FINITE ELEMENT APPROXIMATIONS FOR SOLVING THE ELASTIC PROBLEM

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Finite element approximations for the first boundary value problem of elasticity are given which allow to use subspaces of functions not vanishing on the boundary. L_2 and L_∞ error estimates are derived.

1. The boundary value problem, variational formulation

Let $\Omega \subset R^2$ be a bounded domain with boundary $\partial\Omega$ sufficiently smooth. We will work with vectors $\underline{v} = (v_1, v_2)$. In case $v_i \in L_2 = L_2(\Omega)$ we write $\underline{v} \in \underline{L}_2 = L_2 \times L_2$. The meaning of $\underline{W}_2^1$ etc. is analogue. For simplicity we will also use the notation $\underline{H}_1 = \overset{\circ}{\underline{W}}{}_2^1$, $\underline{H}_2 = \underline{H}_1 \cap \underline{W}_2^2$. Correspondingly we define

$$(\underline{u}, \underline{v}) = (u_i, v_i) \qquad \|\underline{u}\| = (\underline{u}, \underline{u})^{1/2} \quad .$$

(The summation convention is used throughout the paper). To a displacement-vector $\underline{v}$ are associated the two tensors:

$$2\varepsilon_{ik}(\underline{v}) = v_{i,k} + v_{k,i} \quad ,$$

$$\sigma_{ik}(\underline{v}) = \lambda\,(v_{j,j})\,\delta_{ik} + 2\mu\,\varepsilon_{ik} \quad .$$

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Here $\#_{,i}$ denotes the partial derivatives, δ_{ik} is the Kronecker symbol and $\lambda \geq 0$, $\mu > 0$ are the Lame-constants. The first boundary value problems of elasticity is

given $\underline{f} \in \underline{L}_2$, find $\underline{u} \in \underline{H}_2$ such that

(1) $$- \nabla \sigma(\underline{u}) = \underline{f} \quad : \quad - \sigma_{ik,k}(\underline{u}) = f_i \quad \text{in } \Omega .$$

We mention the shift theorem

THEOREM 1: For $f \in L_2$ the solution $u \in \underline{H}_2$ exists uniquely and

(2) $$\|\underline{u}\|_{\underline{W}_2^2} \leq c\|\underline{f}\|_{\underline{L}_2} .$$

Here and later c is a numerical constant which may differ at different places.

The solution of (1) is equivalently characterized by

(3) $$\underline{u} \in \underline{H}_1 \quad : \quad a_o(\underline{u},\underline{v}) = (\underline{f},\underline{v}) \quad \text{for } \underline{v} \in \underline{H}_1$$

with

(4) $$a_o(\underline{v},\underline{w}) = (\sigma_{ik}(\underline{v}) \; , \; \varepsilon_{ik}(\underline{w}))$$
$$= \iint_\Omega \left\{\lambda(v_{i,i})(w_{k,k}) + 2\mu\varepsilon_{ik}(\underline{v})\varepsilon_{ik}(\underline{w})\right\}dx .$$

The form a_o is symmetric, bounded and because of Korn's inequality coercive in $\underline{H}_1$. As long as we are in $\underline{H}_1 = \overset{o}{\underline{W}}{}_2^1$ a_o in (3) can be modified without influencing the solution $\underline{u}$ by - $\underline{n}$ is the normal vector of $\partial\Omega$ -

(5) $$a_1(\underline{v},\underline{w}) = a_o(\underline{v},\underline{w}) - \oint_{\partial\Omega} n_i\left\{\sigma_{ik}(\underline{v})w_k + \sigma_{ik}(\underline{w}) \; v_k\right\} ds ,$$
$$a_2(\underline{v},\underline{w}) = a_1(\underline{v},\underline{w}) + K h^{-1} \oint v_i w_i \, ds .$$

These terms are motivated because of

LEMMA 1: Let $\underline{u}$ be the solution of (1) and $\underline{w} \in \underline{W}_2^1$. Then for $i = 1,2$

(6) $$a_i(\underline{u},\underline{w}) = (\underline{f},\underline{w}) .$$

This relation is essential in deriving L_2 and L_∞ estimates, it is not true for the form a_o .

2. Finite elements

By Γ_h a γ-regular subdivision of Ω with mesh-seize h into generalized triangles will be denoted: For any $\Delta \in \Gamma_h$ there are two spheres $\underline{K}, \overline{K}$ with radii $\underline{r}, \overline{r}$ such that $\underline{K} \subset \Delta \subset \overline{K}$ and $\gamma^{-1}h \le \underline{r} < \overline{r} \le h$ (for more details see CIARLET-RAVIART [1]).

Besides the usual Sobolev-norms we will need certain weighted norms. Let $x_o \in \overline{\Omega}$ and $\rho > 0$. We use the weight-factor

$$p_\alpha(x) = \mu(x)^{-\alpha} \quad \text{with} \quad \mu(x) = |x-x_o|^2 + \rho^2$$

and define for any $\Omega' \subseteq \Omega$

$$\text{(7)} \qquad \begin{aligned} \|v\|_{\alpha.\Omega'} &= \Big\{\iint_{\Omega'} p_\alpha \, v^2 \, dx\Big\}^{1/2} \\ \|\nabla^k v\|_{\alpha.\Omega'} &= \Big\{\sum_{|\varkappa|=k} \|D^\varkappa v\|^2_{\alpha.\Omega'}\Big\}^{1/2} \end{aligned} \quad .$$

In case $\Omega' = \Omega$ we simply write $\|\cdot\|_\alpha$. The scalar-products are denoted by $(\cdot,\cdot)_\alpha$. If $T \subseteq \overline{\Omega}$ is a curve we use for the corresponding integrals the notation $|\cdot|_{\alpha.T}$ resp. $\langle\cdot,\cdot\rangle_{\alpha.T}$ and drop T in case of $T = \partial\Omega$.

The functions we work with will have a reduced regularity across the edges of Γ_h . Therefore we introduce the spaces ${}^hW_2^k$ of functions v with $v_{|\Delta} \in W_2^k(\Delta)$ for $\Delta\in\Gamma_h$ and define

$$\text{(8)} \qquad \|\nabla^k v\|^h_\alpha = \Big\{\sum_{\Delta\in\Gamma_h} \|\nabla^k v\|^2_{\alpha.\Delta}\Big\}^{1/2} \quad .$$

For simplicity we will consider in this paper only linear finite element spaces S_h , i.e. any $\chi \in S_h$ is continuous in Ω and piece-wise linear in $\Delta \in \Gamma_h$.

$\overset{\circ}{S}_h \subseteq S_h$ is the subspace of functions vanishing in the nodes of Γ_h which are on $\partial\Omega$. The standard properties of S_h resp. $\overset{\circ}{S}_h$ used in the next sections are summarized in

THEOREM 2: There is a constant γ_1 such that for any γ-regular subdivision Γ_h and any ρ with $\rho \geq \gamma_1 h$ the propositions hold:

(i) To any $v \in W_2^1 \cap {}^hW_2^k$ $(k=1,2)$ there is a $\chi \in S_h$ with

$$\|v-\chi\|_\alpha + h\,\|\nabla(v-\chi)\|_\alpha \leq c_1(\alpha)\, h^k\, \|\nabla^k v\|_\alpha^h \quad . \tag{9}$$

(ii) For any $\chi \in S_h$

$$\|\nabla\chi\|_\alpha \leq c_2(\alpha)\, h^{-1}\, \|\chi\|_\alpha \quad , \tag{10}$$

$$|\nabla\chi|_\alpha \leq c_3(\alpha)\, h^{-1/2} \left\{\|\chi\|_\alpha + \|\nabla\chi\|_\alpha\right\} \quad .$$

(iii) For any $\chi \in \overset{\circ}{S}_h$

$$|\chi|_\alpha \leq c_4(\alpha)\, h^{3/2} \left\{\|\chi\|_\alpha + \|\nabla\chi\|_\alpha\right\} \quad . \tag{11}$$

The bounds $c_i(\alpha)$ depend only on α, γ, γ_1 and a bound of the curvature of $\partial\Omega$.

Remark: If $v \in H_1$ then the choice $\chi \in \overset{\circ}{S}_h$ is possible in assertion (i). In addition χ may be chosen according to

$$|\chi|_\alpha \leq c_5(\alpha)\, h^k \|\nabla^k v\|_\alpha^h \quad . \tag{12}$$

For more details see NATTERER [1], NITSCHE [1], [2] .

3. Finite element approximations, H_1- and L_2- error estimates

The solution $\underline{u}$ of the boundary value problem (1) will be approximated by an element $\underline{u}_h \in \underline{S}_h = S_h \times S_h$. Though the functions in S_h are not exactly zero on $\partial\Omega$ the forms a_o, a_1, a_2 are positive definite in $\overset{o}{S}_h$. The finite element approximations $\underline{u}_h^{(i)}$ are defined by

$$
(13) \qquad
\begin{aligned}
&\underline{u}_h^{(i)} \in \overset{o}{\underline{S}}_h \;:\; a_i(\underline{u}_h^{(i)}, \underline{\chi}) = (\underline{f}, \underline{\chi}) \quad \text{for } \chi \in \overset{o}{\underline{S}}_h \quad (i=0,1) \\
&\underline{u}_h^{(2)} \in \underline{S}_h \;:\; a_2(\underline{u}_h^{(2)}, \underline{\chi}) = (\underline{f}, \underline{\chi}) \quad \text{for } \underline{\chi} \in S_h \quad .
\end{aligned}
$$

For K - see (5) - sufficiently large $a_2(\underline{\chi},\underline{\chi})^{1/2}$ is in $\underline{S}_h$ a norm equivalent to

$$
\|\underline{\chi}\|_{\underline{W}_2^1} + h^{-1/2}\, |\underline{\chi}| \quad ,
$$

therefore also $\underline{u}_h^{(2)}$ is well-defined.

By standard arguments we get immediately for the errors $e^{(i)} = e_h^{(i)} = \underline{u} - \underline{u}_h^{(i)}$:

THEOREM 3: Assume $f \in L_2$ resp. $u \in H_2$. The errors in the energy norm are bounded by

$$
(14) \qquad \|\underline{e}_h^{(i)}\|_{\underline{W}_2^1} \le c\, h\, \|\underline{f}\| \qquad (i = 0,1,2) \quad ,
$$

in the L_2-norm the bounds differ

$$
(15) \qquad
\begin{aligned}
&\|\underline{e}_h^{(o)}\| \le c\, h^{3/2}\, \|\underline{f}\| \quad , \\
&\|e_h^{(i)}\| \le c\, h^2\, \|\underline{f}\| \qquad (i = 1,2) \quad .
\end{aligned}
$$

The approximation $u_h^{(1)}$ seems to be of most interest. In this case we have in addition

$$|\underline{e}_h^{(1)}| \leq c\, h^2\, \|\underline{f}\| \quad . \tag{16}$$

4. Error-estimates in weighted norms

In this and the next section we restrict ourselves to the bilinear form a_1 and drop here as well as in $\underline{u}_h^{(1)}$ the index 1 . We will need

LEMMA 2: Let $\underline{v},\underline{w} \in \underline{H}_1 \cup \overset{\circ}{\underline{S}}_h$. Then for any $\alpha \in R$

$$|a(\underline{v},\underline{w})| \leq c\|\nabla v\|_\alpha \, \|\nabla w\|_{-\alpha} \quad .$$

LEMMA 3: Let $\underline{v} \in \underline{H}_1$ resp. $\underline{v} \in \overset{\circ}{\underline{S}}_h$. Then for any $\alpha \in R$

$$\|\nabla \underline{v}\|_\alpha^2 \leq c\left\{a(\underline{v},\mu^{-\alpha}\underline{v}) + \|\underline{v}\|_{\alpha+1}^2\right\} \quad .$$

The proof of Lemma 2 is straight-forward. Korn's inequality applied to $\underline{w} = \mu^{-\alpha/2}\underline{v}$ and standard estimates give Lemma 3.

By definition of $\underline{u}_h = \underline{u}_h^{(1)}$ we have for $\underline{e} = \underline{e}_h^{(1)}$

$$a(\underline{e},\underline{\chi}) = 0 \quad \text{for } \underline{\chi} \in \overset{\circ}{\underline{S}}_h \quad . \tag{17}$$

Now let $\underline{U}_h$ be an appropriate approximation on $\underline{u}$ according to Theorem 2 with error $\underline{E} = \underline{E}_h = \underline{u} - \underline{U}_h$. Then we have $\underline{e} = \underline{E} - \underline{\Phi}$ with $-\underline{\Phi} = \underline{U}_h - \underline{u}_h \in \overset{\circ}{\underline{S}}_h$ and

$$a(\underline{\Phi},\underline{\chi}) = a(\underline{E},\underline{\chi}) \quad \text{for } \underline{\chi} \in \overset{\circ}{\underline{S}}_h \quad . \tag{18}$$

Using Lemma 3 we derive with $\alpha \in R$ and any $\underline{\chi} \in \overset{\circ}{\underline{S}}_h$

$$\|\nabla\underline{\Phi}\|^2_\alpha \le c\ \Big\{a(\underline{\Phi},\mu^{-\alpha}\underline{\Phi}-\underline{X}) - a(\underline{E},\mu^{-\alpha}\underline{\Phi}-\underline{X}) + a(\underline{E},\mu^{-\alpha}\underline{\Phi}) + $$

$$+ \|\underline{\Phi}\|^2_{\alpha+1}\Big\}$$

(19)

$$\le c\ \Big\{\|\nabla\underline{\Phi}\|_\alpha + \|\nabla\underline{E}\|_\alpha\Big\}\ \|\nabla(\mu^{-\alpha}\underline{\Phi}-\underline{X})\|_{-\alpha}$$

$$+ c\ \|\nabla\underline{E}\|_\alpha\ \|\nabla(\mu^{-\alpha}\underline{\Phi})\|_{-\alpha} + c\ \|\underline{\Phi}\|^2_{\alpha+1} \quad .$$

Application of $2|ab| \le \delta a^2 + \delta^{-1}b^2$ in a proper way gives

(20) $$\|\nabla\underline{\Phi}\|^2_\alpha \le c\ \Big\{\|\nabla\underline{E}\|^2_\alpha + \|\underline{\Phi}\|^2_{\alpha+1} + \|\nabla(\mu^{-\alpha}\underline{\Phi}-\underline{X})\|^2_{-\alpha}\Big\} \quad .$$

Since $\underline{\Phi}$ is piecewise linear we have by means of Theorem 2 with $\underline{X}$ properly chosen

$$\|\nabla(\mu^{-\alpha}\underline{\Phi}-\underline{X})\|_{-\alpha} \le c\ h\ \|\nabla^2(\underline{\mu}^{-\alpha}\underline{\Phi})\|^h_{-\alpha}$$

(21)

$$\le c\ h\ (\|\underline{\Phi}\|_{\alpha+2} + \|\nabla\underline{\Phi}\|_{\alpha+1}\Big\}$$

$$\le c\ h\ \rho^{-1}(\|\Phi\|_{\alpha+1} + \|\nabla\underline{\Phi}\|_\alpha) \quad .$$

Now we impose the condition $\rho \ge \gamma_2 h$ with $\gamma_2 \ge \gamma_1$ and such that the constant in (20) is less than γ_2 . Then we get

(22) $$\|\nabla\underline{\Phi}\|_\alpha \le c\ \Big\{\|\nabla\underline{E}\|_\alpha + \|\underline{\Phi}\|_{\alpha+1}\Big\} \quad .$$

Now let $\underline{w} \in \underline{H}_2$ be the solution of

(23) $$-\ \nabla\sigma(\underline{w}) = \mu^{-\alpha-1}\underline{\Phi}\ :\ -\sigma_{ik,k}(\underline{w}) = \mu^{-\alpha-1}\varphi_i \quad .$$

Then we have with an arbitrary $\underline{\chi} \in \overset{\circ}{\underline{S}}_h$

$$\|\underline{\Phi}\|^2_{\alpha+1} = a(\underline{\Phi},\underline{w})$$

$$= a(\underline{\Phi},\underline{w}-\underline{\chi}) - a(\underline{E},\underline{w}-\underline{\chi}) + a(\underline{E},\underline{w}) \quad .$$

The last term may be estimated by

$$a(\underline{E},\underline{w}) = (\underline{E},\mu^{-\alpha-1}\underline{\Phi})$$

$$\leq \|\underline{E}\|_{\alpha+1} \, \|\Phi\|_{\alpha+1} \quad .$$

The function $\underline{\chi}$ is now chosen to be an approximation on $\underline{w}$. Then

$$\|\nabla(\underline{w}-\underline{\chi})\|_{-\alpha} \leq c \; h \; \|\nabla^2\underline{w}\|_{-\alpha} \quad .$$

After standard estimates and transformations we come to

$$(24) \quad \|\underline{\Phi}\|_{\alpha+1} \leq \delta \; \|\nabla\underline{\Phi}\|_{\alpha} + c \; \delta^{-1} \left\{\|\underline{E}\|_{\alpha+1} + \|\nabla\underline{E}\|_{\alpha} + h\|\nabla^2\underline{w}\|_{-\alpha}\right\}.$$

Here $\delta > 0$ is arbitrary.

If δ is chosen such that with the constant in (22) $\delta c < 1$ then the combination of (22), (24) gives

$$(25) \quad \|\underline{\Phi}\|_{\alpha+1} + \|\nabla\underline{\Phi}\|_{\alpha} \leq c \left\{\|\underline{E}\|_{\alpha+1} + \|\nabla\underline{E}\|_{\alpha} + h\|\nabla^2\underline{w}\|_{-\alpha}\right\} \quad .$$

From now we specialize $\alpha = 1$. Applying the shift theorem to the functions $x_i\underline{w}$ and $\rho\underline{w}$ gives after some computations

LEMMA 4: *Let* $\underline{w}$ *be the solution of (23) with* $\alpha = 1$. *Then*

$$\|\nabla^2 \underline{w}\|_{-1}^2 \le c \left\{\rho^{-2} \|\underline{\Phi}\|_2^2 + \|\nabla \underline{w}\|^2\right\}$$

(26)

$$\le c \left\{\rho^{-2} \|\underline{\Phi}\|_2^2 + a(\underline{w},\underline{w})\right\} \quad .$$

It remains to estimate the last term by $\|\Phi\|_2^2$ respective by

$$\|\nabla\sigma(\underline{w})\|_{-2}^2 = \iint \mu^2 \, \Sigma \, |\sigma_{ik,k}(\underline{w})|^2 \quad .$$

If we define

$$(27) \quad K = K_\rho = \sup \left\{a(\underline{w},\underline{w}) \;\middle|\; \|\nabla\sigma(\underline{w})\|_{-2} = 1\right\} \quad ,$$

then we have with (26)

$$(28) \qquad \|\nabla^2 \underline{w}\|_{-1}^2 \le c(\rho^{-2} + K_\rho) \, \|\underline{\Phi}\|_2^2 \quad .$$

In the appendix we will sketch the proof of

LEMMA 5: *Let* K_ρ *be defined by (27). Then*

$$K_\rho \le c \, \rho^{-2} \, |\ln \rho| \quad .$$

With the help of this estimate we get combining (28) with (25)

$$\|\underline{\Phi}\|_2 + \|\nabla\underline{\Phi}\|_1 \le c \left\{\|\underline{E}\|_2 + \|\nabla\underline{E}\|_1\right\}$$
$$+ c \, h_\Phi^{-1} \, |\ln\rho|^{1/2} \, \|\underline{\Phi}\|_2 \quad .$$

If we take $\rho \ge \gamma_3 \, h|\ln h|$ with γ_3 properly chosen the imposed conditions on ρ will hold and the coefficient of $\|\underline{\Phi}\|_2$ in the last inequality is smaller than 1 . Remembering the meaning of $\underline{E} = \underline{u} - \underline{U}_h$ we get

LEMMA 6: If the parameter ρ in the weight-factor μ is connected with h by $\rho \geq \gamma_2\, h|\ln h|^{1/2}$ then

(29) $$\|\underline{\Phi}\|_2 + \|\nabla\underline{\Phi}\|_1 \leq c \inf_{\underline{\chi}\in\overset{\circ}{S}_h} \left\{\|\underline{u}-\underline{\chi}\|_2 + \|\nabla(\underline{u}-\underline{\chi})\|_1\right\} \quad .$$

5. L_∞-error-estimates

Let us now assume that the solution $\underline{u}$ of the boundary value problem (1) has bounded second derivatives. Then

$$\inf_{\chi\in\overset{\circ}{S}_h} \left\{\|\underline{u}-\underline{\chi}\|_2 + \|\nabla(\underline{u}-\underline{\chi})\|_1\right\}$$
$$\leq c\left\{h^2\rho^{-1} + h\,|\ln \rho|^{1/2}\right\} \|\nabla^2\underline{u}\|_{\underline{L}_\infty}$$
$$\leq c\ h\ |\ln h|^{1/2}\ \|\nabla^2\underline{u}\|_{\underline{L}_\infty} \quad .$$

The point x_0 in μ is now chosen to be in a $\Delta \in \Gamma_h$ with

$$\|\nabla\underline{\Phi}\|_{\underline{L}_\infty} = |\nabla\underline{\Phi}(x_0)| \quad .$$

Then we have

$$\|\nabla\underline{\Phi}\|_1 \geq c\ \frac{h}{\rho}\ \|\nabla\underline{\Phi}\|_{\underline{L}_\infty}$$

and therefore from (29)

$$\|\nabla\underline{\Phi}\|_{\underline{L}_\infty} \leq c\ h\ |\ln h|\ \|\nabla^2\underline{u}\|_{\underline{L}_\infty} \quad .$$

Because of $\underline{e} = \underline{E} - \underline{\Phi}$ we have got

THEOREM 4: If $\underline{u} \in \underline{W}^2_\infty$ then

$$\|\nabla(\underline{u}-\underline{u}_h)\|_{\underline{L}_\infty} \leq c\ h\ |\ln h|\ \|\nabla^2\underline{u}\|_{\underline{L}_\infty}$$

In order to get an error estimate for $\underline{e}$ in $\underline{L}_\infty$ we consider a $\Lambda_0 \in \Gamma_h$ with

$$\|\Phi\|_{\underline{L}_\infty} = \|\Phi\|_{\underline{L}_\infty(\Lambda_o)} \quad .$$

Since Φ is linear in Λ_o we find with $K_{\underline{r}} \subset \Delta_o$ - $\underline{r} \geq \kappa^{-1} h$ -

(30) $$\|\underline{\Phi}\|_{\underline{L}_\infty} \leq c\, h^{-2} \iint\limits_{K_{\underline{r}}} \underline{\Phi}^2 \, dx \quad .$$

Now let $\underline{w} \in \underline{H}_2$ be the solution - compare with (23) - of

(31) $$- \nabla\sigma(\underline{w}) = \begin{cases} h^{-2}\underline{\Phi} & \text{in } K_{\underline{r}} \\ 0 & \text{else} \end{cases} \quad .$$

By arguments similar to those on pp. 8,9 we come to

$$\begin{aligned} h^{-2} \iint\limits_{K_{\underline{r}}} \underline{\Phi}^2 \, dx &= a(\underline{\Phi}, \underline{w}-\underline{\chi}) - a(\underline{E}, \underline{w}-\underline{\chi}) + a(\underline{E}, \underline{w}) \\ &\leq c\, h^{-2} \iint\limits_{K_{\underline{r}}} \underline{E}^2 \, dx + \\ &+ c\, h \left\{ \|\nabla\underline{\Phi}\|_1 + \|\nabla\underline{E}\|_1 \right\} \|\nabla^2 \underline{w}\|_{-1} \end{aligned}$$

and using (29)

(32) $$h^{-2} \iint\limits_{K_{\underline{r}}} \underline{\Phi}^2 \, dx \leq c\, h^2 \, |\ln h|^{1/2} \, \|\nabla^2 \underline{u}\|_{\underline{L}_\infty} \, \|\nabla^2 \underline{w}\|_{-1} \quad .$$

Using the counterparts of Lemmata 4 and 5 for the function w defined by (31) we get

$$\|\nabla^2 \underline{w}\|_{-1}^2 \leq c\, \rho^2 \, h^{-4} \iint\limits_{K_{\underline{r}}} \underline{\Phi}^2 \, dx$$

and therefore we derive from (32)

$$h^{-2} \iint\limits_{K_{\underline{r}}} \underline{\Phi}^2 \, dx \leq c\, h^4 \, |\ln h|^2 \, \|\nabla^2 \underline{u}\|_{\underline{L}_\infty}^2 \quad .$$

In connection with (30) we have

Theorem 5: If $u \in W^2_\infty$ then

$$\|\underline{u}-\underline{u}_h\|_{\underline{L}_\infty} \leq c\ h^2\ |\ln h|\ \|\nabla^2 u\|_{\underline{L}_\infty} \quad .$$

6. Appendix: Proof of Lemma 5

There exists (at least) one solution $\underline{w} \in \underline{H}_2$ with

$$a(\underline{w},\underline{w}) = K\ \|\nabla\sigma(\underline{w})\|^2_{-2} \quad .$$

For any $\underline{v} \in \underline{H}_2$ the variational equations

$$a(\underline{w},\underline{v}) = K \iint \mu^2(\nabla\sigma(\underline{w})\cdot(\nabla\sigma(\underline{v}))\ dx$$

hold. Since

$$a(\underline{w},\underline{v}) = -\iint \underline{w}(\nabla\sigma(\underline{v})\ dx$$

and $\nabla\sigma(\underline{v}) \in \underline{L}_2$ is arbitrary the function $\underline{w}$ satisfies

(33) $$-\nabla\sigma(\underline{w}) = \lambda\mu^{-2}\ \underline{w}$$

with $\lambda = K^{-1}$. In order to estimate K we need a lower bound of the eigenvalues of (33). Multiplication of (33) with $\underline{w}$ and integration gives

$$K = \lambda^{-1} = \|\underline{w}\|^2_{-2}\ /\ a(\underline{w},\underline{w}) \quad .$$

Because of Korn's inequality we have

$$K \leq c\ \sup\left\{\|\underline{w}\|^2_{-2}\ \middle|\ \|\nabla\underline{w}\| \leq 1\right\}$$

and the right hand side is bounded up to a factor by

(34) $$\bar{K} = \sup\left\{\|w\|^2_{-2}\ \middle|\ w \in \overset{\circ}{W}{}^1_2 \wedge \|\nabla w\| \leq 1\right\} \quad .$$

The extremal function w of (34) is the solution of

$$-\Delta w = \overline{\lambda}\, \mu^{-2} w \tag{35}$$

with $\overline{\lambda} = \overline{K}^{-1}$ being the smalles eigenvalue. Because of the maximum principle w as well as $-\Delta w$ are not negative. From this the monotony of $\overline{K}$ with respect to the domain follows: Let Ω_1, Ω_2 be two domains and $\overline{K}_1$, $\overline{K}_2$ be the corresponding values (34). If $\Omega_1 \subset \Omega_2$ then $\overline{K}_1 \leq \overline{K}_2$. Now let $\hat{\Omega}$ be the circle with center x_o and radius $\hat{r} = \operatorname{diam}(\Omega)$. Then $\Omega \subseteq \hat{\Omega}$ and it suffices to bound the corresponding value of $\hat{K}$. Since μ depends only on $|x-x_o|$ and $w \geq 0$ there is a solution of (34) depending also only on $|x-x_o|$ (actually the smalles eigenvalue is simple). Therefore problem (35) can be handled as 1-dimensional. By direct computation then we get the bound for $\hat{K}$ and hence for K given in Lemma 5.

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