemma only for  $t = t_0$ .

In order to shorten the notations we will write

$$(3.15) \quad \|z\|_{\beta}^2 = \int_0^{t_0} \|z\|_{\beta}^2 dt$$

with  $t_0 > 0$  fixed. The constants in the estimates will not depend on  $t_0$ .

The counterpart of Theorem 3.1 is

Theorem 3.2: Let  $A$  be defined by (3.8) and  $\beta \in \mathbb{R}$ . Then for  $k \geq 2$  and  $z \in L_2(H_k) \cap \{v \mid \dot{v} \in L_2(H_{k-2})\}$

$$(3.16) \quad \begin{aligned} \|\nabla^k z\|_\beta &\leq c \left\{ \|\nabla^{k-2} Az\|_\beta + \|\nabla^{k-3} Az\|_{\beta+1} + \right. \\ &\quad \left. \dots + \|Az\|_{\beta+k-2} + \|\nabla z\|_{\beta+k-1} + \|z\|_{\beta+k} \right\} . \end{aligned}$$

The proof is straight-forward by induction. We will give the details only for  $k = 2, 3$ .

By comparing - for  $t$  fixed - the  $\beta$ -norm of a second derivative of  $z$  with the  $L_2$ -norm of the corresponding derivative of  $\mu^{-\beta/2} z$  we get

$$(3.17) \quad \|\nabla^2 z\|_\beta^2 \leq 2 \|\nabla^2 (\mu^{-\beta/2} z)\|_{L_2}^2 + c \left\{ \|\nabla z\|_{\beta+1}^2 + \|z\|_{\beta+2}^2 \right\}$$

leading to

$$(3.18) \quad \|\nabla^2 z\|_\beta^2 \leq c \left\{ \|\mu^{-\beta/2} z\|_{L_2(H_2)}^2 + \|\nabla z\|_{\beta+1}^2 + \|z\|_{\beta+2}^2 \right\} .$$

By Theorem 3.1 we have

$$(3.19) \quad \|\mu^{-\beta/2} z\|_{L_2(H_2)} \leq \|A(\mu^{-\beta/2} z)\|_{L_2(H_0)} .$$

Differentiation gives

$$(3.20) \quad \begin{aligned} A(\mu^{-\beta/2} z) &= \mu^{-\beta/2} \ddot{z} + \beta/2 \mu^{-\beta/2-1} \dot{z} - \mu^{-\beta/2} \Delta z - \\ &\quad - \nabla(\mu^{-\beta/2}) \nabla z - (\Delta \mu^{-\beta/2}) z \\ &\leq \mu^{-\beta/2} Az + c \left\{ \mu^{-\beta/2-1/2} |\nabla z| + \right. \\ &\quad \left. + \mu^{-\beta/2-1} |z| \right\} . \end{aligned}$$

In this way we get

$$(3.21) \quad \|A(\mu^{-\beta/2}z)\|_{L_2(H_0)} \leq c \left\{ \|Az\|_{\beta} + \|\nabla z\|_{\beta+1} + \|z\|_{\beta+2} \right\}.$$

Thus (3.16) is proven for  $k = 2$ .

Instead of (3.18) we get in case of  $k = 3$

$$(3.22) \quad \|\nabla^3 z\|_{\beta} \leq c \left\{ \|\mu^{-\beta/2}z\|_{L_2(H_3)} + \|\nabla^2 z\|_{\beta+1} + \|\nabla z\|_{\beta+2} + \|z\|_{\beta+3} \right\}.$$

Application of Theorem 3.1 to the first term on the right hand side leads similar to above to

$$(3.23) \quad \|\nabla^3 z\|_{\beta} \leq c \left\{ \|\nabla Az\|_{\beta} + \|Az\|_{\beta+1} + \|\nabla^2 z\|_{\beta+1} + \|\nabla z\|_{\beta+2} + \|z\|_{\beta+3} \right\}.$$

The third term on the right hand side is covered by the others because of Theorem 3.2 for  $k = 2$ . In this way the theorem is proven for  $k = 3$ .

The term with the first derivatives of  $z$  in (3.16) is always covered by the other terms because of

Lemma 3.3: Let  $A$  be defined by (3.8) and  $\gamma \neq 0$ .

Then

$$(3.24) \quad \|\nabla z\|_{\gamma} \leq c \left\{ \|Az\|_{\gamma-1} + \|z\|_{\gamma+1} \right\}.$$

Proof: It is

$$(3.25) \quad \begin{aligned} (Az, z)_{\gamma} &= (z, \dot{z})_{\gamma} + D(z, \mu^{-\gamma}z) \\ &= \partial_t \left( \frac{1}{2} \|z\|_{\gamma}^2 \right) - \frac{\gamma}{2} \|z\|_{\gamma+1}^2 + \|\nabla z\|_{\gamma} - \\ &\quad - \frac{1}{2} \iint z^2 \Delta \mu^{-\gamma} \quad . \end{aligned}$$

Integration with respect to  $t$  gives

$$(3.26) \quad \frac{1}{2} \|z(t_0)\|_Y^2 + \|vz\|_Y^2 \leq \int_0^{t_0} (Az, z)_Y dt + c \|z\|_{Y+1}^2 \\ \leq c \left\{ \|Az\|_{Y-1}^2 + \|z\|_{Y+1}^2 \right\} .$$

#### 4. A PRIORI ESTIMATES FOR THE GALERKIN SOLUTION IN WEIGHTED NORMS

It will be advantageous to derive the estimates for the difference

$$(4.1) \quad \phi = u_h - P_h u$$

of the Galerkin solution  $u_h$  and the  $L_2$ -projection of  $u$  - see (0.4). We mention the initial condition

$$(4.2) \quad \phi_{t=0} = 0$$

because of (0.3). The right hand side of (0.2) can be rewritten in the form

$$(4.3) \quad (f, \chi) = (\dot{u}, \chi) + D(u, \chi) \\ = ((P_h u)^\cdot, \chi) + D(P_h u, \chi) + D(u - P_h u, \chi) .$$

Thus we get the defining relation for  $\phi$

$$(4.4) \quad (\dot{\phi}, \chi) + D(\phi, \chi) = D(\varepsilon, \chi) \quad \text{for } \chi \in \overset{0}{S}_h \quad \text{and} \\ t \in (0, T]$$

with the abbreviation

$$(4.5) \quad \varepsilon = u - P_h u .$$

Our aim is to derive a series of inequalities for  $\dot{\phi}$  resp.  $\nabla\dot{\phi}$  in weighted norms. The weight function  $\mu$  is the one defined by (3.14) with some  $t_0$  fixed.  $t$  will be in the range  $0 < t < t_0$ .

In view of Lemma 2.2 we try to find a bound for  $\|\dot{\phi}\|_{\alpha}^2$  at  $t = t_0$ . In order to do this we consider

$$(4.6) \quad (\|\dot{\phi}\|_{\alpha}^2)' = 2(\dot{\phi}, \mu^{-\alpha}\dot{\phi}) + \alpha\|\dot{\phi}\|_{\alpha+1}^2 \quad ..$$

Let  $\chi = P_h(\mu^{-\alpha}\dot{\phi})$  be the  $L_2$ -projection of  $\mu^{-\alpha}\dot{\phi}$ . By the aid of (4.4) we get

$$(4.7) \quad \begin{aligned} (\dot{\phi}, \mu^{-\alpha}\dot{\phi}) &= (\dot{\phi}, \chi) \\ &= -D(\dot{\phi}, \chi) + D(\varepsilon, \chi) \\ &= -D(\dot{\phi}, \mu^{-\alpha}\dot{\phi}) + D(\varepsilon, \mu^{-\alpha}\dot{\phi}) + \\ &\quad + D(\dot{\phi}, \mu^{-\alpha}\dot{\phi} - \chi) - D(\varepsilon, \mu^{-\alpha}\dot{\phi} - \chi) \\ &=: J_1 + J_2 + J_3 + J_4 \quad . \end{aligned}$$

We will analyze the four terms separately. By partial integration we get

$$(4.8) \quad \begin{aligned} J_1 &= -\|\nabla\dot{\phi}\|_{\alpha}^2 - \iint \dot{\phi} \nabla\dot{\phi} \nabla\mu^{-\alpha} \\ &= -\|\nabla\dot{\phi}\|_{\alpha}^2 + \frac{1}{2} \iint \dot{\phi}^2 \Delta\mu^{-\alpha} \\ &\leq -\|\nabla\dot{\phi}\|_{\alpha}^2 + c\|\dot{\phi}\|_{\alpha+1}^2 \quad . \end{aligned}$$

Since we want to avoid derivatives of  $u$  resp.  $\varepsilon$  we have to use partial integration for  $J_2$ :

$$(4.9) \quad \begin{aligned} J_2 &= \sum_{\Delta \in \Gamma_h} \left\{ \langle \varepsilon, (\mu^{-\alpha}\dot{\phi})_n \rangle_{\Delta} - (\varepsilon, \Delta(\mu^{-\alpha}\dot{\phi}))_{\Delta} \right\} \\ &=: J_{2.1} + J_{2.2} \quad . \end{aligned}$$

The counterpart of (1.5) leads to

$$(4.10) \quad |J_{2.1}| \leq \sum_{\Delta} |\varepsilon|_{\alpha, \Delta} |(\mu^{-\alpha_{\Phi}})_n|_{-\alpha, \Delta} .$$

Because of

$$(4.11) \quad \mu^{\alpha} (\mu^{-\alpha_{\Phi}})_n^2 \leq c \{ \mu^{-\alpha_{\Phi}}_n^2 + \mu^{-\alpha-1}_{\Phi}^2 \}$$

we get by the aid of Schwarz' inequality

$$(4.12) \quad |J_{2.1}| \leq c |\varepsilon|_{\alpha} \{ |\Phi|_{\alpha+1} + |\nabla \Phi|_{\alpha} \} .$$

Finally using the inverse properties (Lemma and Corollary 2.4) we get with  $0 < \delta < 1$

$$(4.13) \quad |J_{2.1}| \leq \delta \{ \|\Phi\|_{\alpha+1}^2 + \|\nabla \Phi\|_{\alpha}^2 \} + c \delta^{-1} h^{-1} |\varepsilon|_{\alpha}^2 .$$

In the same way we come to

$$(4.14) \quad \begin{aligned} |J_{2.2}| &\leq \|\varepsilon\|_{\alpha} \|\Delta(\mu^{-\alpha_{\Phi}})\|_{-\alpha} \\ &\leq c \|\varepsilon\|_{\alpha} \{ \|\Phi\|_{\alpha+2} + |\nabla \Phi|_{\alpha+1} + \|\nabla^2 \Phi\|_{\alpha} \} . \end{aligned}$$

Once more the inverse relations give rise to

$$(4.15) \quad |J_{2.2}| \leq \delta \{ \|\Phi\|_{\alpha+1}^2 + \|\nabla \Phi\|_{\alpha}^2 \} + c \delta^{-1} h^{-2} \|\varepsilon\|_{\alpha}^2 .$$

In view of Lemma 2.6 the super-approximability expressed in Lemma 2.5 is valid also with  $\chi = P_h(\mu^{-\alpha_{\Phi}})$ . This gives

$$(4.16) \quad \begin{aligned} |J_3| &\leq \|\nabla \Phi\|_{\alpha} \|\nabla(\mu^{-\alpha_{\Phi}} - \chi)\|_{-\alpha} \\ &\leq c \frac{h}{p} \|\nabla \Phi\|_{\alpha} \{ \|\Phi\|_{\alpha+1} + \|\nabla \Phi\|_{\alpha} \} \\ &\leq c \frac{h}{p} \{ \|\Phi\|_{\alpha+1}^2 + \|\nabla \Phi\|_{\alpha}^2 \} . \end{aligned}$$



The remaining term  $J_4$  has to be treated like  $J_2$ .  
The result is

$$(4.17) \quad |J_4| \leq \frac{h}{\rho} \left\{ \|\phi\|_{\alpha+1}^2 + \|\nabla \phi\|_{\alpha}^2 \right\} + c \left\{ h^{-1} |\epsilon|_{\alpha}^2 + h^{-2} \|\epsilon\|_{\alpha}^2 \right\}.$$

Summarizing we come from (4.6) to

$$(4.18) \quad (\|\phi\|_{\alpha}^2)^{\cdot} + 2\|\nabla \phi\|_{\alpha}^2 \leq c_1(\delta + h/\rho) \|\nabla \phi\|_{\alpha}^2 + c_2 \|\phi\|_{\alpha+1}^2 + c \delta^{-1} \left\{ h^{-1} |\epsilon|_{\alpha}^2 + h^{-2} \|\epsilon\|_{\alpha}^2 \right\}.$$

Remark: The two numbered constants will be of special relevance.

Now we choose  $\delta = 1/(2c_1)$  and have

Step 1: Let  $\rho \geq 2c_1 h$ . Then

$$(4.19) \quad (\|\phi\|_{\alpha}^2)^{\cdot} + \|\nabla \phi\|_{\alpha}^2 \leq c_2 \|\phi\|_{\alpha+1}^2 + c \left\{ h^{-1} |\epsilon|_{\alpha}^2 + h^{-2} \|\epsilon\|_{\alpha}^2 \right\}.$$

In the next step we treat the term  $\|\phi\|_{\alpha+1}^2$  on the right hand side. Similar to the  $L_{\infty}$ -analysis for elliptic problems we apply standard duality arguments (see Dobrowolski [ ]): In view of (4.6) we are actually interested in the term  $\|\phi\|_{\alpha+1}^2$ . In order to treat it we would like to introduce an auxiliary function  $w$  by

$$(4.20) \quad A^* w = \mu^{-\alpha-1} \phi$$

with  $A^*$  the adjoint of  $A$  (3.8). Then we would get

$$\begin{aligned}
 (4.21) \quad \|\phi\|_{\alpha+1}^2 &= \int_0^t (\phi, A^* w) dt \\
 &= \int_0^t (A\phi, w) dt
 \end{aligned}$$

which could be handled by using the defining relation (4.4) for  $\phi$ . For technical reasons we need  $w \in L_2(H_4)$  but the shift theorem gives because of our assumption  $S_h^0 \subseteq H_1$  only  $w \in L_2(H_3)$ . Therefore an additional smoothing is necessary. Postponing the proof to Section 5 we will use

Lemma 4.1: Let  $\theta$  be in the range  $|\beta| \leq 2N$ . For  $\theta > 0$  given to any  $\phi \in S_h^0$  there are approximations  $\tilde{\phi} = \phi_\theta$  according to:  $\tilde{\phi} \in H_2$  and

$$\begin{aligned}
 (4.22) \quad \|\phi - \tilde{\phi}\|_\beta &\leq \theta \|\phi\|_\beta, \\
 \|\nabla(\phi - \tilde{\phi})\|_\beta &\leq \theta \|\nabla\phi\|_\beta, \\
 \|\nabla^2\phi\|_\beta &\leq c(h\theta)^{-1} \|\nabla\phi\|_\beta.
 \end{aligned}$$

In our further analysis any  $\theta < 1$  will be sufficient. We fix  $\theta = 1/2$  and write  $\tilde{\phi} = \phi_{1/2}$ .

In accordance to (4.20) we define  $w$  by

$$\begin{aligned}
 (4.23) \quad -\dot{w} - \Delta w &= \mu^{-\alpha-1} \tilde{\phi} \quad \text{in } \Omega \times (0, t_0), \\
 w &= 0 \quad \text{on } \partial\Omega \times (0, t_0), \\
 w_{t=t_0} &= 0 \quad \text{in } \Omega.
 \end{aligned}$$

Then we have

$$\begin{aligned}
 (4.24) \quad \|\dot{\Phi}\|_{\alpha+1}^2 &= (\dot{\Phi}, \dot{\Phi} - \tilde{\Phi})_{\alpha+1} + (\dot{\Phi}, \tilde{\Phi})_{\alpha+1} \\
 &\leq \|\dot{\Phi}\|_{\alpha+1} \|\dot{\Phi} - \tilde{\Phi}\|_{\alpha+1} + (\dot{\Phi}, \tilde{\Phi})_{\alpha+1}
 \end{aligned}$$

and because of the choice of  $\vartheta$

$$(4.25) \quad \|\dot{\Phi}\|_{\alpha+1}^2 \leq 2(\dot{\Phi}, \tilde{\Phi})_{\alpha+1}$$

and further

$$\begin{aligned}
 (4.26) \quad \frac{1}{2} \|\dot{\Phi}\|_{\alpha+1}^2 &\leq -(\dot{w}, \dot{\Phi}) + D(w, \dot{\Phi}) \\
 &\leq -(w, \dot{\Phi})' + (\dot{\Phi}, w) + D(\dot{\Phi}, w) \quad .
 \end{aligned}$$

Because of the defining relation (4.4) we get for  $\chi \in S_h$  arbitrary

$$\begin{aligned}
 (4.27) \quad \frac{1}{2} \|\dot{\Phi}\|_{\alpha+1}^2 &\leq -(\dot{\Phi}, w) + (\dot{\Phi}, w - \chi) + D(\dot{\Phi}, w - \chi) + \\
 &\quad + D(\varepsilon, w) - D(\varepsilon, w - \chi) \\
 &=: J_5 + J_6 + J_7 + J_8 + J_9 \quad .
 \end{aligned}$$

At the end we will integrate with respect to  $t$  from 0 to  $t_0$ . Because of (4.2) and (4.23<sub>3</sub>) we have

$$(4.28) \quad \int_0^{t_0} J_5 \, dt = 0 \quad .$$

Next we choose  $\chi = P_h w$ . Then

$$(4.29) \quad J_6 = 0$$

is the consequence. In the standard way - remembering that the degree  $m$  of the spaces  $S_h^0$  is at least 4 - we get

$$\begin{aligned}
 |J_7| &\leq \|\nabla \phi\|_{\alpha} \|\nabla(w-\chi)\|_{-\alpha} \\
 (4.30) \quad &\leq \delta \|\nabla \phi\|_{\alpha}^2 + c \delta^{-1} h^6 \|\nabla^4 w\|_{-\alpha} .
 \end{aligned}$$

In order to get a bound for  $J_8$  we apply partial integration:

$$(4.31) \quad J_8 = (\varepsilon, -\Delta w) .$$

$\varepsilon = u - P_h u$  (4.5) is orthogonal to  $S_h^0$ . Therefore with  $\psi \in S_h^0$  arbitrary

$$(4.32) \quad J_8 = (\varepsilon, -\Delta w + \psi) .$$

For  $t$  fixed  $\Delta w$  is an element of  $W_2^1$  and thus can be approximated according to Lemma 2.3 giving with  $\psi$  chosen appropriately

$$\begin{aligned}
 |J_8| &\leq c \|\varepsilon\|_{\alpha} h^2 \|\nabla^2(\Delta w)\|_{-\alpha} \\
 (4.33) \quad &\leq c h^{-2} \|\varepsilon\|_{\alpha}^2 + h^6 \|\nabla^4 w\|_{-\alpha}^2 .
 \end{aligned}$$

The treatment of  $J_9$  is parallel to that of  $J_4$  and not repeated in detail. We come to

$$(4.34) \quad |J_9| \leq c \left\{ h^{-1} \|\varepsilon\|_{\alpha}^2 + h^{-2} \|\varepsilon\|_{\alpha}^2 \right\} + h^6 \|\nabla^4 w\|_{-\alpha}^2 .$$

Summarizing we have derived

$$\begin{aligned}
 \|\phi\|_{\alpha+1}^2 &\leq 2\delta \|\nabla \phi\|_{\alpha}^2 + \\
 (4.35) \quad &c_{\delta} \left\{ h^{-1} \|\varepsilon\|_{\alpha}^2 + h^{-2} \|\varepsilon\|_{\alpha}^2 + h^6 \|\nabla^4 w\|_{-\alpha}^2 \right\} .
 \end{aligned}$$

By comparison of (4.35) with (4.19) we choose  $\delta = 1/(4c_2)$  and get

Step 2: Let  $\rho \geq 2c_1 h$ . Then

$$(4.36) \quad \begin{aligned} & (\|\phi\|_{\alpha}^2)^{\cdot} + \|\nabla\phi\|_{\alpha}^2 + \|\phi\|_{\alpha+1}^2 \\ & \leq c \left\{ -(w, \phi)^{\cdot} + h^{-1} |\epsilon|_{\alpha}^2 + h^{-2} \|\epsilon\|_{\alpha}^2 + h^6 \|\nabla^4 w\|_{-\alpha}^2 \right\}. \end{aligned}$$

In the next section we will give the proof of

Lemma 4.2: Let  $\alpha$  be in the range  $0 < \alpha < 3$ . Then

$$(4.37) \quad \|\nabla^4 w\|_{-\alpha} \leq c h^{-1} \rho^{-2} \left\{ \|\phi\|_{\alpha+1} + \|\nabla\phi\|_{\alpha} \right\}.$$

By the aid of this lemma we get finally

Step 3: For  $\rho = \gamma h$  with  $\gamma$  chosen appropriately

$$(4.38) \quad \begin{aligned} & \|\phi(t_0)\|_{\alpha}^2 + \|\nabla\phi\|_{\alpha}^2 + \|\phi\|_{\alpha+1}^2 \\ & \leq c \int_0^{t_0} \left\{ h^{-1} |\epsilon|_{\alpha}^2 + h^{-2} \|\epsilon\|_{\alpha}^2 \right\} dt. \end{aligned}$$

Now let us assume

$$(4.39) \quad \alpha > N/2 + 1.$$

Then we get

$$\begin{aligned}
(4.40) \quad ||| \epsilon |||_{\alpha}^2 &= \int_0^{t_0} dt \iint_{\Omega} \mu^{-\alpha} \epsilon^2 dx \\
&\leq \| \epsilon \|_{L_{\infty}(L_{\infty})}^2 \int_0^{t_0} dt \iint_{\mathbb{R}^N} \frac{r^{N-1} dr d\theta}{(r^2 + \rho^2 + t_0 - t)^{\alpha}} \\
&\leq c \| \epsilon \|_{L_{\infty}(L_{\infty})}^2 \int_0^{t_0} \frac{dt}{(\rho^2 + t_0 - t)^{\alpha - N/2}} \\
&\leq c \rho^{N+2-2\alpha} \| \epsilon \|_{L_{\infty}(L_{\infty})}^2 .
\end{aligned}$$

The estimate of the first term on the right hand side of (4.38) follows the same lines giving the common bound

$$(4.41) \quad c h^{-2} \rho^{N+2-2\alpha} \| \epsilon \|_{L_{\infty}(L_{\infty})}^2 .$$

It remains to apply Lemma 2.2. Since we have coupled  $\rho$  and  $h$  by  $\rho = \gamma h$  we finally have

$$(4.42) \quad \| \hat{\epsilon}(t_0) \|_{L_{\infty}(\Omega)} \leq c \| \epsilon \|_{L_{\infty}(0, t_0; L_{\infty}(\Omega))} .$$

$P_h u$  is bounded in  $L_{\infty}$  by  $u$  and thus also  $\epsilon = u - P_h u$ . Therefore the main theorem is proven.

## 5. PROOF OF LEMMATA 4.1 AND 4.2

The smoothing process we will work with is the standard one (see Gilbarg-Trudinger [5], p. 140). Let  $w(x)$  be defined by

$$(5.1) \quad \omega(x) = \begin{cases} ce^{-1/(1-|x|^2)} & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases},$$

with  $c$  according to

$$(5.2) \quad \iint \omega(x) \, dx = 1.$$

To any  $v \in L_1(\mathbb{R}^N)$  we define

$$(5.3) \quad v_k(x) = J_k v(x) = k^{-N} \iint \omega\left(\frac{x-y}{k}\right) v(y) \, dy.$$

Obviously we have  $v_k \in C^\infty(\mathbb{R}^N)$  and for  $v \in W_\infty^1(\mathbb{R}^N)$

$$(5.4) \quad |v(x) - v_k(x)| \leq k \|\nabla v\|_{L_\infty(B_k(x))}$$

with

$$(5.5) \quad B_k(x) = \{y \mid |x-y| < k\}.$$

Further we mention for  $v \in W_1^1$

$$(5.6) \quad \partial_i v_k = J_k(\partial_i v)$$

and

$$(5.7) \quad |\partial_i v_k| \leq c k^{-1} \|v\|_{L_\infty(B_k(x))}.$$

Now we turn over to the case  $v = \varphi \in \overset{\circ}{S}_h$ . Because of  $\overset{\circ}{S}_h \subseteq \overset{\circ}{W}_2^1(\Omega)$  we extend  $\varphi$  to a function of  $\overset{\circ}{W}_2^1(\mathbb{R}^N)$  by defining  $\varphi = 0$  outside of  $\Omega$ .  
is valid.

Because of the  $\kappa$ -regularity of  $\Gamma_h$  the number of  $\Delta' \in N_\Delta$  is uniformly bounded resp.  $\{\Omega_\Delta \mid \Delta \in \Gamma_h\}$  is a finite covering. This leads from (5.14) to

$$(5.15) \quad \|\varphi - \varphi_k\|_{\beta} \leq c(k/h) \|\varphi\|_{\beta} .$$

The same arguments give

$$(5.16) \quad \|\nabla(\varphi - \varphi_k)\|_{\beta} \leq c(k/h) \|\nabla\varphi\|_{\beta} .$$

Finally these arguments in connection with (5.7) show

$$(5.17) \quad \|\nabla^2 \varphi_k\|_{\beta} \leq c k^{-1} \|\nabla\varphi\|_{\beta} .$$

In order to get the estimates of Lemma 4.1 we have to choose  $k = c \cdot h$  in the proper way.

In order to prove Lemma 4.2 we apply Theorem 3.2 in connection with Lemma 3.3:

$$(5.18) \quad \begin{aligned} \|\nabla^4 w\|_{-\alpha} \leq c \{ & \|\nabla^2(\mu^{-\alpha-1}\tilde{\varphi})\|_{-\alpha} + \|\nabla(\mu^{-\alpha-1}\tilde{\varphi})\|_{-\alpha+1} + \\ & + \|\mu^{-\alpha-1}\tilde{\varphi}\|_{-\alpha+2} + \|w\|_{-\alpha+4} \} . \end{aligned}$$

We estimate the first term on the right hand side in the way - for  $t$  fixed:

$$(5.19) \quad \begin{aligned} \|\nabla^2(\mu^{-\alpha-1}\tilde{\varphi})\| & \leq c \int \int_{\Omega} \mu^{\alpha} \{ \mu^{-\alpha-1} |\nabla^2 \tilde{\varphi}| + \\ & + \mu^{-\alpha-1} |\nabla \tilde{\varphi}| + \mu^{-\alpha-2} |\tilde{\varphi}| \}^2 \\ & \leq c \{ \|\nabla^2 \tilde{\varphi}\|_{\alpha+2}^2 + \|\nabla \tilde{\varphi}\|_{\alpha+3}^2 + \|\tilde{\varphi}\|_{\alpha+4}^2 \} . \end{aligned}$$

Let  $\Delta \in \Gamma_h$  be fixed for the moment. We define the neighbour-set

$$(5.8) \quad N_{\Delta} = \{ \Delta' \mid \Delta' \in \Gamma_h \text{ and } \bar{\Delta}' \cap \bar{\Delta} \neq \emptyset \}$$

and

$$(5.9) \quad \Omega_{\Delta} = \cup \{ \bar{\Delta}' \mid \Delta' \in N_{\Delta} \} .$$



If we impose the restriction

$$(5.10) \quad k < \kappa^{-1}h$$

then for  $x \in \Delta$

$$(5.11) \quad B_k(x) \subseteq \Omega_\Delta \quad .$$

Restricted to any  $\Delta' \in \Gamma_h$  the function  $\varphi$  is a polynomial of degree less than  $m$ . Any two norms in finite dimensional spaces are equivalent. Since  $\Delta'$  is  $\kappa$ -regular and of 'seize'  $h$  we have

$$(5.12) \quad c^{-1} \|\varphi\|_{L_2(\Delta)}^2 \leq h^N \|\varphi\|_{L_\infty(\Delta)}^2 \leq c \|\varphi\|_{L_2(\Delta)}^2$$

with  $c$  depending only on  $N, m$  and  $\kappa$ .

Straight-forward we come from (5.4) to

$$(5.13) \quad \begin{aligned} \|\varphi - \varphi_k\|_{L_2(\Delta)}^2 &\leq c k^2 \|\nabla \varphi\|_{L_2(\Omega_\Delta)}^2 \\ &\leq c(k/h)^2 \|\varphi\|_{L_2(\Omega_\Delta)}^2 \quad . \end{aligned}$$

The weight function  $\mu$  does not change too fast - see Lemma 1 of [ ]. Therefore also

$$(5.14) \quad \iint_{\Delta} \mu^{-\beta} |\varphi - \varphi_k|^2 \, dx \leq c(k/h)^2 \iint_{\Omega_\Delta} \mu^{-\beta} \varphi^2 \, dx$$

Because of the choice of  $\tilde{\chi}$  and of (1.6) we get

$$(5.20) \quad \|\nabla^2(\mu^{-\alpha-1}\tilde{\chi})\|_{-\alpha}^2 \leq c h^{-1} \rho^{-2} \{ \|\tilde{\phi}\|_{\alpha+1}^2 + \|\nabla \tilde{\phi}\|_{\alpha}^2 \} \quad .$$

The second and third term on the right hand side of

(5.18) are bounded in the same way. Integration with respect to  $t$  shows that these terms are bounded according to Lemma 4.2.

It remains to analyze  $\|w\|_{\alpha+4}$ . Let  $\Gamma(x,y;t)$  be the fundamental solution of the heat equation, i.e. the solution of the initial-boundary value problem

$$(5.21) \quad \begin{aligned} \dot{z} - \Delta z &= f & \text{in } \Omega \times (0,T) &, \\ z &= 0 & \text{on } \partial\Omega \times (0,T) &, \\ z_{t=0} &= 0 & \text{in } \partial\Omega & \end{aligned}$$

is given by

$$(5.22) \quad z(x,t) = \int_0^t d\tau \iint_{\Omega} \Gamma(x,y;t-\tau) f(y,\tau) dy .$$

Because of the maximum principle  $\Gamma$  is positive in  $\Omega \times (0,T)$ . The function  $w$  defined by (4.23) has the representation

$$(5.23) \quad w(x,t) = \int_t^{t_0} d\tau \iint_{\Omega} \Gamma(x,y;\tau-t) \mu^{-\alpha-1} \tilde{\varphi} dy .$$

We apply Schwarz' inequality in the form

$$(5.24) \quad \begin{aligned} w^2 &\leq \left\{ \int_t^{t_0} d\tau \iint_{\Omega} \Gamma \mu^{-\alpha-1} dy \right\} \times \left\{ \int_t^{t_0} d\tau \iint_{\Omega} \Gamma \mu^{-\alpha-1} \tilde{\varphi}^2 dy \right\} \\ &= \sigma_{\alpha}(x,t) \int_t^{t_0} d\tau \iint_{\Omega} \Gamma \mu^{-\alpha-1} \tilde{\varphi}^2 dy . \end{aligned}$$

The factor  $\sigma_{\alpha}$  is the solution of

$$(5.25) \quad \begin{aligned} -\dot{\sigma}_{\alpha} - \Delta \sigma_{\alpha} &= \mu^{-\alpha-1} & \text{in } \Omega \times (0,t_0) &, \\ \sigma_{\alpha} &= 0 & \text{on } \partial\Omega \times (0,t_0) &, \\ \sigma_{\alpha}|_{t=t_0} &= 0 & \text{in } \Omega &. \end{aligned}$$

Now

$$(5.26) \quad \hat{\sigma}_\alpha = \frac{1}{\alpha} \left\{ \rho^{-2\alpha} - (\rho^2 + t_0 - t)^{-\alpha} \right\}$$

is a super-solution and therefore

$$(5.27) \quad \sigma_\alpha(x, t) \leq \hat{\sigma}_\alpha(t) < \bar{\sigma}_\alpha = \frac{1}{\alpha} \rho^{-2\alpha} .$$

Further we get by interchanging the order of integration

$$(5.28) \quad \begin{aligned} \|w\|_{\alpha+4}^2 &= \int_0^t dt \iint_{\Omega} \mu^{\alpha-4} w^2 dx \\ &\leq \bar{\sigma}_\alpha \int_0^t d\tau \iint_{\Omega} \mu^{-\alpha-1} \bar{\varphi}^2 dy \quad \times \\ &\quad \int_0^\tau dt \iint_{\Omega} \Gamma(x, y; \tau-t) \mu^{\alpha-4} dx \\ &:= \bar{\sigma}_\alpha \bar{\Sigma}_\alpha \| \bar{\varphi} \|_{\alpha+1}^2 \end{aligned}$$

with

$$(5.29) \quad \bar{\Sigma}_\alpha \geq \sup \Sigma_\alpha(y, \tau)$$

and

$$(5.30) \quad \Sigma_\alpha(y, \tau) = \int_0^\tau dt \iint_{\Omega} \Gamma(x, y; \tau-t) \mu^{\alpha-4} dt .$$

The function  $\Sigma_\alpha$  is the solution of

$$(5.31) \quad \begin{aligned} \dot{\Sigma}_\alpha - \Delta \Sigma_\alpha &= \mu^{\alpha-4} && \text{in } \Omega \times (0, t_0) , \\ \Sigma_\alpha &= 0 && \text{on } \partial\Omega \times (0, t_0) , \\ \Sigma_\alpha|_{t=0} &= 0 && \text{in } \Omega . \end{aligned}$$

In case  $\alpha < 3$  then

$$(5.32) \quad \hat{\Sigma}_\alpha = \frac{1}{3-\alpha} (\rho^2 + t_0 - t)^{\alpha-3}$$

is a supersolution and according to this

$$(5.33) \quad \bar{\Sigma}_\alpha := \frac{1}{3-\alpha} \rho^{2\alpha-6}$$

can be chosen according to (5.29). Thus

$$(5.34) \quad \|w\|_{\alpha-4} \leq c \rho^{-3} \|\tilde{\phi}\|_{\alpha+1}$$

is proven. Because of (4.22)  $\tilde{\phi}$  may be replaced by  $\phi$  what finishes the proof.

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