IX. Weighted Norms and $L_\infty$-Analysis I

The aim of this chapter is to derive $L_\infty$-error estimates for approximation methods as described in the Chapters VI and VII. It will turn out that the use of certain 'weighted' norms is advantageous. They are defined by means of the weight-function

\[ u(x) = |x-x_0|^2 + \epsilon^2 \]

(respective powers of $u$). Here $x_0$ is a point in $\Omega$, fixed but chosen appropriately and $\epsilon > 0$ is a parameter later on coupled with $h$. We define for any $v \in L_2$ resp. $v \in W_2^1$ and $\alpha \in \mathbb{R}$

\[ \|v\|_\alpha = \left\{ \iint u^{-\alpha} |v|^2 \, dx \right\}^{1/2} \]

(resp.

\[ \|\eta v\|_\alpha = \left\{ \iint u^{-\alpha} |\eta v|^2 \, dx \right\}^{1/2} \].

We suppress the dependence on $x_0$ since there will be no confusion. The connection of weighted norms with the $L_\infty$-norm is clarified by the next two propositions.

**Proposition IX.1:** Let $\alpha > N/2$. There is a constant $c$ depending only on $\alpha$ and $N$ such that for $v \in L_\infty$ and $x_0$ arbitrary

\[ \|v\|_\alpha \leq c \, e^{N/2-N} \|v\|_{L_\infty} \]

holds true.
Proposition IX.2: Let $S_h^{k,L}$ be a finite element space as discussed in Chapter V according to a $x$-regular triangulation. Further assume the one-sided constraint $a = \gamma h$. There is a constant $c$ depending only on $a, N$ and $k, x, v, K$ such that for any $\varphi \in S_h^{k,L}$

\begin{equation}
\|\varphi\|_{L^\infty} \leq c \gamma^2 h^{-N/2} \sup_{x_0 \in \Omega} \|\varphi\|_{a}
\end{equation}

holds true.

Remark: We will couple $\varphi$ with $h$ by a condition $a = \gamma h$.

Then the $a$- and the $L^\infty$-norms are up to a factor $c(\gamma)h^{N/2-a}$ equivalent in the spaces $S_h$.

Proof of Proposition IX.1:

It is

\begin{equation}
\|\nu\|_a^2 \leq \|\nu\|_{L^\infty}^2 \int_\Omega u^{-a} \, dx
\end{equation}

and

\begin{equation}
\int_\Omega u^{-a} \, dx \leq \int_\Omega u^{-a} \, dx < \nu_N N(N-1) \frac{r^{N-1}}{r^2 \gamma^2 h^2 \gamma}.
\end{equation}

Here $r$ denotes the distance $|x-x_0|$ and $\nu_N$ is the volume of the $N$-dimensional unit sphere. We may replace

\begin{equation}
r^{N-1} \leq r(r^2 + \delta^2)^{N/2-1}
\end{equation}

giving in case of $a > N/2$

\begin{equation}
\int_\Omega u^{-a} \, dx \leq \frac{1}{2} \nu_N \int_0^\infty \int_{1-a}^{N/2-1-a} \frac{r^{N-1} \, dr}{r^2 h^2 \gamma} < \nu_N \gamma \gamma^2 h^2 \gamma.
\end{equation}

Proof of Proposition IX.2: Let $\varphi \in S_h$ be fixed and let $\Lambda_0$ be one triangle of the subdivision $\Gamma_h$ such that $|\varphi|$ attains its maximum in $\Lambda_0$, i.e. such that

\begin{equation}
\|\varphi\|_{L^\infty} = \|\varphi\|_{L^\infty}(\Lambda_0) = \|\varphi\|_{L^\infty}(\Lambda_0).
\end{equation}

Now by the arguments of Chapter V, see page V.11, we find

\begin{equation}
\|\tau\|_{L^\infty}(\Lambda_0) \leq c(x, t, N, K) h^{-N/2} \|\tau\|_{L^2}(\Lambda_0).
\end{equation}

For $x \in \Lambda_0$, we have because of $x_0 \in \Lambda_0$

\begin{equation}
\gamma^2 \leq u(x) \leq \gamma^2 + 4h^2 \leq (1+\gamma^2)h^2.
\end{equation}

This gives us for both cases $\gamma \geq 0$ resp. $\gamma < 0$ an estimate of the type

\begin{equation}
\|\varphi\|_{L^2}(\Lambda_0) \leq c(\gamma, \gamma) \gamma^2 \int_\Lambda_0 u^{-a} \, dx \leq c(\gamma, \gamma) \gamma^2 \gamma \tau_a^2.
\end{equation}

It remains to combine (IX.10), (IX.11), and (IX.13).

In the sequel it will be essential that the weight function $u$ does not change too fast in any triangle:

Lemma IX.3: Let $\Lambda$ be a triangle of diameter not exceeding $2h$ and assume at least $\gamma \geq 2h$. Then

\begin{equation}
\overline{u}_{\Lambda} := \max\{u(x) \mid x \in \Lambda\} \leq \delta \frac{\Lambda}{\min\{u(x) \mid x \in \Lambda\}}.
\end{equation}

IX.3
Proof: Let \( \vec{x}, \vec{x} \in \Delta \) be chosen according to

\[(IX.15)\quad u(\vec{x}) = u_\Delta, \quad u(\vec{x}) = \bar{u}_\Delta.\]

Then we have

\[(IX.16)\quad \bar{u}_\Delta - u_\Delta = (\vec{x} - \vec{x}) \cdot v_u.\]

The (euclidian) norm of \( v_u \) is bounded by \( 2\sqrt{u_\Delta} \) and the distance \( |\vec{x} - \vec{x}| \) by \( 2h \). Schwarz's inequality gives then

\[(IX.17)\quad \bar{u}_\Delta \leq u_\Delta + 4h \sqrt{\bar{u}_\Delta} = u_\Delta + \frac{1}{2} \bar{u}_\Delta + 8h^2.\]

Finally we have

\[(IX.18)\quad 8h^2 \leq 2\sigma \leq 2u_\Delta.\]

Next we will show the validity of the counterparts of Theorem V.4, Corollary V.5 and Theorem V.7. We will use the notations of Chapter V, the weighted 'broken' norms \( \| \cdot \|_\alpha \) are defined in the natural way. Further the assumptions 1.-3. of Chapter V should hold.

Theorem IX.4: In the space \( S_h = S_h^{1,\tau}(\Gamma_h) \) to any \( u \) with \( \tau \leq t \) assumed

\[(IX.19)\quad u \in W_h^1 \cap C^\infty(\Omega)\]

there is a \( \bar{u} \in S_h \) according to

\[(IX.20)\quad \| u^{k,u}(u) \|_\alpha \leq c h^{t-k} \| \bar{u} \|^2_{\alpha} \]

for all \( 0 \leq k < t \) with \( c \) depending on \( N, n, x, t \) and \( K \) (the curvature of \( \Gamma_0 \)).

Proof: Because of \( S_h^{1,\tau} \subseteq S_h^{1,\tau} \) we may look for the interpolation of \( u \) in the space \( S_h^{1,\tau} \). In any \( \Delta \in \Gamma_h \) we have - see (V.28)

\[(IX.21)\quad \| u^{k,u}(u) \|^2_{L^2(\Delta)} \leq c h^{2(t-k)} \| \bar{u} \|^2_{L^2(\Delta)}.\]

Because of Lemma IX.3 we get at once

\[(IX.22)\quad \int_{\Delta} \int_{\Delta} u^{k,u}(u) \|^2_{L^2(\Delta)} \leq c h^{2(t-k)} \int_{\Delta} \int_{\Delta} \bar{u} \|^2_{L^2(\Delta)} dx.\]

and by summation over all \( \Delta \in \Gamma_h \) finally (IX.20).

The proof of the next theorem follows the same lines and is omitted here.

Theorem IX.5: In the spaces \( S_h^{k,\tau} \) inverse relations of the type

\[(IX.23)\quad \| u^{m,u} \|_{L^2(\Gamma_h)} \leq c h^{n} \| \bar{u} \|^2_{\alpha} \]

for \( \chi \in S_h^{k,\tau} \) and \( 0 \leq n < m \leq t-1 \) hold true. The constant \( c \) differs from that in (V.36) by the factor \( 6|\alpha|/2 \).
In order to make the technique more transparent we will consider the case of the $L_2$-projection firstly. It is defined by:

Find $\varphi := P_h u \in S_h$ such that

\begin{equation}
(\varphi, \chi) = (u, \chi) \quad \text{for} \quad \chi \in S_h.
\end{equation}

(IX.24)

In view of Propositions IX.1,2 we try to get estimates of $\varphi$ in weighted norms.

By the aid of (IX.24) we get with $\chi \in S_h$ arbitrary

\begin{equation}
\|\varphi\|_a^2 = (\varphi, \varphi - \chi) - (u, \varphi - \chi) + (u, \varphi).
\end{equation}

(IX.25)

Now we will make use of the following modified versions of Schwarz's inequality

\begin{equation}
|\langle v, w \rangle_a| = |\langle v, u - \varphi \rangle| \leq \|v\|_a \|w\|_a,
\end{equation}

(IX.26)

\begin{equation}
|\langle v, w \rangle_a| \leq \|v\|_a \|w\|_{-a}.
\end{equation}

(IX.27)

Applying these we come to

\begin{equation}
\|\varphi\|_a^2 \leq (\|\varphi\|_a + \|u\|_a)\|\varphi - \chi\|_a - \|\varphi\|_a \|u\|_a.
\end{equation}

(IX.28)

The right hand side still depends on $\chi$ to be chosen properly in $S_h$. Essential for the method of weighted norms is a superapproximability property of finite elements - see also Lemma VIII.5.

Lemma IX.6: There is a constant independent of $h, \rho$ such that

\begin{equation}
\inf_{\chi \in S_h} \|u - \varphi - \chi\|_{-a} \leq c \frac{h}{\rho} \|\varphi\|_a.
\end{equation}

(IX.29)

Proof: Let us think of $S_h = S_h^{1,t}$. Then Theorem IX.4 gives with $\chi = I_h u$

\begin{equation}
\inf_{\chi \in S_h} \|u - \varphi - \chi\|_{-a} \leq c \frac{h}{\rho} \|\varphi - (u - \varphi)\|_{-a}.
\end{equation}

(IX.30)

In any $A \in \Gamma_h$ since $\varphi$ restricted to $A$ is a polynomial of degree less than $t$ we have

\begin{equation}
|\varphi(x) - \varphi(0)| \leq c \sum_{0}^{t-1} |\varphi^\tau|_{1,\tau} |u^{\tau-t}|_0.
\end{equation}

(IX.31)

Since evidently

\begin{equation}
|\varphi(x) - \varphi(0)| \leq c \frac{1}{2} |u^{\tau-t}|_0,
\end{equation}

(IX.32)

we have

\begin{equation}
\sum_{A} \int_A |\varphi(x) - \varphi(0)|^2 \leq c \sum_{A} \int_A |u^{\tau-t}|^2 \ dx.
\end{equation}

(IX.33)

and after summation

\begin{equation}
\|\varphi\|_{-a} \leq c \sum_{A} |\varphi(0)|^2.
\end{equation}

(IX.34)

On the one hand we may apply Theorem IX.5 and on the other hand the estimate

\begin{equation}
\|v\|_{a+a} \leq c^{-a} \|v\|_a
\end{equation}

(IX.35)

valid for $a > 0$. This leads to
By the aid of Propositions IX.1,2 and (IX.40) we get

**Theorem IX.8:** The $L_2$-projection has bounded norm in $L_\infty$, i.e.

\[(IX.41)\quad \|P_hu\|_{L_\infty} \leq c \|u\|_{L_\infty} \]

for any $u \in L_\infty$.

Remark: Considering Theorem II.3 we have as a consequence

\[(IX.42)\quad \|u-P_hu\|_{L_\infty} \leq c \inf_{\chi \in \mathcal{D}_h} \|u-\chi\|_{L_\infty} ,\]

i.e. the $L_2$-projection is almost best approximating in $L_\infty$.

The $L_\infty$-analysis of Ritz- resp. Galerkin-procedures follows the same lines but of course then more technical details are necessary. Especially shift theorems in weighted norms are needed. They will be given in the next chapter together with the $L_\infty$-analysis in the general case. Here we will restrict ourselves to the situation discussed at the beginning of Chapter VII: We consider the Dirichlet problem for the Laplace equation in a convex polygonal domain in the plane and the corresponding Ritz method using linear finite elements.

The defining relation of the Ritz approximation $\varphi := u_h = \sum n \in \mathcal{D}_h$ is - see (VI.3)

\[(IX.43)\quad D(\varphi, \chi) = D(u, \chi) \quad \text{for} \quad \chi \in \mathcal{D}_h .\]

Similar to the treatment of the $L_2$-projection we will derive bounds in weighted norms but this time for $\varphi$ as well as for the gradient of $\varphi$. From

\[(IX.44)\quad \|\varphi\|_a^2 = D(\varphi, \mu^{-a}\varphi) - \iint \varphi \mu\varphi \mu^{-a} .\]
we get by partial integration because of \( \psi = 0 \) on \( \partial \Omega \)

\[
\|\psi\|^2_\alpha = D(\phi, \mu^{-1}\phi) + \frac{1}{2} \iint \phi^2_\alpha \mu^{-1} \tag{IX.45}
\]

and further because of \( |\Delta \mu^{-1}| < c_1 \mu^{-1-\alpha} \)

\[
\|\psi\|^2_\alpha \leq D(\phi, \mu^{-1}\phi) + c_1 \|\psi\|^2_{\alpha+1} \tag{IX.46}
\]

**Remark:** In order to have the constants under control we will number them. They are independent of \( h, \rho \) but may depend on \( \kappa \).

With \( \chi \in C^\infty_0 \) arbitrary we have

\[
D(\phi, \mu^{-1}\phi) = D(\phi, \mu^{-1}\phi - \chi) - D(u, \mu^{-1}\phi - \chi) + D(u, \mu^{-1}\phi) \tag{IX.47}
\]

leading to - see (IX.28)

\[
\|\psi\|^2_\alpha \leq \left( \|\psi\|_\alpha + \|\nu\|_\alpha \right) \|\nu^2(\mu^{-1}\phi - \chi)\|_{-\alpha} + \|\nu\|_\alpha \|\nu(\mu^{-1}\phi)\|_{-\alpha} + c_1 \|\psi\|^2_{\alpha+1} \tag{IX.48}
\]

By simple calculation we find

\[
\|\nu(\mu^{-1}\phi)\|_{-\alpha} \leq 2 \|\nu\|_\alpha + c_2 \|\psi\|_{\alpha+1} \tag{IX.49}
\]

and hence for the middle term

\[
\|\nu\|_\alpha \|\nu(\mu^{-1}\phi)\|_{-\alpha} \leq \frac{1}{2} \|\psi\|^2_\alpha + 3 \|\nu\|^2_\alpha + c_3 \|\psi\|^2_{\alpha+1} \tag{IX.50}
\]

leading to

\[
\|\psi\|^2_\alpha \leq 2\left( \|\psi\|_\alpha + \|\nu\|_\alpha \right) \|\nu(\mu^{-1}\phi - \chi)\|_{-\alpha} + 6 \|\nu\|^2_\alpha + c_4 \|\psi\|^2_{\alpha+1} \tag{IX.51}
\]

Now we need the counterpart of Lemma IX.6:

**Lemma IX.91** There is a constant \( c_5 \) independent of \( h, \rho \) such that for \( 2h < \rho \)

\[
\inf_{\chi \in C^\infty_0} \|\psi(\mu^{-1}\phi - \chi)\| \leq c_5 \frac{h}{\rho} \left( \|\psi\|_\alpha + \|\psi\|_{\alpha+1} \right) \tag{IX.52}
\]

**Proof:** We have - see (IX.30) - (IX.34)

\[
\inf_{\chi \in C^\infty_0} \|\psi(\mu^{-1}\phi - \chi)\| \leq c_5 h^{-1} \sum \|\psi\|_{\alpha+t-\tau} \tag{IX.53}
\]

The term with \( \tau = 0 \) can be estimated by

\[
\|\psi\|_{\alpha+t} \leq \rho^{-t} \|\psi\|_{\alpha+1} \tag{IX.54}
\]

whereas

\[
\|\psi\|_{\alpha+t} \leq c \|\psi\|_{\alpha} \tau^{-1} \|\psi\|_{\alpha+1} \tag{IX.55}
\]

After these preparations we go back to (IX.51). If we impose the condition

\[
\rho \geq \gamma_1 h \text{ with } \gamma_1 = \frac{4c_5}{5} \tag{IX.56}
\]

then the quadratic term \( \|\psi\|^2_\alpha \) on the right hand side has a factor less than \( 1/2 \). This finishes

**Step 1:** Let \( \rho \geq \gamma_1 h \). Then

\[
\|\psi\|^2_\alpha \leq c_6 \left( \|\nu\|^2_\alpha + \|\psi\|^2_{\alpha+1} \right) \tag{IX.57}
\]

Next we try to eliminate \( \|\psi\|_{\alpha+1} \) on the right hand side by a duality argument. Let the function \( w \) be defined by
Then we have

\[(IX.59)\]
\[\|\psi\|_{a+1}^2 = \langle \psi', \Delta \psi \rangle = D(\psi, w)\]

and with any \(x \in S_h\)

\[(IX.60)\]
\[\|\psi\|_{a+1}^2 = D(\psi, w - x) - D(u, w - x) + D(u, w).\]

The last term may be rewritten in the form

\[(IX.61)\]
\[D(u, w) = (u, \Delta w) = (u, \varphi)_{a+1}\]

and estimated by

\[(IX.62)\]
\[D(u, w) \leq \frac{1}{2} \|\psi\|_{a+1}^2 + \frac{1}{2} \|u\|_{a+1}^2.\]

In this way we get from (IX.60)

\[(IX.63)\]
\[\|\psi\|_{a+1}^2 \leq 2 \left( \|\psi\|_{a}^2 + \|u\|_{a}^2 \right) \|\varphi(w)\|_{-a} + \|u\|_{a+1}^2\]

and with \(\delta > 0\) arbitrary

\[(IX.64)\]
\[\|\psi\|_{a+1}^2 \leq \delta \|\psi\|_{a}^2 + \delta \|\psi\|_{a}^2 + \|u\|_{a+1}^2 + \delta^{-1} \|\varphi(w)\|_{-a}^2.\]

Now we compare this with (IX.57). We choose \(\delta = 1/(2c_\delta)\).

An obvious consequence is

**Step 2:** Let \(\rho = \gamma h\). Then

\[(IX.65)\]
\[\|\psi\|_{a}^2 + \|\psi\|_{a+1}^2 \leq c_\gamma \left( \|\psi\|_{a}^2 + \|u\|_{a+1}^2 + \|\varphi(w)\|_{-a}^2 \right).\]

Since the function \(w\) depends on \(\varphi\) an additional step is necessary. We work with linear finite elements. Then \(x \in S_h\) may be chosen according to

\[(IX.66)\]
\[\|\varphi(x)\|_{-a} \leq c_\varphi \|\varphi\|_{-a}.\]

At the end of this chapter we will prove the remaining

**Step 3:** Let \(a = 1\). For any \(w \in S_2(n) \cap V_2(n)\) the inequality holds

\[(IX.67)\]
\[\|\varphi w\|_{-a} \leq c_\varphi s^{-1} |\ln p|^{1/2} \|w\|_{a-1}^2.\]

Since we have by the definition of \(w\)

\[\|\varphi w\|_{a-1} = \|\varphi\|_{a+1}\]

we get from (IX.65)

\[(IX.68)\]
\[\|\psi\|_{a}^2 + \|\psi\|_{a+1}^2 \leq c_\gamma \left( \|\psi\|_{a}^2 + \|u\|_{a+1}^2 \right) + c_\gamma c_\varphi s^2 h^2 s^{-2} \|\ln p\| \|\psi\|_{a+1}^2.\]

Finally we impose the condition

\[(IX.69)\]
\[\rho = \gamma h \left| \ln h \right|^{1/2}.\]

We may choose \(\gamma\) such that

\[(IX.70)\]
\[c_\gamma c_\varphi h^2 s^{-2} |\ln p| < \frac{1}{20}\]

and therefore

\[(IX.71)\]
\[\|\psi\|_{a}^2 + \|\psi\|_{a+1}^2 \leq 2c_\gamma \left( \|\psi\|_{a}^2 + \|u\|_{a+1}^2 \right).\]

This relation corresponds to (IX.40). With the help of Propositions IX.1, IX.2 we get this time since \(a = 1 = N/2\)
and \( p \) is defined by (IX.69)
\[
\|\varphi\|_{L_1}^2 \leq c \rho^2 \|\varphi\|_{L_1} + \frac{\rho^2}{3} \|\varphi\|_{L_1}^2 
\]
and
\[
\|u\|_{\alpha+1}^2 \leq c \rho^2 \|u\|_{L_1}^2 
\]
\[
\|\varphi u\|_{\alpha}^2 \leq c |\ln\rho| \|\varphi u\|_{L_1}^2 .
\]
This leads us to

**Theorem IX.10**: The \( L_\infty \)-norm of the Ritz approximation is bounded by those of the solution and its gradient in the form
\[
\|u - R_h u\|_{L_\infty} \leq c |\ln h|^{3/2} \left\{ \|u\|_{L_1} + h \|\varphi u\|_{L_1} \right\} .
\]

**Remark**: As before we get at once
\[
\|u - R_h u\|_{L_\infty} \leq c |\ln h|^{3/2} \inf_{u \in B_h} \left\{ \|u - u_x\|_{L_1} + h \|\varphi (u - u_x)\|_{L_1} \right\} .
\]

It remains to prove the assertions of Step 3. Firstly we give a special shift-theorem in weighted norms.

**Lemma IX.11**: Let \( a = 1 \). For any \( w \in \tilde{H}_2 = \tilde{H}_2^1(\Omega) \cap \tilde{H}_2^2(\Omega) \) then
\[
\|v w\|_{L_1}^2 \leq c \left\{ \|\Delta w\|_{L_1}^2 + \|w\|_{H_1}^2 \right\} .
\]

**Proof**: We remark that the standard shift theorem - see (II. )
\[
\|v w\| \leq c \|\Delta w\|
\]
holds for convex polygons.

We use a translation such that \( x_0 = 0 \). Then we have (with \( x = x_1, y = x_2 \))
\[
\|v^2 w\|_{L_1}^2 = \int \left( \rho^2 + x^2 + y^2 \right) \left( w^2 + 2 x^2 y^2 + w_{xx}^2 \right) .
\]
We will consider only the terms with \( w_{xx} \) in detail. Obviously we have
\[
\rho^2 \int w_{xx}^2 \leq c \rho^2 \|v^2 w\|_{L_1}^2
\]
\[
\leq c \rho^2 \|\Delta w\|_{L_1}^2 \leq c \|\Delta w\|_{L_1}^2 .
\]
Next because of
\[
\|x w_{xx}\| = \|x w\|_{L_1}^2 + 2 w_x^2
\]
we have
\[
\int \int x^2 w_{xx}^2 \leq \int \int (x w_{xx})^2 + 2 \int \int w_x^2
\]
\[
\leq 2 \|\| \left( x w_{xx} \right) \|_{L_1}^2 + 8 \|\| w_x^2 \|
\]
\[
\leq 2 c^2 \|\Delta (x w_{xx})\|_{L_1}^2 + 8 \|\| w_x^2 \|
\]
and this leads because of
\[
\|x w_{xx}\| = x \|\Delta w\| + 2 w_x
\]
similarly to
\[
\|x w_{xx}\|_{L_1}^2 \leq c \left\{ \|x w\|_{L_1}^2 + \|w_x\|_{L_1}^2 \right\}
\]
\[
\leq c \left\{ \|w\|_{L_1}^2 + \|w_x\|_{L_1}^2 \right\} .
\]
Since

\[ \| \omega \|_{-\alpha} \leq \rho^{-1} \| \omega \|_{-\alpha-1} \]

we have with Lemma IX.11

\[ \| \nu \|_{-\alpha}^2 \leq \rho^{-2} \| \omega \|_{-\alpha-1}^2 + \| \nu \|_{-\alpha}^2 \]

Therefore in order to prove (IX.67) it is sufficient to show the a priori estimate

\[ \| \nu \|_{-\alpha}^2 \leq \alpha_0 \rho^{-2} |\ln \rho| \| \omega \|_{-\alpha-1}^2 \]

for all functions \( \nu \in \mathcal{H}_2 \).

Let us introduce the quantity

\[ \lambda^{-1} = \lambda^{-1}(\Omega) = \sup_{\nu \in \mathcal{H}_2} \frac{\| \nu \|_{-2}^2}{\| \nabla \nu \|_{-2}^2} \]

Our aim is to bound \( \lambda^{-1} \) up to a numerical constant by \( \rho^{-2} |\ln \rho| \). This is done in the following steps: (i) We characterize \( \lambda \) being the smallest eigen-value of the problem

\[ -\Delta v = \lambda \mu^{-2} v \quad \text{in} \quad \Omega, \]

\[ v = 0 \quad \text{on} \quad \partial \Omega \]

and show that the eigen-function \( v \) and \( -\Delta v \) are necessarily non-negative. (ii) We prove the monotonicity of \( \lambda \) with respect to domains, i.e. we prove that \( \lambda(\Omega_1) \leq \lambda(\Omega_2) \) is a consequence of \( \Omega_2 \subseteq \Omega_1 \). (iii) We derive an explicit bound for \( \lambda \) for a circle with center in \( x_0 \) and radius \( d = \text{diam}(\Omega) \) thus getting because of (ii) the needed estimate (IX.67).

\[ \{ \} \]

There is an extremal sequence \( v_k = \hat{v}_k \), i.e. such that (for convenience)

\[ \| \nabla v_k \|_{-2} = 1 \]

and

\[ \| \nu_k \|_{-2}^2 \rightarrow \lambda^{-1} \]

The norm \( \| \Delta \|_{-2} \) is equivalent to the Sobolev-norm in \( \mathcal{H}_2 \).

Since \( H_1 = k_2^1(\Omega) \) is compactly embedded in \( H_2 \) there is a subsequence converging to an element \( \hat{v} \in \mathcal{H}_2 \) weakly in the norm of \( H_2 \) but strongly in the norm of \( H_1 \):

Renumbering we have

\[ v_k \rightarrow \hat{v} \quad \text{in} \quad H_1 \]

\[ v_k \rightarrow \hat{v} \quad \text{in} \quad H_2 \]

Because of - see (I.)

\[ \| \nabla \hat{v} \|_{-2} = \lim \| \nabla v_k \|_{-2} = 1 \]

and

\[ \| \nu \|_{-2}^2 = \lim \| \nu_k \|_{-2}^2 = \lambda^{-1} \]

we come to

\[ \| \nu \|_{-2}^2 \rightarrow \lambda^{-1} \]

But since \( \lambda^{-1} \) is the supremum the equality sign must hold in

\[ (IX.94) \]
Now let \( z \in \mathcal{H}_2 \) be arbitrary and put \( v = \bar{v} + tz \) with \( t \in \mathbb{R} \).

The function

\[
\psi(t) = \frac{\|v(\bar{v} + tz)\|^2}{\|\Delta(\bar{v} + tz)\|^2}
\]

attains its maximum for \( t = 0 \). A simple calculation gives

\[
\psi'(0) = \frac{d\psi}{dt}|_{t=0} = \frac{2}{\|\Delta \mathcal{V}\|^2} \left\{ (\mathcal{V}, v) - \frac{\|\mathcal{V}\|^2}{\|\Delta \mathcal{V}\|^2} (\Delta \mathcal{V}, \Delta \mathcal{V}) \right\}
\]

With (IX.94) in mind and because of

\[
(\mathcal{V}, v) = (\mathcal{V}, -\Delta \mathcal{V})
\]

we get from \( \psi'(0) = 0 \)

\[
(\bar{v} + \lambda^{-1} \mu \Delta \mathcal{V}, \Delta \mathcal{V}) = 0
\]

Since \( z \in \mathcal{H}_2 \) resp. \( \Delta \mathcal{V} \in L_2 \) is arbitrary \( \mathcal{V} \) is a solution of the eigen-value problem (IX.88).

Now assume \( \mathcal{V} \) and hence \( \Delta \mathcal{V} \) attain values of different sign. Then let us consider the function \( \bar{\mathcal{V}} \) defined by

\[
\Delta \bar{\mathcal{V}} = |\Delta \mathcal{V}| \quad \text{in} \quad \Omega
\]

\[
\bar{\mathcal{V}} = 0 \quad \text{on} \quad \partial \Omega
\]

Because of the maximum principle (II. ) then

\[
0 \leq |\bar{\mathcal{V}}| \leq \mathcal{V}
\]

and \( \bar{\mathcal{V}} \neq \mathcal{V} \). On the one hand we have

\[
\|\Delta \mathcal{V}\|^2_{\mathcal{H}_2} = \|\Delta \mathcal{V}\|^2_{\mathcal{H}_2}
\]

but on the other

\[
\|v \mathcal{V}\|^2 = (\mathcal{V}, -\Delta \mathcal{V}) > (\mathcal{V}, -\Delta \mathcal{V}) = \|\mathcal{V}\|^2
\]

\( \therefore \). In order to prove the monotonicity of \( \lambda \) let \( \Omega_1, \Omega_2 \) be domains with \( \Omega_2 \subset \Omega_1 \). Further let \( \bar{\mathcal{V}}_2 \) be the eigen-function corresponding to the smallest eigen-value \( \lambda_2 = \lambda(\Omega_2) \).

This time we consider the function \( \mathcal{V} \) defined by

\[
-\Delta \mathcal{V} = \begin{cases} |\Delta \bar{\mathcal{V}}_2| & \text{in} \quad \Omega_2, \\ 0 & \text{in} \quad \Omega_1 - \Omega_2, \\ \end{cases}
\]

\[
\mathcal{V} = 0 \quad \text{on} \quad \partial \Omega_1
\]

Because of the maximum principle it is

\[
\mathcal{V} > 0 \quad \text{in} \quad \Omega_1
\]

The difference \( \mathcal{V} = \mathcal{V} - \bar{\mathcal{V}}_2 \) is harmonic in \( \Omega_2 \) and positive on \( \partial \Omega_2 \) and hence also positive in \( \Omega_2 \). This leads to

\[
\|\mathcal{V} \|^2_{\mathcal{H}_1} = \iint_{\Omega_2} \mathcal{V}(-\Delta \bar{\mathcal{V}}_2)
\]

\[
> \iint_{\Omega_2} \bar{\mathcal{V}}_2(-\Delta \bar{\mathcal{V}}_2) = \|\bar{\mathcal{V}}_2\|^2_{\mathcal{H}_2}
\]

whereas by construction

\[
\|\Delta \mathcal{V}\|^2_{\mathcal{H}_2} = \|\Delta \mathcal{V}\|^2_{\mathcal{H}_2}
\]

\#
Now let \( K = K_0(x_0) \) be the circle with center \( x_0 \) and radius \( d = \text{diam}(\Omega) \). Then \( \bar{\Omega} \subseteq K \) and by (ii) we know the \( \lambda = \lambda(\Omega) \geq 1 \) is a solution of \( (\text{IX.107}) \)

\[
\lambda = \lambda(\Omega) = \lambda(K) = \lambda.
\]

To \( \lambda \) there corresponds a solution \( v \) of

\[
-\Delta v = \lambda v^2 \quad \text{in} \quad K,
\]

\[
v = 0 \quad \text{on} \quad \partial K
\]

which is positive. Actually \( v \) is unique and depends only on \( r = |x - x_0| \). It is sufficient for us to show that there is a \( v \) depending only on \( r \) and being also a solution of \( (\text{IX.108}) \). In order to get this let us introduce polar-coordinates \( r, \theta \) by

\[
x = x_0 + r \cos \theta, \quad y = y_0 + r \sin \theta
\]

and define

\[
\varphi(r) = \frac{1}{2\pi} \int_0^{2\pi} v(x_0 + r \cos \theta, y_0 + r \sin \theta) \, d\theta.
\]

Now integrating the eigenvalue equation \( (\text{IX.108}) \) with respect to \( \theta \) gives because of

\[
\Delta \varphi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2},
\]

and the periodicity of \( v \) resp. \( v |_{\bar{\Omega}} \) the desired result since of course \( \mu \) depends only on \( r \).

Now we can go back to the original extremum problem \( (\text{IX.87}) \) but this time for the circle \( K \) and for functions \( w \in H_0(K) \) depending only on \( r \). Such functions have the property

\[
(\text{IX.112}) \quad w'(0) = 0.
\]

We denote for the moment \( -\Delta w \) by \( z \), i.e.

\[
(\text{IX.113}) \quad - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial z}{\partial r} \right) = z.
\]

Then we find

\[
(\text{IX.114}) \quad -w'(r) = \frac{1}{r} \int_0^r z(s) \, ds
\]

Since we have

\[
(\text{IX.115}) \quad \|\Delta w\|_2^2 = 2\pi \int_0^d r \mu^2 z^2(r) \, dr
\]

we apply Schwarz' inequality for the integral \( (\text{IX.114}) \) in the form

\[
(\text{IX.116}) \quad |w'(r)|^2 \leq \frac{1}{r^2} \left\{ \int_0^r \mu^2 z^2(s) \, ds \right\} \left\{ \int_0^r \mu^{-2} \, ds \right\}
\]

Because of

\[
(\text{IX.117}) \quad \int_0^r s \mu^{-2} \, ds = \frac{1}{2} (\mu^{-1}(0) - \mu^{-1}(r)) = \frac{r^2}{2p \mu}
\]

we get

\[
(\text{IX.118}) \quad |w'|^2 \leq \frac{1}{2p \mu} \int_0^d r \mu^2 z^2 \, ds
\]

This leads to

\[
\|w\|_2^2 = 2\pi \int_0^d r w^2 \, dr
\]

\[
\leq \frac{\pi}{p} \int_0^d r \mu^{-1} \, dr \int_0^d s \mu^2 z^2 \, ds
\]
and by interchanging the order of integration

\[ \|w\|_2^2 \leq \frac{H}{\mu} \int_0^d \int_0^{\mu^2} r^2 \, ds \int_0^d r \mu^{-1} \, dr \]

The inner integral is bounded by

\[ \int_0^d r \mu^{-1} \, dr = \frac{1}{2} \ln \frac{H(d)}{\mu(0)} \]

(IX.121)

\[ \leq c(d)|\ln \rho| \]

for \( \rho < 1 \). Together with (IX.115) we get the essential a priori estimate (IX.67) of Step 3.