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**INTERIOR ERROR ESTIMATES  
FOR SEMI-DISCRETE GALERKIN APPROXIMATIONS  
FOR PARABOLIC EQUATIONS (\*) (\*\*)**

by J A NITSCHÉ (1)

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*Abstract* — The initial boundary value problem for the heat equation in a domain  $\Omega$  and the corresponding standard Galerkin method is considered. A certain regularity of the initial data in some sub-domain  $\Omega_1$  leads to the same regularity of the solution in  $\Omega_1$  and for all times. It is shown that the error between the exact solution and the Galerkin approximation is also of (almost) optimal order in the interior of  $\Omega_1$ . Of course certain properties of the underlying approximation spaces are needed, they are typical for finite elements.

*Resume* — On considere le probleme aux limites avec conditions initiales pour l'equation de la chaleur dans un domaine  $\Omega$ , ainsi que l'approximation habituelle de Galerkin correspondante. Une certaine regularite des donnees initiales dans un sous-domaine  $\Omega_1$  conduit a la même regularite de la solution dans  $\Omega_1$  pour tous les temps. On montre que l'erreur entre la solution exacte et l'approximation de Galerkin est aussi d'ordre (presque) optimal dans l'interieur de  $\Omega_1$ . Naturellement, certaines proprietes des espaces d'approximations sont utilisees, qui sont caracteristiques des espaces d'elements finis.

1. In order to avoid technical details we restrict ourselves to the model problem

$$\begin{aligned} \dot{u} &= \Delta u & \text{in } \Omega \times (0, T], \\ u &= 0 & \text{on } \partial\Omega \times (0, T], \\ u_{t=0} &= v & \text{in } \Omega \end{aligned} \tag{1}$$

The boundary of  $\Omega \subseteq \mathbb{R}^N$  is assumed to be sufficiently smooth. With the help of a finite element space  $S_h \subseteq H_1(\Omega)$  the Galerkin approximation  $u_h = u_h(t) \in S_h$  is defined by

$$\begin{aligned} (\dot{u}_h, \chi) + D(u_h, \chi) &= 0 \quad \text{for } \chi \in S_h \wedge t > 0, \\ u_h|_{t=0} &= P_h v \end{aligned} \tag{2}$$

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Here  $(\cdot, \cdot)$  resp.  $D(\cdot, \cdot)$  is the  $L_2$ -inner product resp. the Dirichlet integral and (for simplicity)  $P_h$  is the  $L_2$ -projector onto  $S_h$ .

For the corresponding elliptic problem

$$\begin{aligned} -\Delta u &= f & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega \end{aligned} \quad (3)$$

interior estimates of the error  $e = u - u_h$  of the Ritz approximation  $u_h \in S_h$  defined by

$$D(u_h, \chi) = (f, \chi) \quad \text{for } \chi \in S_h \quad (4)$$

were derived in [1], [5], [6], [7]. They are of the following type (\*):

Assume  $f \in L_2(\Omega)$  and in addition  $f \in H_{k-2}(\Omega_1)$  for some domain  $\Omega_1 \subseteq \Omega$  and  $k > 2$ . Further let  $\Omega_2$  be contained properly in  $\Omega_1$ . If  $S_h$  is of degree  $r$  with  $r \geq k$  then the error  $e$  is of order  $k$  in  $\Omega_2$ , i.e.

$$\|e\|_{\Omega_2} \leq ch^k \{ \|f\|_{k-2, \Omega_1} + \|f\| \}. \quad (5)$$

The norms of  $f$  could be replaced by the appropriate norms of  $u$  because of the shift theorem.

Now let us assume the regularity

$$v \in L_2(\Omega) \cap H_k(\Omega_1) \quad (6)$$

and let  $\Omega_2 \subset \subset \Omega_1$ . Then the solution of (1) has the regularity

$$u \in L_\infty(L_2(\Omega)) \cap L_2(H_1(\Omega)) \cap L_\infty(H_k(\Omega_1)) \quad (7)$$

with the abbreviation  $L_p(Z) = L_p(0, T; Z)$  for some  $T > 0$  fixed. Corresponding to (5) we would expect in the parabolic case an estimate of the type

$$\|e\|_{L_\infty(L_2(\Omega_2))} \leq ch^k \{ \|v\|_{k, \Omega_1} + \|v\| \}. \quad (8)$$

In the next two sections we prove this error bound for  $k < r$  (being the degree of  $S_h$ ). In case  $k = r$  the factor  $ch^k$  is to be replaced by  $c_\varepsilon h^{r-\varepsilon}$  with  $\varepsilon > 0$  arbitrary small.

This problem was already treated in Thomee [9]. There the local error in the  $L_\infty(L_2(\Omega_2))$  norm is bounded by the  $L_2(L_2(\Omega_1))$  norm besides of a remainder. Although this result does not give the final answer it turns out to be the main step. With respect to the notations as already mentioned as well as to the

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(\*) We use the notations of [7] resp. [9].

assumptions on  $S_h$  we refer to [9]. Since we do not extend our result to difference quotients the uniformity of the subdivisions in defining  $S_h$  is not necessary of course.

2. We start with an interior estimate for  $u_h$ , see lemma 3.3 in [9] :

LEMMA 1 (Thomée) : Let  $u_h$  be the solution of (2). Further assume

$$\hat{\Omega}_2 \subset \subset \Omega_1 \subseteq \Omega$$

and let  $q > 0$  be fixed. Then

$$\| u_h(t) \|_{\hat{\Omega}_2}^2 \leq c \left\{ \| P_h v \|_{\Omega_1}^2 + \int_0^t [ \| u_h \|_{\Omega_1}^2 + h^q \| \dot{u}_h \|_{\Omega_1}^2 ] d\tau \right\}. \tag{9}$$

Because of

$$\| \dot{u}_h \|_{\Omega_1}^2 \leq \| \dot{u}_h \|^2 = -\frac{1}{2} \partial_t \| u_h' \|^2 \tag{10}$$

we get

$$\int_0^t \| \dot{u}_h \|_{\Omega_1}^2 d\tau \leq \frac{1}{2} \| u_h'(0) \|^2. \tag{11}$$

Further we have

$$\begin{aligned} \| u_h'(0) \| &\leq ch^{-1} \| u_h(0) \| \\ &\leq ch^{-1} \| v \|. \end{aligned} \tag{12}$$

In this way the last term in (9) may be replaced by  $h^{q-2} \| v \|^2$ .

Next let us assume there is an additional domain  $\Omega_0$  according to

$$\Omega_1 \subset \subset \Omega_0 \subseteq \Omega.$$

Using the result of [6] or of Douglas *et al.* [3] we may estimate

$$\| P_h v \|_{\Omega_1} \leq c \{ \| v \|_{\Omega_0} + h^r \| v \| \}. \tag{13}$$

By replacing  $\Omega_0$  by  $\Omega_1$  we get with  $q = 2r + 2$ .

LEMMA 2 : Let  $\Omega_2 \subset \subset \Omega_1 \subseteq \Omega$ . Then

$$\| u_h(t) \|_{\Omega_2}^2 \leq c \left\{ \| v \|_{\Omega_1}^2 + h^{2r} \| v \|^2 + \int_0^t \| u_h \|_{\Omega_1}^2 d\tau \right\}. \tag{14}$$

Now we use an induction argument. Let  $\Omega_1, \Omega_2$  with  $\Omega_2 \subset \subset \Omega_1$  and  $p \in \mathbb{N}$  be given. We can choose domains  $\Omega^\nu$  according to

$$\Omega_2 = \Omega^p \subset \subset \Omega^{p-1} \subset \subset \dots \subset \subset \Omega^0 = \Omega_1. \quad (15)$$

Repeated application of (14) and the interchange of the order of integration leads to

$$\|u_h(t)\|_{\Omega_2}^2 \leq c \left\{ \|v\|_{\Omega_1}^2 + h^{2r} \|v\|^2 + \int_0^t (t-\tau)^{p-1} \|u_h\|_{\Omega_1}^2 d\tau \right\}. \quad (16)$$

Since anyway

$$\|u_h\|_{\Omega_1} \leq \|u_h\| \leq \|v\| \quad (17)$$

we get :

LEMMA 3 : Assume  $\Omega_2 \subset \subset \Omega_1 \subseteq \Omega$  and let  $p > 0$  be arbitrary. Then

$$\|u_h(t)\|_{\Omega_2} \leq c \{ \|v\|_{\Omega_1} + (t^p + h^r) \|v\| \}. \quad (18)$$

This is the counterpart of the a priori estimate

$$\|u(t)\|_{\Omega_2} \leq c \{ \|v\|_{\Omega_1} + t^p \|v\| \} \quad (19)$$

for the solution of (1) which is easily derived (for instance using the exponential decay of the fundamental solution).

3. Now we are ready to prove (8). Let  $\Omega_2 \subset \subset \Omega_1$  be fixed. We choose an  $\Omega'$  according to

$$\Omega_2 \subset \subset \Omega'_0 \subset \subset \Omega_1. \quad (20)$$

Next let  $\omega$  be a cut-off function with respect to  $\Omega', \Omega_1$ , i.e.  $\omega \in C^\infty(\Omega)$  with  $0 \leq \omega \leq 1$  and

$$\omega = \begin{cases} 1 & \text{in } \Omega' \\ 0 & \text{in } \Omega - \Omega_1 \end{cases}. \quad (21)$$

We will use the splitting

$$v = v^1 + v^2 := \omega v + (1 - \omega) v \quad (22)$$

and denote by  $u^i, u_h^i$  the solutions of (1), (2) with the initial data  $v^i$ .

We have

$$u = u^1 + u^2, \quad u_h = u_h^1 + u_h^2. \tag{23}$$

The regularity assumption (6) leads to  $v^1 \in H_k(\Omega)$  and

$$\|v^1\|_k \leq c \|v\|_{k,\Omega_1} \tag{24}$$

Therefore (see e.g. Bramble *et al.* [2])

$$\|u^1 - u_h^1\| \leq ch^k \|v\|_{k,\Omega_1} \tag{25}$$

is guaranteed. On the other hand  $v^2$  is in  $L_2(\Omega)$  with  $\|v^2\| \leq \|v\|$  and  $v^2$  vanishes in  $\Omega'$ . The estimates (18), (19) for  $u^2$  and  $u_h^2$  with  $\Omega_1$  replaced by  $\Omega'$  give

$$\|u^2 - u_h^2\|_{\Omega_2} \leq c(t^p + h^r) \|v\|. \tag{26}$$

In this way

$$\|e(t)\|_{\Omega_2} = \|u - u_h\|_{\Omega_2} \leq c \{ h^k \|v\|_{k,\Omega_1} + (t^p + h^r) \|v\| \} \tag{27}$$

is shown. In order to eliminate the time dependence we make use of the time dependent error estimate

$$\|e(t)\| \leq ch^r t^{-r/2} \|v\| \tag{28}$$

due to Helfrich [4], Thomee [8] which in connection with (27) gives

$$\|e(t)\|_{\Omega_2} \leq c \{ h^k \|v\|_{k,\Omega_1} + [h^r + \text{Min}(t^p, h^r t^{-r/2})] \|v\| \}. \tag{29}$$

The minimum is maximal for  $t = c_p h^\alpha$  with  $\alpha = 2r/(r + 2p)$  leading to

$$\text{Min}(t^p, h^r t^{-r/2}) = ch^\beta \tag{30}$$

with

$$\beta = \frac{2pr}{2p+r} = r - \frac{r^2}{2p+r}. \tag{31}$$

For any  $k < r$  we may choose  $p$  such that  $\beta \geq k$ .

Then (29) gives (8). In case of  $k = r$  and  $\varepsilon > 0$  fixed we can choose  $p$  such that  $\beta \geq r - \varepsilon$ . Since the choice of  $p$  influences the number of iterations the constant in (8) then depends on  $p$  resp.  $\varepsilon$ .

*Remark :* In the previous estimates we did not control the constants. The-

refore we did not take into account a contribution of  $(p - 1)^{-1}$  in the denominator of the integral in (16) By a careful analysis the "garbage" term  $h^{-\varepsilon}$  in (8) in case of  $k = r$  could be replaced by a logarithmic one

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