On the Discrete Analogues of Some Generalizations of Gronwall's Inequality

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1. We are concerned here with some discrete generalizations of the following result of *Gronwall* [1], which has been very useful in the study of ordinary differential equations:

Lemma (Gronwall). *If* $u(t)$ *and* $v(t)$ *are non-negative measurable functions for* $t \geq 0$ *and* u_0 *is a non-negative constant, then the inequality*

$$u(t) \leq u_0 + \int_0^t v(s)\, u(s)\, ds \qquad (t \geq 0) \tag{1.1}$$

implies that

$$u(t) \leq u_0 \exp\left(\int_0^t v(s)\, ds\right) \qquad (t \geq 0). \tag{1.2}$$

Stronger forms of *Gronwall's* Lemma are produced by replacing (1.1) with more general inequalities, which usually fit the form

$$u(t) \leq u_0(t) + h\left(\int_0^t g(t, s, u(s))\, ds\right). \tag{1.3}$$

Bihari [2] determines an upper bound and *Langenhop* [3] determines a lower bound for $u(t)$ when $u_0(t)$ is constant, $h(x) = x$, and $g(t, s, u) = v(s)\, q(u)$, where $q(u)$ is a monotone increasing function. *Lakshmikantham* [4] extends the results in [2] and [3] by assuming $g(t, s, u) = q(s, u)$ and by giving conditions under which u is bounded between the maximal and minimal solutions of $r'(t) = \pm\, q(t, r)$, $r(0) = |u(0)|$, respectively. Further results in this direction with applications to ordinary differential equations are given by *Brauer* [5] and *Lakshmikantham* [6]. *Willett* [7] has considered the so-called L_p case $(1 \leq p < +\infty)$, i. e., the case when $h(x) = x^{1/p}$, $g(t, s, u) = w^p(t)\, u^p$, and all functions are p^{th} power Lebesgue integrable. (See Theorem 1 of this paper.)

The discrete analogue of the result of *Bihari* [2] was partially given by *Hull* and *Luxemburg* [8] and was used by them for the numerical treatment of ordinary differential equations. In this note we will state and prove discrete analogues to the L_p case considered by *Willett* [7] and to the interesting special case of *Lakshmikantham* [4] when $q(s, u) = v(s)\,u(s) + w(s)\,u^p(s)$ $(p \geq 0,\ p \neq 1)$.

2. In the following two theorems we assume that all functions are real-valued, non-negative, defined on a given interval I with zero as left endpoint, and Lebesgue measurable on I. Such a function $x = x(t)$ is said to be *locally integrable on* I if for each $t \in I$, the Lebesgue integral $\int_0^t x(s)\,ds$ is finite.

Theorem 1. *Let the functions* $v(t)\,u^p(t)$, $v(t)\,w^p(t)$, *and* $v(t)\,u_0^p(t)$ *be locally integrable non-negative functions on* I. *Then the following inequality for* $1 \leq p < \infty$

$$u(t) \leq u_0(t) + w(t) \left(\int_0^t v(s)\,u^p(s)\,ds \right)^{1/p} \qquad (t \in I) \tag{2.1}$$

implies that

$$\left(\int_0^t v(s)\,u^p(s)\,ds \right)^{1/p} \leq \frac{\left(\int_0^t v(s)\,u_0^p(s)\,e(s)\,ds \right)^{1/p}}{1 - (1 - e(t))^{1/p}} \qquad (t \in I), \tag{2.2}$$

where

$$e(t) = \exp\left(- \int_0^t v(s)\,w^p(s)\,ds\right). \tag{2.3}$$

Theorem 1 with $v(t) = 1$ is proven as Lemma 2.2 in [7]. The case for general $v(t)$ follows easily from this case by multiplying inequality (2.1) by $v^{1/p}(t)$ and identifying $v^{1/p}(t)\,u(t)$ with $u(t)$. A bound on $u(t)$, which is independent of $u(t)$, can be obtained now by substituting for $\left(\int_0^t v(s)\,u^p(s)\,ds \right)^{1/p}$ in equation (2.1).

Theorem 2. *Let the functions* $v(t)$, $w(t)$, $v(t)\,u(t)$, *and* $w(t)\,u^p(t)$ *be locally integrable non-negative functions on* I. *If* $u_0 > 0$ *and* $p \geq 0$, $p \neq 1$, *then the following inequality*

$$u(t) \leq u_0 + \int_0^t v(s)\,u(s)\,ds + \int_0^t w(s)\,u^p(s)\,ds \qquad (t \in I) \tag{2.4}$$

implies that

$$u(t) \exp(-\int_0^t v(s)\, ds) \leq (u_0^q + q \int_0^t w(s) \exp(-q \int_0^s v(r)\, dr)\, ds)^{1/q}$$
$$(q = 1 - p;\ t \in I). \tag{2.5}$$

To prove Theorem 2, let $\psi(t)$ be defined as the right member of (2.4); so

$$\psi'(t) \leq v(t)\, \psi(t) + w(t)\, \psi^p(t) \qquad (t \in I), \tag{2.6}$$

since $p \geq 0$. By Theorem 1 of [4] we know that $\psi(t)$ is bounded by the maximal solution $r(t)$ of

$$r'(t) = v(t)\, r(t) + w(t)\, r^p(t),\ r(0) = u_0; \tag{2.7}$$

and we can solve (2.7) explicitly as a Bernoulli equation. However, we need not refer to the result in [4] at all for this special case, but can obtain directly from (2.6) that

$$\theta'(t) \leq w(t)\, \theta^p(t) \exp(-q \int_0^t v(s)\, ds) \qquad (q = 1 - p), \tag{2.8}$$

where

$$\theta(t) = \psi(t) \exp(-\int_0^t v(s)\, ds). \tag{2.9}$$

Since $\theta(t) > 0$ on I, we can divide (2.8) by $\theta^p(t)$ and integrate to obtain equation (2.5) for all $p \geq 0$, $p \neq 1$ $(q = 1 - p)$.

If $u_0 = 0$, then equation (2.4) is valid for all positive constants u_* in place of u_0. By letting $u_* \to 0$ in the corresponding equation (2.5), we get that equation (2.5) as it now stands is also valid when $u_0 = 0$, if we agree to first write the right hand side with a factor u_0 when $q < 0$.

3. We will prove in this section the discrete versions of Theorems 1 and 2.

Theorem 3. *Suppose that $u_0(n)$, $w(n)$, $v(n-1)$, and $u(n)$ $(n = 1, 2, \ldots)$ are non-negative sequences of numbers with $v(0) = 0$. Then the following inequality for $1 \leq p < \infty$*

$$u(n+1) \leq u_0(n+1) + w(n+1)\, (\sum_{j=0}^{n} v(j)\, u^p(j))^{1/p} \quad (n = 0, 1, \ldots) \tag{3.1}$$

implies that

$$(\sum_{j=0}^{n} v(j)\, u^p(j))^{1/p} \leq \frac{(\sum_{j=0}^{n} v(j)\, u_0^p(j)\, e(j))^{1/p}}{1 - (1 - e(n))^{1/p}} \quad (n = 0, 1, \ldots), \tag{3.2}$$

where

$$e(n) = \prod_{i=0}^{n} (1 + v(i)\, w^p(i))^{-1} \qquad (n = 0, 1, \ldots). \tag{3.3}$$

Proof: The proof is the direct analogue of the proof of Theorem 1 as given in [7]. Define a sequence of numbers

$$\psi(k) = e(k) \sum_{j=0}^{k} v(j)\, u^p(j) \qquad (k = 0, 1, \ldots, n), \tag{3.4}$$

where

$$e(k) - e(k-1) = -\, v(k)\, w^p(k)\, e(k) \qquad (k = 1, 2, \ldots, n), \tag{3.5}$$
$$e(0) = 1.$$

The solution $e(k)$ of (3.5) is given by (3.3). It follows from (3.1), (3.4), and (3.5) that

$$\psi(k) - \psi(k-1) \leq \left(v^{1/p}(k)\, u_0(k)\, e^{1/p}(k) + \frac{v^{1/p}(k)\, w(k)\, \psi^{1/p}(k-1)}{(1 + v(k)\, w^p(k))^{1/p}} \right)^p$$
$$- \frac{v(k)\, w^p(k)\, \psi(k-1)}{1 + v(k)\, w^p(k)} \qquad (k = 1, 2, \ldots, n). \tag{3.6}$$

Next, sum (3.6) from $k = 1$ to $k = n$, transpose the second sum in the right member, form the p^{th} root of both sides, and apply *Minkowski*'s inequality for sums to the right member to obtain

$$\left(\psi(n) + \sum_{k=1}^{n} \frac{v(k)\, w^p(k)\, \psi(k-1)}{1 + v(k)\, w^p(k)} \right)^{1/p} \leq$$
$$(\sum_{k=1}^{n} v(k)\, u_0^p(k)\, e(k))^{1/p} + \left(\sum_{k=1}^{n} \frac{v(k)\, w^p(k)\, \psi(k-1)}{1 + v(k)\, w^p(k)} \right)^{1/p}. \tag{3.7}$$

Transpose the second term of the right member of (3.7) to obtain a left member of the form $f(x) = (c + x)^{1/p} - x^{1/p}$ $(c \geq 0,\ p \geq 1)$. Since $f'(x) \leq 0$ for all $x \geq 0$, we may replace x by a larger quantity without destroying inequality (3.7). In this regard, we note that

$$\sum_{k=1}^{n} \frac{v(k)\, w^p(k)\, \psi(k-1)}{1 + v(k)\, w^p(k)} = \sum_{k=1}^{n} \frac{v(k)\, w^p(k)\, e(k-1)}{1 + v(k)\, w^p(k)} \sum_{j=0}^{k-1} v(j)\, u^p(j) \leq$$
$$\leq \sum_{k=1}^{n} v(k)\, w^p(k)\, e(k) \sum_{k=0}^{n} v(j) u^p(j) = (1 - e(n)) \sum_{j=0}^{n} v(j)\, u^p(j). \tag{3.8}$$

(3.2) follows by substituting from (3.8) and (3.4) into (3.7).

Theorem 4. *Suppose that $v(n)$, $w(n)$, and $u(n+1)$ $(n = 0, 1, 2, \ldots)$ are non-negative sequences of numbers with $v(0) = w(0) = 0$, and that u_0 and p are constants with $u_0 > 0$ and $p \geq 0$, $p \neq 1$. Then the inequality*

$$u(n+1) \leq u_0 + \sum_{j=0}^{n} v(j)\, u(j) + \sum_{j=0}^{n} w(j)\, u^p(j) \qquad (n = 0, 1, \ldots) \qquad (3.9)$$

implies that

$$e(n)\, u(n+1) \leq \Big(u_0^q + q \sum_{k=0}^{n} w(k)\, e^q(k)\Big)^{1/q} \quad (q = 1-p; n = 0, 1, \ldots), \quad (3.10)$$

where

$$e(n) = \prod_{j=0}^{n} (1 + v(j))^{-1} \qquad (n = 0, 1, \ldots). \qquad (3.11)$$

Proof: Define a sequence of numbers $\psi(k)$ by the right member of (3.9), i. e.,

$$\psi(k) = u_0 + \sum_{j=0}^{k} v(j)\, u(j) + \sum_{j=0}^{k} w(j)\, u^p(j) \qquad (k = 0, 1, \ldots). \qquad (3.12)$$

Then,

$$\psi(k+1) - \psi(k) \leq v(k+1)\, \psi(k) + w(k+1)\, \psi^p(k), \qquad (3.13)$$

since $u^p(k+1) \leq \psi^p(k)$ for $p \geq 0$.

Transpose $v(k+1)\, \psi(k)$ in (3.13) and multiply by $e(k+1)$, where

$$\begin{aligned} e(k+1) - e(k) &= -\, v(k+1)\, e(k+1), \\ e(0) &= 1, \end{aligned} \qquad (3.14)$$

to produce

$$\psi(k+1)\, e(k+1) - \psi(k)\, e(k) \leq w(k+1)\, e^q(k+1)\, [\psi(k)\, e(k+1)]^p. \quad (3.15)$$

Because $\psi(k)$ is monotone increasing, $e(k)$ is monotone decreasing, and $q - 1 \leq 0$, we know that

$$[\psi(k)\, e(k+1)]^{q-1} \geq x^{q-1}$$

for all values x between $\psi(k)\, e(k)$ and $\psi(k+1)\, e(k+1)$. So if we apply the mean value theorem to the function $f(x) = x^q/q$, we see that

$$\frac{[\psi(k+1)\, e(k+1)]^q - [\psi(k)\, e(k)]^q}{q} \leq$$
$$[\psi(k)\, e(k+1)]^{q-1}\, [\psi(k+1)\, e(k+1) - \psi(k)\, e(k)]. \qquad (3.16)$$

From (3.15), (3.16), and $q = 1 - p$, we obtain

$$[\psi(k+1)\, e(k+1)]^q - [\psi(k)\, e(k)]^q \leq w(k+1)\, e^q(k+1),$$

from which (3.10) follows for all values $q \leq 1$, $q \neq 0$ by summing.

4. Most of the results above have been used in their weak forms in several aspects of differential equations, as we have already indicated to some

extent. The corresponding strengthening of these results may be achieved in most cases through the stronger forms of *Gronwall's* inequality. To be more specific, the results in [7] and [9] may be slightly improved with the use of Theorems 1 and 2, respectively, and Theorem 4 may be used to extend the discussions given in [10]. Other applications of these results in the study of slightly perturbed differential equation systems will be considered in a later paper.

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