FINITE ELEMENT APPROXIMATIONS FOR SOLVING
THE ELASTIC PROBLEM

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Finite element approximations for the first boundary
value problem of elasticity are given which allow to use
subspaces of functions not vanishing on the boundary. \( L^2 \)
and \( L^\infty \) error estimates are derived.

1. The boundary value problem, variational formulation

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with boundary \( \partial \Omega \)
sufficiently smooth. We will work with vectors \( \mathbf{v} = (v_1, v_2) \).
In case \( v_1 \in L^2 = L^2(\Omega) \) we write \( \mathbf{v} \in L^2 = L^2 \times L^2 \). The
meaning of \( W^1_2 \) etc. is analogue. For simplicity we will
also use the notation \( H_1 = W^1_2 \), \( H_2 = H_1 \cap L^2 \). Correspond-
ingly we define

\[
(u, v) = (u_1, v_1) \quad \|u\| = (u, u)^{1/2}.
\]

(The summation convention is used throughout the paper).
To a displacement-vector \( \mathbf{v} \) are associated the two
tensors:

\[
\varepsilon_{ik}(\mathbf{v}) = v_{1,k} + v_{k,1},
\]

\[
\sigma_{ik}(\mathbf{v}) = \lambda (v_{j,j}) \delta_{ik} + 2\mu \varepsilon_{ik}.
\]
Here \( \#_1 \) denotes the partial derivatives, \( \delta_{ik} \) is the Kronecker symbol and \( \lambda \geq 0 \), \( \mu > 0 \) are the Lame-constants. The first boundary value problems of elasticity is given \( f \in L^2 \), find \( u \in H^2 \) such that

\[
(1) \quad - \nabla \sigma(u) = f \quad : \quad - \sigma_{1k,k}(u) = f_1 \quad \text{in } \Omega .
\]

We mention the shift theorem

**THEOREM 1:** For \( f \in L^2 \) the solution \( u \in H^2 \) exists uniquely and

\[
(2) \quad \|u\|_{H^2} \leq c \|f\|_{L^2} .
\]

Here and later \( c \) is a numerical constant which may differ at different places.

The solution of (1) is equivalently characterized by

\[
(3) \quad u \in H^1 : \quad a_o(u,v) = (f,v) \quad \text{for } v \in H^1
\]

with

\[
(4) \quad a_o(v,w) = (\sigma_{1k}(v), \epsilon_{1k}(w))
\]

\[
= \iint_{\Omega} \left\{ \lambda (v_{1,1}) (w_{k,k}) + 2\mu \epsilon_{1k}(v) \epsilon_{1k}(w) \right\} dx .
\]

The form \( a_o \) is symmetric, bounded and because of Korn's inequality coercive in \( H^1 \). As long as we are in \( H^1 = W^1_0 \) \( a_o \) in (3) can be modified without influencing the solution \( u \) by - \( n \) is the normal vector of \( \partial \Omega \) -

\[
(5) \quad a_1(v,w) = a_o(v,w) - \oint_{\partial \Omega} n_1 \{ \sigma_{1k}(v) w_k + \sigma_{1k}(w) v_k \} ds ,
\]

\[
(6) \quad a_2(v,w) = a_1(v,w) + K h^{-1} \oint v_1 w_1 ds .
\]

These terms are motivated because of

**LEMMA 1:** Let \( u \) be the solution of (1) and \( w \in W^1_0 \).

Then for \( i = 1,2 \)

\[
(6) \quad a_i(u,w) = (f,w) .
\]
This relation is essential in deriving $L_2$ and $L_\infty$ estimates, it is not true for the form $a_0$.

2. Finite elements

By $\Gamma_h$ a $\gamma$-regular subdivision of $\Omega$ with mesh-seize $h$ into generalized triangles will be denoted: For any $\Delta \in \Gamma_h$ there are two spheres $K, \bar{K}$ with radii $r, \bar{r}$ such that $K \subset \Delta \subset \bar{K}$ and $\gamma^{-1} h \leq r < \bar{r} \leq h$ (for more details see CIARLET-RAVIART [1]).

Besides the usual Sobolev-norms we will need certain weighted norms. Let $x_0 \in \Omega$ and $\rho > 0$. We use the weight-factor

$$p_\alpha(x) = \mu(x)^{-\alpha} \quad \text{with} \quad \mu(x) = |x-x_0|^2 + \rho^2$$

and define for any $\Omega' \subseteq \Omega$

$$\|v\|_{\alpha, \Omega'} = \left\{ \int_{\Omega'} p_\alpha v^2 \, dx \right\}^{1/2}$$

$$(7)  \quad \|v^k\|_{\alpha, \Omega'} = \left\{ \sum_{|\kappa|=k} \|D^\kappa v\|_{\alpha, \Omega'}^2 \right\}^{1/2}.$$  

In case $\Omega' = \Omega$ we simply write $\|v\|_{\alpha}$. The scalar-products are denoted by $(\ldots)_\alpha$. If $T \subseteq \Omega$ is a curve we use for the corresponding integrals the notation $|\cdot|_{\alpha, T}$ resp. $<\ldots,>_{\alpha, T}$ and drop $T$ in case of $T = \partial \Omega$.

The functions we work with will have a reduced regularity across the edges of $\Gamma_h$. Therefore we introduce the spaces $\mathbb{W}_2^k$ of functions $v$ with $v|_{\Delta} \in \mathbb{W}_2^k(\Delta)$ for $\Delta \in \Gamma_h$ and define

$$(8)  \quad \|v^k\|_{\alpha}^h = \left\{ \sum_{\Delta \in \Gamma_h} \|v^k\|_{\alpha, \Delta}^2 \right\}^{1/2}.$$  

For simplicity we will consider in this paper only linear finite element spaces $S_h$, i.e. any $x \in S_h$ is continuous in $\Omega$ and piece-wise linear in $\Delta \in \Gamma_h$.  

$S_h \subseteq S^0_h$ is the subspace of functions vanishing in the
nodes of $\Gamma_h$ which are on $\partial \Omega$. The standard properties of
$S^0_h$ resp. $S_h$ used in the next sections are summarized
in

**THEOREM 2:** There is a constant $\gamma_1$ such that for any
$\gamma$-regular subdivision $\Gamma_h$ and any $\rho$ with $\rho \geq \gamma_1 h$ the
propositions hold:

1. To any $v \in W^1 \cap W^2_h (k=1,2)$
   there is a $\chi \in S^0_h$ with

   \[ \|v - \chi\|_\alpha + h \|v(v - \chi)\|_\alpha \leq c_1(\alpha) h^k \|v^k v\|_\alpha. \]

2. For any $\chi \in S_h$

   \[ \|v \chi\|_\alpha \leq c_2(\alpha) h^{-1} \|\chi\|_\alpha, \]
   \[ |v \chi|_\alpha \leq c_3(\alpha) h^{-1/2} \left\{ \|\chi\|_\alpha + \|v\chi\|_\alpha \right\}. \]

3. For any $\chi \in S^0_h$

   \[ |\chi|_\alpha \leq c_4(\alpha) h^{3/2} \left\{ \|\chi\|_\alpha + \|v\chi\|_\alpha \right\}. \]

The bounds $c_i(\alpha)$ depend only on $\alpha, \gamma, \gamma_1$ and a bound of
the curvature of $\partial \Omega$.

**Remark:** If $v \in H^1$ then the choice $\chi \in S^0_h$ is possible in
assertion (1). In addition $\chi$ may be chosen according to

\[ |\chi|_\alpha \leq c_5(\alpha) h^k \|v^k v\|_\alpha. \]

For more details see NATTERER [1], NITSCHE [1], [2].
3. Finite element approximations, $H_1$- and $L_2$- error estimates

The solution $u$ of the boundary value problem (1) will be approximated by an element $u_h \in S_h = S_h \times S_h$. Though the functions in $S_h$ are not exactly zero on $\partial \Omega$, the forms $a_0, a_1, a_2$ are positive definite in $S_h$. The finite element approximations $u_h^{(i)}$ are defined by

$$u_h^{(i)} \in S_h^o : a_i(u_h^{(i)}, \chi) = (f, \chi) \quad \text{for} \quad \chi \in S_h^o \quad (i=0,1)$$

(13)

$$u_h^{(2)} \in S_h^o : a_2(u_h^{(2)}, \chi) = (f, \chi) \quad \text{for} \quad \chi \in S_h.$$ 

For $K$ - see (5) - sufficiently large $a_2(x,x)^{1/2}$ is in $S_h$ a norm equivalent to

$$\|x\|_{W_2} + h^{-1/2} |x|,$$

therefore also $u_h^{(2)}$ is well-defined.

By standard arguments we get immediately for the errors $e_h^{(1)} = e_h^{(1)}(u - u_h)$ :

THEOREM 3: Assume $f \in L_2$ resp. $u \in H_2$. The errors in the energy norm are bounded by

$$\|e_h^{(i)}\|_{W_1^1} \leq c h \|f\| \quad (i = 0,1,2),$$

(14)

in the $L_2$-norm the bounds differ

$$\|e_h^{(0)}\| \leq c h^{3/2} \|f\|,$$

(15)

$$\|e_h^{(i)}\| \leq c h^2 \|f\|^i \quad (i = 1,2).$$
The approximation \( u_h^{(1)} \) seems to be of most interest. In this case we have in addition

\[
|e_h^{(1)}| \leq c h^2 \|e\|.
\]

4. Error-estimates in weighted norms

In this and the next section we restrict ourselves to the bilinear form \( a_1 \) and drop here as well as in \( u_h^{(1)} \) the index 1. We will need

**LEMMA 2:** Let \( v, w \in H_1 \cup \hat{S}_h \). Then for any \( \alpha \in \mathbb{R} \)

\[
|a(v, w)| \leq c \|v\|_\alpha \|w\|_{-\alpha}.
\]

**LEMMA 3:** Let \( v \in H_1 \) resp. \( v \in \hat{S}_h \). Then for any \( \alpha \in \mathbb{R} \)

\[
\|v\|^2_\alpha \leq c \{a(v, u - \alpha v) + \|v\|^2_{\alpha+1}\}.
\]

The proof of Lemma 2 is straight-forward. Korn's inequality applied to \( w = u - \alpha/\sqrt{v} \) and standard estimates give Lemma 3.

By definition of \( u_h = u_h^{(1)} \) we have for \( e = e_h^{(1)} \)

\[
a(e, \chi) = 0 \quad \text{for} \quad \chi \in \hat{S}_h.
\]

Now let \( U_h \) be an appropriate approximation on \( u \) according to Theorem 2 with error \( E = E_h = u - U_h \). Then we have \( e = E - \hat{u} \) with \( \hat{u} = U_h - u_h \in \hat{S}_h \) and

\[
a(\hat{u}, \chi) = a(E, \chi) \quad \text{for} \quad \chi \in \hat{S}_h.
\]

Using Lemma 3 we derive with \( \alpha \in \mathbb{R} \) and any \( \chi \in \hat{S}_h \)
\[ \|v_{\delta}\|^2_{\alpha} \leq c \left\{ a(\delta, \mu^{-\alpha} \delta - \chi) - a(E, \mu^{-\alpha} \delta - \chi) + a(E, \mu^{-\alpha} \delta) + \|v\|^2_{\alpha+1} \right\} \]

(19)

\[ \leq c \left\{ \|v_{\delta}\|_{\alpha} + \|v\|_{\alpha}\right\} \|v(\mu^{-\alpha} \delta - \chi)\|_{-\alpha} \]

\[ + c \|v\|_{\alpha} \|v(\mu^{-\alpha} \delta - \chi)\|_{-\alpha} + c \|v\|^2_{\alpha+1} \cdot \]

Application of \( 2|ab| \leq \delta a^2 + \delta^{-1}b^2 \) in a proper way gives

(20) \[ \|v_{\delta}\|^2_{\alpha} \leq c \left\{ \|v\|_{\alpha}^2 + \|v\|_{\alpha+1}^2 + \|v(\mu^{-\alpha} \delta - \chi)\|_{-\alpha}^2 \right\} . \]

Since \( \delta \) is piecewise linear we have by means of Theorem 2 with \( \chi \) properly chosen

\[ \|v(\mu^{-\alpha} \delta - \chi)\|_{-\alpha} \leq c h \|v^2(\mu^{-\alpha} \delta)\|_{h} \]

(21)

\[ \leq c h (\|v\|_{\alpha+2} + \|v\|_{\alpha+1}) \]

\[ \leq c h \rho^{-1}(\|v\|_{\alpha+1} + \|v\|_{\alpha}) . \]

Now we impose the condition \( \rho \geq \gamma_2 h \) with \( \gamma_2 \geq \gamma_1 \) and such that the constant in (20) is less than \( \gamma_2 \). Then we get

(22) \[ \|v_{\delta}\|_{\alpha} \leq c \left\{ \|v\|_{\alpha} + \|v\|_{\alpha+1} \right\} . \]

Now let \( \mathbf{w} \in \mathcal{H}_2 \) be the solution of

(23) \[ -v\sigma(\mathbf{w}) = \mu^{-\alpha-1} \mathbf{i} : -\mathbf{i} k(\mathbf{w}) = \mu^{-\alpha-1} \mathbf{q}_1 . \]
\[ \| \hat{\Delta} \|_{\alpha+1}^2 = a(\hat{\Delta}, \hat{\Delta}) \]
\[ = a(\hat{\Delta}, \hat{\Delta} - \hat{\Delta}) - a(E, \hat{\Delta} - \hat{\Delta}) + a(E, \hat{\Delta}) \]

The last term may be estimated by
\[ a(E, \hat{\Delta}) = (E, \mu^{-1} \frac{\alpha-1}{\alpha} \cdot \hat{\Delta}) \]
\[ \leq \|E\|_{\alpha+1} \|\hat{\Delta}\|_{\alpha+1} \]

The function \( \hat{x} \) is now chosen to be an approximation on \( \hat{w} \). Then
\[ \|\nabla (\hat{w} - \hat{x})\|_{-\alpha} \leq C \|\nabla^2 \hat{w}\|_{-\alpha} \]

After standard estimates and transformations we come to

(24) \[ \|\hat{\Delta}\|_{\alpha+1} \leq \delta \|\nabla \hat{\Delta}\|_{\alpha} + C \delta^{-1} \left( \|E\|_{\alpha+1} + \|\nabla^2 \hat{w}\|_{-\alpha} + \|\nabla^2 \hat{w}\|_{-\alpha} \right) \]

Here \( \delta > 0 \) is arbitrary.

If \( \delta \) is chosen such that with the constant in (22) \( \delta < 1 \) then the combination of (22), (24) gives

(25) \[ \|\hat{\Delta}\|_{\alpha+1} + \|\nabla \hat{\Delta}\|_{\alpha} \leq C \left( \|E\|_{\alpha+1} + \|\nabla^2 \hat{w}\|_{-\alpha} + \|\nabla^2 \hat{w}\|_{-\alpha} \right) \]

From now we specialize \( \alpha = 1 \). Applying the shift theorem to the functions \( x_{1,\hat{w}} \) and \( \omega_{\hat{w}} \) gives after some computations
LEMMA 4: Let \( w \) be the solution of (23) with \( \alpha = 1 \).

Then

\[
\| v^2 w \|_{-1}^2 \leq c \left\{ \rho^{-2} \| \tilde{x} \|_2^2 + \| v w \|_2^2 \right\}
\]

\[
\leq c \left\{ \rho^{-2} \| \tilde{x} \|_2^2 + a(w, w) \right\}.
\]

(26)

It remains to estimate the last term by \( \| \tilde{x} \|_2^2 \) respective by

\[
\| v \sigma(w) \|_{-2}^2 = \int \int \Sigma \Sigma |\sigma_{1k}, k(w)|^2 .
\]

If we define

(27) \( K = K_\rho = \sup \left\{ a(w, w) \mid \| v \sigma(w) \|_{-2} = 1 \right\} \),

then we have with (26)

(28) \( \| v^2 w \|_{-1}^2 \leq c(\rho^{-2} + K_\rho) \| \tilde{x} \|_2^2 \).

In the appendix we will sketch the proof of

LEMMA 5: Let \( K_\rho \) be defined by (27). Then

\( K_\rho \leq c \rho^{-2} |\ln \rho| \).

With the help of this estimate we get combining (28) with (25)

\[
\| \tilde{x} \|_2 + \| v \tilde{x} \|_1 \leq c \left\{ \| \tilde{x} \|_2 + \| v \tilde{x} \|_1 \right\}
\]

\[
+ c h^{-1} |\ln \rho|^{1/2} \| \tilde{x} \|_2 .
\]

If we take \( \rho \geq \gamma_3 h|\ln h| \) with \( \gamma_3 \) properly chosen the imposed conditions on \( \rho \) will hold and the coefficient of \( \| \tilde{x} \|_2 \) in the last inequality is smaller than 1. Remembering the meaning of \( \tilde{x} = u - U_h \) we get
LEMMA 6: If the parameter $\phi$ in the weight-factor $\mu$ is connected with $n$ by $\phi \approx \gamma_2 n|\ln n|^{1/2}$ then

\begin{equation}
\|\hat{\mathbf{e}}\|_2 + \|\hat{\mathbf{v}}\|_1 \leq c \inf_{\chi \in \mathcal{S}_h} \left\{ \|u-\chi\|_2 + \|\mathbf{v}(u-\chi)\|_1 \right\}.
\end{equation}

5. $L_\infty$-error-estimates

Let us now assume that the solution $u$ of the boundary value problem (1) has bounded second derivatives. Then

\begin{align*}
\inf_{\chi \in \mathcal{S}_h} \left\{ \|u-\chi\|_2 + \|\mathbf{v}(u-\chi)\|_1 \right\} \\
\leq c \left\{ n^{1/2} e^{-1} + h |\ln n|^{1/2} \right\} \|\nabla^2 u\|_{L_\infty} \\
\leq c h |\ln n|^{1/2} \|\nabla^2 u\|_{L_\infty}.
\end{align*}

The point $x_0$ in $\mu$ is now chosen to be in a $\Delta \in \Gamma_h$ with

\begin{align*}
\|\hat{v}\|_{L_\infty} = |\hat{v}(x_0)|.
\end{align*}

Then we have

\begin{align*}
\|v\|_1 &\geq \frac{n}{\delta} \|v\|_{L_\infty},
\end{align*}

and therefore from (29)

\begin{align*}
\|v\|_{L_\infty} &\leq c h |\ln n| \|\nabla^2 u\|_{L_\infty}.
\end{align*}

Because of $e = E - \hat{v}$ we have got

THEOREM 4: If $u \in W_2^\infty$ then

\begin{align*}
\|v(u-u_h)\|_{L_\infty} &\leq c h |\ln n| \|\nabla^2 u\|_{L_\infty}.
\end{align*}

In order to get an error estimate for $e$ in $L_\infty$ we consider a $x_0 \in \Gamma_h$ with
\[ \| \phi \|_{L_\infty} = \| \phi \|_{L_\infty}(\Lambda_0) \].

Since \( \phi \) is linear in \( \Lambda_0 \), we find with \( K_r \subset \Lambda_0 - r^2 \leq -1 \):

\[ (30) \quad \| \phi \|_{L_\infty} \leq c n^{-2} \iint_{K_r} \phi^2 \, dx. \]

Now let \( w \in H_2 \) be the solution - compare with (23) - of

\[ (31) \quad - \nabla \sigma(w) = \begin{cases} n^{-2} \phi & \text{in } K_r \\ 0 & \text{else} \end{cases} \]

By arguments similar to those on pp. 8,9 we come to

\[ n^{-2} \iint_{K_r} \phi^2 \, dx = a(\phi, w - x) - a(E, w - x) + a(E, w) \]
\[ \leq c n^{-2} \iint_{K_r} \phi^2 \, dx + \]
\[ + c n \left\{ \| \nabla \phi \|_1 + \| \nabla E \|_1 \right\} \| \nabla^2 w \|_1 \]

and using (29)

\[ (32) \quad n^{-2} \iint_{K_r} \phi^2 \, dx \leq c n^2 | \ln n |^{1/2} \| \nabla^2 w \|_{L_\infty} \| \nabla^2 w \|_1. \]

Using the counterparts of Lemmata 4 and 5 for the function \( w \) defined by (31) we get

\[ \| \nabla^2 w \|_1 \leq c n^2 \iint_{K_r} \phi^2 \, dx \]

and therefore we derive from (32)

\[ n^{-2} \iint_{K_r} \phi^2 \, dx \leq c n^4 | \ln n |^2 \| \nabla^2 w \|_{L_\infty}^2. \]
In connection with (30) we have

Theorem 5: If \( u \in \mathcal{W}_2^2 \) then
\[
\|u - u_h\|_{L_\infty} \leq c h^2 \ln h \|\sigma^2 u\|_{L_\infty}.
\]

6. Appendix: Proof of Lemma 5

There exists (at least) one solution \( w \in \mathcal{H}_2 \) with
\[
a(w,w) = K \|\sigma(w)\|_{-2}^2.
\]

For any \( v \in \mathcal{H}_2 \) the variational equations
\[
a(w,v) = K \iint \mu^2 (\sigma(w) \cdot (\sigma(v))) \, dx
\]
hold. Since
\[
a(w,v) = - \iint w (\sigma(v)) \, dx
\]
and \( \sigma(v) \in L_2 \) is arbitrary the function \( w \) satisfies
\[
(33) \quad -\sigma(w) = \lambda \mu^{-2} w
\]
with \( \lambda = K^{-1} \). In order to estimate \( K \) we need a lower bound of the eigenvalues of (33). Multiplication of (33) with \( w \) and integration gives
\[
K = \lambda^{-1} = \frac{\|w\|_{-2}^2}{a(w,w)}.
\]

Because of Korn's inequality we have
\[
K \leq c \sup \left\{ \|w\|_{-2}^2 \mid \|\sigma w\| \leq 1 \right\}
\]
and the right hand side is bounded up to a factor by
\[
(34) \quad \overline{K} = \sup \left\{ \|w\|_{-2}^2 \mid w \in \mathcal{W}_2^1 \land \|\sigma w\| \leq 1 \right\}.
\]
The extremal function \( w \) of (34) is the solution of

(35) \[-\Delta w = \lambda \mu^{-2} w\]

with \( \lambda = k^{-1} \) being the smallest eigenvalue. Because of the maximum principle \( w \) as well as \( -\Delta w \) are not negative. From this the monotony of \( K \) with respect to the domain follows: Let \( \Omega_1, \Omega_2 \) be two domains and \( K_1, K_2 \) be the corresponding values (34). If \( \Omega_1 \subset \Omega_2 \) then \( K_1 \leq K_2 \).

Now let \( \hat{\Omega} \) be the circle with center \( x_0 \) and radius \( \hat{a} = \text{diam} (\Omega) \). Then \( \Omega \subset \hat{\Omega} \) and it suffices to bound the corresponding value of \( K \). Since \( \mu \) depends only on \( |x-x_0| \) and \( w > 0 \) there is a solution of (34) depending also only on \( |x-x_0| \) (actually the smallest eigenvalue is simple). Therefore problem (35) can be handled as 1-dimensional. By direct computation then we get the bound for \( \hat{K} \) and hence for \( K \) given in Lemma 5.
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